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OPTIMAL AND SUBOPTIMAL FILTERING FOR TOEPLITZ SYSTEMS*

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ABSTRACT

Efficient, on-line implementations of filters for Toeplitz systems are presented. For the optimal filter, the information transfer among subsystems is shown to be relaizable by a spatial transfer function. Suboptimal filters are proposed in order to overcome two difficulties associated with the optimal filter.

I. INTRODUCTION

Toeplitz systems are infinite-dimensional, spatially-invariant linear systems. They are composed of an infinite number of identical subsystems having stationary dynamics. Two particular areas where such systems arise are the control of a string of vehicles [1], [2] and the recursive estimation of images [3], [4].

Hager and Horowitz [5] have rigorously solved the optimal filtering problem for Toeplitz systems. The goal of this paper is to build upon the optimal centralized filter to obtain optimal and suboptimal filters having efficient on-line implementations. The key observation in achieving this goal is that the information transfer among subsystems can be decomposed into a spatial transfer function. This spatial transfer function can be recursively realized by two linear systems moving up and down the line of subsystems. When the output operator is block diagonal, these two linear systems are simply two spatial Kalman filters. The two spatial filters immediately suggest efficient, suboptimal schemes for filtering of Toeplitz systems.

Much of the motivation for this development comes from the work of Attasi [3] in recursive processing of noisy images. In searching for computationally tractable estimation formulas, Attasi introduced a particular two-parameter model for images. On the basis of this model, Attasi considered "line-by-line" filtering of an image, i.e. the i^{th} line of an image is estimated from all the lines to the left of line i. Attasi then showed that the measurement update step of this filter was essentially a smoothing problem and so could be solved by operating up and down the line. We use Attasi's ideas in much of our development of filtering for Toeplitz systems.

The organization of this paper now follows. After some preliminary results in Section II, the optimal solution to the Toeplitz filtering problem is presented in Section III. Then, motivated by work in the image processing field [3], [4], Section IV proposes an implementation of the optimal filter that involves only very limited inter-subsystem communication. Section V considers time-invariant filters having a finite-dimensional spatial realization. Decentralized filters having only finite (spatial) memory and their corresponding implementation are given in Section VI. Conclusions are detailed in Section VII.

II. PRELIMINARIES

A. Toeplitz Systems

The dynamics of the k^{th} subsystem in a stochastic, discrete-time, Toeplitz system are given by

$$x_k(i+1) = \sum_{\ell=-\infty}^{+\infty} A_{k-\ell} x_\ell(i) + D_{k-\ell} w_\ell(i) \tag{2.1}$$

where the state of the k^{th} subsystem at time i is $x_k(i) \in \mathbb{R}^n$ and $k=0,\pm1,\ldots$. The k^{th} output is

$$y_k(i) = \sum_{\ell=-\infty}^{+\infty} C_{k-\ell} x_\ell(i) + v_k(i) \tag{2.2}$$

where $y_k(i) \in \mathbb{R}^p$. The noises $w_\ell(i)$ and $v_\ell(i)$ are independent, zero-mean, Gaussian white noise processes,

$$E\{w_k(i) w_\ell'(j)\} = Q_o \delta_{i,j} \delta_{k,\ell} \tag{2.3}$$

$$E\{v_k(i) v_\ell'(j)\} = R_o \delta_{i,j} \delta_{k,\ell} \tag{2.4}$$

The initial state $x_k(0)$ is also assumed to be zero-mean and Gaussian and is independent of the driving and observation noises.

For notational simplicity, the infinite-dimensional state, output, and noise vectors are defined.

$$x(i) = \begin{pmatrix} \vdots \\ x_{-1}(i) \\ x_0(i) \\ x_1(i) \\ \vdots \end{pmatrix}, \quad w(i) = \begin{pmatrix} \vdots \\ w_{-1}(i) \\ w_0(i) \\ w_1(1) \\ \vdots \end{pmatrix}, \quad y(i) = \begin{pmatrix} \vdots \\ y_{-1}(i) \\ y_0(i) \\ y_1(i) \\ \vdots \end{pmatrix}, \quad v(i) = \begin{pmatrix} \vdots \\ v_{-1}(i) \\ v_0(i) \\ v_1(i) \\ \vdots \end{pmatrix}$$

In terms of these vectors, the infinite set of equations represented by (2.1) and (2.2) are written as:

$$x(i+1) = A\,x(i) + D\,w(i) \tag{2.5}$$

$$y(i) = C\,x(i) + v(i) \tag{2.6}$$

where the Toeplitz system matrix is

$$A = \begin{pmatrix} \ddots & & & & \iddots \\ & A_0 & A_{-1} & A_{-2} & A_{-3} \\ & A_1 & A_0 & A_{-1} & A_{-2} \\ & A_2 & A_1 & A_0 & A_{-1} \\ & A_3 & A_2 & A_1 & A_0 \\ \iddots & & & & \ddots \end{pmatrix}$$

Matrics C and D have the same form.

The mutual interaction between subsystems k and ℓ depends only on $k-\ell$, as can be seen from (2.1). It is in this sense that Toeplitz systems are spatially invariant. The z-transform can be used formally to decouple Toeplitz systems. The z-transform of the state vector x(i) is

$$\begin{aligned} \bar{x}(i,z) &= Z[x_k\ (i)] \\ &= \sum_{k=-\infty}^{+\infty} x_k(i)\ z^{-k} \end{aligned} \tag{2.7}$$

The transforms y(i,z), w(i,z), v(i,z) are defined similarly. The transform of the system matrix is

$$\begin{aligned} A(z) &= Z[A_k] \\ &= \sum_{k=-\infty}^{+\infty} A_k z^{-k} \end{aligned} \tag{2.8}$$

Likewise, the matrices C(z) and D(z) are defined from C and D.

The system dynamics (2.1) and output (2.2) equations can now be rewritten in the z-domain. Using the property of the z-transform for convolution sums [6] yields

$$x(i+1,z) = A(z)\, x(i,z) + D(z)\, w(i,z) \tag{2.9}$$

$$y(i,z) = C(z)\, x(i,z) + v(i,z) \tag{2.10}$$

where the noises w and v are white noises in both i and z, i.e.

$$E[w(i,z_1)w'(j,z_2)] = Q_o\delta(z_1-z_2)\delta_{i,j} \tag{2.11}$$

$$E[v(i,z_1)v'(j,z_2)] = R_o\delta(z_1-z_2)\delta_{i,j} \tag{2.12}$$

These equations, of course, are purely formal. But in the transform domain, the dynamics and observations are decoupled, and the noises are independent. The transformed subsystems, however, are indexed by the continuous complex variable z.

The Toeplitz system matrix A is an operator mapping the sequence $\{x_k(i)\}$ into the sequence $\{x_k(i+1)\}$. For the operator A to be a bounded operator on the space of square summable sequences, the induced norm of A must be finite. Widom [7] has shown that the operator norm of A is related to the z-transform A(z) by

$$||A|| = \operatorname*{ess\,sup}_{z \in U} ||A(z)||_2$$

where the set U is the unit circle in the complex plane. Since only bounded operators A are of interest as the system matrix for a Toeplitz system, it is assumed that the z-transform of A does exist and that the region of convergence includes the unit circle. This does not imply that the discrete-time system is itself stable. We are only assuming stability of the spatial interactions among subsystems. The region of convergence of A(z) is an annulus in the complex plane consisting of all z for which the defining sum (2.8) is absolutely convergent. Likewise, the transforms B(z) and C(z) are also assumed to exist and to have regions of convergence containing the unit circle.

A further assumption that will be useful throughout this paper is that all the z-transforms of the matrices used in the Toeplitz model are rational functions of z. This assumption is not necessary for much of the development, and it will be explicitly noted when the rationality assumption is used. What this assumption allows, however, is the construction of an efficient procedure for optimal linear filtering. In summary,

Assumption 1. All the z-transforms of matrices used in Toeplitz models are assumed to exist, to have regions of convergence including the unit circle, and to be rational functions of the complex variable z.

B. Recursive Realization of Toeplitz Operators

It will be useful in the sequel to have a recursive spatial realization of a Toeplitz operator. What we have in mind here is the realization of a Toeplitz operator such as the system matrix A by a recursion over the spatial index. Schoute et al. [4] have given the following realization of such an operator.

Consider the operator L mapping $\{u_k\}$ into $\{y_k\}$,

$$y_k = \sum_{\ell=-\infty}^{+\infty} L_{k-\ell} u_\ell \tag{2.14}$$

Also, let S be a linear dynamical system having impulse response $\{L_k\}$, i.e., the output of S is $\{y_k\}$ when the input is $\{u_k\}$. The system S may be non-causal since L_k is not necessarily zero for all negative k. The impulse response of S may be expressed as the sum of a causal and an anticausal part,

$$\{L_k\} = \{h_k^+\} + \{h_k^-\} \tag{2.15}$$

where

$$\begin{aligned} \{h_k^+\} &= \{\ldots,0,0,\alpha L_0, L_1, L_2, \ldots\} \\ \{h_k^-\} &= \{\ldots, L_{-2}, L_{-1}, (1-\alpha)L_0, 0, 0, \ldots\} \end{aligned} \tag{2.16}$$

and α is an arbitrary scalar. Let S^+ (respectively S^-) be a causal (anticausal) dynamical system with impulse response $\{h_k^+\}$ ($\{h_k^-\}$). Then

$$y_k = \varepsilon_k^+ + \varepsilon_k^- \tag{2.17}$$

where $\varepsilon_k^+(\varepsilon_k^-)$ is the output of $S^+(S^-)$ when the input is $\{u_k\}$. The system S^- is anticausal for increasing k, but causal for decreasing k. That is, if S^- is viewed as a dynamical system running backward over k, then it is causal.

Thus the output $\{y_k\}$ is obtained as the sum of the outputs of two linear dynamical systems -- one (S^+) running forward over k and one (S^-) running backward. This provides a recursive spatial relaization of (2.14) as desired. Moreover, if L has a rational z-transform, then the sequences $\{h_k^+\}$ and $\{h_k^-\}$ have rational z-transforms also. The linear systems S^+ and S^-, therefore, have finite-dimensional realizations.

III. SOLUTION OF THE OPTIMAL FILTERING PROBLEM

The filtering problem is to obtain the minimum variance estimate $\hat{x}_k(i|i)$ of $x_k(i)$ given all the observations up to and including time i, $\{y_\ell(j) | 0 \le j \le i$ and all $\ell\}$. Under the condition of detectability for the system (2.1) and (2.2), Hager and Horowitz [5] have shown that the solution to this problem is a Kalman filter,

$$\hat{x}(i|i) = \hat{x}(i|i-1) + K(i)\,[y(i) - C\,\hat{x}(i|i-1)] \tag{3.1}$$

$$\hat{x}(i|i-1) = A\,\hat{x}(i-1|i-1) \tag{3.2}$$

$$K(i) = P(i|i-1)C'[C\,P(i|i-1)C' + R]^{-1} \tag{3.3}$$

$$P(i|i) = [I - K(i)C]P(i|i-1) \tag{3.4}$$

$$P(i|i-1) = A\,P(i-1|i-1)A' + DQD' \tag{3.5}$$

where the matrices $P(i|i)$ and $P(i|i-1)$ are covariance matrices of the filtered error $\hat{x}(i|i) - x(i)$ and the predicted error $\hat{x}(i|i-1) - x(i)$, respectively. Equations (3.1), (3.3), (3.4) define the measurement update step; the prediction step is given by (3.2) and (3.5).

If the initial state covariance matrix $P(0|-1)$ is Toeplitz, then $K(i)$, $P(i|i)$, and $P(i|i-1)$ will be Toeplitz matrices for all i. Thus the Kalman filter can be written in the z-transform domain,

$$\hat{x}(i,z|i) = \hat{x}(i,z|i-1) + K(i,z)\,[y(i,z) - C(z)\hat{x}(i,z|i-1)] \tag{3.6}$$

$$\hat{x}(i,z|i-1) = A(z)\,\hat{x}(i-1,z|i-1) \tag{3.7}$$

$$K(i,z) = P(i,z|i-1)C^*(z)[C(z)P(i,z|i-1)C^*(z) + R(z)]^{-1} \tag{3.8}$$

$$P(i,z|i) = [I - K(i,z)C(z)]P(i,z|i-1) \tag{3.9}$$

$$P(i,z|i-1) = A(z)\ P(i-1,z|i-1)\ A^*(z) + D(z)Q(z)D^*(z) \tag{3.10}$$

The z-transforms $P(i,z|i)$ and $P(i,z|i-1)$ are the spectra of the filtered and predicted errors, respectively.

It is to be noted that the Kalman filtering equations (3.6) - (3.10) can be though of formally as defining the optimal filter for the z-transformed system (2.9) and (2.10). The filtering problems for different values of z are independent of one another, e.g., the estimate $\hat{x}(i,z_o|i)$ depends on only the observations $y(j,z_o)$, $j \leq i$. The solution to the filtering problem, therefore, is just a set of finite-dimensional Kalman filters (3.6) - (3.10) indexed by the continuous transform variable z.

IV. IMPLEMENTATION OF THE OPTIMAL FILTER

The objective in this section is to demonstrate how the optimal filter (3.1) - (3.5) can be efficiently implemented. To this end, we fix the time index i and address separately the update and prediction cycles of the filter.

A. The Measurement Update Cycle

The on-line portion of the measurement update cycle is given by (3.6). We are viewing the filter gain K(i,z) as a dynamical system in the transform variable z with i constant. From this perspective, the realization procedure proposed by Schoute et al. can be used to implement the filter gain K(i,z) by the linear systems $S^+(i)$ and $S^-(i)$ (see Section II.A). This realization is illustrated in Figure 1. The linear system $S^+(i)$ (respectively, $S^-(i)$) propagates a forward (backward) "wave of information" along the subsystems.

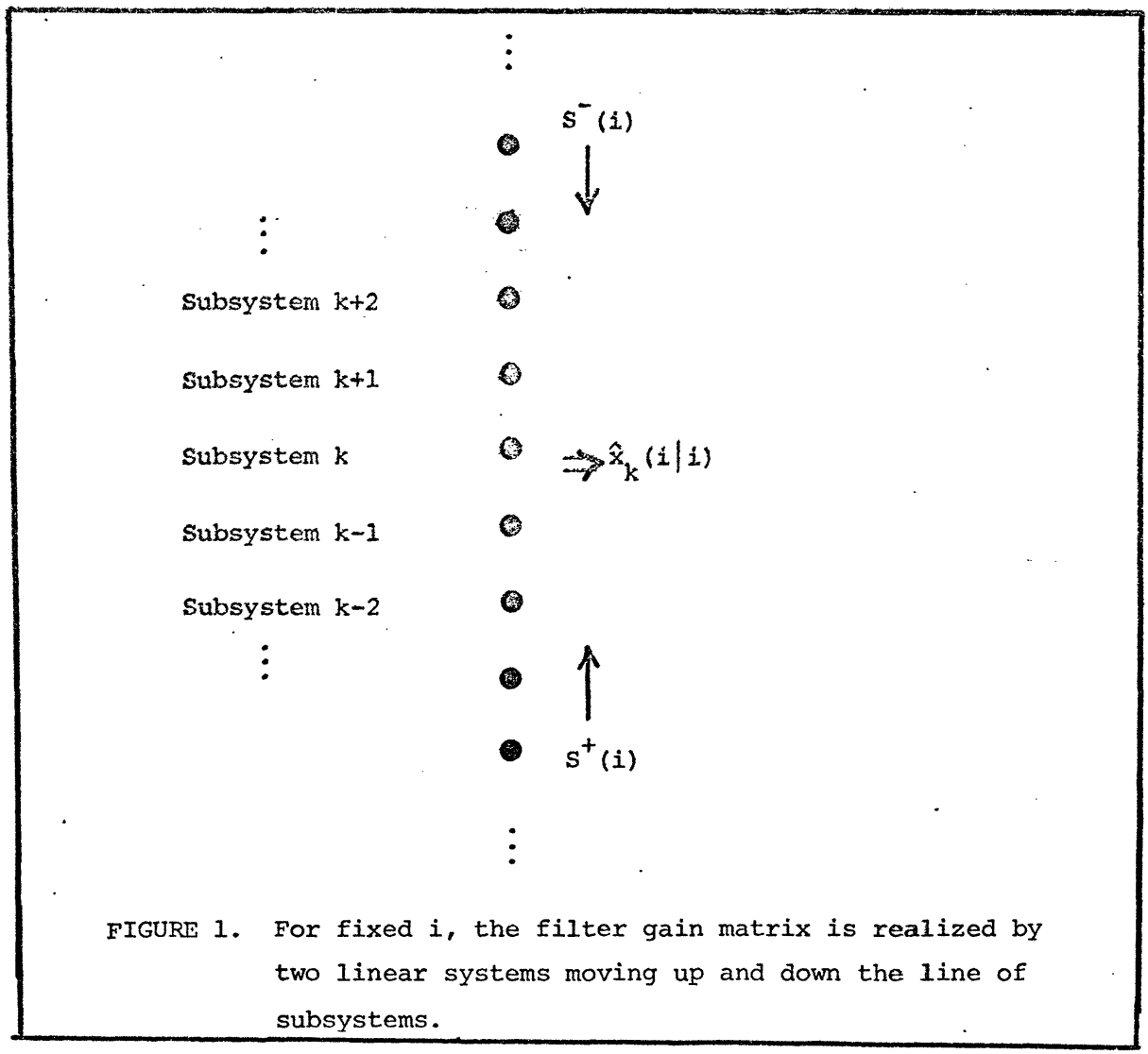

FIGURE 1. For fixed i, the filter gain matrix is realized by two linear systems moving up and down the line of subsystems.

Together, these two waves summarize all the information about $x_k(i)$ contained in the measurement at time i. Furthermore, under Assumption 1 that A(z), C(z), D(z), and $P(0,z|-1)$ are all rational, the transformed filter gain K(i,z) is also rational. This implies that the linear systems $S^+(i)$ and $S^-(i)$ will have finite-dimensional realizations.

The role of these two spatial systems can be explored further by defining the predicted error of substate k at time i,

$$e_k(i) = x_k(i) - \hat{x}_k(i|i-1) \tag{4.1}$$

The innovations process $I_k(i)$ is related to the predicted error by

$$\begin{aligned} I_k(i) &= y_k(i) - \sum_{\ell=-\infty}^{+\infty} C_{k-\ell}\hat{x}_\ell(i|i-1) \\ &= \sum_{\ell=-\infty}^{+\infty} C_{k-\ell}e_\ell(i) + v_k(i) \end{aligned} \tag{4.2}$$

By the Orthogonal Projection Theorem of Hilbert Space Theory [8], the estimate $\hat{x}_k(i|i)$ is simply the sum of $\hat{x}_k(i|i-1)$ and $\hat{e}_k(i)$, the optimal estimate of $e_k(i)$. Thus by (3.1), the optimal estimate of $e_k(i)$ is just the (spatial) convolution of the filter gains $\{K_k(i)\}$ with the innovations process $\{I_k(i)\}$,

$$\hat{e}_k(i) = \sum_{\ell=-\infty}^{+\infty} K_{k-\ell}(i)\ I_\ell(i) \tag{4.3}$$

The filter gain, however, can be realized by the systems $S^+(i)$ and $S^-(i)$. Then

$$\hat{e}_k(i) = \varepsilon_k^+(i) + \varepsilon_k^-(i) \tag{4.4}$$

where $\varepsilon_k^+(i)$ and $\varepsilon_k^-(i)$ are the outputs of $S^+(i)$ and $S^-(i)$, respectively, when the input is $\{I_k(i)\}$.

Summarizing the development this far, the on-line aspects of the filter update cycle consist of the following steps:

1.) Run the forward and backward systems $S^+(i)$ and $S^-(i)$ with input $I_k(i)$.

2.) Combine the two outputs $\varepsilon_k^+(i)$ and $\varepsilon_k^-(i)$ by (4.4) to get $\hat{e}_k(i)$.

3.) The updated estimate is $\hat{x}_k(i|i) = \hat{x}_k(i|i-1) + \hat{e}_k(i)$.

The off-line computations are propagation of the error covariance by (3.8) - (3.10) and obtaining realizations for $S^+(i)$ and $S^-(i)$. It is clear that we could realize the observation operator C by two (spatial) linear systems and, thereby, obtain $I_k(i)$ from $y_k(i)$ and the $\hat{x}_\ell(i|i-1)$.

From (4.2), the innovations are nothing but linear observations of the $e_k(i)$ corrupted by additive noise. In general, each $I_k(i)$ depends on all the $e_\ell(i)$. However, in the important special case when each subsystem observes a suboutput which is a function of only the local substate (i.e., $C(z) = C_o$), the innovation $I_k(i)$ is simply a noisy observation of $e_k(i)$,

$$I_k(i) = C_o e_k(i) + v_k(i) \tag{4.2'}$$

Assumption 2. $C(z) = C_o$, i.e., $C_k = C_o \delta_{k,0}$.

In this case, we have a great deal of intuition about the role of the systems $S^+(i)$ and $S^-(i)$. In fact, it will be shown that these two systems are spatial Kalman filters. This provides an interpretation of the preceding development and leads directly to the suboptimal filters presented in the next two sections.

Recall that the time index i is fixed in this discussion. Then since each innovation is an observation of a local error, determining the estimates $\{e_k(i)\}$ from the innovations $\{I_k(i)\}$ may be viewed as a smoothing problem. The optimal estimate $\hat{e}_k(i)$ cannot be made on the basis of $I_k(i)$ alone because of the cross-correlation among the predicted errors. The cross-correlation between $e_k(i)$ and $e_\ell(i)$ is given by

$$P_{k-\ell}(i|i-1) = E[e_\ell(i) e_k'(i)] \tag{4.5}$$

and so $P(i,z|i-1)$ is the spectrum of the predicted error process. $P(i,z|i-1)$ will be a rational function of z if Assumption 1 is valid. In other words, the stationary discrete process $e_k(i)$, as a function of k, has a rational spectrum.

Since the process $e_k(i)$ does have a rational spectrum, the spectral factorization theorem [9] guarantees that one can obtain a finite-dimensional, time (space) - invariant linear system

$$\xi_{k+1}(i) = \Phi(i)\ \xi_k(i) + \Gamma(i)\mu_k(i) \tag{4.6}$$

$$\zeta_k(i) = \Theta(i)\ \xi_k(i) \tag{4.7}$$

driven by the vector white noise process $\mu_k(i)$ having identity covariance, such that the spectrum of $\zeta_k(i)$ equals $P(i,z|i-1)$. That is to say, the sequence $e_k(i)$ can be identified with the sequence $\zeta_k(i)$ and viewed as the output of the above system. Then the innovations $I_k(i)$ are simply noisy linear observations of the state $\xi_k(i)$,

$$I_k(i) = C_o\Theta(i)\ \xi_k(i) + v_k(i) \tag{4.8}$$

and can be used to estimate it. The optimal estimate $\hat{e}_k(i)$ is now $\Theta(i)\ \hat{\xi}_k(i)$ where $\hat{\xi}_k(i)$ is the smoothed estimate of $\xi_k(i)$.

There are many ways to obtain the smoothed estimate $\hat{\xi}_k(i)$; the discrete-time two-filter smoother described in [10] is most appropriate for our purposes. Using this smoother, the estimate is

$$\hat{\xi}_k(i) = P_s(i)\ [P_f^{-1}(i)\ \hat{\xi}^f_{k|k}(i) + P_b^{-1}(i)\ \hat{\xi}^b_{k|k+1}(i)] \tag{4.9}$$

$$P_s(i) = [P_f^{-1}(i) + P_b^{-1}(i) - \sigma^{-1}(i)]^{-1} \tag{4.10}$$

where

$$\hat{\xi}^f_{k|k}(i) = E\{\xi_k(i)\,|\,I_\ell(i),\ \ell \leq k\}$$

$$= \text{the forward Kalman filter estimate of } \xi_k(i)$$

$\hat{\xi}^{b}_{k|k+1}(i) = E\{\xi_k(i) \mid I_\ell(i),\ \ell>k\}$

= the backwards Kalman filter one-step-ahead predicted estimate of $\xi_k(i)$

$P_f(i)$ = the steady-state covariance of the forward Kalman filter estimate

$P_b(i)$ = the steady-state covariance of the backwards Kalman filter predicted estimate

$P_s(i)$ = the smoothed error covariance

$\sigma(i)$ = the steady-state *a priori* system covariance of (4.6)

Figure 2 illustrates this smoother employing two Kalman filters - the forward Kalman filter operates in the positive k direction; the backward Kalman filter operates in the negative k direction. Both filters are in the steady-state since the k index extends to plus and minus infinity.

With Assumptions 1 and 2, the update cycle of the filter has the following steps:

1.) Propagate the error spectrum by (3.8) - (3.10).

2.) Obtain a realization $(\Phi(i), \Gamma(i), \Theta(i))$ of $P(i,z|i-1)$.

3.) Perform forward and backward Kalman filtering for this system.

4.) Combine the filtered estimates according to (4.9) to get the smoothed estimate of $\xi_k(i)$.

5.) The updated estimate is $\hat{x}_k(i|i) = \hat{x}_k(i|i-1) + \Theta(i)\ \hat{\xi}_k(i)$.

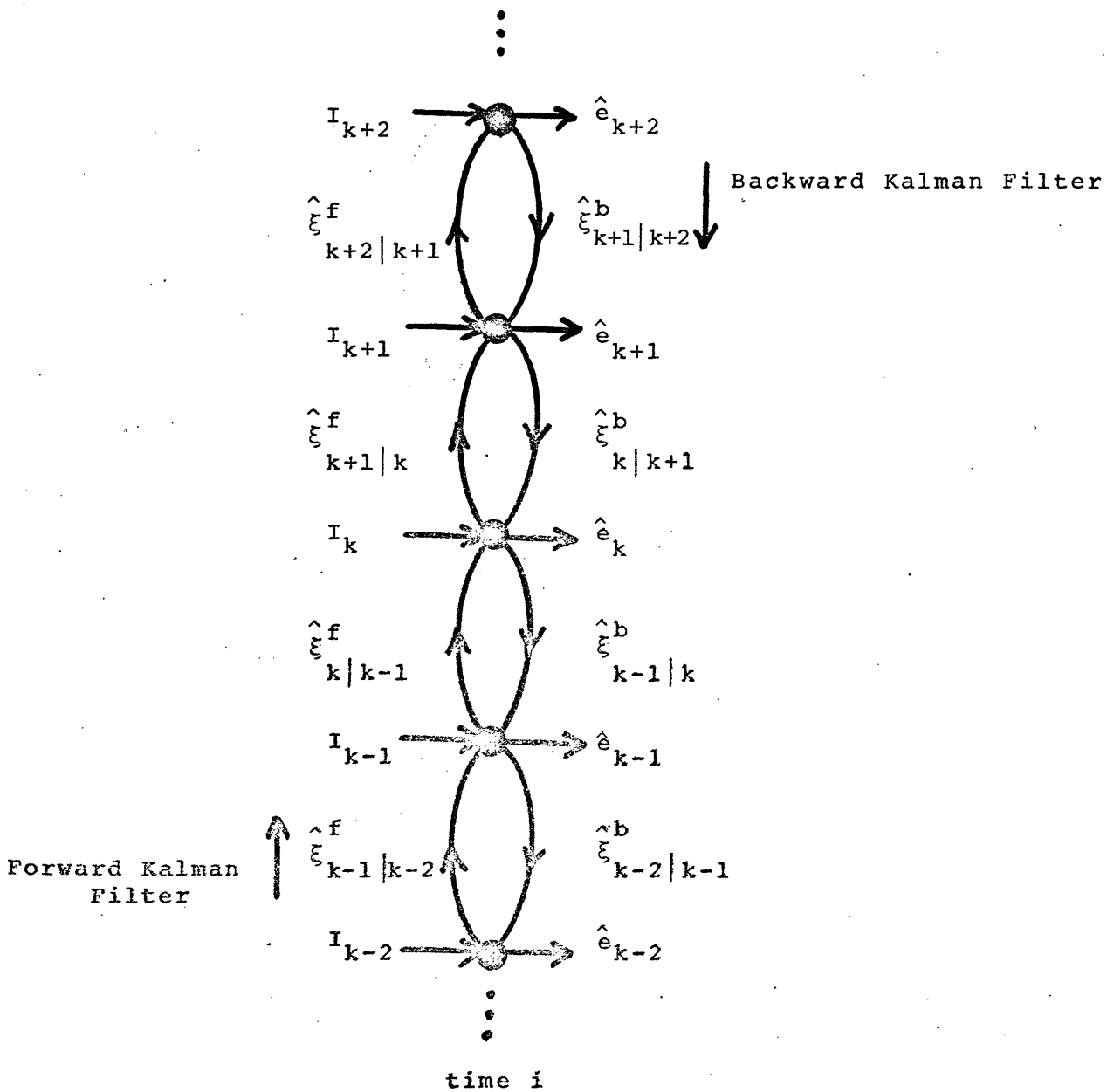

FIGURE 2. Under Assumption 2, the measurement update step is equivalent to a smoothing problem and so can be solved by forward and backward spatial Kalman filters.

The first two steps, of course, can be done off-line.

The actual procedure will now be explained by considering subsystem k as it estimates $e_k(i)$ (See Figure 2). The estimate $\hat{e}_k(i)$ is obtaining by combining the outputs of two spatial Kalman filters. Subsystem k-1 computes the forward estimate $\hat{\xi}^f_{k-1|k-1}(i)$ and communicates it to subsystem k. The estimate $\hat{\xi}^f_{k|k}(i)$ can now be computed as

$$\hat{\xi}^f_{k|k}(i) = \Phi(i)\hat{\xi}^f_{k-1|k-1}(i) + K_f(i)\ [I_k(i) - C_o\Theta(i)\ \Phi(i)\hat{\xi}^f_{k-1|k-1}(i)] \tag{4.11}$$

where $K_f(i)$ is the steady-state gain of the forward filter. This estimate is then furnished to subsystem k+1, and the forward filter continues up the line. Meanwhile, the backwards filter is operating down the line independently of the forward filter. At some point, the backwards filter reaches subsystem k+1. The one-step-ahead predicted estimate for the backwards filter, $\hat{\xi}^b_{k|k+1}(i)$, is then furnished to subsystem k from subsystem k+1. The next estimate is computed from

$$\hat{\xi}^b_{k-1|k}(i) = \Phi_b(i)\ \{\hat{\xi}^b_{k|k+1}(i) + K_b(i)\ [I_k(i) - C_o\Theta(i)\ \hat{\xi}_{k|k+1}(i)]\ \} \tag{4.12}$$

where $K_b(i)$ is the steady-state gain of the backwards filter and the backwards system matrix is $\Phi_b(i) = \sigma(i)\ \Phi'(i)\ \sigma^{-1}(i)$. The backwards filter then continues in the negative k direction. Subsystem k can now use (4.9) and (4.10) to obtain the smoothed estimate $\hat{\xi}_k(i)$ and then $\hat{e}_k(i) = \Theta(i)\ \hat{\xi}_k(i)$.

The striking aspect of the update step of this Kalman filter is the very limited communication between adjacent subsystems. All that is required is that each subsystem furnish its two nearest neighbors with estimates of the process $\xi_k(i)$. When Assumption 2 is not valid, the update step can be realized by a causal and an anticausal dynamical system. These two systems do not have the interpretation of Kalman filters, but they can be implemented by the same inter-subsystem communication pattern described above. Therefore, the update step requires only limited communication between adjacent subsystems, regardless of whether Assumption 2 holds.

B. The Prediction Cycle

The remaining step in the filtering algorithm is prediction. In the spatial domain, the prediction step will be a convolution sum,

$$\hat{x}_k(i|i-1) = \sum_{\ell=-\infty}^{+\infty} A_{k-\ell}\hat{x}_\ell(i-1|i-1) \tag{4.13}$$

One noteworthy feature of this expression is that if the dynamics of each subsystem depend directly on only a finite number of other subsystems, then the convolution sum (4.13) will be a finite sume. For example, suppose the Toeplitz system has only nearest neighbor interactions, i.e., $A_k=0$, $k\neq -1,0,1$. The predicted estimate of substate k is then simply

$$\hat{x}_k(i|i-1) = A_{-1}\hat{x}_{k+1}(i-1|i-1) + A_o\hat{x}_k(i-1|i-1) + A_1\hat{x}_{k-1}(i-1|i-1) \tag{4.14}$$

This means that subsystem k can form the predicted estimate of its own state from the state estimates of its two nearest neighbors. That is, each subsystem can predict optimally with only nearest neighbor communication of local estimates. Of course, whenever the subsystem interactions are localized spatially, a similar result holds.

In the general case, the Toeplitz operator A can be realized by the procedure introduced in Section II.A. Namely, consider the sequence $\{A_k\}$ to be the impulse response of a linear system, and then express this response as the sum of a causal and an anti-causal part. In this way the predicted estimate $\hat{x}_k(i|i-1)$ is obtained as the sum of the outputs of a forward and a backward linear system. These systems have inputs $\hat{x}_k(i-1|i-1)$ and have a finite-dimensional realization if A(z) is rational.

In summary, the prediction and update steps of the Toeplitz Kalman filter can both be realized by forward and backward linear dynamical systems. Under Assumption 1, these systems are finite-dimensional for all i. The prediction step can be obtained as a finite sum whenever there are only a finite number of subsystem interactions. Under Assumption 2, the update step is equivalent to a smoothing problem and can be realized by forward and backward Kalman filters.

V. TIME-INVARIANT FILTERS AND FINITE-DIMENSIONAL APPROXIMATIONS

The optimal centralized Kalman filter discussed in Sections III and IV has a very appealing inter-subsystem communication pattern but is time-varying. Since the transformed filter gain $K(i,z)$ is recursively computed from the discrete-time Riccati equation (3.8)-(3.10), the degree of $K(i,z)$ as a rational function of z grows rapidly with increasing i. The dimension of the linear systems $S^+(i)$ and $S^-(i)$ which implement $K(i,z)$, therefore, will be growing. Thus even though the realization problems for $S^+(i)$ and $S^-(i)$ can be solved off-line, the growing dimension of the realizations leads one to consider a time-invariant sub-optimal filter. Even with a time-invariant filter, we note that each estimate $\hat{e}_k(i)$ requires infinite computation.

The time-invariant steady-state Kalman filter is specified by the steady-state predicted error covariance P. The corresponding spectrum $P(z)$ is given by the discrete-time algebraic Riccati equation in the transform domain,

$$P(z) = A(z) \{P(z) - P(z)C^*(z) [C(z)P(z)C^*(z) + R]^{-1} C(z)P(z)\} A^*(z) + D(z) Q D^*(z) \tag{5.1}$$

The difficulty here is that even though $P(i,z|i-1)$ is a rational function of z for all i, in general, the limiting value $P(z)$ will <u>not</u> be a rational function. It is possible [10] for the steady-state error spectrum $P(z)$, and hence the filter gain $K(z)$, to be a rational function of z. Although characterizing precisely when this is the case is still an open problem, it is expected that the occurrence of a rational $P(z)$ will be quite rare.

When $P(z)$ is irrational, some approximation must be made in order to obtain a finite-dimensional realization of $K(z)$. Let $P_a(z)$ be a rational approximation of $P(z)$. The corresponding approximate filter gain $K_a(z)$ is given by

$$K_a(z) = P_a(z)C^*(z) [C(z) P_a(z)C^*(z) + R(z)]^{-1} \tag{5.2}$$

Alternatively, one could directly approximate $K(z)$ by some rational $K_a(z)$ without the intermediate step of obtaining $P_a(z)$. In either case, the steady-

state covariance that results when $K_a(z)$ is used as the filter gain is of interest.

The predicted error is given by

$$\begin{aligned} e(i+1) &= x(i+1) - \hat{x}(i+1|i) \\ &= \{A\,x(i) + Dw(i)\} - A\{\hat{x}(i|i-1) + K_a[y(i) - C\hat{x}(i|i-1)]\} \\ &= A[I - K_aC]e(i) + Dw(i) - AK_av(i) \end{aligned} \tag{5.3}$$

If the filter system matrix $A[I-K_aC]$ is stable, then the predicted error covariance reaches a steady-state value. The resulting predicted error spectrum $P_p(z)$ is given by

$$P_p(z)=A(z)\,[I-K_a(z)C(z)]P_p(z)\,[I-K_a(z)C(z)]^*A^*(z) + D(z)Q(z)D^*(z) + \\ + A(z)K_a(z)R(z)K_a^*(z)A^*(z) \tag{5.4}$$

Equation (5.4) is just the steady-state Lyapunov equation for the system described by (5.3). In terms of $P_p(z)$, the filtered error spectrum $P_f(z)$ is simply

$$P_f(z) = [I-K_a(z)C(z)]\,P_p(z)\,[I-K_a(z)C(z)]^* + K_a(z)R(z)K_a^*(z) \tag{5.5}$$

The suboptimality of the approximate filter gain $K_a(z)$ can be evaluated, therefore, from (5.4) and (5.5).

The update step when Assumption 2 holds, i.e. $C(z)=C_o$, will be briefly considered further. Recall that in this case the update step is equivalent to a smoothing problem and can be realized by two Kalman filters. The irrational steady-state error spectrum $P(z)$ can be approximated arbitrarily closely by a rational function $P_a(z)$. Assume that $P_a(z)$ is a sufficiently accurate approximation, and let $(\Phi_a, \Gamma_a, \Theta_a)$ be the corresponding finite-dimensional linear system. The dimension of this realization, while finite, might well be quite high - too high, in fact, to directly design a filter from it.

Rather than using $(\Phi_a, \Gamma_a, \Theta_a)$ to implement the smoothing of the predicted errors, we propose using it as a benchmark against which reduced-order smoothers

can be compared. The suboptimality of smoothing with the reduced-order model $(\Phi_r, \Gamma_r, \Theta_r)$ instead of the high-order model can be evaluated from the results in [10] - assuming that the spectrum $P_a(z)$ is the actual error spectrum. Since $P_a(z)$ is an accurate representation of the error spectrum, this approach provides a useful measure of the suboptimality of the reduced-order smoother.

In this section, time-invariant filters employing only finite-dimensional realizations S^+ and S^- have been discussed. A second approximation will be required, however, in order to obtain an estimate $\hat{x}_k(i|i)$ with only <u>finite</u> computation.

VI. FINITE MEMORY FILTERS

In addition to the growing dimension of the realization of the filter gain, the optimal Kalman filter has a second undesirable property. The estimate $\hat{e}_k(i)$ of the error at subsystem k depends, in general, on the innovations all along the line, i.e., $\forall \ell$. This centralized estimate is obtained from two linear systems, both starting infinitely far away from subsystem k. The resulting delay in computing $\hat{e}_k(i)$ must be infinite.

One is led to consider a suboptimal filter which estimates $e_k(i)$ from only a finite number of the innovations at neighboring subsystems. This corresponds to using a filter gain K_a which has only a finite number of nonzero elements. Any such gain K_a has a transform $K_a(z)$ which is a rational function of z, and hence (5.4) and (5.5) can be used to evaluate the suboptimality of using only a finite number of measurements to update each substate estimate $\hat{x}_k(i|i)$.

In general, it is not particularly clear how such a K_a, with only a finite number of nonzero elements, should be chosen. But under Assumption 2, it is quite obvious how to handle this situation. Suppose the estimate $\hat{e}_k(i)$ is restricted to being a function of $I_\ell(i)$ for $k-N_1 \leq \ell \leq k+N_2$. This is nothing but a finite interval smoothing problem. Thus, the estimate $\hat{e}_k(i)$ can be obtained as the output of two Kalman filters -- the forward one starting at $k-N_1$ and the backwards one starting at $k+N_2$. The two filters are not in the steady state, but are time (space)-varying. The covariance of the estimate $\hat{e}_k(i)$ can be obtained from the reduced order smoother results of [10] assuming that $P_a(z)$ is the spectrum of the process $e_k(i)$.

The implementation of this suboptimal finite-interval smoother will now be considered. Since the estimate $\hat{e}_k(i)$ is obtained from two Kalman filters, only the forward filter will be handled explicitly. Similar comments, of course, apply to the backwards filter. Once again, the discussion has the time index i fixed.

The forward estimate of $e_k(i)$ is based on $I_\ell(i)$ for $k-N_1 \leq \ell \leq k$. This estimate is obtained from a forward Kalman filter starting at subsystem $k-N_1$. If the forward estimate of $e_{k+1}(i)$ is based on the same number of innovations, i.e.,

$I_\ell(i)$ for $k-N_1 + 1 \leq \ell \leq k+1$, then another Kalman filter starting at subsystem $k-N_1 + 1$ is required. Continuing this argument, it is clear that if each estimate uses exactly N_1+1 of the innovations, then it is necessary to start a forward Kalman filter at each subsystem. Also, each innovation $I_\ell(i)$ is used to update N_1+1 different forward Kalman filters. Figure 3 illustrates the use of a separate filter to obtain each estimate. The number of updates required to compute the forward estimate of $e_k(i)$, of course, is also $N_1 + 1$.

Since each estimate is computed from a separate forward Kalman filter, this is a totally parallel computation scheme. Consider now the processing that occurs at each subsystem as this parallel filtering is performed. The innovations $I_k(i)$ at subsystem k are used N_1+1 times to update N_1+1 different filters, as previously noted. Subsystem k transmits N_1 of these estimates to subsystem $k + 1$; the other estimate yields $\hat{e}_k(i)$. The computational burden at each subsystem under this parallel filtering procedure, therefore, is very much different from the burden when all the forward estimates are computed from one filter starting at minus infinity -- essentially a totally serial computational scheme. In the latter case, the innovation $I_k(i)$ is used to update only one filter, and only one estimate is transmitted from subsystem k to subsystem k+1. The conclusion is that the parallel scheme greatly increases both the number of times each subsystem must update a filter, and the number of transmissions between subsystems.

By altering the requirement that each forward estimate must be based on exactly N_1+1 of the innovations, it is possible to obtain a tradeoff between: (i) the computations and transmissions required at each subsystem, and (ii) the delay in computing all the forward estimates. Delay here refers to the maximum number of updates and transmissions needed to compute any particular estimate. This will become more clear shortly. Suppose the requirement is that <u>at least</u> N_1+1 innovations are used for every forward estimate and a new Kalman filter is started every M_1 subsystems. This implementation is shown in Figure 4. Under the totally parallel scheme, M_1 was equal to one.

If a new filter happens to be started at subsystem k_o, this filter provides its first estimate (based on N_1+1 innovations) at subsystem $k_o + N_1$. The next

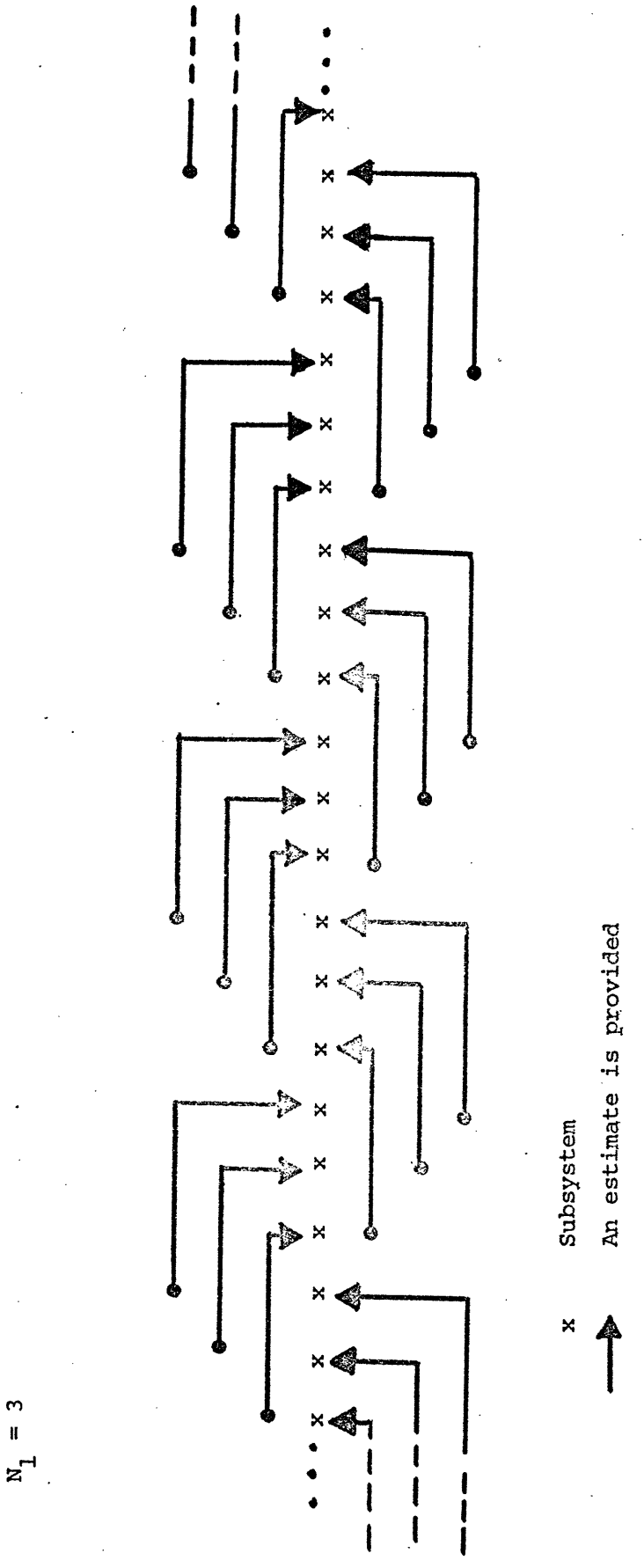

FIGURE 3. Each forward estiamte uses exactly 4 innovations and is obtained from a separate Kalman filter.

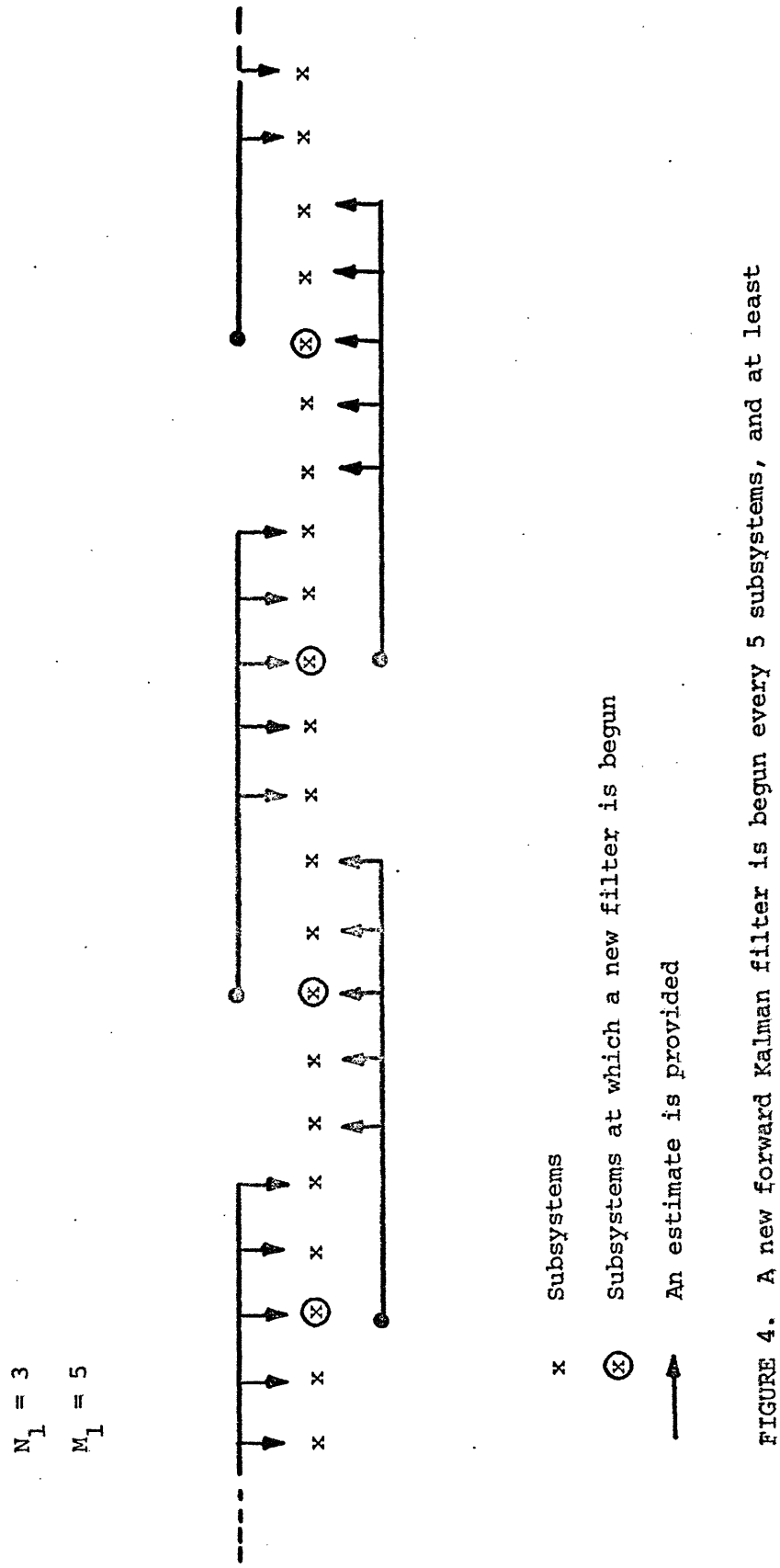

FIGURE 4. A new forward Kalman filter is begun every 5 subsystems, and at least 4 innovations are used in every estimate

filter starts at subsystem $k_o + M_1$ and provides its first estimate at subsystem $k_o + M_1 + N_1$. The filter that started at subsystem k_o, therefore, must provide estimates at subsystems k_o+M_1, $k_o+M_1+1,\ldots,k_o+M_1+N_1-1$ -- a total of M_1 estimates. This last estimate (the one at subsystem $k_o + M_1 + N_1-1$) is based on N_1+M_1 innovations. This means that the delay in obtaining this estimate is due to updates at N_1+M_1 subsystems and transmissions between $N_1 + M_1-1$ subsystems. When a new Kalman filter is begun at every subsystem, i.e., $M_1 = 1$, its last (in fact only) estimate requires just N_1+1 updates and N_1 transmissions. Choosing M_1 greater than one, therefore, increases the delay in obtaining the forward estimates.

However, what is the effect of increasing M_1 on the computational burden of each subsystem? Consider the choice $N_1=3$ and $M_1=5$ as depicted in Figure 4. If a new filter is begun at subsystem k_o, then subsystems k_o, k_o+1, and k_o+2 must update two different filters, and subsystems k_o and k_o+1 must transmit two estimates apiece. The remaining subsystems update only one filter and transmit only one estimate. For general N_1 and M_1, simple counting arguments may be used to determine how many updates and transmissions must be performed by the various subsystems. The point here is that the computational burden at the individual subsystems is reduced by increasing M_1, the separation between adjacent Kalman filters. Increasing M_1, however, was previously observed to result in an increased delay before the last estimate of a Kalman filter was available. Therefore the parameter M_1 can be used to perform a tradeoff between (i) computations and transmissions, and (ii) delay, as was desired.

It should be remembered that this discussion has taken the time index i to be fixed. There is in fact a clock rate for the Toeplitz system. This means that all the filtering computations must take place between i and i+1. This clock rate limits the acceptable delay in computing each estimate.

VII. CONCLUSIONS

In this paper, an efficient implementation of the optimal centralized Kalman filter of Hager and Horowitz [5] for Toeplitz systems has been presented. The key observation here was that the information transfer among subsystems could be realized by a spatial transfer function. Implementation of the resulting spatial transfer functions requires only nearest neighbor communications.

There were two disadvantages associated with the optimal filter, however. The first was that the order of the spatial realizations may grow unbounded with increasing time. This led to the time-invariant filters and their finite-dimensional approximation in Section V. The second disadvantage of the optimal filter was that infinite computation is required to compute the estimate of any substate. This led to an examination of finite (spatial) memory filters in Section VI. In both Sections V and VI, the fact that the measurement update step is equivalent to a smoothing problem was a key ingredient in the formulation of suboptimal filters.

Recall that the discussion of Section VI dealt with a suboptimal update step that uses only a finite number of the innovations to update each subsystem. How many innovations are required to yield a good estimate of $e_k(i)$? The process $\{e_k(i)\}$ is being viewed as the output of the linear system $[\Phi(i), \Gamma(i), \Theta(i)]$. Thus, if the innovations used to estimate $e_k(i)$ extend about subsystem k for several of the slowest "time" (space) constants of $\Phi(i)$, then the resulting estimate should be quite good. We see that there are two "time" scales of interest here. First, of course, is the actual time index of the system. For example, one may ask how large i must be before the system can be considered to be in the steady state. The second "time" scale is actually a spatial scale of the spatial linear system $[\Phi(i), \Gamma(i), \Theta(i)]$. The question here is how many neighboring subsystems must be used to provide an accurate estimate of a substate.

It is important to realize that as more neighboring subsystems are used to compute a substate estimate, i.e., as the number of space constants is increased, the time required to perform the update step is also increased. The time required, however, must be less than the difference between time i+1 and time i. This means the time and space indices of the system directly interact in the filtering

process. There is an interesting and important tradeoff, therefore, between (i) the actual time period at which the observations are sampled and (ii) the number of neighboring subsystems used to update a substate estimate. We point to the work in filtering for systems with multiple time scales, such as [11], as one area which might be useful here in making this tradeoff.

We also note that the measurement update step is being interpreted as a "vertical" propagation of information, while the prediction step is interpreted as a "horizontal" propagation of information. An intriguing possibility is the use of "diagonal" propagations of information in the filtering process. What we have in mind here is incorporating into the estimate of $e_k(i)$ not only observations taken at time i by subsystems near k, but also observations taken at (say) time i-1 by subsystems farther from k. It is to be noted that this suggestion is purely speculative.

In closing, we point out that there is, of course, a control problem dual to the filtering problem considered here. The control gain can be implemented by two linear systems moving up and down the line of subsystems just like the filter gain. Again by duality, the two spatial Kalman filters used in the measurement update step have corresponding control problem duals. It is not clear, however, exactly what are the forward and backward control problems. Also, it is not clear what is the finite memory controller, dual to the finite memory filter of Section VI. These issues are touched upon in [10] and will be the subject of a future publication.

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