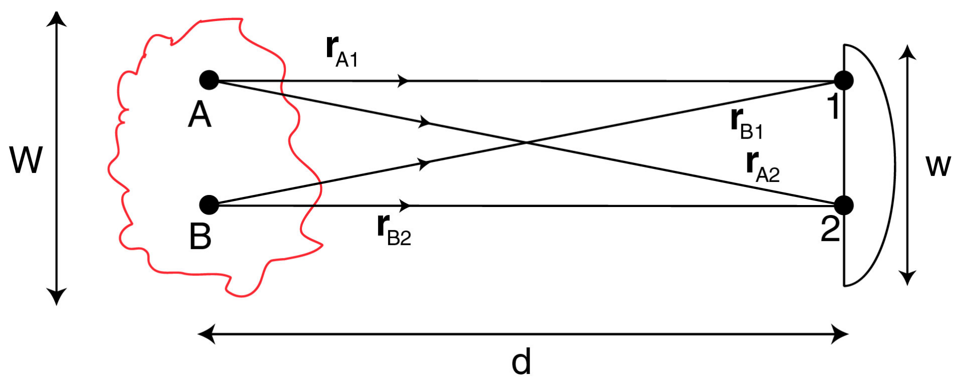
**Homework Assignment #4**

Physics 8.422, Spring 2005, Prof. W. Ketterle

Due Friday, March 11.

1. **Hanbury Brown and Twiss Experiment with Atoms**
This problem illustrates the coherence and collimation requirements for performing a HBT experiment with atoms. (Note: Ignore gravity in this problem.)
If a free particle starts at point $A$ at time $t = 0$ with an amplitude (wavefunction) $\psi_A$, then the amplitude at another point 1 and time $t = \tau$ is proportional to $\psi_A e^{i(\mathbf{k}\cdot\mathbf{r}_{A1} - \omega\tau)}$, where $\mathbf{r}_{A1}$ is the vector from $A$ to 1, $\mathbf{k}$ is the particle's wavevector, and $\hbar\omega$ is its total energy. This can be regarded as Huygen's principle for matter waves, and is a special case of the Feynman path integral formulation of quantum mechanics.




(Based on figure 19-5, in G. Baym, *Lectures on Quantum Mechanics*)

   (a) <u>Correlation function</u> Assume we have a particle at $A$ with amplitude $\psi_A$ and one at $B$ with amplitude $\psi_B$. The joint probability, $P$, of finding one particle at 1 and one at 2 is

$$P = \left|\psi_A e^{i\phi_{A1}} \psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}} \psi_B e^{i\phi_{B1}}\right|^2$$

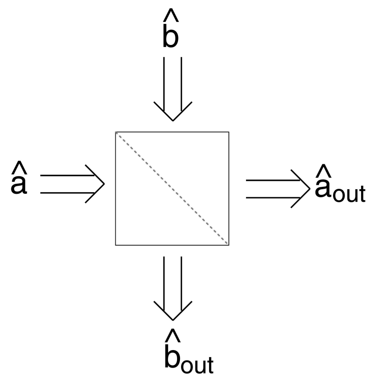
   and is proportional to the second-order coherence function $g^{(2)}(1,2)$. The $\pm$ is for bosons/fermions and makes the two-particle wavefunction symmetric/antisymmetric under the exchange of particles. Here, $\phi_{A1} = \mathbf{k}_A \cdot \mathbf{r}_{A1} - \omega\tau$ is the phase factor for the path from point $A$ to detector 1, etc.
   Calculate $P$ as a function of $\mathbf{r}_{21}$, the vector from point 2 to point 1 on the detector.

   (b) <u>Transverse Collimation</u>
   Assume you are given a source (e.g. a ball of trapped atoms) with transverse dimension $W$ and detector with transverse dimension $w$ where $|\mathbf{r}_{21}| \leq w$. The distance between source and detector, $d$, is much greater than all other distances.
   The transverse component of the phase factor in part (a) can be written: $\phi_t = (\mathbf{k}_A - \mathbf{k}_B)_t (\mathbf{r}_{21})_t$. Assume that the signal at the detector is mainly due to atoms with wavevectors distributed around $\mathbf{k}_0$. Argue that the transverse collimation required to see second order correlation effects can be expressed as $Ww \ll d\lambda_{dB}$, where $\lambda_{dB}$ is the deBroglie wavelength corresponding to $\mathbf{k}_0$. (Hint: How does $\phi_t$ vary for atoms originating at different points in the source and being detected at different points on the detector?) Assuming a source and detector of approximately equal size ($W \approx w$), make an order of magnitude estimate of $W$ and $w$ using $d = 10$cm and a $\lambda_{dB} = 100nm$ (appropriate for trapped atoms).

(c) Longitudinal Collimation
The longitudinal component of the phase factor in part (a) can be written: $\phi_l = (\mathbf{k}_A - \mathbf{k}_B)_l (\mathbf{r}_{21})_l$. Assume a Gaussian distribution of wavevector differences $p(\mathbf{k}_A - \mathbf{k}_B) = e^{-|\mathbf{k}_A - \mathbf{k}_B|^2\gamma^2}$ where the width, $\gamma$, is related to the temperature of the atoms. Calculate $\langle P \rangle$ using this distribution and your result from part (a). Sketch $\langle P \rangle$ for both fermions and bosons, indicating the extent of $(\mathbf{r}_{21})_l$ over which the second order correlation effect can be seen.
Now assume you have a pulsed source of atoms with longitudinal dimension $L$ (the detector is assumed to have zero longitudinal extent). Atoms are released at time $t = 0$ and detected at some later time $t = \tau$. Give geometric arguments to show that the wavevectors of detected atoms must obey $(\mathbf{k}_A - \mathbf{k}_B)_l \leq \frac{mvL}{\hbar d}$, where the velocity $v = \frac{d}{\tau}$. This implies that the different velocity groups separate during the expansion, narrowing (by a factor $\frac{L}{d}$) the velocity distribution of atoms detected at any particular time.

(d) Phase-Space Volume Enhancement
We now pull all the pieces together. The peak in $g^{(2)}(1,2)$ is visible for $(\mathbf{k}_A - \mathbf{k}_B) \cdot \mathbf{r}_{21} \leq 2\pi$. This is equivalent to saying that we must detect atoms from within a single phase space cell, defined by $\delta p_x \delta x \leq h$ (and likewise for $y$ and $z$). In our trapped atom sample, the 3D volume of a phase space cell is $\delta x \delta y \delta z = (\lambda_{dB})^3$. Liouville's theorem says that as our ball of atoms expands, the number of phase space cells remains constant. Verify that, by using this pulsed source, the volume of a coherent phase space cell is increased by a factor $d^3/W^2L$ by the time atoms reach the detector. What is the order of magnitude of this increase (assuming $L \approx W$)?

2. **Beam splitter**




In last weeks homework you used and measured entangled states. This problem will demonstrate one of the ways experimentalists can create and manipulate such states using simple optics such as a beam splitter. An ideal beam splitter transforms two input field modes, $\hat{a}$ and $\hat{b}$, according to

$$\hat{a}_{out} = \hat{a}\cos\theta + i\hat{b}\sin\theta \tag{1}$$
$$\hat{b}_{out} = \hat{b}\cos\theta + i\hat{a}\sin\theta \tag{2}$$

where $\theta$ expresses the reflectivity of the beam splitter which transmits with probability $T = \cos^2\theta$ and reflects with probability $R = \sin^2\theta$.

(a) Show that the action of the beam splitter can be described by a unitary transformation $U = \exp\left(i\theta\left(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}\right)\right)$. In other words, show that $\hat{a}_{out} = U^{-1}\hat{a}U$ and $\hat{b}_{out} = U^{-1}\hat{b}U$.

(b) Using simple optics, describe how you would physical demonstrate the unitary nature of beam splitter. What effect should $\theta$ have in your optical setup in this case?

(c) Calculate how the Bell states ( $|\Psi_A\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$, $|\Psi_B\rangle = (|10\rangle - |01\rangle)/\sqrt{2}$, $|\Psi_C\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $|\Psi_D\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$ ) are transformed by passing through a beamsplitter. (Hint: Calculate the effect on the component Fock states first)

(d) What are the Schmidt numbers (a measure of entanglement used in PS 3, Problem 1) for

i. The Bell states
ii. The Fock states which make up the Bell states
iii. The Bell states after passing through beam splitters with T=50% or T=25%.
iv. The Fock states after passing through beamsplitters with T=50% or T=25%.

Which of these processes create or increase the amount of entanglement in the system?

(e) Assume that the input field modes $\hat{a}$ and $\hat{b}$ are in coherent states such that the total state of the field is $|\Psi_{in}\rangle = |\alpha\rangle\,|\beta\rangle$. Give an expression for the quantum state of the field $|\Psi_{out}\rangle$ after the beam splitter in terms of $\hat{a}_{out}$ and $\hat{b}_{out}$. Do you recognize this state and if so what kind of state is it?

(f) Your research adviser has recently purchased a single photon (Fock state) source. During shipping the label fell off and both of you are worried that the manufacturer may have instead shipped you a highly attenuated laser. A friendly visiting scientist suggests that you should be able to determine if you have a Fock state source or a coherent source using a beamsplitter and two photon counters. Do the calculations to show how this would work.

(g) The success of your analysis has emboldened your research adviser to ask how you might be able to distinguish any two single photon wavefunctions using the same apparatus.
See Nature **410**, 1067 (2001) for nontrivial applications of such simple manipulations.

Extra point: Consider the more realistic situation in which two multi-mode 1-photon wavepackets interfere on a beamsplitter. Suppose that the two photons are in pure Gaussian wavepackets of duration T, delayed with respect to each other by a time $\tau$. What is the coincidence probability of detecting at least one photon at both outputs of the beamsplitter?
*References:* Science, **290** 2282 (2000), Nature **419** 594 (2002).