Assignment #2

Due: Friday, February 25, 2005

1. **Classical Coherence of Light**

Consider a classical light field. The classical expressions for first-order and second-order coherence are

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}$$

$$g^{(2)}(\tau) = \frac{\langle \bar{I}(t)\bar{I}(t+\tau)\rangle}{\langle \bar{I}(t)\rangle^2} = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle^2}$$

where the $\langle\rangle$ denotes a statistical averaging over many measurements.

a) Prove that for zero time-delay, the second order coherence obeys the inequality $g^{(2)}(0) \geq 1$. [Hint: Start with Cauchy's inequality which states that, for two measurements of the intensity at times t_1 and t_2,

$$2\langle \bar{I}(t_1)\bar{I}(t_2)\rangle \leq \bar{I}(t_1)^2 + \bar{I}(t_2)^2.$$

Using this inequality, show that for N measurements,

$$\left[\frac{\bar{I}(t_1) + \bar{I}(t_2) + \cdots + \bar{I}(t_N)}{N}\right]^2 \leq \frac{\bar{I}(t_1)^2 + \bar{I}(t_2)^2 + \cdots + \bar{I}(t_N)^2}{N}$$

This implies that light in a number state, with $g^{(2)}(0) < 1$, has no classical analog.

b) Using a similar argument, show that:

$$g^{(2)}(\tau) \leq g^{(2)}(0)$$

This implies that anti-bunched light, with $g^{(2)}(\tau) \geq g^{(2)}(0)$, has no classical analog.

c) Consider chaotic classical light generated by an ensemble of ν atoms. The total electric field can be expressed as $E(t) = \sum_{i=1}^{\nu} E_i(t)$, where the phases of the E_i are random. Show that when ν is large,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 \qquad (1)$$

2. **Quantum Coherence of Light**

Consider light in a single mode of the radiation field. The quantum mechanical expressions for first-order and second-order coherence are

$$g^{(1)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) = \frac{\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^+(\mathbf{r}_2 t_2)\rangle}{[\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^+(\mathbf{r}_1 t_1)\rangle\langle \hat{E}^-(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_2 t_2)\rangle]^{1/2}}$$

$$g^{(2)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2; \mathbf{r}_2 t_2, \mathbf{r}_1 t_1) = \frac{\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^-(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_1 t_1)\rangle}{\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^+(\mathbf{r}_1 t_1)\rangle\langle \hat{E}^-(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_2 t_2)\rangle}$$

where

$$\hat{E}^+(\mathbf{r}t) = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \epsilon a e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

$$\hat{E}^-(\mathbf{r}t) = -i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \epsilon a^\dagger e^{+i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

Using these expressions, show that the second order coherence may be written as

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2}$$

a) Using the expression you just derived, show that, for light in a number state $|n\rangle$ where $n > 2$, $|g^{(1)}| = 1$ and $g^{(2)} = 1 - 1/n$, independent of space-time separation. What is $g^{(1)}$ and $g^{(2)}$ for $n = 0$ and $n = 1$?

b) Show that, for light in a coherent state $|\alpha\rangle$, $|g^{(1)}| = 1$ and $|g^{(2)}| = 1$.

c) Show that, for chaotic light with density matrix

$$\hat{\rho} = (1 - e^{-\hbar\omega/k_B T})\sum_n e^{-n\hbar\omega/k_B T}|n\rangle\langle n|$$

$|g^{(1)}| = 1$ and $g^{(2)} = 2$.

Note that the values of the first and second-order coherence functions in part (c) satisfy the classical relation you derived in problem 1 for chaotic light (equation 1). It can be proved that equation 1 holds quantum mechanically for multi-mode chaotic light.

3. **The Squeezed State**

The squeezing operator is defined as

$$S(\epsilon) = \exp\left[\frac{1}{2}\epsilon^* a^2 - \frac{1}{2}\epsilon a^{\dagger 2}\right]$$

where $\epsilon = re^{i\phi}$. Let $|\alpha\rangle$ be a coherent state and $S(\epsilon)|0\rangle \equiv |0_\epsilon\rangle$ be the squeezed vacuum state.

a) Find the expected number of photons in the squeezed vacuum state as a function of r and ϕ. (In other words, evaluate $\langle 0_\epsilon|a^\dagger a|0_\epsilon\rangle$.

b) Find the expected number of photons in a coherently displaced squeezed state $|\alpha, 0_\epsilon\rangle \equiv D(\alpha)|0_\epsilon\rangle$ as a function of α, r and ϕ.

[Hint: Evaluate $S^\dagger(\epsilon)aS(\epsilon)$ and $S^\dagger(\epsilon)a^\dagger S(\epsilon)$, then use them to evaluate $\langle \alpha, 0_\epsilon|a^\dagger a|\alpha, 0_\epsilon\rangle$.]

c) Consider the two-mode squeezed state

$$|\Psi^s_{EPR}\rangle_{23} \propto \sum_n s^n |n\rangle_2 |n\rangle_3$$

Show that it may be rewritten in the coherent-state representation as

$$|\Psi^g_{EPR}\rangle_{23} \propto \int_C d^2\alpha\; e^{-|\alpha|^2/g^2} |\alpha\rangle_2 |\alpha^*\rangle_3$$

where the integral is taken over the complex plane. Neglect normalization factors for this problem. What is g in terms of s?

[Note: Using such two-mode squeezed states, we may perform teleportation of arbitrary pure states. Moreover, this particular $|\Psi^s_{EPR}\rangle_{23}$ is easily generated via ideal down-conversion.]

d) The Q-function is defined as $Q_\rho(\alpha) \equiv \langle\alpha|\rho|\alpha\rangle$. Compute and plot

(i) $Q_1(\alpha) = |\langle\alpha|0_\epsilon\rangle|^2$

(ii) $Q_2(\alpha) = |\langle\alpha|D(\beta)S(\epsilon)|0\rangle|^2$

(iii) $Q_3(\alpha) = |\langle\alpha|S(\epsilon)D(\beta)|0\rangle|^2$

Are $Q_2(\alpha)$ and $Q_3(\alpha)$ different? Why?

[Hint: If you get into very ugly math for parts (i) - (iii), you can just compute $Q(\alpha)$ numerically and plot. The important point is to be able to visualize these states.]

[Extensive hints: If you really want to compute a closed form for the Q function, you may use the following relations (and take r to be real for simplicity) :

$$\begin{aligned} S(r)|0\rangle &= \frac{1}{\pi}\frac{e^{r/2}}{\sqrt{e^{2r}-1}} \int_{-\infty}^{\infty} d\alpha\; e^{-[\alpha^2/(e^{2r}-1)]}|\alpha\rangle \\ D(\gamma)S(z) &= S(z)D(\gamma_+) \\ S(z)D(\gamma) &= D(\gamma_-)S(z) \end{aligned}$$

where

$$\begin{aligned} \gamma_\pm(z) &= (\cosh r)\gamma \pm (e^{i\theta}\sinh r)\gamma^* \\ z &= re^{i\theta} \end{aligned}$$

Some more interesting states...

(iv) Compute $Q_4(\alpha)$ for $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ and plot for various values of ϕ and θ. Is this a minimum uncertainty state? Is it a squeezed state?

(v) For $|\psi\rangle = |0\rangle + \frac{1}{\sqrt{2!}}|2\rangle + \frac{1}{\sqrt{4!}}|4\rangle + \frac{1}{\sqrt{6!}}|6\rangle + \cdots$, compute and plot $Q_5(\alpha)$. What is this state?