Atomic Physics II (8.422) Spring 2005
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Problem Set 6

Due Friday, April 1 - no fooling

1. Optical Bloch Equations

The time independent form of the optical Bloch equations [ref: Cohen-Tannoudji, *Atom-Photon Interactions*, p. 359], including spontaneous emission and the rotating wave approximations, are written:

$$\frac{d\hat{\sigma}_{bb}}{dt} = i\frac{\Omega_1}{2}\left(\hat{\sigma}_{ba} - \hat{\sigma}_{ab}\right) - \Gamma\hat{\sigma}_{bb}$$

$$\frac{d\hat{\sigma}_{ab}}{dt} = i\left(\omega_0 - \omega_L\right)\hat{\sigma}_{ab} - i\frac{\Omega_1}{2}\left(\hat{\sigma}_{bb} - \hat{\sigma}_{aa}\right) - \frac{\Gamma}{2}\hat{\sigma}_{ab}$$

$$\hat{\sigma}_{aa} + \hat{\sigma}_{bb} = 1$$

$$\hat{\sigma}_{ab} = \hat{\sigma}_{ba}^*$$

a) Weak limit: Show that the solution of these equations to lowest order in $|\Omega_1|$ in the limit $|\Omega_1| \ll \Gamma$, with the initial conditions $\hat{\sigma}_{bb} = 0$ and $\hat{\sigma}_{ab} = 0$, gives

$$\hat{\sigma}_{bb} = \frac{\frac{1}{4}\left|\Omega_1\right|^2}{\left(\omega_0 - \omega_L\right)^2 + \left(\frac{\Gamma}{2}\right)^2}\left\{1 + e^{-\Gamma t} - 2\cos\left[\left(\omega_0 - \omega_L\right)t\right]e^{-\frac{\Gamma t}{2}}\right\}.$$

What does this solution reduce to in the limit of an infinitely narrow linewidth ($\Gamma \to 0$)?

b) Short time limit: Show that the solution of these equations to lowest order in $|\Omega_1|$ in the limit $|\Omega_1|\, t \ll 1$, with the initial conditions $\hat{\sigma}_{bb} = 0$ and $\hat{\sigma}_{ab} = 0$, gives

$$\hat{\sigma}_{bb} = \frac{1}{4}\left|\Omega_1\right|^2 t^2$$

irrespective of the values of $\left(\omega_0 - \omega_L\right)$ and Γ.

2. Van Der Waals Scattering and the Refractive Index for Matter Waves
This problem addresses several concepts discussed in class including the van der Waals potential, elastic cross section, and partial waves . The elastic cross section leads to attenuation of the atomic beam and is expressed by the imaginary part of an index of refraction of matter waves. This problem nicely demonstrates analogies between ordinary optics and matter wave optics.
When a wave passes through a medium, two things happen. The wave is attenuated and its phase is shifted. Both of these effects are accounted for by introducing a complex index of refraction n. This is true whether it is a light wave passing through glass or a matter wave passing through a dilute gas. The attenuation (related to the imaginary part of n) of particle beams passing through gas samples has been studied extensively. The main difficulty is to accurately know, in absolute terms, the density of the gas sample. Recently, the phase shift (related to $\mathrm{Re}(n)$) of a matter wave passing through a gas sample has been measured. The measurement requires an interferometer so that the wave coming out of the sample can be compared (interfered) with a wave that did not pass through the sample. This is a probe of the interatomic potentials governing the scattering process that, on the microscopic scale, determine the macroscopic quantity n. In this problem we will estimate the quantities $\mathrm{Re}(n)$ and $\mathrm{Im}(n)$ for a system that turns out to be described quite well by the Van der Waals potential: the scattering of sodium (Na) on Xenon (Xe). [ref: Schmiedmayer *et al.*, Phys. Rev. Lett., (74), 1043 (1995); J.J. Sakurai, *Modern Quantum Mechanics*, Chapter 7]

The wavefunction for a wave can be written $\Psi(x) = \Psi(0)e^{ik_{lab}x}$ where k_{lab} is the wavevector in the lab frame. Suppose that at $x = 0$, the wave enters a medium with complex index of refraction

$$n = 1 + \frac{\Delta\phi(x)}{k_{lab}x}$$

where

$$\Delta\phi(x) = \frac{2\pi}{k}Nxf(k, \theta = 0).$$

N is the density of the medium, k is the wavevector in the center-of-mass frame, and $f(k, \theta = 0)$ is the (complex) amplitude of the forward scattered wave (hence $\theta = 0$).

a) Show that the wave function is

$$\Psi(x) = \Psi(0)e^{ik_{lab}x}e^{i\frac{2\pi}{k}NxRe[f(k,0)]}e^{-\frac{2\pi}{k}NxIm[f(k,0)]}.$$

Verify, using $\Delta\phi(x)$ as chosen above, the optical theorem for the total scattering cross-section. (Look at the intensity of the wave.)

b) The forward scattering amplitude is written as a sum over angular momentum as

$$f(k, 0) = \frac{1}{k}\sum_{l=0}^{\infty}(2l + 1)e^{i\delta_l(k)} \sin \delta_l(k).$$

$\delta_l(k)$ is the phase shift for a partial wave with angular momentum l and center-of-mass wavevector k. This experiment used a beam of atoms at thermal energies so several hundred partial waves contribute to the sum. It is therefore valid to replace the sum over l with an integral over the classical impact parameter b.
Use the correspondence relation $\hbar(l+\frac{1}{2}) = b \cdot \hbar k$ to express $f(k,0)$ in integral form.

c) To do the integral, we first need to calculate the phase shift $\delta(b,k)$. This is easier when the collision energy is much greater than the potential, that is, when the potential varies only a small amount over a wavelength. Then we can use the Eikonal approximation, in which δ is calculated as the phase accumulated by the sodium atom along a straight-line path with impact parameter b through the potential. Thus

$$\delta(b,k) = -\frac{m_{Na}}{2k\hbar^2}\int_{-\infty}^{+\infty} V(r)dz.$$

Calculate $\delta(b,k)$ for the case $V(r) = -C_6 r^{-6}$.
(It is analytic. Use $\int_0^\infty \frac{dx}{(a^2+x^2)^n} = \frac{\pi}{2}\frac{(2n-3)!!}{(2n-2)!!}\frac{1}{a^{2n-1}}, \quad [a>0, n=2,3,\cdots]$.)

d) Separate the integral for $f(k,0)$ into real and imaginary parts and use the result from c) to calculate them.

(The integrals are analytic. You will find it convenient to make a change of variables and integrate over δ instead of b. In each case, paying close attention to [ ], first use
$\int_0^\infty \frac{\sin^p(x)dx}{x^m} = \frac{p}{m-1}\int_0^\infty \frac{\sin^{p-1}(x)\cos(x)dx}{x^{m-1}}, \quad [p > m-1 > 0]$, then
$\int_0^\infty \frac{\sin(ax)dx}{x^m} = \frac{1}{a^{1-m}}\Gamma(1-m)\cos(\frac{m\pi}{2}), \quad [a>0, 0<m<1]$ or
$\int_0^\infty \frac{\cos(ax)dx}{x^m} = \frac{1}{a^{1-m}}\Gamma(1-m)\sin(\frac{m\pi}{2}), \quad [a>0, 0<m<1]$ as needed.)

e) Now plug in the numbers. The value of C_6 is needed to evaluate Re(f) and Im(f). In Homework Assignment #5 you derived $C_6 = \frac{3}{2}\hbar\frac{\bar{\omega}_1\bar{\omega}_2}{\bar{\omega}_1+\bar{\omega}_2}\alpha_1\alpha_2$ using the dominant level approximation. By using $\alpha = \frac{ne^2}{m_e\bar{\omega}^2}$ to substitute for $\bar{\omega}_1$ and $\bar{\omega}_2$, where n is the number of valence electrons (also equal to the sum over oscillator strengths), derive an expression for C_6 in terms of the polarizabilities (the Slater-Kirkwood formula). Use $\alpha_{Na} = 24.1\ \mathring{A}^3$ and $\alpha_{Xe} = 4.1\ \mathring{A}^3$ and $n = 6$ for xenon to calculate C_6 for the Na-Xe system.

f) Using a velocity of 1000 m/s for the sodium atom, calculate k_{lab} using the deBroglie wave relation. The center-of-mass wavevector for this system is $k = 0.85\ k_{lab}$. Evaluate Re(f) and Im(f). Then, for a 1 mTorr xenon gas, evaluate Re($n-1$) and Im($n-1$).

g) Note the order of magnitude of $n-1$. Why is this measurable in an atom interferometer, that is, why would it be much harder to measure an equivalent magnitude of $n-1$ in an optical interferometer?