# Problem Set 3

**Due Friday, March 4th 2005**

1. **Measures of pure state entanglement**

Entanglement is a property of a composite quantum system that cannot be changed by local operations and classical communications. How do we mathematically determine if a given state is entangled or not? And if a state is entangled, how entangled is it? For a bipartite system (composed of two sub-systems with independent Hilbert spaces), the Schmidt number provides one measure of entanglement. You should review the proof of the Schmidt decomposition on page 109 of Nielsen and Chuang (2005).

> By virtue of the Schmidt decomposition, a pure state $|\psi\rangle$ in the Hilbert space of systems $A$ and $B$ can be written as
>
> $$|\psi\rangle = \sum_k \lambda_k |k_A\rangle |k_B\rangle \,, \tag{1}$$
>
> where $|k_A\rangle$ and $|k_B\rangle$ are orthonormal states of systems $A$ and $B$, respectively, and $\sum_k \lambda_k^2 = 1$. The *Schmidt number* $\mathrm{Sch}(|\psi\rangle)$ is the number of nonzero $\lambda_k$ and $\mathrm{Sch}(|\psi\rangle) = \mathrm{Rank}[\mathrm{Tr}_B(|\psi\rangle\langle\psi|)]$.

(a) Prove that $|\psi\rangle$ is a product state, that is $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$, if and only if $\mathrm{Sch}(|\psi\rangle) = 1$. This corresponds to no entanglement.

(b) Prove that the Schmidt number cannot be changed by local unitary transforms (transforms to one qbit of a pair) and classical communication. (The Schmidt number is strictly nonincreasing under more general conditions, for *arbitrary* local operations, but you don't need to prove that here).

(c) Give the Schmidt numbers for each of the following states:

$$|\phi_1\rangle = \frac{|00\rangle + |11\rangle + |22\rangle + |33\rangle}{2} \qquad |\phi_2\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{2}$$

$$|\phi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \qquad |\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} \,. \tag{3}$$

(d) Prove that if a bipartite state $|\psi\rangle$ can be expressed as any state of the form $|\psi\rangle = \sum_k |\phi_k\rangle|k_B\rangle$, where $|k_B\rangle$ are orthonormal states of $B$ and $|\phi_k\rangle$ are arbitrary (possibly un-normalized) states of $A$, then the number of terms in the sum is at least as great as the Schmidt number of $|\psi\rangle$. Recall that $\mathrm{Rank}[\mathbf{A}+\mathbf{B}] \leq \mathrm{Rank}[\mathbf{A}] + \mathrm{Rank}[\mathbf{B}]$.

2. **Measures of mixed state entanglement**

The entanglement of mixed states is much more difficult to characterize than for pure states, and many simple questions concerning mixed state entanglement remain unsolved. However, some simple facts are known, one of which is explored in this problem.

A density matrix $\rho$ of a composite system $AB$ is separable (i.e. represents an unentangled state) if and only if it can be separated into a sum of direct products,

$$\rho = \sum_k p_k [\rho_k^A \otimes \rho_k^B] \,, \tag{4}$$

where $\rho_k^A$ and $\rho_k^B$ are states of $A$ and $B$, respectively, and $p_k$ are nonnegative weights satisfying $\sum_k p_k = 1$.

(a) Recall that $\rho$ is a valid density matrix if and only if $\text{tr}(\rho) = 1$ and $\rho$ is positive, meaning that $\langle\phi|\rho|\phi\rangle \geq 0$ for any state $|\phi\rangle$ (i.e. its eigenvalues are non-negative). Prove that if $\rho$ is a density matrix, then $\rho^T$ (the transpose of $\rho$) is also a density matrix.

(b) In general, $\rho$ can be written as the matrix

$$\rho = \sum_{m,n,\mu,\nu} \rho_{m,\mu,n,\nu} [|m_A\rangle\langle n_A| \otimes |\mu_B\rangle\langle\nu_B|] \,, \tag{5}$$

where the pure states are orthonormal basis vectors of the Hilbert spaces of $A$ and $B$. Define $\sigma$ as the *partial transpose* state of $\rho$,

$$\sigma = \sum_{m,n,\mu,\nu} \rho_{m,\mu,n,\nu} [|n_A\rangle\langle m_A| \otimes |\mu_B\rangle\langle\nu_B|] \,, \tag{6}$$

such that if $\rho$ were separable, we would have

$$\sigma = \sum_k p_k [(\rho_k^A)^T \otimes \rho_k^B] \,. \tag{7}$$

Prove that a *necessary* condition for separability is that $\sigma$ has non-negative eigenvalues. It turns out that positivity of the eigenvalues of the partial transpose of $\rho$ is necessary *and* sufficient for separability, but proof of this fact is beyond the scope of this problem set.

(c) Why is it important (and interesting!) to study entanglement of mixed states? Consider a two-qubit system governed by the Hamiltonian $H = \hbar\omega(S_Z \otimes I + I \otimes S_Z)$, where $S_Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $I = |0\rangle\langle 0| + |1\rangle\langle 1|$. The state of this system in thermal equilibrium is the Boltzmann distribution,

$$\rho = e^{-H/k_B T}/\mathcal{Z} \,, \tag{8}$$

where $\mathcal{Z}$ is a normalization factor such that $\text{tr}(\rho) = 1$. Suppose we apply the unitary transform

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \tag{9}$$

to this state, and obtain $\rho' = U\rho U^\dagger$. Compute the minimum eigenvalue $\lambda_{\min}$ of the partial transpose of $\rho'$ as a function of $\alpha \equiv \hbar\omega/k_B T$, and plot. Show that there is a distinct transition around $\alpha \approx 1/2$.

(Optional) Discuss: is this some kind of *phase* transition? What would happen with an $n$-qubit system with Hamiltonian that is symmetric with respect to the $n$ spins?

3. **Entanglement and communication complexity**

Alice is in Amsterdam and Bob is in Boston, and they share an EPR pair in the state $|Q_A Q_B\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Alice chooses some uniformly random bit $x$ (0 or 1 with equal probability) and independently, Bob chooses $y$. Define the rotation operator

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{10}$$

If $x = 1$ Alice applies $R(\pi/4)$ to her qubit $Q_A$; otherwise she does nothing. Bob applies $R(-3\pi/8)$; also, if $y = 1$, Bob applies $R(-\pi/4)$ to his qubit $Q_B$. Both Alice and Bob then measure their qubits in the computational basis (along the z axis), obtaining bits $a$ and $b$, respectively.

(a) Show that prob$[a \oplus b = \overline{x \wedge y}] > 0.853$, where $\oplus$ denotes addition modulo two (1+1=0),$\wedge$ is the logical AND operation and the bar indicates negation (x NAND y).

(b) Now suppose that Alice has a two bit number $x = x_1 x_0$ and Bob has $y = y_1 y_0$, and let $z = z_2 z_1 z_0 = x + y$ be their sum. Alice and Bob desire to obtain the middle bit of the sum, $z_1$, with high probability. Give a protocol using one EPR pair and only two bits of classical communication between Alice and Bob which allows them to obtain $z_2$ with probability better than 0.853.

(c) Show that classically, the best probability achievable with two bits of communication (and no EPR pairs) is 0.75.

4. **Generation of Squeezed States by Two-Photon Interactions**

Consider a mode $\vec{k}\vec{\varepsilon}$ of the electromagnetic field with frequency $\omega$ whose Hamiltonian $H$ is given by

$$H = \hbar\omega a^\dagger a + i\hbar\Lambda\left((a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right) \tag{11}$$

where $a^\dagger$ and $a$ are the creation and annihilation operators of the mode.

The first term of (11) is the energy of the mode for the free field. The second describes a two-photon interaction process such as parametric amplification (a classical wave of frequency $2\omega$ generating two photons with frequency $\omega$). $\Lambda$ is a real quantity characterizing the strength of the interaction.

(a) Write, using the Heisenberg point of view, the equation of motion for $a(t)$. Take

$$a(t) = b(t)e^{-i\omega t}. \tag{12}$$

What are the equations of motion for $b(t)$ and $b^\dagger(t)$ ?

(b) Using the Heisenberg picture, the contribution of the mode $\vec{k}\vec{\varepsilon}$ to the electric field is written

$$\vec{E}(\vec{r}, t) = i\mathcal{E}_\omega \vec{\varepsilon}\left(a(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger(t)e^{-i\vec{k}\cdot\vec{r}}\right) \tag{13}$$

where $a(t)$ is the solution of Equation 12. Show that

$$b_P(t) = \frac{b(t) + b^\dagger(t)}{2} \qquad \text{and} \qquad b_Q(t) = \frac{b(t) - b^\dagger(t)}{2i} \tag{14}$$

(where b(t) is defined in (12)) represent physically two quadrature components of the field. Find the equations of motion of $b_P(t)$ and $b_Q(t)$ and give their solutions, assuming that $b_P(0)$ and $b_Q(0)$ are known.

(c) Assume that at $t = 0$, the electromagnetic field is in the vacuum state. Calculate at time $t$ the mean number of photons, $\langle N \rangle$, in the mode $\vec{k}\vec{\varepsilon}$ as well as the dispersion $\Delta b_P(t)$ and $\Delta b_Q(t)$ on the two quadrature components of the field. Explain the results.