Sensitivity of polynomial composition and decomposition for signal processing applications

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Sefa Demirtas, Guolong Su, and Alan V. Oppenheim
Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge MA, 02139
Email: sefa@mit.edu

***Abstract*—Polynomial composition is well studied in mathematics but has only been exploited indirectly and informally in signal processing. Potential future application of polynomial composition for filter implementation and data representation is dependent on its robustness both in forming higher degree polynomials from ones of lower degree and in exactly or approximately decomposing a polynomial into a composed form. This paper addresses robustness in this context, developing sensitivity bounds for both polynomial composition and decomposition and illustrates the sensitivity through simulations. It also demonstrates that sensitivity can be reduced by exploiting composition with first order polynomials and commutative polynomials.**

I. Introduction

Functional composition of two functions $F(x)$ and $G(x)$, denoted by $F(G(x))$ or $F \circ G(x)$, corresponds to replacing the independent variable of $F(\cdot)$ by $G(\cdot)$. Conversely, functional decomposition is the process of obtaining two or more functions which, when composed, yield the original function. There are a number of examples in which functional composition has been exploited in signal processing applications. For example a method for computing the DFT of a signal on a nonuniform frequency grid using the FFT was introduced in [1], where functional composition was used to warp the frequency axis. Following the main idea in [1], an audio equalizer design technique was proposed in [2].

As another example, linear phase FIR filter design in one or two dimensions represents the frequency response as a trigonometric polynomial which is in effect functional composition [3], [4].

An example of polynomial composition as a special case of functional composition is *filter sharpening* [5] where multiple instances of a given filter with Fourier transform $G(e^{j\omega})$ are used in different configurations in order to improve the passband and stopband characteristics of that given filter. Since, in general, squaring can be represented as composition with ω^2, the Fourier transform of a filter cascaded wih itself corresponds to $\omega^2 \circ G(e^{j\omega})$. More sophisticated addition and multiplication operations on $G(e^{j\omega})$ are described in [5] that correspond to the composition of high order polynomials of ω with $G(e^{j\omega})$ to obtain better performance in both the passband and the stopband. Polynomial composition and decomposition deserve particular attention since polynomials are ubiquitous in the form of the z-transform representation of discrete-time FIR filters and signals. Utilizing polynomial decomposition, a discrete time signal $h[n]$ can be represented by fewer parameters than its nonzero coefficients if its z-transform $H(z)$ is decomposable as $F \circ G(z)$ since in general the degree of $H(z)$ is larger than the sum of the degrees of $F(z)$ and $G(z)$.

The main focus of this paper is the sensitivity of the polynomial composition and the decomposition operations. This is useful in understanding the types of signal processing applications in which these operations can be used and the extent to which they remain reliable. For example, such an analysis can suggest when a decomposable signal can be faithfully represented in terms of its components in the presence of quantization noise. Similarly, this analysis can quantify the performance of the sharpened filter described in [5] in the presence of error in the multiplier coefficients.

In Section II, polynomial composition and decomposition are introduced. A discussion of sensitivity is given in Section III followed by several simulations in Section IV. Equivalent compositions with lower sensitivity are discussed in Section V.

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II. Polynomial Composition and Decomposition

Consider $F(x)$, the polynomial that represents a length-M sequence f_n,

$$F(x) = \sum_{n=0}^{M} f_n x^n. \tag{1}$$

Composing $F(x)$ with another polynomial that represents a length-N sequence g_n, we obtain

$$H(x) = F(G(x)) = \sum_{n=0}^{M} f_n G^n(x). \tag{2}$$

Hence h_n, the sequence represented by $H(x)$ becomes

$$h_n = f_0(g^{(0)}) + f_1(g^{(1)}) + f_2(g^{(2)}) + f_3(g^{(3)}) + \dots \tag{3}$$

where $g^{(i)}$ corresponds to i self-convolutions of the sequence g_n. Equivalently

$$\mathbf{h} = \mathbf{C}\mathbf{f} \tag{4}$$

where the kth column of matrix $\mathbf{C}$ consists of $g^{(k-1)}$; and $\mathbf{f}$ and $\mathbf{h}$ are the coefficient vectors of $F(x)$ and $G(x)$ in the ascending order, respectively. It is relatively straightforward to obtain the coefficients of the composition polynomial $H(x)$ given the coefficients of its components $F(x)$ and $G(x)$. The inverse problem is, however, more difficult.

Given a polynomial $H(x)$ that is known to be decomposable as $F \circ G(x)$ with $deg(F) = M$ and $deg(G) = N$, several methods have been proposed in the literature to obtain $F(x)$ and $G(x)$ [6], [7], [8]. Decomposition methods in [6], [7] do not require knowledge of the degrees of the composing polynomials. However this information is usually not critical since $deg(F)$ and $deg(G)$ are restricted to be factors of $deg(H)$. On the other hand, the algorithm given in [8] employs a more systematic implementation than the methods presented in [6] and [7].

III. Sensitivity

In this section, the sensitivity for polynomial composition and decomposition are formally defined. Explicit expressions as well as upper and lower bounds for certain sensitivity measures are obtained.

A. Composition Sensitivity

The coefficients of a decomposable polynomial $H(x)$ that is given as in equation (2) are linearly dependent on the coefficients of $F(x)$ and nonlinearly dependent on the coefficients of $G(x)$. The sensitivity of composition for a given decomposable polynomial $H(x)$ can be defined as the maximum magnification of a small perturbation Δu in its composing polynomials, i.e.

$$S_{U\to H} = \max_{\Delta \mathbf{u}} \frac{E_{\Delta \mathbf{h}}/E_{\mathbf{h}}}{E_{\Delta \mathbf{u}}/E_{\mathbf{u}}} \tag{5}$$

where U is either F or G depending on which is being perturbed, $E_{\mathbf{h}} = ||\mathbf{h}||_2^2$ is the energy of the coefficient vector $\mathbf{h}$, $E_{\mathbf{u}} = ||\mathbf{u}||_2^2$ is the energy of the coefficient vector $\mathbf{u}$ and $||\cdot||_2^2$ is the square of the two norm of a vector. The relative increase in perturbation depends on the direction of the perturbation vector $\Delta\mathbf{u}$ when its magnitude is arbitrarily small; and sensitivity is defined at the direction of maximum magnification.

1) Formulation of $S_{F\to H}$: Due to the linear relationship given in equation (4), a perturbation $\Delta\mathbf{f}$ in the coefficient vector of $F(x)$ will result in a change in the coefficients of $H(x)$ given by

$$\Delta\mathbf{h} = \mathbf{C}\Delta\mathbf{f}. \tag{6}$$

The sensitivity of composition with respect to $F(x)$ becomes, by equation (4), (5) and (6)

$$S_{F\to H} = \max_{\Delta \mathbf{f}} \frac{||\mathbf{C}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2}\frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2}. \tag{7}$$

For a given decomposition of a polynomial $H(x)$ as $F\circ G(x)$, the factor $\frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2}$ is constant. The maximum value of $\frac{||\mathbf{C}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2}$ is equal to $\sigma_{\mathbf{C},max}^2$, where $\sigma_{\mathbf{C},max}$ is the maximum singular value of $\mathbf{C}$. Therefore equation (7) becomes

$$S_{F\to H} = \sigma_{\mathbf{C},max}^2 \frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2}. \tag{8}$$

Furthermore, $\frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2}$ is bounded above by $\sigma_{\mathbf{C},min}^{-2}$ and bounded below by $\sigma_{\mathbf{C},max}^{-2}$ for any $\mathbf{f}$. Hence, regardless of $F(x)$, the sensitivity $S_{F\to H}$ satisfies

$$1 \leqslant S_{F\to H} \leqslant \frac{\sigma_{\mathbf{C},max}^2}{\sigma_{\mathbf{C},min}^2} \tag{9}$$

where $\frac{\sigma_{\mathbf{C},max}^2}{\sigma_{\mathbf{C},min}^2}$ is the square of the condition number of $\mathbf{C}$.

2) Formulation of $S_{G\to H}$: Perturbing a single coefficient g_k in $G(x)$ by Δg_k does not affect the coefficient of x^n in $H(x)$ for $k > n$. Such a perturbation results in the composition

$$\tilde{H}(x) = F(G(x) + \Delta g_k x^k) = H(x) + \Delta H(x) \tag{10}$$

where

$$\Delta H(x) \approx F'(G(x))\Delta g_k x^k \tag{11}$$

assuming Δg_k is small and only the first term in the Taylor series for equation (10) is considered. For $k \leqslant n$, equation (11) implies

$$\Delta h_n = \Delta g_k d_{n-k} \tag{12}$$

where d_{n-k} is the coefficient of x^{n-k} in the polynomial $D(x)$ defined as

$$D(x) = F'(G(x)). \tag{13}$$

Perturbation of all the coefficients $g_k,\ k = 0, 1, \ldots, N$ results in the addition of error terms in equation (12), i.e.

$$\Delta h_n = \sum_{k\leqslant n} \Delta g_k d_{n-k}. \tag{14}$$

Equivalently,

$$\Delta\mathbf{h} = \mathbf{D}\Delta\mathbf{g} \tag{15}$$

where $\mathbf{D}$ is an $(MN+1)\times(N+1)$ Toeplitz matrix the first column of which consists of the coefficients of the polynomial $D(x)$, namely $[d_0\ d_1\ d_2 \ldots d_{MN+1-N}]^T$, with zero padding of length N. The sensitivity of composition with respect to $G(x)$ becomes, by equations (5) and (15),

$$S_{G\to H} = \max_{\Delta \mathbf{g}} \frac{||\mathbf{D}\Delta\mathbf{g}||_2^2}{||\Delta\mathbf{g}||_2^2}\frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2}. \tag{16}$$

As in the previous section, for a given decomposition of $H(x)$ as $F\circ G(x)$, $\frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2}$ is constant. The maximum value of $\frac{||\mathbf{D}\Delta\mathbf{g}||_2^2}{||\Delta\mathbf{g}||_2^2}$ is $\sigma_{\mathbf{D},max}^2$, where $\sigma_{\mathbf{D},max}$ is the maximum singular value of $\mathbf{D}$. Therefore equation (16) becomes

$$S_{G\to H} = \sigma_{\mathbf{D},max}^2 \frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2}. \tag{17}$$

An upper bound for $S_{G\to H}$ can be obtained by an alternative representation of the coefficient vector of $D(x)$ given in equation (13) in the form of equation (4), i.e.

$$\mathbf{d} = \mathbf{C}\tilde{\mathbf{f}} = \mathbf{C}\mathbf{V}\mathbf{f} \tag{18}$$

where $\tilde{\mathbf{f}}$ is the coefficient vector of $F'(x)$ and $\mathbf{V}$ is the $(M+1)\times(M+1)$ matrix with superdiagonal elements $1, 2, \ldots M$ and zeros elsewhere, corresponding to the derivative operator. For vectors $\mathbf{d}$, $\Delta\mathbf{h}$ and $\Delta\mathbf{g}$, which are related through equation (14), a general result given in the appendix for the convolution of two sequences implies

$$\frac{E_{\Delta\mathbf{h}}}{E_{\Delta\mathbf{g}}} \leqslant (N+1)E_{\mathbf{d}}. \tag{19}$$

Therefore, from the definition in equation (5), $S_{G\to H}$ can be bounded as

$$S_{G\to H} \leqslant (N+1)E_{\mathbf{g}}\frac{E_{\mathbf{d}}}{E_{\mathbf{h}}} = (N+1)||\mathbf{g}||_2^2 \frac{||\mathbf{C}\mathbf{V}\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2} \tag{20}$$

Defining

$$\mathbf{w} = \mathbf{C}\mathbf{f}, \tag{21}$$

it can be shown $\mathbf{f} = \left(\mathbf{C}^{\mathbf{T}}\mathbf{C}\right)^{-1}\mathbf{C}^{\mathbf{T}}\mathbf{w}$ since $\mathbf{C}$ is full rank. Therefore equation (20) becomes

$$\begin{aligned} S_{G\to H} &\leqslant (N+1)||\mathbf{g}||_2^2 \frac{||\mathbf{C}\mathbf{V}\left(\mathbf{C}^{\mathbf{T}}\mathbf{C}\right)^{-1}\mathbf{C}^{\mathbf{T}}\mathbf{w}||_2^2}{||\mathbf{w}||_2^2} \\ &\leqslant (N+1)||\mathbf{g}||_2^2\,\sigma_{\mathbf{T},max}^2 \end{aligned} \tag{22}$$

where the matrix $\mathbf{T} = \mathbf{C}\mathbf{V}\left(\mathbf{C}^{\mathbf{T}}\mathbf{C}\right)^{-1}\mathbf{C}^{\mathbf{T}}$ and $\sigma_{\mathbf{T},max}$ is the maximum singular value of $\mathbf{T}$.

B. Decomposition Sensitivity

A small perturbation $\Delta\mathbf{h}$ on the coefficients of a decomposable polynomial $H(x) = F\circ G(x)$ will render it nondecomposable in general. In this case, defining the sensitivity of decomposition is not meaningful. In other cases, $H(x)$ may remain decomposable but the new components $\hat{F}(x)$ and $\hat{G}(x)$ may have different degrees than $F(x)$ and $G(x)$, respectively. These cases are excluded from a discussion regarding their sensitivity here as well since the decomposition process may be regarded as having failed by not predicting the orders of the components correctly. Consequently, the definition for sensitivity of the decomposition will be restricted to cases in which the perturbation preserves decomposability with components of the same order.

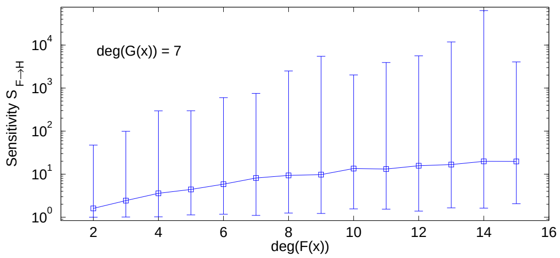

Fig. 1. Sensitivity of the coefficients of $H(x)$ with respect to the coefficients of $F(x)$. The order of $G(x)$ is seven in all compositions. Each point is the median of $S_{F\to H}$ obtained from ten thousand compositions, where the vertical bars indicate the range from the maximum to the minimum values attained.

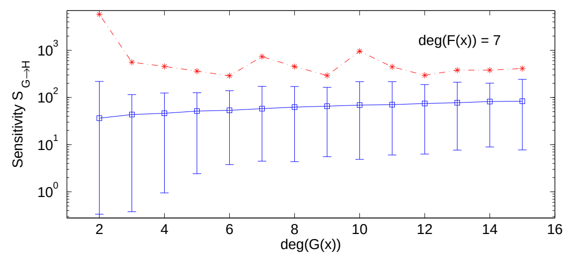

Fig. 2. Sensitivity of the coefficients of $H(x)$ with respect to the coefficients of $G(x)$. The order of $F(x)$ is seven in all compositions. Each point is the median of $S_{G\to H}$ obtained from ten thousand compositions, where the vertical bars indicate the range from the maximum to the minimum values attained. The dashed line indicates the upper bound given in equation (22).

Perturbations in composing polynomials $F(x)$ and $G(x)$ may lead to much smaller perturbations in the coefficients of $H(x)$. This implies that decomposition under this perturbation in $H(x)$ will yield larger relative perturbations in $F(x)$ and $G(x)$. The sensitivity of decomposition hence can reasonably be defined as

$$S_{H\to U} = \max_{\Delta\mathbf{u}} \left(\frac{E_{\Delta\mathbf{h}}/E_{\mathbf{h}}}{E_{\Delta\mathbf{u}}/E_{\mathbf{u}}} \right)^{-1} \quad (23)$$

where again U is either F or G. $S_{H\to U}$ corresponds to the case where the perturbation on the components occurs in the direction of maximum attenuation.

1) Formulation of $S_{H\to F}$: The sensitivity associated with obtaining $F(x)$ from a decomposable polynomial $H(x)$ becomes, by equations (4), (23) and (6)

$$S_{H\to F} = \left(\min_{\Delta\mathbf{f}} \frac{||\mathbf{C}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2} \frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2} \right)^{-1} = \left(\sigma^2_{\mathbf{C},min} \frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2} \right)^{-1}. \quad (24)$$

Furthermore, $\left(\frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2}\right)^{-1}$ is bounded above by $\sigma^2_{\mathbf{C},max}$ and bounded below by $\sigma^2_{\mathbf{C},min}$. Hence similar to equation (9), for any $F(x)$, the sensitivity $S_{H\to F}$ is bounded by the square of the condition number of $\mathbf{C}$, which only depends on $G(x)$, i.e.

$$1 \leqslant S_{H\to F} \leqslant \frac{\sigma^2_{\mathbf{C},max}}{\sigma^2_{\mathbf{C},min}}. \quad (25)$$

2) Formulation of $S_{H\to G}$: The sensitivity associated with obtaining $G(x)$ from a decomposable polynomial $H(x)$ becomes, by equations (23) and (15),

$$S_{H\to G} = \left(\min_{\Delta\mathbf{g}} \frac{||\mathbf{D}\Delta\mathbf{g}||_2^2}{||\Delta\mathbf{g}||_2^2} \frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2} \right)^{-1} = \left(\sigma^2_{\mathbf{D},min} \frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2} \right)^{-1}. \quad (26)$$

IV. Simulations

In the following subsections, several simulation results are provided to illustrate the sensitivity of the polynomial composition and decomposition operations. The vectors of coefficients of the composing polynomials $F(x)$ and $G(x)$ were selected from a standard normal distribution by the `randn` function of MATLAB and were normalized to have unit energy. The effect of normalization and scaling will be discussed in Section V.

A. Simulations for composition sensitivity

1) Evaluation of $S_{F\to H}$: In Section III-A1, $S_{F\to H}$ was shown to be bounded by the square of the condition number of $\mathbf{C}$ as given in equation (9) regardless of the specific value of $F(x)$. This bound is in fact attained if $\mathbf{f}$ is aligned with the right singular vector of $\mathbf{C}$ that corresponds to its smallest singular value. Since the condition number can be made as large as desired by different choices of $G(x)$, for example when the leading coefficient $G(x)$ is made arbitrarily small, the composition sensitivity $S_{F\to H}$ is unbounded. Hence composition can be very ill-conditioned with respect to the coefficients of $F(x)$. However for a fixed $F(x)$, it is not obvious that $S_{F\to H}$ as given in equation (8) can be made arbitrarily large with different choices for matrix $\mathbf{C}$. This follows from the fact that matrix $\mathbf{C}$ is restricted to have a certain structure, namely its columns has to be self convolutions of the coefficients of some $G(x)$.

The sensitivity $S_{F\to H}$, as defined in equation (8), is shown in Fig. 1 as a function of the degree of $F(x)$. In Fig. 1, each point shows the median value of $S_{F\to H}$ obtained from composing one hundred instances of $F(x)$ of the corresponding order with each one of one hundred instances of $G(x)$ of order seven. The vertical bars show the maximum and minimum values attained in these ten thousand compositions. For consistency, the same set of $G(x)$ were used for each degree of $F(x)$. The simulation results are consistent with the lower and upper bounds given in equation (9), namely 1 and the square of the condition number of $\mathbf{C}$, respectively. However the upper bound has been omitted from this figure due to very large values that exceed the display scale by multiple orders although it is tight, i.e. attainable for certain choices of $F(x)$.

2) Evaluation of $S_{G\to H}$: The sensitivity $S_{G\to H}$, as defined in equation (17), is shown in Fig. 2 as a function of the degree of $G(x)$. In Fig. 2, each point indicates the median value of $S_{G\to H}$ obtained from composing one hundred instances of $G(x)$ of the corresponding order with each one of one hundred instances of $F(x)$ of order seven. The dashed line indicates the upper bound given in equation (22) where $||\mathbf{g}||_2^2 = 1$ and $\sigma_{\mathbf{T},max}$ is evaluated for the $G(x)$ that attains the maximum value of $S_{G\to H}$ in the simulations for each degree.

B. Simulations for decomposition sensitivity

1) Evaluation of $S_{H\to F}$: Fig. 3 illustrates the sensitivity of the coefficients of $F(x)$ with respect to the perturbations in $H(x)$, namely $S_{H\to F}$ as described in equation (24). The values are extracted from the experiments performed in Section IV-A1.

2) Evaluation of $S_{H\to G}$: $S_{H\to G}$, as described in equation (26) the sensitivity of the coefficients of $G(x)$ with respect to the perturbations in $H(x)$ is illustrated in Fig. 4. The values are extracted from the experiments performed in Section IV-A2.

V. Equivalent compositions with lower sensitivity

A. Compositions with first order polynomials

From a decomposition $F \circ G(x)$ for $H(x)$, equivalent decompositions can be obtained through basic operations on $F(x)$ and

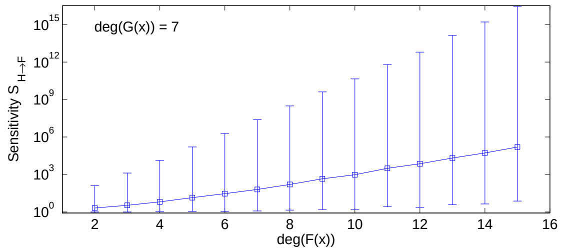

Fig. 3. Sensitivity of the coefficients of $F(x)$ with respect to the coefficients of $H(x)$. The order of $G(x)$ is seven in all compositions. Each point is the median of $S_{H\to F}$ obtained from ten thousand compositions, where the vertical bars indicate the range from the maximum to the minimum values attained.

$G(x)$ [9], [10]. For example, given any first order polynomial $\lambda(x) = ax + b,\ \ a \geqslant 0$ with its inverse with respect to composition $\lambda^{-1}(x) = \frac{x}{a} - \frac{b}{a}$, the composition can be represented as

$$F(G(x)) = (F \circ \lambda^{-1}) \circ (\lambda \circ G)(x) = \bar{F}(\bar{G}(x)). \qquad (27)$$

This implies that the composition $H(x)$ and the orders of $F(x)$ and $G(x)$ can be preserved while the sensitivity can be lowered by appropriate choices for the coefficients of $\lambda(x)$. Similar to equation (4), equation (27) corresponds to the matrix equation

$$\mathbf{h} = \mathbf{Cf} = (\mathbf{CA})(\mathbf{A^{-1}f}) \qquad (28)$$

where $\mathbf{A}$ is a square, upper triangular and invertible matrix kth column of which consists of $k-1$ self convolutions of the sequence $\{b, a\}$ or equivalently the coefficients of $(ax+b)^{k-1}$ in the ascending order. From equation (7), the sensitivity $S_{\bar{F}\to H}$ becomes

$$S_{\bar{F}\to H} = \max_{\Delta \mathbf{f}} \frac{||\mathbf{CA}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2} \frac{||\mathbf{A^{-1}f}||_2^2}{||\mathbf{Cf}||_2^2}. \qquad (29)$$

Although matrix $\mathbf{A}$ can be further decomposed into the product of two simpler matrices that depend only on a and b, respectively, it is not obvious how $S_{\bar{F}\to H}$ will behave as a joint function of a and b in general. The effect of pure scaling, which corresponds to the case $a > 0$, $b = 0$ and A is diagonal, can be inferred by examining the extremal values of a. More specifically, as a tends to infinity, the term $\max_{\Delta \mathbf{f}} \frac{||\mathbf{CA}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2}$ also tends to infinity whereas the term $\frac{||\mathbf{A^{-1}f}||_2^2}{||\mathbf{Cf}||_2^2}$ tends to a constant number if the constant term of $F(x)$ is nonzero. The roles of these two terms are reversed as a tends to zero. In both cases, $S_{\bar{F}\to H}$ becomes infinity, which suggests the existence of a minimum at a finite value of $a > 0$.

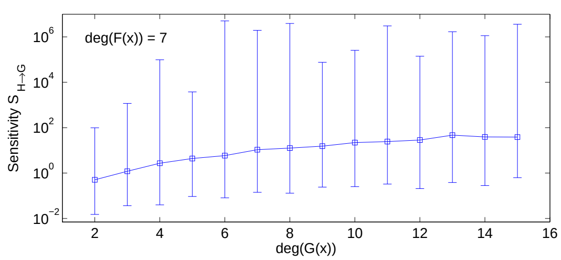

Fig. 4. Sensitivity of the coefficients of $G(x)$ with respect to the coefficients of $H(x)$. The order of $F(x)$ is seven in all compositions. Each point is the median of $S_{H\to G}$ obtained from ten thousand compositions, where the vertical bars indicate the range from the maximum to the minimum values attained.

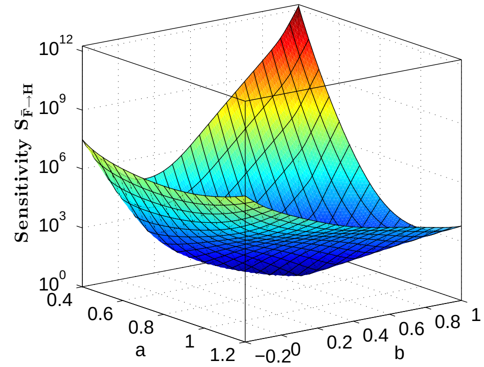

Fig. 5. $S_{\bar{F}\to H}$ as a function of a and b. $F(x)$and $G(x)$ are chosen to be the pair of polynomials that attained the largest sensitivity in Fig. 1 with $F(x)$ of order fourteen and $G(x)$ of order seven.

Fig. 5 illustrates the effectiveness of choosing different values for a and b in order to reduce $S_{F\to H}$. Here, $F(x)$and $G(x)$ are chosen to be the pair of polynomials that attained the largest sensitivity of 6.3×10^4 in Fig. 1 with $F(x)$ of order fourteen and $G(x)$ of order seven. The simulation results in Fig. 5 indicate that $S_{\bar{F}\to H}$ gets larger as b tends to infinity in either direction for this pair of $F(x)$ and $G(x)$. $S_{\bar{F}\to H}$ attains its minimum at $a^* = 0.73$ and $b^* = 0.57$. Table I displays the values of all four sensitivities associated with this composition before and after composition with $\lambda(x) = 0.73x + 0.57$.

The effect of compositions with first order polynomials on $S_{G\to H}$ is relatively straightforward. Since $D(x) = F'(G(x))$ as given in equation (13), introducing a first order polynomial and its inverse into the composition yields

$$(F \circ \lambda^{-1})' \circ (\lambda \circ G(x)) = ((\lambda^{-1})'F' \circ \lambda^{-1}) \circ \lambda \circ G(x) = \frac{1}{a}D(x), \qquad (30)$$

which corresponds to simply scaling $D(x)$. From equation (16), the sensitivity $S_{\bar{G}\to H}$ becomes,

$$S_{\bar{G}\to H} = \max_{\Delta \mathbf{g}} \frac{||\frac{1}{a}\mathbf{D}\Delta\mathbf{g}||_2^2}{||\Delta\mathbf{g}||_2^2} \frac{||a\mathbf{g} + b\mathbf{e}||_2^2}{||\mathbf{h}||_2^2} = \frac{||\mathbf{g} + \frac{b}{a}\mathbf{e}||_2^2}{||\mathbf{g}||_2^2} S_{G\to H} \qquad (31)$$

where $\mathbf{e} = [1, 0, \ldots, 0]^T$ and it is the same size as $\mathbf{g}$. This implies that if $\left|g_0 + \frac{b}{a}\right| < |g_0|$ where g_0 is the constant term in $G(x)$, $S_{G\to H}$ will also be improved. This is indeed the case for the optimal point in Fig. 5 and introducing a linear composition to improve $S_{F\to H}$ has decreased $S_{G\to H}$. Due to its relationship with $S_{G\to H}$, the effect is reversed on $S_{H\to G}$ in such a way that their product remains the same. On the other hand, the effect on $S_{H\to \bar{F}}$ can be only described at extreme values of a and b similarly to the case of $S_{\bar{F}\to H}$. Fig. 6 illustrates the behavior of these sensitivities as a function of a and b for same pair of polynomials $F(x)$ and $G(x)$. Since the optimal points are not the same for all sensitivities, a^* and b^* can be chosen depending on the application.

TABLE I
SENSITIVITY BEFORE AND AFTER COMPOSITION WITH $a^*x + b^*$

Sensitivity	Original	at (a^*, b^*)
$S_{F\to H}$	6.3×10^4	2.2×10^0
$S_{G\to H}$	1.7×10^2	1.5×10^2
$S_{H\to F}$	1.1×10^8	3.5×10^1
$S_{H\to G}$	1.5×10^4	1.7×10^4

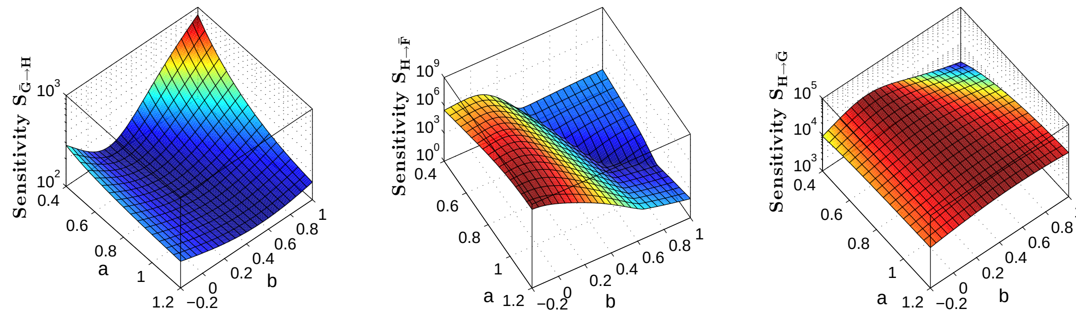

Fig. 6. The behavior of $S_{\bar{G}\to H}$, $S_{H\to\bar{F}}$ and $S_{H\to\bar{G}}$ as a function of a and b for same pair of polynomials $F(x)$ and $G(x)$ as in Fig. 5.

B. Commutative polynomials

Equivalent decompositions from a given decomposition can also be obtained when the components are commutative. For instance, monomials have the commutative property, i.e. $x^p \circ x^q = x^q \circ x^p = x^{pq}$ for any nonnegative integers p and q. Another class of polynomials that has the commutative property is Chebyshev polynomials which are defined as $T_n(x) = \cos(n \cos^{-1}(x))$ where n is a nonnegative integer. This property follows easily since

$$\begin{aligned} T_m \circ T_n(x) &= \cos(m \cos^{-1}(\cos(n \cos^{-1}(x)))) \\ &= \cos(mn \cos^{-1}(x)) \\ &= \cos(n \cos^{-1}(\cos(m \cos^{-1}(x)))) \\ &= T_n \circ T_m(x). \end{aligned} \tag{32}$$

An *entire set of commutative polynomials* is defined in [10] as a set of polynomials which contains at least one of each positive degree, and any two members commute with each other. Furthermore, it is shown in [10] that only two such classes exist, which are of the form $\lambda^{-1} \circ P_n \circ \lambda(x)$ where $P_n(x)$ is a monomial or a Chebyshev polynomial and $\lambda(x)$ is any first order polynomial. The commutative property allows reordering the components in a decomposition in a way to minimize the sensitivity of interest among the four different definitions in Section III along with first order compositions as discussed in Section V-A.

VI. Conclusion

In this paper, the sensitivities associated with polynomial composition and decomposition have been studied in order to quantify their robustness for signal processing applications. Expressions for sensitivities as well as their bounds are obtained and the consistency of these bounds are validated through simulations. It is also empirically shown that sensitivity can be improved significantly using equivalent compositions by utilizing first order polynomials or commutativity of certain class of polynomials.

Appendix

Lemma: Denote $s_3[n]$ as the convolution of the finite length signals $s_1[n]$ which is non-zero only for $0 \leqslant n \leqslant L_1$ and $s_2[n]$ which is non-zero only for $0 \leqslant n \leqslant L_2$. Assume $L_1 \geqslant L_2$, then the energy of these signals satisfy

$$E_{s_3} \leqslant (L_2 + 1) E_{s_1} E_{s_2},$$

where the energy is given by $E_{s_i} = \sum_{n=-\infty}^{\infty} s_i^2[n], \ i = 1, 2, 3.$

Proof: For $0 \leqslant n \leqslant L_1 + L_2$, Cauchy-Schwarz inequality implies

$$\begin{aligned} s_3^2[n] &= \left(\sum_{m=\max(0,n-L_2)}^{\min(L_1,n)} s_1[m] s_2[n-m] \right)^2 \\ &\leqslant \left(\sum_{m=\max(0,n-L_2)}^{\min(L_1,n)} s_1^2[m] \right) \left(\sum_{m=\max(0,n-L_2)}^{\min(L_1,n)} s_2^2[n-m] \right) \\ &\leqslant \left(\sum_{m=\max(0,n-L_2)}^{\min(L_1,n)} s_1^2[m] \right) E_{s_2}. \end{aligned}$$

Summing for $n = 0, 1, \ldots, (L_1 + L_2)$

$$\begin{aligned} E_{s_3} &\leqslant \sum_{n=0}^{L_1+L_2} \left(\sum_{m=\max(0,n-L_2)}^{\min(L_1,n)} s_1^2[m] \right) E_{s_2} \\ &= \sum_{m=0}^{L_1} \left(\sum_{n=m}^{m+L_2} s_1^2[m] \right) E_{s_2} = (L_2 + 1) E_{s_1} E_{s_2}. \end{aligned}$$

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