Homework Assignment #7

Physics 8.422, Spring 2005, Prof. W. Ketterle

Due Friday, April 8.

1. **Classical Model of the Light Force**
   Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency ω_0 and damping coefficient γ. The nucleus is fixed at position $\mathbf{x}_0$ while the electron's position is denoted by $\mathbf{x}$. Now suppose the atom is illuminated with an electromagnetic wave of the form

$$\mathbf{E}(\mathbf{x}, t) = \hat{\epsilon} E_0(\mathbf{x}) \cos(\theta(\mathbf{x}) + \omega t) \tag{1}$$

   where $\theta(\mathbf{x})$ is the phase of the wave as a function of position $\mathbf{x}$ at time $t = 0$. The dipole moment of the electron may be written as

$$\mathbf{p}(\mathbf{x}, t) = \hat{p}(u \cos(\theta(x) + \omega t) - v \sin(\theta(x) + \omega t) \tag{2}$$

   Then the force of the light on the atom is

$$\mathbf{F} = (\mathbf{p} \cdot \hat{\epsilon}) \nabla E(\mathbf{x}, t) \tag{3}$$

   (a) **Time averaged force**
   Make the dipole approximation that $\mathbf{E}(\mathbf{x}) \approx \mathbf{E}(\mathbf{x}_0)$. Show that the time averaged force is

$$\langle \mathbf{F} \rangle = \frac{1}{2}(\hat{p} \cdot \hat{\epsilon})(u \nabla E_0(\mathbf{x}_0) + v E_0(\mathbf{x}_0) \nabla \theta(\mathbf{x}_0)) \tag{4}$$

   This expression is exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.

   (b) **The potential picture**
   Recalculate the time averaged force on the atom by first calculating the instantaneous force from the potential. How does this answer different from that of 1a? Speculate as to why.

   (c) **Dipole moment of electron**
   Now we will solve explicitly for the dipole moment of the electron. In complex notation, the equation of motion is

$$m\frac{\partial^2 \mathbf{r}}{\partial t^2} + \gamma \frac{\partial \mathbf{r}}{\partial t} + m\omega_0^2 \mathbf{r} = -e\hat{\epsilon} E_0(\mathbf{x}_0) e^{i(\theta(\mathbf{x}_0) + \omega t)} \tag{5}$$

   where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. Solve this equation to find $\mathbf{p} = -e\mathbf{r}$. Substitute the quadrature components of $\mathbf{p}$ into the force equation from part (a) to find that

$$\mathbf{F} = -\frac{e^2}{2m\omega_0} \frac{\delta \nabla E_0^2 + \Gamma E_0^2 \nabla \theta}{4\delta^2 + \Gamma^2} \tag{6}$$

   where $\delta = \omega - \omega_0$ and $\Gamma = \gamma/m$. Make the approximation that $\omega \approx \omega_0$.

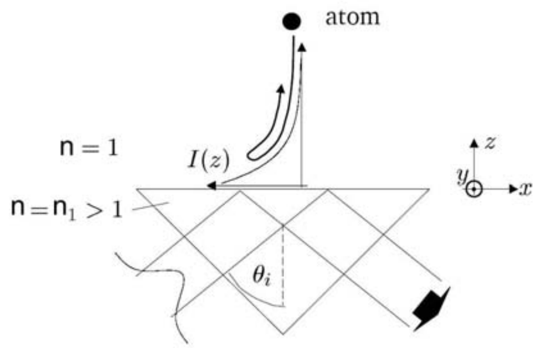
   (d) **Force on a two-level atom**
   The quantum mechanical expression for the light force on a two-level atom in the low intensity limit is

$$\mathbf{F} \approx -\frac{\hbar\delta \nabla \omega_R^2 + \hbar\Gamma \omega_R^2 \nabla \theta}{4\delta^2 + \Gamma^2} \tag{7}$$

where ω_R is the Rabi frequency. Show that if we introduce the oscillator strength f_{fi} between the two levels, we may write

$$\mathbf{F}_{\text{quantum}} = f_{fi}\mathbf{F}_{\text{classical}} \tag{8}$$

2. **An Atomic Trampoline**



It is possible to construct an effective wall for atoms using dipole forces induced by light. A realization of such a wall is the short range repulsive potential formed by an evanescent wave. This wave is produced by total internal reflection of light at a dielectric interface. The interface is mounted horizontally and the atoms are dropped onto it.
Assume the atoms you are dropping are ideal 2-level systems. The evanescent wave can be described in units of the Rabi frequency as:

$$\Omega(x) = \Omega_0 e^{-z/l},$$

where Ω_0 is the maximum Rabi frequency at the interface, z is the distance from the interface, and l is the characteristic extension of the evanescent wave (typically $\approx \lambda/2\pi$).
NOTE: Do not consider gravity in 2a through 2e.

(a) For a repulsive interaction, what is the sign of the detuning δ?

(b) Calculate the classical turning point near the interface, z_{tp}, for a bouncing atom in terms of the detuning δ, the line width Γ, the peak Rabi frequency Ω_0, and l, if its incident velocity is v. (i.e. v is the velocity with which the atom would hit the interface without the evanescent wave.)

(c) What is the minimum peak Rabi frequency Ω_0 for which the atom would bounce? Express the result in terms of δ, l, and Γ as a function of incident velocity v.

(d) Assume that the atom moves perpendicularly to the mirror on a classical trajectory. Find an exact expression for the number N of spontaneously scattered photons per bounce of the form:

$$N = \int g(z)dz,$$

where g is a function of the distance z. (You can assume that $N \ll 1$, so that the spontaneous scattering of photons does not perturb the trajectory.)

(e) Let's assume that $\delta \gg \Gamma$ and $\delta \gg \Omega_0$. In this case, the result of 2d can be simplified, yielding an integral that is exactly solvable. Obtain an expression for N in terms of δ, Γ, v, and l.
Useful relations:

$$ln(1+x) \approx x \text{ for } |x| \ll 1$$
$$\int_0^1 \frac{dx}{\sqrt{1-x}} = 2$$

(f) Assume that confinement in the plane parallel to the mirror is somehow provided. Estimate the vertical extension of the ground "bounce" state by using the position-momentum uncertainty relation. Calculate this length for sodium and compare it to $\lambda_{Na}/2\pi$=94 nm. **Hint:** You can treat the problem as an atom moving in a 1-dimensional potential which is linear on one side (the gravitational potential) and has an infinitely steep wall on the other side (the atom-wall potential).

(g) At what 1-dimensional temperature would you expect a significant population in this ground state in the case of "bouncing" sodium atoms? Use this temperature to quickly estimate a vertical trap frequency.

Figure courtesy of D. Schneble, SUNY Stony Brook.
Attachment PRL **71** p. 3083 (1993).