Problem Set 5 – Van der Waals Interaction and Casimir Effect

Due: March 18, 2005

Problem 1. Long-range (Van der Waals) interaction between ground-state atoms

The electrostatic interaction between atoms a and b is described to first order by the dipole-dipole term:

$$H_{el}(R) = \frac{\vec{d_a} \cdot \vec{d_b} - 3(\vec{d_a} \cdot \hat{R})(\vec{d_b} \cdot \hat{R})}{R^3} \tag{1}$$

where
$\vec{d_a} = e\vec{r_a}$ is the electric dipole operator of atom a
$\vec{d_b} = e\vec{r_b}$ is the electric dipole operator of atom b
$\vec{R} = \vec{R_{nb}} - \vec{R_{na}}$ is a position vector pointing from the nuclei of a to the nuclei of b.

We will use time-*independent* perturbation theory to calculate the effect of H_{el}. This is simpler than the time-*dependent* perturbation expansion discussed in class and does not lead to a virtual-photon picture of the Van der Waals interaction.
Notation:
Let $|g_a g_b\rangle$ denote atom a and atom b in the ground state.
Let $|i_a g_b\rangle$ denote atom a in an excited state i and atom b in the ground state.
$\vdots$

(a) What is the first non-vanishing term in the series for the perturbed ground state energy of the system?

(b) Dipole matrix elements in atomic physics are often discussed in terms of "oscillator strength," $f_{ig} = \frac{2m\omega_{ig}}{\hbar}|\langle i|x|g\rangle|^2$.
Note: $\omega_{ig} = \frac{E_i - E_g}{\hbar}$, so f_{ig} is positive for absorption and negative for emission. Also, $\sum_i f_{ig} = 1$, the Thomas-Reiche-Kuhn sum rule.

Express your result from (a) in terms of oscillator strengths. You will have to make some arguments (non-mathematical if you prefer) about the symmetry of photon emission to get rid of annoying cross terms.

(c) We can estimate C_6 using the approximation that the oscillator strength f_{ig} is large for only one transition, $|g\rangle \to |i\rangle$. The $|nS\rangle \to |(n+1)P\rangle$ transitions in alkalai atoms are the classic examples, with $f \approx 0.98$. Use this in combination with the sum rule above and the definition of the static polarizability of the ground state:

$$\alpha_g = 2e^2 \sum_i \frac{|\langle i|z|g\rangle|^2}{E_i - E_g} \tag{2}$$

and express your result for C_6 from (b) in terms of polarizabilities $\alpha_g^{(a)}$ and $\alpha_g^{(b)}$. (You should not have any summation signs in your final answer.)

Problem 2. Long-range interaction between an excited atom and a ground-state atom

Consider the case where one atom is excited and the other atom is in its ground state. For simplicity model each atom as a two level system with one ground state and one excited state.

(a) Assume you have two atoms a and b with almost (but not quite) degenerate ground $\leftrightarrow$ excited state transition energies $(E_i^{(a)} - E_g^{(a)}) \approx (E_i^{(b)} - E_g^{(b)})$. How does the energy of the state $|i_a g_b\rangle$ change as a function of the separation R for large distances? What about state $|g_a i_b\rangle$? For what separation does perturbation theory become invalid?

(b) Now assume you have two identical (i.e. same transition energy) atoms. Calculate the long-range interaction potential curves for the case of one excited atom and one ground state atom.

(c) For case (b) what is the relation between the spontaneous decay rate of the atom and its long-range interaction coefficient?

Problem 3. Casimir model of the electron

Model the electron as two parallel plates of area a^2, separated by distance a and carrying charge $q = \frac{e}{2}$. Balance the Casimir and electrostatic forces and from this determine a value for the fine-structure constant $\alpha \equiv \frac{e^2}{\hbar c}$ (cgs units).