Non-Equilibrium Fluctuation Induced-Phenomena in Quantum Electrodynamics

by

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Abstract

We study fluctuation-induced phenomena in systems out of thermal equilibrium, resulting from the stochastic nature of quantum and thermal fluctuations of electromagnetic currents and waves. Specifically, we study radiative heat transfer and Casimir forces by applying the scattering formalism that expresses results solely in terms of the classical scattering matrices of the objects. For example, we obtain exact formulas for the heat radiation emitted by long cylindrical objects, as well as for Casimir forces that arise between them. We apply our results to explore the dependence of these phenomena on size and material properties of cylinders.

While the scattering formalism is very general and technically can be employed for arbitrary shapes, in practice it is very time-consuming to apply it to the most experimentally-relevant and complex case of objects at close proximity. We examine easier ways to compute the heat transfer in such case. In particular, we develop a small distance expansion for the heat transfer between gently curved objects, in terms of the ratio of distance to radius of curvature. This expansion allows us to rigorously justify the widely used approach of “proximity transfer approximation”, and to quantify corrections to it in the limit of small separation. Moreover, we study the role of surface roughness, and show that it may change the distance dependence of the heat transfer as well as Casimir forces between curved objects at proximity.

Finally, as an alternative approach we construct general Green-Kubo relations that connect radiative heat transfer, non-equilibrium Casimir forces and vacuum friction between arbitrary objects to fluctuations in equilibrium which may be easier to consider from the perspective of experiment and simulations.

Thesis Supervisor: Mehran Kardar
Title: Professor

Thesis Supervisor: Robert Jaffe
Title: Professor
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Getting accepted into MIT and successfully obtaining my PhD would have been impossible without my parents, Sasha and Lilya. From very early age, they supported my interest in math and physics and were willing to give away all that they had so that I would be able to get the best education, tutors, books, and parental support. My other family members also played a vital role in my education. My father Sasha, paternal grandfather Slava and his father (my grandfather) Roman were physicists and my interest towards the subject was probably ingrained in my genes. When I was six, my aunt, Raya, signed me up to attend classes at the mathematical club "Evrika" at Khrakiv National University (at which I obtained B.S. and M.S. degrees 15 years later). That was my first, and perhaps, my most important step in the world of science. A couple of years later, my grandmother Galya and my now deceased grandfather Borya were helping my parents financially so that I could attend English classes that my parents could not afford due to a terrible economic situation in Ukraine during the nineties. My brother Dima, cousin Jenya, uncle Valya, grandmother Katya - all of them were essential to my academic success and were sharing with me all the happy and sad moments since my childhood.

This thesis is dedicated to them.

Before coming to MIT I had been working in the field of theoretical hard condensed matter physics and was planning to stay in the same field at MIT without having a specific idea of whose supervision I would be under. I was interested in several groups at MIT, but (luckily) they were not accepting new members that year and after my first semester at MIT I felt disappointed that I was without a clear idea of what to do next. I was ready to deviate from my initial desire to work in hard condensed matter theory and decided to contact Prof. Mehran Kardar in December of 2010. We met to discuss a potential collaboration in the area of his interest – Casimir forces and heat transfer. After the meeting I e-mailed him again saying that I was interested in working with him on the topics he suggested, but he did not reply back (now, when I know him much better, I realize it was an exception. He is one of the most
punctual and organized people I have ever met). I thought he was not impressed by me or found another student. In the end of January of 2011, after I got even more desperate to find a research group, I decided to try and contact him again, just in case. Luckily, he replied immediately apologizing to me that he did not notice my previous e-mail due to a problem with his inbox. Thus, we met and he provided me right away with an office space, a computer, Dr. Matthias Krüger as an officemate, and an invitation to attend his group’s meetings. After all of the disappointment I endured trying to find a research group, I felt like the happiest man on Earth, as someone did actually believe in me, and was interested in working with me. Now, having worked with Prof. Mehran Kardar for the last 3.5 years, I can responsibly claim that if I had to choose again any other advisor at MIT or elsewhere – there is nobody who I would even consider. On a personal level, throughout my stay at MIT, he was very supportive, responsible towards me, understanding, responsive, and willing to do what was best for me given my desires and interests. I could not thank him enough for that. On the scientific side of things, I learned a lot from him not only through taking three absolutely amazing courses he taught and TA-ing two of them, but also much more through research collaboration. His expertise and knowledge are immense and I humbly hope that I grasped at least a small portion of it. Moreover, he was a fantastic example for me regarding how to think and formulate and present things clearly – something I was not good at before meeting him.

As I mentioned above, when I joined Prof. Mehran Kardar’s group I was sharing an office with Dr. Matthias Krüger. At first, I think he was not extremely happy about sharing an office with me that he used to have all to himself. Also, I was bothering him with a lot of questions as the problem I was given to study was in his field of expertise. Luckily, he was kind enough to answer a huge amount of questions I had and that helped me to progress very quickly. That was the beginning of a very fruitful collaboration and friendship. Our numerous discussions and brainstorming sessions resulted into a set of papers that are the skeleton of this thesis. He was a co-author of every paper that was published during my stay at MIT and without him none of them would be as well done and well written as they were. He was not
only the best collaborator I could dream of, he also became a good friend of mine. During difficult times (especially during the "gradient expansion" project) he was supporting and pushing me when I was ready to give up. Many times I thought he was saying nonsense and we would not be able to accomplish anything, but he pushed me, motivated me, and in the end he was almost every time right. I just cannot thank him enough for all of the effort and support he put into me. Without him, I would never be able to publish five papers in such a short period of time and graduate from MIT in just four years.

As I started working for Prof. Mehran Kardar and attending the group meetings, my interactions with Prof. Robert Jaffe were established and he became my co-adviser. He was always very friendly to me and his sincere curiosity in the work I was doing made it a joy to discuss it with him. I found his questions very deep and meaningful, and the suggestions he made very insightful. His vast knowledge of physics enriched my understanding of things.

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Chapter 1

Introduction

Quantum thermal fluctuations of electromagnetic waves are a cornerstone of modern statistical physics, and inherent to such phenomena as thermal radiation and Casimir forces. The topic of thermal radiation traces back to the beginnings of quantum mechanics more than a century ago [1], when Planck employed thermal fluctuations to describe the spectral radiation intensity of a black body at a given temperature. Planck’s law directly yields the well-known Stefan-Boltzmann law [2] (see Eq. (2.41) below). In turn, the equilibrium Casimir force between two perfectly conducting parallel plates also arises due to the quantum fluctuations of the electromagnetic waves in the intervening vacuum (or, equivalently, due to the charge and current fluctuations in the plates) [3]. Typically, at small separations zero-point fluctuations shape the force, whereas at separations large compared to the thermal wavelength \( \lambda_T = \frac{h c}{k_B T} \) (approximately 7.6 \( \mu m \) at room temperature \( T = 300K \)), thermal effects dominate and give rise to non-equilibrium Casimir forces [4-6].

Over the last decade, there has been progress in the precision measurements of heat transfer and Casimir forces at sub-micron scale. This growing interest can be attributed to the fact that heat transfer measurements are directly connected to scanning tunnelling microscopy and scanning thermal microscopy under ultra-high vacuum conditions [7, 8], whereas Casimir forces must be accounted for in the fabrication of micro and nano electromechanical devices actuated by electrical bias (where this force is dominant). Thus, the development of theoretical tools that allow treat-
The study of systems out of thermal equilibrium is of particular relevance. One approach is to apply fluctuational electrodynamics (FE) introduced by Rytov [9] to such systems [10, 11]. Importantly, Rytov's formalism makes an additional simplification that there is local equilibrium within each object. The main difficulty of this, and other approaches, compared to the equilibrium case is that such powerful tools of statistical physics such as entropy and Helmholtz free energy (that contains information about forces and torques) cannot be applied to systems out of thermal equilibrium. In turn, the application of FE to equilibrium Casimir force within the framework of scattering formalism allows to compactly represent results in terms of scattering operators of the objects [12–14]. Later, the description of non-equilibrium Casimir force in terms of scattering operators of the objects has been developed [15], and some of the results presented in this thesis are a part of that work.

Heat radiation by objects with sizes smaller or comparable to the thermal wavelength are of particular interest as it cannot be accurately described by Planck’s and Stefan-Boltzmann laws. This happens due to interference effects between the object and the emitted radiation, so that the emissivity and absorptivity of object depend on its size, shape, and separation from other object. Additionally, if the object is smaller than the penetration (skin) depth, the emitted power is proportional to the object’s volume, rather than its surface area. Theoretical studies of these effects have been carried out for spheres, plates and cylinders, where the scattering formalism was extensively exploited [9, 16–22] and allowed to obtain results in terms of scattering matrices of the objects. Also, effects of excitations [22] and electric currents [23] on the radiation have been studied. Furthermore, recent studies on superscattering properties of subwavelength nanostructures (e.g. nanorods) [24] make such systems potential candidates for efficient heat transfer applications.

While heat radiation by an isolated object is already a complicated topic, heat transfer between multiple objects is even more challenging. The existence of another scale in a system, the separation between objects, yields non-trivial effects if it is

1 While for plates and spheres the exact result for the heat radiation has been obtained, the radiation by cylinder has been treated with some approximations [16–18]
smaller or comparable to the thermal wavelength. Indeed, more than 40 years ago Van Hove and Polder used Rytor's FE to predict that radiative heat transfer between objects separated by a vacuum gap can exceed the blackbody limit [25]. This is due to evanescent electromagnetic fields decaying exponentially into the vacuum. The enhancement of heat transfer in the near-field regime (generally denoting separations small compared to the thermal wavelength) has only recently been verified experimentally [26, 27]. Theoretically, heat transfer has been considered for a limited number of shapes: parallel plates [7, 25, 28, 29], a dipole or sphere in front a plate [19, 30, 31], two dipoles or spheres [30, 32, 33], and a cone in front of a plate [34]. The scattering formalism has been successfully exploited [5, 9, 19, 35, 36] in this context. Although powerful numerical techniques [34, 37] exist for arbitrary geometries, analytical computations are limited to planar, cylindrical and spherical cases [5, 15].

Rytov’s formalism is also very useful to study out of equilibrium Casimir interactions. Recently, out of equilibrium Casimir forces have been considered in several systems: parallel plates [11, 36, 38], modulated plates [35], plate and an atom in different setups [36, 39 42], two spheres and sphere and a plate [5]. Also, there is extensive literature on the topic of non-equilibrium interactions between two atoms (or molecules) [43-49]. Formalisms for arbitrary objects were presented in Refs. [15, 19, 36]. A common feature for setups involving compact objects is the need to account for the contribution of the environment to the force, which depends on a possibly different ambient temperature. Importantly, thermal non-equilibrium Casimir forces can be repulsive [5, 11, 15, 36, 42, 50] and yield stable zero force points [5, 15, 50].

Alternatively, heat transfer and Casimir forces between closely spaced curved objects can be estimated by use of the proximity transfer approximation (PTA) [19, 26, 27, 31, 51] and the proximity force approximation (PFA) [10, 52–55], respectively. The proximity approximation (PA) has long served as a useful guide for estimating interactions between closely spaced objects with curved surfaces. Originally introduced by Derjaguin [56] to compute van der Waals forces between colloidal particles, the PA relates the interaction between curved objects at close separations to the
corresponding interaction between two flat surfaces, over an area determined by the local radii of curvature. In addition to the cases of radiative heat transfer (PTA) and Casimir forces (PFA), the PA has been successfully used in experimental studies in other fields, such as classical Casimir forces in a fluid near a critical point [57], interactions among nuclei [58] and gravitational forces [59–61]. In case of heat transfer, PTA works as follows: the heat transfer between two parallel plates (per unit area), which is a function of separation, is averaged over one of the (projected) curved surfaces. PTA is generally assumed to hold asymptotically for small separations. However, in contrast to the case of PFA for equilibrium Casimir forces, there had been no rigorous proof for PTA available in the literature before our work.

The theoretical analysis, simulation or experimental measurement of out of equilibrium quantities are important and challenging. The inability to employ fundamental concepts of equilibrium statistical physics, e.g. free energy or entropy, can make theoretical analysis laborious as discussed above. Experimentally, it can be tedious to maintain the system in a well controlled non-equilibrium state. In contrast to that, equilibrium quantities are often easier to access. For example, the mean square displacement of a Brownian particle in equilibrium is well amenable to measurements, whereas the direct evaluation of the Brownian particle’s mobility is generally more difficult [62]. The two quantities are linked by the well known Einstein linear response relation. More generally, linear response relations are helpful in understanding and quantifying non-equilibrium properties in terms of equilibrium fluctuations. The Green-Kubo relation [63, 64] allows to obtain e.g. thermal [65] and electric [66] transport coefficients or the shear viscosity [67], by connecting linear transport coefficients to time integrals of equilibrium correlation functions of fluxes associated with conserved densities. Such relations have found applications in the context of molecular dynamics simulations. Thus, an interesting question that arises in the context of FE is on developing corresponding relations that could point to easier ways in computations and measurements of non-equilibrium quantities such as radiative heat transfer and Casimir forces.

In this thesis, we consider systems of one and multiple static objects held at temp-
peratures different from one another and from the ambient environment. In the very end, we briefly consider a system in thermal equilibrium that consists of one or multiple objects moving at constant velocity relative to one another and the environment. We use and develop different approaches based on the scattering formalism and Rytov's FE that allow us to treat and better understand heat radiation, heat transfer, non-equilibrium Casimir forces and vacuum friction. Here is a list of problems that we address in this thesis:

- Exploiting the framework of FE in thermal non-equilibrium, we present a detailed derivation of thermal radiation by cylindrical tubes as well as of non-equilibrium Casimir forces between them, expressing the results in terms of the scattering matrices of the objects [68, 69]. We study the dependencies of the obtained results on the sizes, material properties and temperatures of the objects.

- We develop a small distance expansion for the radiative heat transfer between gently curved objects that allows us to go beyond the lowest order PA and rigorously justify it [70]. Additionally, using PTA we show that roughness or surface modulations change the distance dependence of (power-law) interactions between curved objects at proximity [71]. We concentrate on the heat transfer between a dielectric sphere in front of a dielectric plate.

- Developing a method for computation of higher order correlation functions in FE, we derive a set of linear response relations in FE [72], e.g. analogues of Green-Kubo [63, 64], Kirkwood [73-75] and Onsager reciprocity [76] relations.

In the following, we discuss these points in more detail. In the first part of this thesis, Chapter 2, we start by studying heat radiation of a long cylindrical object (wire). This is an experimentally relevant topic as the radiation of thin cylindrical objects with thickness in the range of the thermal wavelength is very well accessible. It has e.g. been studied using metal wires with interesting findings made: the radiation was measured to be polarized in the direction perpendicular to the wire [77, 78] when the wires’ thickness was larger or comparable to the thermal wavelength. These findings triggered a number of studies on the properties of thermal radiation of sources of various designs including carbon nanotubes [79-84]. For wires with thickness smaller
or comparable to the thermal wavelength, the radiation was found (e.g. for platinum) to be polarized in the direction parallel to the wire becoming fully polarized as the diameter approaches zero [79, 80]. Polarization effects have also been observed for radiation of bundles of carbon nanotubes [82] and are considered a simple way of finding the degree of alignment inside the bundle. Recent work on the heat radiation of thin metal wires [18] provides experimental as well as theoretical results, albeit restricted to emission perpendicular to the cylindrical axis. Moreover, a series of works [17, 85, 86] discuss radiation emitted by individual incandescent carbon nanotubes. From our side, we attempt to describe these experimental findings by deriving an exact formula for heat radiation in terms of classical scattering properties of arbitrary cylindrical objects. We do so by exploiting a more general framework [19] for non-equilibrium electromagnetic fluctuations involving multiple objects and arrays [5, 31, 34–36, 42, 50, 87]. Thereby we provide derivation of the corresponding results presented in Refs. [19, 68] that include the possibility of dielectric or magnetic losses, locality or non-locality. We do so by starting from quantum thermal fluctuations inside the object following the methodology of Rytov’s FE. We apply our results to study the dependence of heat radiation on material properties, temperatures and radius as well as discuss polarization effects for various cases. In addition to conductors and dielectrics, we apply our formula to introduce a simple model for the heat radiation of multi-walled carbon nanotubes (MWCNT). While our results agree with previous experimental work, we additionally predict new effects: for example, the metallic wire’s emissivity may significantly exceed the black-body limit for the same geometry. Also, it is important to mention that our theoretical predictions for the heat radiation published in Ref. [68] have already been verified experimentally for the case of a silicon dioxide nanowire Ref. [88].

Further in Chapter 2, using the same formalism we consider non-equilibrium forces between two parallel arbitrary cylinders. We are interested in this particular configuration because cylindrical geometries are amenable to high precision experiments [89–94], providing good contrast to the more widely studied spherical geometry [95]. We obtain general expressions for forces between wires maintained at different tempera-
tures from each other and the environment. These expressions are then analytically and numerically studied in a number of cases. We find that the non-equilibrium forces are generally larger than their equilibrium counterpart at separations greater than the thermal wavelength. They may also exhibit oscillations as function of separation, leading to stable points of zero net force. These effects are particularly pronounced for thin conducting cylinders (e.g. 40nm diameter nanowires of tungsten) due to their large emissivity, which make such wires potential candidates for future experiments.

In Chapter 3, we study heat transfer between objects for the experimentally most relevant case of small separations. Given the difficulty (by numerics or analysis) of finding results by the scattering formalism, we introduce an alternative route by developing a gradient expansion for heat transfer between closely spaced objects. This enables us to rigorously justify PTA and to quantify corrections to it in the limit of small separations. We find that the expansion converges faster for the derivative of the heat transfer than for the transfer itself, and we use this finding by introducing a near-field adjusted plot that allows direct comparison of experimental data to our theoretical prediction (without a free parameter). We concentrate on the case of a sphere and a plate, and find that the logarithmic correction to the leading term has a very small prefactor for all materials investigated.

Further in Chapter 3, we utilize PTA to investigate scaling of the heat transfer between two surfaces whose local radii of curvature can be decomposed as the sum of components which vary along the surface on well separated length scales. We note that this situation encompasses the experimentally relevant cases of two large rough objects when the correlation length of the roughness is much smaller than the characteristic radius of curvature of the surface, or alternatively the example of a surface with large average radius of curvature modulated by small structures fabricated by the experimenter. We find that at proximity a subtle interplay between roughness/modulation and the global curvature of the surfaces leads to a drastic change in the distance dependence of the heat transfer, compared to that for perfectly smooth (structureless) surfaces. Moreover, we discuss the influence of roughness/modulations on other interactions, such as Casimir forces. The modified asymptotic scaling law of
interaction is found to depend in a simple way on the order of the first non-vanishing term in the Taylor expansion of the height distribution function of the surfaces. The latter is a geometric feature of the surface that can be either computed for well characterized modulations, or experimentally obtained via scanning probes of rough surfaces, and then used to predict the short distance scaling of results in a multitude of experiments. We discuss what our findings imply for the most relevant heat transfer setup – a sphere in front of a plate. Specifically, we show how modulations (of various shapes) or roughness limit the maximum amount of non-contact heat transfer that can be achieved between two surfaces in close proximity.

Motivated by the fact that equilibrium quantities are often easier to access both theoretically and experimentally, in Chapter 4 we study linear response relations for quantum thermal fluctuations of the electromagnetic field, which are related to radiative heat transfer $H$, and the Casimir force $F$. By explicitly computing correlation functions of these quantities for a collection of arbitrary objects in vacuum, we identify them with previously found non-equilibrium expressions for radiative heat transfer and non-equilibrium Casimir force, thereby obtaining a Green-Kubo matrix for heat conductivities. The non-equilibrium force is in turn related to the equilibrium correlations of $F$ and $H$. We also provide a closed form expression for the vacuum friction for a collection of arbitrary objects. We finally give a relation for the change in heat absorption upon changes in velocities, explicitly confirming the Onsager theorem. The experimental relevance of our findings is then discussed.

Finally, in Chapter 5, we expand upon some future directions and open problems.
Chapter 2

Scattering approach to fluctuation-induced phenomena in systems out of thermal equilibrium

In this Chapter we reproduce our results, up to some changes in notation, as published in Refs. [68, 69]. In retrospect, we must note that the results obtained in this Chapter could have been achieved in an easier way if we exploited the framework from Ref. [15] that was developed after this initial work had been completed.

In Sec. 2.1 we consider heat radiation from long cylindrical objects depicted in Fig. 2-1(a). We start by incorporating Green’s function techniques into Rytov’s formalism [9], which relates fluctuations of the electromagnetic field to fluctuating sources within the bodies, and accordingly to the material’s dispersive properties, via the fluctuation-dissipation theorem. This allows us to give the general formalism for heat radiation of arbitrary objects, and to apply it to the case of a cylindrical object. We apply our results to study the dependence of heat radiation on material properties, temperatures and radius as well as discuss polarization effects for various cases. We consider such materials as dielectrics, conductors, and multi-walled carbon nanotubes (MWCNT).

In Sec. 2.2 we consider non-equilibrium Casimir force between two parallel long cylindrical objects depicted in Fig. 2-1(b). We start by extending our formalism to
Figure 2-1: The setups of the problems considered in Chapter 2: (a) In Sec. 2.1 we study the heat radiation from long cylindrical objects made up of uniaxial or isotropic dielectric material. The cylinder with radius $R$ is held at temperature $T$ and is surrounded by environment at temperature $T_{env}$. Red arrows display heat radiation emitted by the sources inside the cylinder. (b) In Sec. 2.2 we study the non-equilibrium parts of Casimir forces $F^{(1)}$ and $F^{(2)}$ acting on two long cylindrical objects made up of uniaxial or isotropic dielectric materials. The cylinders are described by the corresponding radii $R_1$ and $R_2$, and are held at the temperatures $T_1$ and $T_2$, respectively. The temperature of the surrounding environment is $T_{env}$. In general, the forces $F^{(1)}$ and $F^{(2)}$ can be either attractive (as depicted in the figure) or attractive.
the case of two objects and find a general expression for non-equilibrium force be-
 tween parallel cylinders characterized by an arbitrary dielectric function. We then analytically and numerically study these expressions for specific cases such as dielectric/conducting materials in cold/hot environment.

2.1 Heat radiation from long cylindrical objects

2.1.1 Heat radiation in terms of scattering (U) operator

General formalism for arbitrary objects

In this Chapter, for any two generally non-commuting field operators \( \hat{A} \) and \( \hat{B} \), we consider the symmetrized expectation value,

\[
\langle \{ \hat{A}(t,r), \hat{B}(t',r') \}_s \rangle = \frac{1}{2} \langle \hat{A}(t,r)\hat{B}(t',r') + \hat{B}(t',r')\hat{A}(t,r) \rangle.
\]  

(2.1)

We denote averages in equilibrium by \( \langle \cdots \rangle^q \), while non-equilibrium averages are indicated as \( \langle \cdots \rangle \). In a general case of non-commuting quantum operators, sym-
 metrization ensures the reality of \( \langle \{ \hat{A}(t,r), \hat{B}(t',r') \}_s \rangle \). Under stationary conditions, the expectation value depends only on the time difference \( t - t' \), and one can define the spectral density \( \langle A(r)B^*(r') \rangle_{s,\omega} \):

\[
\langle \{ \hat{A}(t,r)\hat{B}(t',r') \}_s \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \langle A(r)B^*(r') \rangle_{s,\omega}.
\]  

(2.2)

The reality of the symmetrized correlation function on the left hand side guarantees that the real and imaginary parts of the spectral density are even and odd functions of frequency respectively.

Consider an object of uniform temperature \( T_a \) placed in vacuum, enclosed by an environment (e.g. a cavity much larger than all other scales in the system) at temperature \( T_{env} \). A basic quantity of interest is the spectral density \( C \equiv C_{ij} \) of the
electric field $\mathbf{E}$,

$$C_{ij} \equiv \langle E_i(r) E_j^*(r') \rangle_{s,\omega}. \quad (2.3)$$

In global equilibrium, i.e. with $T_\alpha = T_{env} = T$, the spectral density $C_{ij}$ is related to the imaginary part of the dyadic Green’s function $G_{ij}$ of the object by the fluctuation-dissipation theorem (FDT) [9, 96],

$$C_{ij}^{eq}(T; \mathbf{r}, \mathbf{r}') \equiv \langle E_i(r) E_j^*(r') \rangle_{s,\omega}^{eq} \equiv \langle \mathbf{E}(r) \otimes \mathbf{E}^*(r') \rangle_{s,\omega,ij}^{eq} = [a(T) + a_0] \text{Im} G_{ij}(\omega; \mathbf{r}, \mathbf{r}'), \quad (2.4)$$

where $\otimes$ denotes a dyadic product. $a(T)$ describes the thermal contribution to quantum fluctuations,

$$a(T) \equiv \text{sgn}(\omega) \frac{8\pi \hbar \omega^2}{c^2} n_\omega(T), \quad n_\omega(T) = \frac{1}{\exp[\hbar \omega / k_B T] - 1}, \quad (2.5)$$

where $n_\omega(T)$ is the Bose-Einstein weight, $\omega$ is the angular frequency of radiation and $\hbar$, $c$ and $k_B$ are reduced Planck’s constant, the speed of light and Boltzmann’s constant, respectively. The zero point fluctuations, which contribute $a_0 \equiv \text{sgn}(\omega) \frac{4\pi \hbar \omega^2}{c^2}$, are independent of the object’s temperature and do not contribute to non-equilibrium effects. Hereafter we use the operator notation $\mathcal{G} \equiv G_{ij}(\omega; \mathbf{r}, \mathbf{r}')$, where operator multiplication implies an integration over space as well as a $3 \times 3$ spatial matrix multiplication, e.g. for the operators $\mathcal{A}$ and $\mathcal{B}$ (using Einstein summation convention),

$$(\mathcal{A} \mathcal{B})_{ik}(\mathbf{r}, \mathbf{r}'') = \int d^3 r' A_{ij}(\mathbf{r}, \mathbf{r}') B_{jk}(\mathbf{r}', \mathbf{r}''). \quad (2.6)$$

The Green’s function is the solution of [14, 97]

$$\left[ \mathcal{H}_0 - \mathcal{V} - \frac{\omega^2}{c^2} \mathcal{I} \right] \mathcal{G} = \mathcal{I}, \quad (2.7)$$

which follows because the electric field obeys the Helmholtz equation. Here, $\mathcal{H}_0 = \nabla \times \nabla \times$ describes free space, and $\mathcal{V} = \frac{\omega^2}{c^2} (\varepsilon - \mathcal{I}) + \nabla \times \left( \frac{1}{\mu} \mathcal{I} - \frac{1}{\mu} \right) \nabla \times$ is the potential introduced by the object. $\varepsilon$ and $\mu$ are the complex (possibly nonlocal) dielectric
permittivity and magnetic permeability tensor of the object. For isotropic and local materials, they reduce to scalars (e.g. $c = \epsilon \mu$). Let $G_0$ stand for the Green’s function of free space. Using the identities $\text{Im} G = -G \text{Im} G^{-1} G^*$ and $\text{Im} V = -\text{Im} (G^{-1} - G_0^{-1})$ [96], which follow from Eq. (2.7), we obtain

$$C^q(T) = C^0 + C^o(T) + C^{env}(T),$$

(2.8)

$$C^o(T) = a(T_\alpha) \text{Im} V G^* = a(T_\alpha) \int_{r',r'' \in V_\alpha} d^3r' d^3r'' G_{ij}(r, r')$$

$$\times \text{Im} V_{jk}(r', r'') G_{kl}^*(r'', r''''),$$

(2.9)

where the spacial integrals are restricted to the volume of object $\alpha$, $V_\alpha$ (note, $V_\alpha$ is different from the potential $V \equiv V_{ii}$). $C^0 = a_0 \text{Im} G$ is the zero point term. Equation (2.8) shows two different finite temperature contributions to the electric field in equilibrium. $C^o(T)$ contains an explicit integral over the sources within the object, as $\text{Im} V$ is only nonzero inside the object, and we identify it with the desired heat radiation from the object. The expression in Eq. (2.9) can be shown to be identical to expressions in the literature for both complex electric and magnetic permeabilities [9, 96], where, in general, one has two terms, including $\text{Im} \epsilon$ and $\text{Im} \mu$, respectively. The introduction of the potential $V$ appears useful here, as it allows for a compact notation including both terms. The third term in Eq. (2.8),

$$C^{env}(T_{env}) = -a(T_{env}) \text{Im} G_0^{-1} G^*$$

(2.10)

is the contribution sourced by the environment. As a specific model for environment, consider the objects enclosed in a very large black cavity maintained at temperature $T_{env}$. This latter identification can be corroborated on a different route by introducing a cold object into the thermal background field $E$ sourced by the environment, with field correlator given by $\langle E \otimes E^* \rangle^0_{S, \omega} = a(T_{env}) \text{Im} G_0$. At this point, it is useful to introduce the $\mathcal{T}$ operator or scattering amplitude $\mathcal{T}$ [14, 98] of the object. It relates the homogeneous solution of the Helmholtz equation in free space, $E$ (for $V = 0$), to the solution of this equation with the object present, $E^w$. This solution can be stated
in terms of the Lippmann-Schwinger equation,

\[
E^{sc} = (1 + G_0 T)E.
\] (2.11)

With this equation, the above introduction of the cold object into the free environment field is readily done and \(C_{\text{env}}\) is then the correlator of the field \(E^{sc}\),

\[
C_{\text{env}}(T_{\text{env}}) = \langle E^{sc} \otimes E^{sc*}\rangle_{S, \omega} = (1 + G_0 T)\langle E \otimes E^*\rangle^0_{S, \omega}(T^*G^* + 1) = -a(T_{\text{env}})\Im G_0^{-1}G^*,
\] (2.12)

in agreement with Eq. (2.10). Here we used the identity [14],

\[
G = G_0 + G_0 T G_0.
\] (2.13)

Equation (2.12) highlights the physical interpretation of \(C_{\text{env}}\): it is the radiation sourced by the environment and scattered by the object.

Having found the contributions of the different sources (environment and object), one can now vary the temperature of these independently in order to arrive at the field outside the object when its temperature is different from that of the environment. If \(T_{\text{env}} = 0\) this field corresponds to the heat radiation of the object. To this end, we notice that it is not necessary to derive all the terms in Eq. (2.8) as explained in the following: the explicit expression for \(C^o(T_o)\) in Eq. (2.9) contains the Green's function with one argument inside and one argument outside the object. While this function can be in principle derived, we find it more convenient to express \(C^o(T_o)\) in terms of the Green's function with both arguments outside the object, as it is directly linked to the scattering operator by Eq. (2.13). Therefore, it is interesting to note that \(C_{\text{env}}\) has all the sources outside the object and hence can be found in terms of this Green's function. While this is already obvious in Eq. (2.12), we additionally present a more rigorous way to derive \(C_{\text{env}}\). The environment sources, described by \(\varepsilon_{\text{env}}\), can be thought of as being everywhere in the infinite space complementary to the object, infinitesimal in strength (environment "dust" [96]), i.e. \(\varepsilon_{\text{env}} \rightarrow 1\). \(C_{\text{env}}\) in
Eq. (2.10) can hence be written

$$C_{\text{env}}(T_{\text{env}}) = a(T_{\text{env}}) \frac{\omega^2}{c^2} \lim_{\varepsilon_{\text{env}} \rightarrow 1} \int_{\text{outside}} d^3r'' \tilde{G}_{ik}(r, r'') \Im[\varepsilon_{\text{env}}] \tilde{G}_{kj}^*(r'', r'), \quad (2.14)$$

which is identical to Eq. (4) in Ref. [19]. Here, we introduced a Green’s function $\tilde{G}$ with $\nu$ inside the object and $\varepsilon_{\text{env}}$ outside. This is a simple modification of $G$ as a finite $\varepsilon_{\text{env}} - 1$ only changes the external speed of light so that $c$ in $G$ is replaced by $c/\sqrt{\varepsilon_{\text{env}}}$.

Finally the heat radiation of the object at temperature $T_{\alpha}$ can now be found by solving Eq. (2.8) for $C^\alpha(T_{\alpha})$,

$$C^\alpha(T_{\alpha}) = a(T_{\alpha}) \Im G - C_{\text{env}}(T_{\alpha}), \quad (2.15)$$

where $G$ is found using Eq. (2.13). Note that $C_{\text{env}}(T)$ can be derived from either Eq. (2.14) or directly from Eq. (2.12). For the case of the cylinder, we present below the former derivation in detail, and briefly sketch the latter starting from Eq. (2.12).

We emphasize again that the field emitted by the object in Eq. (2.15) is fully expressed in terms of the Green’s function with both arguments outside the object. In case one is only interested in the total heat emitted, the first term, i.e., the equilibrium field need not be derived, as it contains no Poynting vector. If, on the other hand, the radiation of the object is scattered at other objects, e.g. in order to compute heat transfer or nonequilibrium Casimir interactions, the full expression (2.15) has to be kept (it will be discussed in more detail in Sec. 2.2).

**Heat radiation of a cylindrical object**

In order to compute the heat radiation of a cylindrical object, we apply Eq. (2.15), evaluating the environment contribution by use of Eq. (2.14). Afterwards in this subsection, we briefly sketch the derivation via Eq. (2.12). In the cylindrical geometry (with parallel, radial and angular coordinates $z$, $r$ and $\phi$, respectively), the free Green’s function $G_0$ is expanded in cylindrical vector waves, $M_{n,k_z}^{\text{reg}}$ and $N_{n,k_z}^{\text{reg}}$. 

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corresponding to \( M \)-polarized and \( N \)-polarized regular waves \([98]\), see App. A.1. These are indexed by the multipole order \( n \) and \( k \), the component of the wavenumber \( k = \omega/c \) along the cylindrical axis. For outgoing waves we use \( \mathbf{M}_{n,k_z}^{\text{out}} \) and \( \mathbf{N}_{n,k_z}^{\text{out}} \) accordingly. In this basis, the \( \mathbb{T} \) operator of a cylindrical object is diagonal in \( n \) and \( k_z \), but couples different polarizations. Its entry \( T_{n,k_z}^{P',P} \) relates the amplitude of a scattered wave of polarization \( P' \) in response to an incoming wave of unit amplitude and polarization \( P \), with \( P, P' \in \{ M, N \} \). More precisely, the application of the cylinder’s \( \mathbb{T} \) operator, \( \mathbb{T}_c \), in Eq. (2.11) on regular cylindrical functions reads (hereafter we refer to the cylindrical object \( \alpha \) as object \( c \)),

\[
\mathbb{G}_0 \mathbb{T}_c \mathbf{P}^{\text{reg}}_{n,k_z} = \sum_{P'} T_{n,k_z}^{P',P} \mathbf{P}^{\text{out}}_{n,k_z}.
\tag{2.16}
\]

With these definitions and Eq. (A.2), the Green’s function of the cylinder is easily found, by use of Eq. (2.13), as

\[
\mathbb{G}_c = \mathbb{G}_0 + \mathbb{G}_0 \mathbb{T}_c \mathbb{G}_0 = \mathbb{G}_0 + \sum_{P,P'} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{idk_z}{8\pi} \mathbf{P}^{\text{out}}_{n,k_z}(r) \otimes \mathbf{P}^{\text{out}}_{-n,-k_z}(r') T_{n,k_z}^{P',P}.
\tag{2.17}
\]

When performing the integration in Eq. (2.14), we note that \( \mathbb{G}_0(r,r') \) in Eq. (A.2) is separated into two pieces, corresponding to \( r < r' \) and \( r' < r \). The contribution of a finite region vanishes asymptotically in the limit of \( \varepsilon_{\text{env}} \to 1 \) and can thus be neglected without changing the result. We can hence restrict the integration range to \( r' \geq r, r'' \), where we have to use exclusively one of the pieces. In general, one can restrict the integration in Eq. (2.14) to \( \xi(r') \geq \xi(r), \xi(r''), \) where \( \xi \) is the component which distinguishes the two expansions of \( \mathbb{G}_0 \).

Due to the orthogonality of two basis sets of the wave functions, the integrations over polar angle \( \phi' \) and cylindrical axis \( z' \) yield \( 2\pi \delta_{n,n'} \) and \( 2\pi \delta(k_z - k'_z) \), respectively, and we are left with only one term for each polarization in Eq. (2.14),

\[
\lim_{\varepsilon_{\text{env}} \to 1} \int r' dr' \left| \mathbf{P}^{\text{out}}_{-n,-k_z}(r') \right|^2 = \frac{2c^2}{\pi\omega^2} \frac{1}{\text{Im}\varepsilon_{\text{env}}} + \ldots,
\tag{2.18}
\]
where $\tilde{P}_{n,k_z}^{\text{out}}$ has analogous form to $P_{n,k_z}^{\text{out}}$ with the wavenumber $\sqrt{\varepsilon_{\text{env}} \omega / c}$ instead of $\omega / c$. Also, "..." represent higher order terms in $\varepsilon_{\text{env}} - 1$. This equation holds for $k_z^2 < \omega^2 / c^2$; for $k_z^2 > \omega^2 / c^2$ all terms are of order $\varepsilon_{\text{env}}^0$ and do not contribute in Eq. (2.14), a manifestation of the fact that the environment radiation does not contain evanescent waves.

The radiation from the environment after scattering at the cylinder then reads

$$C_{\text{env}}(T_{\text{env}})(\mathbf{r}, \mathbf{r}') = a(T_{\text{env}}) \sum_{P, \tilde{P}} \sum_{n=-\infty}^{\infty} \frac{d k_z}{\omega / c} \left\{ P_{n,k_z}^{\text{reg}}(\mathbf{r}) \otimes P_{n,k_z}^{\text{reg}*}(\mathbf{r}') \delta_{P, \tilde{P}} \right. $$

$$+ P_{n,k_z}^{\text{reg}}(\mathbf{r}) \otimes P_{n,k_z}^{\text{out}*}(\mathbf{r}') T_{n,k_z}^{P,P'} + P_{n,k_z}^{\text{out}}(\mathbf{r}) \otimes P_{n,k_z}^{\text{reg}*}(\mathbf{r}') T_{n,k_z}^{P,P'} + $$

$$+ P_{n,k_z}^{\text{out}}(\mathbf{r}) \otimes P_{n,k_z}^{\text{out}*}(\mathbf{r}') T_{n,k_z}^{P,P'} \right\} \text{ (2.19)}$$

where $\tilde{P}$ stands for the polarization opposite to $P$. Physically, Eq. (2.19) describes the thermal field for the case of an environment at temperature $T_{\text{env}}$ and a cold cylinder ($T_c = 0$). It can also be derived via Eq. (2.12), by noting that the radiation of the environment without cylinder present can be given in closed form,

$$\langle E \otimes E \rangle_0^{\text{env}} = a(T_{\text{env}}) \text{Im} G_0 = a(T_{\text{env}}) \sum_{P} \sum_{n=-\infty}^{\infty} \int_{-\omega / c}^{\omega / c} \frac{d k_z}{8\pi} \left[ P_{n,k_z}^{\text{reg}} \otimes P_{n,k_z}^{\text{reg}*} \right].$$

Applying $(1 + G_0 T_c)$ from both sides (compare to Eq. (2.12)) to the equation above, and using Eq. (2.16) we immediately obtain Eq. (2.19). This simple route towards the environment radiation (and hence the radiation of the object) has not been presented before.

Since we are only interested in the energy emitted by the cylinder, we do not explicitly derive the equilibrium field in Eq. (2.15) as it contains no Poynting vector. Equation (2.15) thus states Kirchhoff’s law, that the energy absorbed by the cylinder in the case $T_{\text{env}} = T, T_c = 0$ is the same as the energy radiated by it for $T_{\text{env}} = 0, T_c = T$. This is a special case of detailed balance in equilibrium, which generally states that the absorption coefficient of an object equals its emission coefficient. The formalism described here hence also provides a convenient route to find the absorption
coefficient of arbitrary objects. Technically, due to these considerations, the Poynting vector
\[ \langle U(r) \rangle = \frac{c}{4\pi} \int \frac{d\omega}{2\pi} \text{Re} \left[ \langle \mathbf{E}(r) \times \mathbf{H}^*(r) \rangle_{S,\omega} \right], \quad (2.20) \]
of the field in Eq. (2.19) gives complete information about the net energy flux for any temperature combinations. It can be derived via \( \mathbf{B}(\omega, r) = \frac{-i\omega}{c} \nabla \times \mathbf{E}(\omega, r) \) as well as relation (A.3). The power \( \langle H_c \rangle \) radiated per length \( L \) of the infinite cylinder in the general case of finite \( T_{env} \) and \( T_C \) is finally given by \([19]\),
\[ \frac{\langle H_c \rangle}{L} = -\frac{\hbar}{\pi^2} \int_0^\infty \omega d\omega (n_\omega(T_C) - n_\omega(T_{env})) \times \sum_{P=M,N} \sum_{n=-\infty}^\infty \int_{-\omega/c}^{\omega/c} dk_z \left( \text{Re} \left| T_{n,k_z}^{PP} \right|^2 + \left| T_{n,k_z}^{PP} \right|^2 \right). \quad (2.21) \]

Obviously, if \( T_C < T_{env} \), there is a net energy flux into the cylinder. In the following, we consider exclusively the case \( T_{env} = 0 \) (and denote \( T_C = T \)), for which case the energy flux is referred to as heat radiation of the cylinder. The expression (2.21) can be split into two terms each representing a different polarization of the corresponding electric field. Specifically, the term which describes polarization parallel to the cylinder is given by the \( N \)-modes,
\[ \frac{\langle H_N \rangle}{L} = \frac{\langle H_\parallel \rangle}{L} = -\frac{\hbar}{\pi^2} \int_0^\infty \omega d\omega n_\omega(T) \sum_{n=-\infty}^\infty \int_{-\omega/c}^{\omega/c} dk_z \left( \text{Re} \left| T_{n,k_z}^{NN} \right|^2 + \left| T_{n,k_z}^{NN} \right|^2 \right), \quad (2.22) \]
whereas the term responsible for the polarization perpendicular to the cylindrical axis is given by the \( M \)-modes,
\[ \frac{\langle H_M \rangle}{L} = \frac{\langle H_\perp \rangle}{L} = -\frac{\hbar}{\pi^2} \int_0^\infty \omega d\omega n_\omega(T) \sum_{n=-\infty}^\infty \int_{-\omega/c}^{\omega/c} dk_z \left( \text{Re} \left| T_{n,k_z}^{NM} \right|^2 + \left| T_{n,k_z}^{NM} \right|^2 \right). \quad (2.23) \]

In the following we use a standard definition of the degree of polarization \( I \) in order
to quantify polarization effects,

\[ I = \frac{\langle H_N \rangle - \langle H_M \rangle}{\langle H_N \rangle + \langle H_M \rangle}. \] (2.24)

In case one prefers a description in terms of the scattering matrix \( S \) [14, 18] with
\[ \mathcal{S}^{p'p}_{n,k_z} = 2T^{p'p}_{n,k_z} + \delta_{p,p'}, \]
the radiation in Eq. (2.21) can equivalently be written

\[ \frac{\langle H_c \rangle}{L} = -\frac{\hbar}{4\pi^2} \int_0^\infty \omega d\omega (n_\omega(T_c) - n_\omega(T_{env})) \sum_{p,p'} \sum_{n=-\infty}^{\infty} \int_{-\omega/c}^{\omega/c} dk_z (|\mathcal{S}^{p'p}_{n,k_z}|^2 - \delta_{p,p'}). \] (2.25)

Furthermore we note that the result for the perpendicular emission of the cylinder given in Ref. [18] can be recovered from Eq. (2.21) by restricting to \( k_z = 0 \), taking into account waves normal to the cylindrical axis only. In this case the \( \mathbb{T} \) operator is diagonal in polarization \( P \).

**Limit of large radius (radiation of a plate of anisotropic material)**

For large radius, the radiation of a cylinder is asymptotically identical to that of a plate (a semi-infinite planar object) of same surface area [19]. For the case of a plate made of isotropic material, the heat radiation is well-known [9]. Nevertheless, we will below study the heat radiation of a cylinder made of uniaxial material (as a simple model for carbon nanotubes), see Fig. 2-2, in which case the limit of large radius is a plate of uniaxial material. Recent literature discusses heat transfer between plates of uniaxial materials [99], as well as Casimir forces between a uniaxial plate and single-walled carbon nanotubes [100], but we have not come across an explicit result for radiation of a plate. For materials with anisotropic electric or magnetic response, the Fresnel coefficients are not diagonal in planar-basis polarization \( M \) and \( N^1 \), or transverse electric (TE) and transverse magnetic (TM) accordingly, but take the general form \( r^{Q'Q} \) for a scattered wave of polarization \( Q' \) in response to an incoming wave of polarization \( Q \) (see Ref. [101] for these lengthy coefficients). Thus, the heat

\(^{1}\)Note, despite the similarity in abbreviations to polarizations in cylindrical coordinates, \( M \) and \( N \) here corresponds to planar harmonics.
radiated per surface area (the Poynting vector \( \langle U \rangle \)) can in this general case easily be found from Eq. (2.15), where, using a plane-waves basis \[98\], the steps are similar to the ones performed for the cylinder (Eqs. (2.17)- (2.19)).

\[
\langle U \rangle = \frac{\hbar}{8\pi^3} \int_0^\infty \omega d\omega n_\omega(T) \int_{k_\perp < \omega/c} d^2k_\perp \sum_{Q=\{M,N\}} \left[ 1 - \left( |r^{QQ}|^2 + |r^{QQ'}|^2 \right) \right],
\]

(2.26)

where \( k_\perp \) is the wave-vector component parallel to the plate and \( k_\perp \equiv |k_\perp| \). This equation simplifies to a well-known result for isotropic materials \[9\] with \( r^{QQ} = 0 \) and Fresnel coefficients \( r^{QQ} \equiv r^Q \) given by,

\[
r^M = r^M(k_\perp) = \frac{\mu \sqrt{\frac{\omega^2}{c^2} - k_\perp^2}}{\mu \sqrt{\frac{\omega^2}{c^2} - k_\perp^2} + \sqrt{\varepsilon \mu \frac{\omega^2}{c^2} - k_\perp^2}},
\]

(2.27)

\( r^N \) is obtained from \( r^M \) by interchanging \( \mu \) and \( \varepsilon \).

The expression (2.26) can be rewritten in terms of \( M \) and \( N \) polarization for cylindrical geometry. If we define \( \phi \) to be the angle between the optical axis (which, in order to describe the radiation of a thick cylinder in terms of the one of a plate, is parallel to the plate surface) and the intersection line between the plane of incidence and plate itself, then we can write the limiting values of the \( M \) and \( N \) components of the cylinder radiation as,

\[
\langle U_M \rangle = \frac{\hbar}{8\pi^3} \int_0^\infty \omega d\omega n_\omega(T) \int_0^{2\pi} d\phi \int_0^{\omega/c} k_\perp dk_\perp \left[ 1 - \frac{1}{2} \left\{ (|r^M|^2 + |r^{MN}|^2) \cos^2 \phi + (|r^M|^2 + |r^{NM}|^2) \sin^2 \phi \right\} \right],
\]

(2.28)

\[
\langle U_N \rangle = \frac{\hbar}{8\pi^3} \int_0^\infty \omega d\omega n_\omega(T) \int_0^{2\pi} d\phi \int_0^{\omega/c} k_\perp dk_\perp \left[ 1 - \frac{1}{2} \left\{ (|r^M|^2 + |r^{MN}|^2) \sin^2 \phi + (|r^N|^2 + |r^{NM}|^2) \cos^2 \phi \right\} \right].
\]

(2.30)

Note that \( r^{NM} = r^{MN} \) when the optical axis is parallel to the plate.
Figure 2-2: An infinitely long cylinder made of a uniaxial material. The symmetry axis of the cylinder coincides with the symmetry axis of the diagonal dielectric tensor (the z-axis). Electromagnetic waves radiated by the cylinder are denoted by $M$ and $N$ for $M$-polarized (perpendicular) and $N$-polarized (parallel) respectively. Note, generally $M$-polarized waves have components along both azimuthal and radial directions. Also, despite $N$-polarized waves have non-zero components along all three basis directions, it is only the component parallel to the cylindrical axis that contributes to the Poynting vector.

2.1.2 $\mathbb{T}$ operator for a cylinder made of uniaxial material

In section 2.1.1, we derived the heat radiation of a cylindrical object expressed in terms of its $\mathbb{T}$ operator which is known analytically [21, 102]. One of our aims is to study the radiation of a cylinder of uniaxial material as a simple model for carbon nanotubes (see Sec. 2.1.3). The corresponding $\mathbb{T}$ operator, which seems unavailable in the literature, will be derived here for the case depicted in Fig. 2-2, where the cylindrical axis coincides with the optical axis. This is done by solving the scattering problem in Eq. (2.11), which amounts to satisfying the boundary conditions for the electromagnetic field at the cylinder’s surface. The setup is described by isotropic local magnetic permeability $\mu(\omega)$ and the following tensor for the anisotropic, but homogeneous and local dielectric function inside the cylinder,

$$
\tilde{\varepsilon}(\omega) = \begin{pmatrix}
\varepsilon_r(\omega) & 0 & 0 \\
0 & \varepsilon_r(\omega) & 0 \\
0 & 0 & \varepsilon_z(\omega)
\end{pmatrix}.
$$

(2.32)
The electric displacement field inside the cylinder can then be written as

\[
D = \varepsilon_r(\omega) (E_r e_r + E_\phi e_\phi) + \varepsilon_z(\omega) E_z e_z,
\]

where \(e_r, e_\phi, \) and \(e_z\) correspond to unit vectors in cylindrical coordinates.

Importantly, uniaxial materials split the incident beam into ordinary and extraordinary ones [103]. In our geometry, cylindrical \(M\)-polarized waves correspond to ordinary ones and propagate according to the dielectric function \(\varepsilon_r\). The \(N\) modes correspond to extraordinary waves and propagate according to an effective dielectric function which depends on the direction of propagation.

The resulting \(\Pi\) operator components \(T_{n,k_z}^{PP}\), as defined in Eq. (2.16), are expressed in terms of Bessel functions, \(J_n\), and Hankel functions of first kind, \(H_n^{(1)}\) (see App. A.2 for the detailed derivation),

\[
T_{n,k_z}^{MM} = \frac{J_n(qR)}{H_n^{(1)}(qR)} \frac{\Delta_1 \Delta_4 - K^2}{\Delta_1 \Delta_2 - K^2},
\]

\[
T_{n,k_z}^{NN} = -\frac{J_n(qR)}{H_n^{(1)}(qR)} \frac{\Delta_2 \Delta_3 - K^2}{\Delta_1 \Delta_2 - K^2},
\]

\[
T_{n,k_z}^{NM} = T_{n,k_z}^{MN} = \frac{2i}{\pi \sqrt{\varepsilon_z \mu (qR)^2}} \frac{K}{[H_n^{(1)}(qR)]^2} \frac{1}{\Delta_1 \Delta_2 - K^2},
\]

where

\[
\Delta_1 = \frac{J_n'(qNR)}{qNRJ_n(qNR)} - \frac{1}{\varepsilon_z} \frac{H_n^{(1)y'}(qR)}{qRH_n^{(1)}(qR)},
\]

\[
\Delta_2 = \frac{J_n'(qMR)}{qMRJ_n(qMR)} - \frac{1}{\mu} \frac{H_n^{(1)y'}(qR)}{qRH_n^{(1)}(qR)},
\]

\[
\Delta_3 = \frac{J_n'(qNR)}{qNRJ_n(qNR)} - \frac{1}{\varepsilon_z} \frac{J_n'(qR)}{qRJ_n(qR)},
\]

\[
\Delta_4 = \frac{J_n'(qMR)}{qMRJ_n(qMR)} - \frac{1}{\mu} \frac{J_n'(qR)}{qRJ_n(qR)}.
\]
and

\[ K' = \frac{n k_z c}{\sqrt{\varepsilon_z \mu R^2 \omega}} \left( \frac{1}{q_M^2} - \frac{1}{q^2} \right). \tag{2.40} \]

Here, \( k = \omega/c \) and \( q = \sqrt{k^2 - k_z^2} \) are the wave-vector magnitude in vacuum and its component perpendicular to the \( z \)-axis respectively; while \( q_M = \sqrt{\varepsilon_r \mu k^2 - k_z^2} \) and \( q_N = \sqrt{\varepsilon_z/\varepsilon_r} \sqrt{\varepsilon_r \mu k^2 - k_z^2} \) are the wave-vector components perpendicular to the \( z \)-axis for the \( M \)-polarized ordinary and \( N \)-polarized extraordinary waves inside the cylinder, respectively.

As required by continuity, the above \( \mathbf{T} \) matrix can be easily reduced to the isotropic case when \( \varepsilon_r = \varepsilon_z = \varepsilon \). Then the expressions simplify to \( q_M = q_N = \sqrt{\varepsilon_\mu k^2 - k_z^2} \equiv q' \) and \( K = n k_z c (1/q^2 - 1/q'^2)/(\sqrt{\varepsilon_\mu R^2 \omega}) \), and our results reduce to the known forms for an isotropic cylinder [21, 102].

### 2.1.3 Examples and asymptotic results

In this section we numerically, and analytically, study the radiation of a cylinder for different material classes. We start with isotropic dielectrics and conductors and finally present the case of a uniaxial material using the “in-layer” and “inter-layer” response of graphite to model MWCNT. We consider \( \mu = 1 \) for all studied materials.

**Dielectric cylinder**

Figure 2-3 illustrates the result of the numerical calculation of the radiation by SiO\(_2\) and SiC cylinders for \( T = 300\)K normalized by the Stefan-Boltzmann result for blackbody,

\[ \langle H \rangle = \sigma T^4 A \equiv \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} T^4 A, \tag{2.41} \]

where \( A \) is the surface area.

For SiO\(_2\) optical data was used\(^2\), whereas for SiC the following dielectric function

\(^2\)The optical data was provided by P. Sambegoro from Prof. Gang Chen’s Nanoengineering Group and had been previously used in Refs. [15, 19, 26].
Figure 2-3: The normalized heat radiation as a function of radius $R$ for (a) SiO$_2$ and (b) SiC cylinders at temperature $T = 300$K. Calculations were performed using Eq. (2.21), and analytical expansions, Eq. (2.43) and Eq. (2.44). The horizontal lines show the radiation of (a) SiO$_2$ and (b) SiC plates. $\lambda_T$ and the smallest skin depths $\delta$ in the relevant frequency range are marked on the horizontal axis.

was taken [104],

$$\varepsilon_{SiC}(\omega) = \varepsilon_\infty \frac{\omega^2 - \omega_{LO}^2 + i \omega \gamma}{\omega^2 - \omega_{TO}^2 + i \omega \gamma},$$  \hspace{1cm} (2.42)

where $\varepsilon_\infty = 6.7$, $\omega_{LO} = 0.12$eV, $\omega_{TO} = 0.098$eV, $\gamma = 5.88 \times 10^{-4}$eV.

We see the effects discussed in Ref. [19] involving the three length scales: radius $R$, thermal wavelength $\lambda_T$, and skin depth $\delta = c/(\text{Im} \sqrt{\varepsilon \omega})$, where the latter depends on frequency. If $R$ is the smallest scale, i.e., much smaller than the smallest relevant skin depth and $\lambda_T$, the radiation is proportional to the volume of the cylinder. In this case, radiation emitted inside the cylinder will hardly be reabsorbed on its way out so that all regions of the cylinder contribute equally to the emission. The other asymptotic behavior is approached when $R$ is the largest scale, i.e. much larger than the largest relevant skin depth and $\lambda_T$. Then only the cylinder surface contributes to the radiation which approaches the values of a plate of equal surface area as seen in the figure.

The length scale $\lambda_T$ sets (via the Boltzmann factor) the range of relevant wavelengths of emission. Nevertheless, for dielectrics there is a fine structure to this range given by the resonances of the material. In case of $R \ll \delta$, the cylinder emits predominantly at the resonance wavelengths of the material (where $\delta$ is the minimal value of
the skin-depth within the range of relevant wavelengths of emission). On the other hand, for $R \gg \{\delta, \lambda_T\}$, the emissivity is strongest in regions where $\varepsilon \approx 1$ (compare the plate emissivity).

In general, one might expect resonance effects when the emitted wavelength is of the order of $R/(2\pi)$ (similar to Mie resonances for a sphere [20, 21]). Due to the contribution of all wavelengths, these are smeared out in the total heat emitted. For SiO$_2$ in Fig. 2-3, the fact that the emissivity exceeds the plate result for $R \approx 20\mu$m might be connected to such resonances.

Figure 2-3 shows imprints of the dielectric function of SiC which has a sharp strong resonance (leading to a small $\delta$), but apart from the resonance, SiC is almost black (i.e. has very large $\delta$) in our frequency range. The two regimes $R \ll \{\delta, \lambda_T\}$ and $R \gg \{\delta, \lambda_T\}$, where the cylinder radiation follows the discussed asymptotic laws are far apart, and very large radii $R$ are necessary to approach the classical plate result. In Fig. 2-3, the radiation for $R = 150\mu$m is still distinctly different from the asymptotic result. This might be advantageous for experiments, as a SiC cylinder does not have to be very thin in order to observe deviations from the Stefan-Boltzmann law.

In the limit of $R \ll \{\delta, \lambda_T\}$, the radiation can be studied analytically, see App. A.3, where we find the following asymptotic laws for the two polarizations,

$$
\lim_{R \ll \{\delta, \lambda_T\}} \frac{\langle H_N \rangle}{A} = \frac{\hbar}{6\pi^2} \int_0^\infty d\omega n_\omega(T) \frac{\omega^4 R}{c^3} \frac{\varepsilon(\omega)^2 + 2\varepsilon(\omega) - 1}{\varepsilon(\omega) + 1}, \quad (2.43)
$$

$$
\lim_{R \ll \{\delta, \lambda_T\}} \frac{\langle H_M \rangle}{A} = \frac{\hbar}{2\pi^2} \int_0^\infty d\omega n_\omega(T) \frac{\omega^4 R}{c^3} \frac{\varepsilon(\omega) - 1}{\varepsilon(\omega) + 1}. \quad (2.44)
$$

Note the linear increase with $R$, corresponding to the proportionality of the unnormalized radiation to the volume. The numerical evaluation of these asymptotic forms has been added to the graphs in Fig. 2-3, where the agreement for small $R$ to the full results is visible. As expected and seen from the equations above, cylinders with purely real dielectric functions (or more general with real potential $\Psi$) will not radiate, because the dissipative properties of the material are responsible for the heat radiation (in accord with FDT). This holds for any $R$. The $N$-polarization given in
Eq. (2.43) dominates over the $M$-polarization in Eq. (2.44) if the condition

$$[\text{Re} \, \varepsilon(\omega) + 1]^2 + [\text{Im} \, \varepsilon(\omega)]^2 \gg \text{Im} \, \varepsilon(\omega)$$  \hspace{1cm} (2.45)$$

holds, which is the case for most materials. Further insight can be gained by additionally requiring the temperature to be so low that one can expand the dielectric function, i.e., $\lambda_T \gg \lambda_0$, where $\lambda_0$ is the wavelength of the lowest resonance of the material. In this case [97],

$$\varepsilon(\omega) = \varepsilon_0 + i \frac{\lambda_m \omega}{c} + \mathcal{O}(\omega^2),$$  \hspace{1cm} (2.46)$$

with $\varepsilon_0$ and $\lambda_m$ real. Plugging (2.46) into Eqs. (2.43) and (2.44) the frequency integration can be done and we have,

$$\lim_{R \ll \{\delta, \lambda_T\}, \lambda_0 \ll \lambda_T} \frac{\langle H_N \rangle}{A} = \frac{4\pi^4}{189} \frac{h c^2 \lambda_m R}{\lambda_T^6} \left[1 + \frac{2}{(\varepsilon_0 + 1)^2}\right],$$  \hspace{1cm} (2.47)$$

$$\lim_{R \ll \{\delta, \lambda_T\}, \lambda_0 \ll \lambda_T} \frac{\langle H_M \rangle}{A} = \frac{8\pi^4}{63} \frac{h c^2 \lambda_m R}{\lambda_T^6} \frac{1}{(\varepsilon_0 + 1)^2}. \hspace{1cm} (2.48)$$

Interestingly, to lowest order in $T$, both components scale as $T^6$ and hence fundamentally different from the Stefan-Boltzmann law, which scales as $T^4$.

The degree of polarization $I$ in Eq. (2.24) is finally given by

$$\lim_{R \ll \{\delta, \lambda_T\}, \lambda_0 \ll \lambda_T} I = \frac{\varepsilon_0^2 + 2\varepsilon_0 - 3}{\varepsilon_0^2 + 2\varepsilon_0 + 9}.$$  \hspace{1cm} (2.49)$$

Although the condition $\lambda_0 \ll \lambda_T$ is not fulfilled in Fig. 2-3, Eq. (2.49) still gives a hint to why SiC has higher degree of polarization compared to SiO$_2$, as $\varepsilon_0$ is tangibly larger for SiC and Eq. (2.49) monotonically increases with $\varepsilon_0$ (for $\varepsilon_0 > 0$).

**Well conducting cylinder**

Conductors differ from insulators by a significantly smaller skin depth $\delta$, leading to very different radiation characteristics. In this section, we first study the radiation
Figure 2-4: The normalized heat radiation as a function of radius $R$ for a gold (Au) cylinder at temperatures (a) $T = 300$K and (b) $T = 30$K. Calculations were performed using Eq. (2.21) with dielectric function (2.50). Full numerics, Eq. (2.21), analytical expansion, Eq. (2.51), and the approximation of Eq. (2.52) were used. Black horizontal lines indicate the radiation of Au plates at corresponding temperatures. $\lambda_T$ and the skin depth $\delta$ in the relevant frequency range are marked on the $R$-axis. In the insets the degree of polarizations, (2.24), are plotted.

Figure 2-5: The normalized heat radiation as a function of radius $R$ for a tungsten (W) cylinder at temperatures (a) $T = 298$K and (b) $T = 2400$K. Calculations were performed using Eq. (2.21) with dielectric function (2.53). Black horizontal lines indicate the radiation of tungsten plates at corresponding temperatures. $\lambda_T$ and the skin depth $\delta$ in the relevant frequency range are marked on $R$-axis. In the insets the degree of polarizations are given using expression (2.24).
using a simple Drude model with the parameters of gold in order to highlight the different limiting behaviors. Then, we turn to tungsten (W) which has been extensively used in experiments [18, 77].

The Drude model for gold [105],

\[ \varepsilon_{Au}(\omega) = \varepsilon_{\infty} - \frac{\omega_p^2}{\omega(\omega + i\omega_T)}, \]  

\[ (2.50) \]

has the parameters \( \varepsilon_{\infty} = 1, \omega_p = 9.03\,\text{eV} \) and \( \omega_T = 2.67 \times 10^{-2}\,\text{eV} \).

Figure 2-4 shows the numerical result for the total radiation by a gold cylinder at 300K and 30K. We observe a behavior drastically different from the ones in Fig. 2-3: the radiation, normalized by surface area, can be many orders larger than expected from the Stefan-Boltzmann law. More precisely, it increases with decreasing \( R \), has a maximum at \( R \approx \delta \) (for Drude model of gold (2.50), \( \delta(\omega) \) has no sharp minimum, and we show the skin depth corresponding roughly to thermal wavelength) and approaches the laws (2.43) and (2.44) only for very small (unphysical) radii. For \( R \gg \lambda_T \), the result of a gold plate is approached. The large values of radiated power, compared to the Stefan-Boltzmann law, can be explained by the interplay of two effects: a large imaginary part of the dielectric function gives rise to strong radiation from every volume element of the conductor. On the other hand, it also leads to a very small skin depth such that most radiation is reabsorbed inside the conductor, and only a thin surface layer contributes to the radiation. In the region where \( R \approx \delta \), one has very strong emitting elements which all contribute to the total radiation, and hence emission normalized by surface area is maximal. Interestingly, when the conductivity goes to infinity, the effect of vanishing skin depth is stronger than the effect of increasing radiation such that the emissivity vanishes as \( 1/\sqrt{\varepsilon} \). In this case, the classical (plate) limit goes to zero and the maximum in the curve shifts to smaller and smaller \( R \).

The insets show the degree of polarization (2.24) as a function of \( R \). For \( R \lesssim \lambda_T \), the radiation from the gold cylinder is fully \( N \)-polarized. For \( R \gtrsim \lambda_T \), the degree of polarization becomes negative and asymptotically approaches zero for \( R/\lambda_T \to \infty \).
These qualitative features agree with experimental [18, 77-80] as well as theoretical [18] studies on radiation of thin wires.

For conductors, the appropriate limit for an analytic expansion is $\lambda_T \gg R \gg \delta$ (where $\delta$ is of the order of nanometers for good conductors). In this case, the leading element of the $T$ operator is $T_{0,k_z}^{NN}$, see App. A.4. The resulting radiation is then completely $N$-polarized and reads,

$$
\lim_{\lambda_T \gg R \gg \delta} \frac{\langle H_c \rangle}{A} = \frac{h}{\pi^2} \int_0^\infty d\omega n_d(T) \frac{\omega^3}{c^2} \times \int_0^{\pi/2} d\theta \frac{2 \text{Re}[1/\sqrt{\varepsilon}] \cos^3 \theta}{|\cos^2 \theta (2\gamma_E - i\pi)\omega R/c - 2\sqrt{\varepsilon} + 2 \cos^2 \theta \omega R/c \log |\cos \theta \omega R/2c|^2|}.
$$

(2.51)

where $\gamma_E \approx 0.577$ is Euler-Mascheroni constant and $\theta$ is the angle of incidence so that $k_z = k \sin \theta$.

As described above, we see here explicitly that the radiation (in the considered range of radii) vanishes for $|\varepsilon| \to \infty$. This expression cannot be further simplified as the integrand diverges at $\theta = \pi/2$ if we omit the small term $-2i/\sqrt{\varepsilon}$ in the denominator. A result almost similar to (2.51) was obtained by Rytov [16] (whose derivation was restricted to $\text{Re}[\varepsilon] = 0$ and $\text{Im}[\varepsilon] \gg 0$), which differs by the absence of the first term in denominator. We emphasize however that this term is necessary for accurate results in the considered limit.

Rytov suggests a further simplification of his expression by setting $\cos \theta = 1$ in $\log |\cos \theta \omega R/2c|$, so that integration can be performed analytically. Omitting $\cos^2 \theta (2\gamma_E - i\pi)\omega R/c$ in denominator, the following result for radiation can be obtained after integration,

$$
\lim_{\lambda_T \gg R \gg \delta} \frac{\langle H_c \rangle}{A} \approx \frac{h}{2\pi^2} \int_0^\infty d\omega n_d(T) \frac{\omega^{3/2}}{c^{1/2}} \text{Re}[1/\sqrt{\varepsilon}] \frac{|\varepsilon|^{1/4}}{|R \log |\omega R/2c|^3/2|}.
$$

(2.52)

Note that Eq. (2.52) was applied to any well-conducting (non magnetic) media, whereas the corresponding result, Eq. (IV.35) in Ref. [16], only considers the case of a purely imaginary dielectric function (and $\mu \neq 1$). As such, the integrand of Eq. (2.52)
differs from Eq. (IV.35) in Ref. [16] by \( \text{Re}[1/\sqrt{\varepsilon}]|\varepsilon|^{1/4} \) instead of \((\mu/4|\text{Im}[\varepsilon]|)^{1/4}\) (which agree for \( \text{Re}[\varepsilon] = 0 \) and \( \mu = 1 \)). Also, we emphasize that Eq. (2.52) is an approximation whereas Eq. (2.51) is an exact asymptotic limit for \( \lambda_T \gg R \gg \delta \).

Figure 2-4 provides a test of these approximations, demonstrating that for \( T = 30\text{K} \), where the ratio of \( \lambda_T \) and \( \delta \) is larger than at \( T = 300\text{K} \), Eq. (2.51) describes the full solution over roughly one decade in \( R \). For \( T = 300\text{K} \), the agreement is not as good as the ratio of \( \lambda_T \) and \( \delta \) is too small. Equation (2.52) gives a rough estimate of the overall dependence on \( R \) in Fig. 2-4, but may differ by roughly a factor of 10 from the exact results. Moreover, above some threshold value of \( R \) we cannot obtain a finite radiation rate from Eq. (2.52) because of the divergent term due to the logarithm of unity in the denominator of the integrand.

After this (more theoretical) discussion of gold, where we used the same \( \varepsilon \) for both temperatures in order to demonstrate the pure effect of temperature via the Boltzmann factor, we turn to the experimentally relevant material tungsten (W), at relevant temperatures of \( T = 298\text{K} \) and \( T = 2400\text{K} \) as shown in Fig. 2-5. We use the corresponding dielectric function for tungsten [106],

\[
\varepsilon_W(\omega) = 1 + \sum_{p=1}^{3} \frac{K_0p\lambda^2}{\lambda^2 - \lambda^2_{sp} + i\eta_p\lambda_{sp}\lambda} - \frac{\lambda^2}{2\pi\varepsilon_0} \sum_{q=1}^{2} \frac{\sigma_q}{\lambda_{rq} - i\lambda},
\]

where \( \lambda \) is the wavelength in vacuum and \( \varepsilon_0 \) is the permittivity of vacuum in SI units. The remaining parameters are listed in Table 2.1.

The overall radiation of tungsten at 298K is very similar to gold at 300K in Fig. 2-4. At high temperature \( T = 2400\text{K} \), the increase of the normalized radiation over the Stefan-Boltzmann law is reduced. We attribute this to a smaller conductivity at \( T = 2400\text{K} \). We also observe in the insets that the zero in the polarization curves shifts by roughly a factor of 10 when comparing the two temperatures. This manifests that the zero in the polarization curves is mostly a function of \( R/\lambda_T \) as \( \lambda_T \) is also shifted by almost a factor of 10. Furthermore, although the polarization in the inset is indistinguishable from unity at small \( R \), we note that the ratio of \( N \) and \( M \) polarizations is a factor of \( 10^3 \) larger for \( T = 298\text{K} \) compared to \( T = 2400\text{K} \).
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Table 2.1: Optical data for tungsten from Ref. [106]. Temperature $T$ is in Kelvins. Conductivities ($\sigma_1$, etc.) are in units of $10^6$ ohm$^{-1}$m$^{-1}$. Wavelengths ($\lambda_{r1}$, etc.) are in $\mu$m. $K_{01}$, etc. and $\eta_1$, etc. are dimensionless.

Multi-walled carbon nanotube

We finally turn to heat radiation of a cylinder made of uniaxial material. This can be considered a simple model for a MWCNT [107] which is of high importance in modern science and technology. As a MWCNT is a wrapped up graphite sheet, we can in a crude approximation regard it as a (solid) cylinder described by two different dielectric response functions: the response along the cylindrical axis is given by the in-layer properties of the graphite sheets, whereas the response perpendicular to this axis is approximately given by the inter-layer response. We also note that most mineral crystals have uniaxial optical properties, and heat emission by these materials might open new possibilities for applications [99].

Figure 2-6 shows the heat radiation of a MWCNT for $T = 300$K using expressions (2.21) and (2.33)–(2.40), normalized as before by the Stefan-Boltzmann law. We used the following form for the graphite dielectric function [108],

$$
\varepsilon_{r,z}(\omega) = 1 - \frac{\Omega_p^2}{\omega(\omega + i\Gamma_0)} - \sum_j \frac{\Theta_j \omega_p^2}{(\omega^2 - \omega_j^2) + i\omega \Gamma_j},
$$

(2.54)
Figure 2-6: The heat radiation of a MWCNT as a function of radius $R$, normalized by the Stefan-Boltzmann result, at $T = 300$K. Contributions from the two polarizations are indicated. Corresponding thin curves without boxes represent the heat radiation for “isotropic graphite” with dielectric function $(\varepsilon_z+\varepsilon_r)/2$. Horizontal lines of different colors indicate the graphite plate classical result for correspondingly colored curves. $\lambda_T$ and the smallest skin depth $\delta$ in the relevant frequency range are marked on the $R$-axis. Note that the smallest skin depths corresponding to both $\varepsilon_r$ and $\varepsilon_z$ are approximately equal and labeled here by $\delta$. In the inset the degree of polarization is given using expression (2.24).
Table 2.2: Optical parameters for the in-layer dielectric function of graphite ($\varepsilon_z$) from Ref. [108]. $\Theta_j$ and $\zeta_j$ are dimensionless, whereas $\omega_{ij}$ and $\Gamma_j$ are in eV.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_j$</td>
<td>0.134</td>
<td>0.072</td>
<td>0.307</td>
<td>0.380</td>
<td>0.065</td>
<td>0.553</td>
<td>1.381</td>
</tr>
<tr>
<td>$\zeta_j$</td>
<td>24.708</td>
<td>0.524</td>
<td>0.217</td>
<td>0.518</td>
<td>0.286</td>
<td>0.248</td>
<td>15.101</td>
</tr>
<tr>
<td>$\omega_{ij}$</td>
<td>2.358</td>
<td>5.149</td>
<td>13.785</td>
<td>10.947</td>
<td>16.988</td>
<td>24.038</td>
<td>36.252</td>
</tr>
<tr>
<td>$\Gamma_j$</td>
<td>9.806</td>
<td>472.7</td>
<td>4.651</td>
<td>1.797</td>
<td>2.418</td>
<td>21.395</td>
<td>37.025</td>
</tr>
</tbody>
</table>

Table 2.3: Optical parameters for the inter-layer dielectric function of graphite ($\varepsilon_r$) from Ref. [108]. $\Theta_j$ and $\zeta_j$ are dimensionless, whereas $\omega_{ij}$ and $\Gamma_j$ are in eV.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_j$</td>
<td>0.073</td>
<td>0.056</td>
<td>0.069</td>
<td>0.005</td>
<td>0.262</td>
<td>0.460</td>
<td>0.200</td>
</tr>
<tr>
<td>$\zeta_j$</td>
<td>0.505</td>
<td>7.079</td>
<td>0.362</td>
<td>7.426</td>
<td>0.00382</td>
<td>1.387</td>
<td>28.963</td>
</tr>
<tr>
<td>$\omega_{ij}$</td>
<td>0.275</td>
<td>3.508</td>
<td>4.451</td>
<td>13.591</td>
<td>14.226</td>
<td>15.550</td>
<td>32.011</td>
</tr>
<tr>
<td>$\Gamma_j$</td>
<td>4.102</td>
<td>7.328</td>
<td>1.414</td>
<td>0.046</td>
<td>1.862</td>
<td>11.922</td>
<td>39.091</td>
</tr>
</tbody>
</table>

where $\Omega_p = \sqrt{\Theta_0}\omega_p$ and a specific form of $\Gamma_j' = \Gamma_j \exp \left[ -\zeta_j \left( \frac{\hbar \omega - \hbar \omega_j}{\Gamma_j} \right)^2 \right]$ is used, which best describes the experimental data. For the axial component $\varepsilon_z$ along the cylindrical axis, i.e., the in-layer response, the parameters are $\omega_p = 19eV$, $\Gamma_0 = 0.091eV$, $\Theta_0 = 0.016$. The remaining parameters are given in Table 2.2. For the radial component $\varepsilon_r$ perpendicular to the cylindrical axis, i.e., the inter-layer response, the parameters are $\omega_p = 27eV$, $\Gamma_0 = 6.365eV$, $\Theta_0 = 0.014$ and the remaining parameters are given in Table 2.3.

These parameterizations apply to the frequency range $0.12 - 40eV$ and $2 - 40eV$ for $\varepsilon_r$ and $\varepsilon_z$ respectively, but we nevertheless use them for the range of roughly $0.004 - 0.2eV$ due to the lack of simple analytic forms for the broader range. Note that $\lambda_T$ corresponds to a frequency of $0.163eV$ for $T = 300K$.

The overall radiation curve is in between the characteristic shapes of dielectrics and conductors (compare Figs 2-3 and 2-4): the regime proportional to volume as in Eqs. (2.43) and (2.44) is visible for small $R$ in contrast to Fig. 2-4. On the other hand, the strong increase over the plate-result and over the Stefan-Boltzmann value, characteristic for conductors, is visible as well. These features follow from the dielectric functions in Eq. (2.54) carrying a smaller conductivity compared to gold.

In order to highlight the effect of material anisotropy, in Fig. 2-6 we also show
the thin curves without boxes for which we use an isotropic dielectric function, given by \((\varepsilon_z + \varepsilon_r)/2\). We see that the influence of anisotropy on the radiation is almost negligible at small \(R\), whereas at intermediate \(R\) it strongly increases the degree of polarization perpendicular to the tube. Interestingly, the asymptotic value of \(I\) in the inset is different from 0 and takes the value \(-0.297\), an effect purely due to the anisotropy which is computed using the result for a plate of anisotropic material, Eqs. (2.28) and (2.30). We note that a very thick MWCNT might in fact be best described by a plate with optical axis perpendicular to the surface. Thus, while Fig. 2-6 gives the correct radiation for a material with the dielectric properties given in Eq. (2.54), it probably only describes MWCNT for small \(R\). Since, at small \(R\), the polarization is hardly dependent on the anisotropy of the material, we conclude that the polarization effects for MWCNT [82] are mainly an effect of cylindrical geometry rather than anisotropy of the material.

2.1.4 Spectral emissivity

In Sec. 2.1.3, we studied the total heat radiation of a cylinder made of different materials. While this is of interest in connection with efficient heating or cooling, another quantity which can be more appropriate for direct comparison to experiments is the spectral emissivity. In this section, we discuss the spectral emissivity for cylinders made of dielectrics (SiO\(_2\)), conductors (tungsten) and uniaxial materials (graphite) for a fixed radius of \(R = 5\mu m\).

The spectral emissivity (density) \(\langle H_\omega \rangle\) is given by the integrand of Eq. (2.21),

\[
\frac{\langle H_\omega \rangle}{L} = -\frac{\hbar}{\pi^2} \omega n_\omega(T) \sum_{P=M,N} \sum_{n=-\infty}^{\infty} \int_{-\omega/c}^{\omega/c} dk_z (\text{Re}[T_{n,k_z}^{PP}] + |T_{n,k_z}^{PP}|^2 + |T_{n,k_z}^{PP}|^2). \tag{2.55}
\]

We denote by \(\langle H_{\omega M} \rangle\) and \(\langle H_{\omega N} \rangle\) the correspondingly polarized components of \(\langle H_\omega \rangle\).
Silicon dioxide (SiO2)

Figure 2-7 illustrates the spectral density for SiO$_2$ at $T = 300$K. The unsteady local fine structure reflects the nature of the optical data which has a number of smaller peaks. For short wavelengths (high frequencies) $M$-polarized radiation mostly dominates, whereas for $\lambda \gtrsim 25\mu$m the $N$-polarized radiation starts to prevail up to the limit of long wavelengths (low frequencies). In the inset the spectral degree of polarization as a function of wavelength is shown, where

$$I_\omega = \frac{\langle H_{\omega N} \rangle - \langle H_{\omega M} \rangle}{\langle H_{\omega N} \rangle + \langle H_{\omega M} \rangle}.$$ 

(2.56)

The two large valleys in this curve are due to the two dominant resonances of SiO$_2$. The fact that the resonances lead to negative valleys in the polarization is a feature specific for the radius chosen. For large wavelengths, the spectral degree of polarization approaches a constant value, which can be computed easily using expressions (2.43) and (2.44),

$$\lim_{\lambda \rightarrow \infty} I_\omega = \frac{\varepsilon(\omega)^2 + 2 \operatorname{Re}[\varepsilon(\omega)] - 3}{\varepsilon(\omega)^2 + 2 \operatorname{Re}[\varepsilon(\omega)] + 9}. \quad (2.57)$$

Furthermore, if additionally $\lambda \gg \lambda_0$ holds, the dielectric function is described by Eq. (2.46) and the spectral degree of polarization is given then by Eq. (2.49).

Note that the spectral degree of polarization is independent of temperature if the dielectric function is independent of temperature.

Tungsten (W)

Figure 2-8 shows the spectral density of radiation for tungsten cylinders at $T = 298$K and $T = 2400$K. The shape of the curves is very similar to Planck's classical law due to the Bose-Einstein statistics factor $n_\omega(T)$. The curves peak at values slightly larger than the corresponding thermal wavelengths. For short wavelengths $M$-polarized radiation is stronger than $N$-polarized one, whereas for $\lambda \gtrsim 2R$, $N$-polarized radiation dominates, strongly suppressing the $M$-polarized radiation in the limit of
Figure 2-7: The spectral density divided by Stefan-Boltzmann law as a function of wavelength $\lambda$ for SiO$_2$ cylinder of radius $R = 5\mu$m at temperature $T = 300$K. In the inset the spectral degree of polarization is provided.

long wavelengths. This transition of polarization is also manifested in the insets where spectral degrees of polarizations are plotted. In the limit of long wavelengths these approach unity, as can be justified by results of Sec. 2.1.3, where it is shown that the $N$ component is dominant in the limit $\lambda T \gg R \gg \delta$. On the other hand, in the limit of short wavelengths the spectral degree of polarization approaches zero as the cylinder radiates as an isotropic plate. We emphasize again that the spectral degree of polarization is independent of temperature (if $\varepsilon(\omega)$ is).

**Graphite**

Figure 2-9 shows corresponding results for a graphite cylinder at $T = 300$K, displaying similar behavior as the case of conductors in Fig. 2-8. The transition point, where the polarization changes sign is at $\lambda \approx 25\mu$m. Analogously to conductors, the spectral degree of polarization for a graphite cylinder tends to unity in the limit of long wavelengths, i.e. spectral density has polarization parallel to the cylinder. The range
Figure 2-8: The spectral density divided by Stefan-Boltzmann law as a function of wavelength $\lambda$ for tungsten cylinder of radius $R = 5\mu m$ at temperatures (a) $T = 298K$ and (b) $T = 2400K$. In the insets the spectral degrees of polarization are provided.

of wavelengths $200 - 300\mu m$ shows an unexpected plateau, the origin of which is unclear.

We note that $I_\omega$ goes to zero for $\lambda \to 0$, although a finite value is approached for $R/\lambda_T \to \infty$ in Fig. 2-6. This is due to the fact that both $\varepsilon_z$ and $\varepsilon_r$ tend to 1 for $\omega \to \infty$, and the material is asymptotically isotropic. Nevertheless, at $\lambda \approx 5\mu m$, the polarization is very strong compared to Fig. 2-8, an effect which we attribute to the anisotropy of the material.

Another model describing the spectral degree of polarization of MWCNT’s is presented in Ref. [17], where we note partly common structure to our description arising from the expansions of Eqs. (A.15)–(A.19). For large $\lambda/R \to \infty$, both the experimental data as well as the theoretical predictions of Ref. [17] give values for $I_\omega$ close to unity, in agreement to Fig. 2-9.

Comparing material classes

Finally, Fig. 2-10 compares the spectral polarization for the different materials discussed, where we used simplified dielectric functions for dielectrics and uniaxial materials in order to illuminate the pure influence of $\lambda/R$.

Figure 2-10 manifests that the overall dependence of the polarization on wave-
Figure 2-9: The spectral density divided by Stefan-Boltzmann law as a function of wavelength $\lambda$ for graphite cylinder of radius $R = 5\mu m$ at temperature $T = 300K$. In the inset the spectral degree of polarization is provided.

Length is quite universal following similar curves for all cases shown. Fundamental differences are seen in the limiting cases. In the limit of large wavelengths all conductors approach unity, whereas dielectrics go to a constant value different from unity when $\lambda \gg \{\delta, R\}$ (see Eq. (2.57)).

In the opposite limit of small wavelengths, Fig. 2-10 manifests strong dependence of the polarization on uniaxiality. We emphasize that by showing additionally artificial materials with very strong anisotropy (factor 199 between $\varepsilon_r$ and $\varepsilon_z$). Varying this factor, a range from roughly 0.4 to -0.9 in polarization at $\lambda = 10\mu m$ can be swept. Figure 2-10 clearly illustrates that the smaller $\varepsilon_r/\varepsilon_z$, the smaller the spectral polarization.
Figure 2-10: Spectral degree of polarization as a function of wavelength $\lambda$ for cylinders made of isotropic dielectric, isotropic conducting, and anisotropic conducting materials for $R = 5\mu m$. Black horizontal and orange lines correspond to long wavelengths limiting values for conductors (isotropic and uniaxial) and the dielectric (using Eq. (2.57)), respectively.
2.2 Casimir forces between cylinders at different temperatures

In this section we study non-equilibrium Casimir forces between nanowires and nanotubes which provide a simple example of an extended shape, with several possible applications [102, 109–111]. Cylinders can also be easier fixed positionally and held at different temperatures for possible experimental study of non-equilibrium effects [17].

2.2.1 Non-equilibrium Casimir forces in terms of $T$ operator of cylinders. Asymptotic results

We start by extending the formalism described in Sec. 2.1 to the case of two objects [19], and apply it to a system of two parallel cylinders of radii $R_\alpha$ ($\alpha = 1, 2$) and axis-to-axis separation $d$ in vacuum. Each object is characterized by the corresponding electric and magnetic responses $\epsilon_\alpha$ and $\mu_\alpha$ held at a constant and homogeneous temperature $\{T_\alpha\}$, and embedded in an environment at temperature $T_{env}$.

General formalism for multiple objects. Cylindrical basis

We start by extending the expression for the equilibrium spectral density, Eq. (2.8), to the case of two objects (generalization to the case of $N$ objects is straightforward) [15],

$$C^{eq}(T) = C^0 + \sum_{\alpha=1,2} C_{sc}^\alpha(T) + C_{env}^\alpha(T),$$  \hspace{1cm} (2.58)

where we have taken into account that now the field sourced by a specific object $\alpha$, $C^\alpha$, gets scattered by the other object, yielding $C_{sc}^\alpha$. $C_{sc}^\alpha$ can be obtained by using Eq. (2.11) or by using Eq. (2.9) and taking into account that now $G$ is the total Green’s function of the system of two objects given in Eq. (4.9).

We can easily identify all the contributions from the difference sources in Eq. (2.58) and change the temperature of these sources independently. Thus, we rewrite Eq. (2.58)
as,
\[
\mathcal{C}_{\text{eq}}^{\text{eq}}(\{T_\alpha\}, T_{\text{env}}) = \mathcal{C}^0 + \sum_{\alpha=1,2} \mathcal{C}_{\text{sc}}^\alpha(T_\alpha) + \mathcal{C}_{\text{env}}^\alpha(T_{\text{env}}). \tag{2.59}
\]

The expression (2.59) is valid for the general field correlator for arbitrary combinations of the temperatures of a two-body system. There are three unknown terms as there is another source coming from the environment. By introducing the equilibrium correlation function at finite temperature, we can simplify Eq. (2.59) [15] to,
\[
\mathcal{C}_{\text{neq}}^{\text{eq}}(\{T_\alpha\}, T_{\text{env}}) = \mathcal{C}_{\text{eq}}^{\text{eq}}(T_{\text{env}}) + \sum_{\alpha=1,2} [\mathcal{C}_{\text{sc}}^\alpha(T_\alpha) - \mathcal{C}_{\text{sc}}^\alpha(T_{\text{env}})]. \tag{2.60}
\]

Thus, we need to evaluate only \(\mathcal{C}_{\text{sc}}^\alpha\) for two objects in order to consider non-equilibrium effects. Having done that, one can obtain the Casimir force between two cylinders (given that the equilibrium part of the force is known) by considering the Maxwell stress–tensor\(^3\).

In a non-equilibrium steady state, the net force on the system may be non-zero since there is momentum carried away by the field radiated to the environment. Thus we need to consider the force acting on each cylinder separately (hereafter we refer to our objects as cylinders). We denote the total force acting on the cylinder 1 by \(\langle \mathbf{F}^{(1)} \rangle\), whereas \(\langle \mathbf{F}^{(2)} \rangle\) can be found directly from the expression for \(\langle \mathbf{F}^{(1)} \rangle\) by interchanging indices 1 and 2 and changing its sign. It follows directly from Eq. (2.60) that [5, 15, 19],
\[
\langle \mathbf{F}^{(1)} \rangle(T_1, T_2, T_{\text{env}}) = \langle \mathbf{F}^{(1)} \rangle_{\text{eq}}(T_{\text{env}}) + \sum_{\alpha=1,2} [\langle \mathbf{F}_{\alpha}^{(1)} \rangle(T_\alpha) - \langle \mathbf{F}_{\alpha}^{(1)} \rangle(T_{\text{env}})], \tag{2.61}
\]

where \(\langle \mathbf{F}^{(1)} \rangle_{\text{eq}}(T_{\text{env}})\) is the Casimir force between the cylinders for the case of global equilibrium at temperature \(T_{\text{env}}\), containing the contribution from zero point fluctuations. This force is not discussed in the present thesis and treated as known (it can be computed using scattering results for cylinders as in Refs. [102, 112]; more

---

\(^3\)The heat transfer can be analogously obtained by computing the Poynting vectors associated with difference in source terms in Eq. (2.60). The general expression for the Casimir force and the heat transfer is given by Eqs. (C.5) and (C.7).
general numerical schemes exist for calculating Casimir forces between 3D objects of arbitrary shapes and dielectric properties [113]). The difference of \( \langle F^{(1)}(T_{env}, T_1, T_2) \rangle \) from \( \langle F^{(1)}_{\alpha}(T_{env}) \rangle \) is due to the deviations of the cylinder temperatures \( T_1 \) and \( T_2 \) from \( T_{env} \). \( \langle F^{(1)}_{\alpha}(T) \rangle \) is the force acting on cylinder 1 due to the sources in cylinders \( \alpha = 1, 2 \) at temperature \( T \). Although dealing with three sources (including the environment), we have thus only to evaluate two terms: \( \langle F^{(1)}_{1}(T) \rangle \) and \( \langle F^{(1)}_{2}(T) \rangle \). Due to symmetry of the system, the net force is parallel to the axis-normal connecting the two cylinders, and we denote its magnitude by \( \langle F^{(1)}_{1}(T) \rangle \) and \( \langle F^{(1)}_{2}(T) \rangle \).

Computation of forces proceeds through integrating appropriate components of the electromagnetic stress–energy tensor around a surface enclosing one (or both) cylinders. In the Rytov formalism [9], correlations of the electric field \( E \) at frequency \( \omega \) are sourced by currents in the two cylinders. The contribution of each cylinder can be straightforwardly computed from Eqs. (2.15), (2.17), and (2.19), and for a pair of points at \( \mathbf{r} \) and \( \mathbf{r}' \) outside the cylinder, the symmetrized correlator has the following spectral density,

\[
C^\alpha(T_\alpha, \omega) = -a(T_\alpha) \frac{\omega^2}{c^2} \sum_{\{P, P'\} = M, N} \sum_{n = -\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_z}{8\pi} A_{\alpha,n,k_z} \mathbf{P}_{n,k_z}(\mathbf{r}) \otimes \mathbf{P}^*_{n,k_z}(\mathbf{r}'),
\]

where we have also introduced the abbreviation

\[
A_{\alpha,n,k_z}^{PP'} = \left( \text{Re} T_{\alpha,n,k_z}^{PP'} + \sum_{P'' = M, N} T_{\alpha,n,k_z}^{P''P''} T_{\alpha,n,k_z}^{P''P''*} \right) \Theta \left( \frac{\omega}{c} - |k_z| \right) + (-1)^n \text{Re} T_{\alpha,n,k_z}^{P''P''} \Theta \left( |k_z| - \frac{\omega}{c} \right),
\]

(2.63)

(describing the contribution of propagating and evanescent waves. Note that only propagating waves contribute to the heat emitted by a single cylinder discussed in Sec. 2.1, whereas evanescent waves also contribute to the interactions of two cylinders.

**Interaction Force**

In this subsection we compute the force exerted on cylinder 1 by the field produced by cylinder 2, i.e., the interaction force \( \langle F^{(1)}_{2}(T) \rangle \). To this end, we scatter the field
in Eq. (2.62) (for \( \alpha = 2 \)) at cylinder 1 and then compute the force by integration of the Maxwell stress tensor on a surface enclosing 1. Since multiple reflections on the cylinders are ignored this is a one reflection approximation [5], valid for large \( d/R \), which results in a force per unit length of

\[
\lim_{d \to R} \frac{F_2^{(1)}}{L} = \frac{\hbar}{2\pi^2} \int_0^\infty d\omega n_\omega (T_2) \times \sum_{P,P^*} \sum_{n,m=-\infty}^\infty \left[ (-1)^{n+m} \int_{|k_z| > \omega/c} dk_z |q| f_{2,n,m,k_z}^{PP'} - \int_{|k_z| < \omega/c} dk_z q f_{2,n,m,k_z}^{PP'} \right],
\]

(2.64)

Note that negative and positive signs of the numerical result correspond to attraction and repulsion respectively (again, the force is parallel to the axis-to-axis separation).

The functions \( f \) and \( \tilde{f} \) in Eq. (2.64) take the rather lengthy forms

\[
\tilde{f}_{2,n,m,k_z}^{PP'} = \text{Re} A_{2,n,m,k_z}^{PP'} \text{Im} \left[ H_{n-m}^{(1)}(qd) H_{n-m-1}^{(1)*}(qd) \left( T_{1,m,k_z}^{PP'} + T_{1,m+1,k_z}^{P*} \right) + 2 \sum_{P''} T_{1,m,k_z}^{P''} T_{1,m+1,k_z}^{P''*} \right] + 2 \text{Im} A_{2,n,m,k_z}^{PP'} \text{Re} \left[ H_{n-m}^{(1)}(qd) H_{n-m-1}^{(1)*}(qd) \sum_{P''} T_{1,m,k_z}^{PP'} T_{1,m+1,k_z}^{P''*} \right],
\]

\[
\tilde{f}_{2,n,m,k_z}^{PP'} = \text{Re} T_{2,n,m,k_z}^{PP'} \text{Re} \left[ H_{n-m}^{(1)}(qd) H_{n-m-1}^{(1)*}(qd) \left( T_{1,m,k_z}^{PP'} + T_{1,m+1,k_z}^{P*} \right) \right].
\]

(2.65)

The interaction force in Eq. (2.64) consists of two distinct terms: Due to propagating (\( |k_z| < \omega/c \)) and evanescent waves (\( |k_z| > \omega/c \)), whose properties will be discussed below. We note that for nanowires with thickness in the range of a few tens of nanometers, Eq. (2.64) gives accurate results down to separations of well below 1 micron.

**Thin cylinders \( \{ R_\alpha \} \ll d, \{ \delta_\alpha \}, \lambda_T \):** The general expression in Eq. (2.64) is complicated as it involves an infinite series; more insightful expressions are obtained in the asymptotic limit of thin cylinders. In this paragraph we analytically study the limit \( \{ R_\alpha \} \ll d, \{ \delta_\alpha \}, \lambda_T \), where \( \delta_\alpha = c/\text{Im}[\sqrt{\varepsilon_\alpha}] \omega \) is the skin-depth of tube \( \alpha \). The additional limits \( \{ R_\alpha \} \ll \{ \delta_\alpha \}, \lambda_T \) allow restriction to a finite number of partial waves whose \( T \)-matrix elements are proportional to \( R_\alpha^2 \) (see App. A.3 and note that we take
Thus all terms with $|n|$ or $|m|$ larger than 1, and the terms with products of two T-matrix elements, can be neglected. The corresponding force can be found analytically in the two limits concerning the ratio of thermal wavelength and separation.

For $d \ll \lambda_{T_2}$, we have

$$
\lim_{\{R_0\} \approx d, \{\delta_0\}, \lambda_{T_2}} \frac{\langle F_2(1) \rangle}{L} = \hbar \int_0^\infty d\omega n_\omega(T_2) R_1^2 R_2^2 \left[ \frac{g_0^{in}(\varepsilon_1(\omega), \varepsilon_2(\omega))}{\omega^5} + \frac{\omega^2 g_4^{in}(\varepsilon_1(\omega), \varepsilon_2(\omega))}{c^2 d^3} \right],
$$

(2.66)

where the auxiliary functions $g_0^{in}$ and $g_4^{in}$, that depend only on the dielectric functions, are given in Eqs. (A.21) and (A.22), respectively. We note that the force in Eq. (2.66) is in most cases attractive and dominated by the evanescent part. (It can nevertheless be made repulsive for certain dielectric functions, as discussed in Ref. [50]). At large separations, i.e., $d \gg \lambda_{T_2}$, the leading term of the interaction force is,

$$
\lim_{\{R_0\} \approx d, \{\delta_0\}, \lambda_{T_2}} \frac{\langle F_2(1) \rangle}{L} = \int_0^\infty d\omega n_\omega(T_2) \frac{\omega^5 R_1^2 R_2^2 g_1^{in}(\varepsilon_1(\omega), \varepsilon_2(\omega))}{c^5 d},
$$

(2.67)

where $g_1^{in}$ is given in Eq. (A.23). This limit always yields a repulsive force which is due to momentum transfer by propagating waves. Evanescent waves do not contribute in this order as they decay too quickly with separation.

Further insight can be gained by additionally requiring the temperature to be so low that one can expand the dielectric function, i.e., $\lambda_{T_2} \gg \lambda_0$, where $\lambda_0$ is the wavelength of the lowest resonance of the dielectric response of cylinder $\alpha$. According to Eq. (2.46) [97],

$$
\lim_{\lambda_0, \alpha \ll \lambda_{T_2}} \varepsilon(\omega) = \varepsilon_0 + i \frac{\lambda_{in,\alpha} \omega}{c} + O(\omega^2),
$$

(2.68)

with real valued static dielectric constant $\varepsilon_0$, and inelastic collision length $\lambda_{in,\alpha}$. With this form of $\varepsilon(\omega)$, the corresponding leading behaviors of the force are

$$
\lim_{d, \{\lambda_0\} \ll \lambda_{T_2}, d, \{\delta_0\}, \lambda_{T_2} \gg \{R_0\}} \frac{\langle F_2(1) \rangle}{L} = -\hbar c \lambda_{in,\alpha} R_1^2 R_2^2 \left[ \frac{f_6^{in}(\varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0)}{d^6} - \frac{f_4^{in}(\varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0)}{\lambda_{T_2}^2 d^4} \right],
$$

(2.69)
and

\[
\lim_{d \gg \lambda_{T_2} \gg \{\lambda_{a_i}\}} \frac{\langle F_2^{(1)} \rangle}{L} = \frac{\hbar c \lambda_{in_1} \lambda_{in_2} R_1^2 R_2^2 \int_1^{\infty} (\varepsilon_{0_1}, \varepsilon_{0_2})}{\lambda_{T_2}^8 d}.
\]  

(2.70)

Note that the dependence of temperature (via powers of \( \lambda_T \)) is quite different in these limits. The corresponding auxiliary functions \( f_0^{in}, f_4^{in} \) and \( f_1^{in} \) are given by Eqs. (A.24), (A.25) and (A.26), respectively.

**Self-force**

We next compute the self-force \( \langle F_1^{(1)} \rangle \), acting on cylinder 1 due to the field emitted by cylinder 1 and reflected from cylinder 2, again restricting to \( d \gg \{R_{\alpha}\} \), where the one reflection approximation is valid. The origin of this force is cylinder 2 acting as a reflector of electromagnetic waves emitted by cylinder 1. The interference of the emitted and reflected waves is expected to give rise to the observed oscillatory behavior of the self-force as a function of separation. Computationally it is easier to first evaluate the force \( \langle F_1^{(1+2)} \rangle = \langle F_1^{(2)} \rangle + \langle F_1^{(1)} \rangle \) acting on both cylinders due to sources on cylinder 1, and to then subtract \( \langle F_1^{(2)} \rangle \). Again, it is a property of non-equilibrium that the net force on the system is not zero. Note that \( \langle F_1^{(2)} \rangle \) is obtained from Eq. (2.64) by interchanging indices 1 and 2 and changing its overall sign.

After tedious algebra we arrive at the following expression for the force \( \langle F_1^{(1+2)} \rangle \),

\[
\lim_{d \gg R} \frac{\langle F_1^{(1+2)} \rangle}{L} = \frac{\hbar}{\pi^2} \int_0^\infty d\omega n_\omega(T_1) \sum_{\{P,P'\}=N,M} \sum_{n,m=-\infty}^\infty \int |k_x|<c/k \ dk_x q \\
\times \text{Re} \ A_{1,n,k_x}^{PP'} \text{Im} \left[ H_{n-m}(qd)J_{n-m-1}(qd)T_{2,m,k_x}^{PP'} + J_{n-m}(qd)H_{n-m-1}^{(1)*}(qd)T_{2,m+1,k_x}^{PP'*} \right].
\]  

(2.71)

As expected, the force on the system is solely due to propagating waves, since evanescent waves do not carry momentum to the environment.

**Thin cylinders** \( \{R_{\alpha}\} \ll d, \{\delta_{\alpha}\}, \lambda_T \): We gain further insight by examining \( \langle F_1^{(1+2)} \rangle \)
in the asymptotic limit of thin dielectric cylinders. For \( d \ll \lambda_{T_1} \),

\[
\lim_{d, \{\delta_\alpha\}, \lambda_{T_1} \gg \{R_\alpha\}} \frac{\langle F_1^{(1+2)} \rangle}{L} = \mathcal{O}(d^{-1})
\]  

i.e. diverging weakly at small \( d \). Compared to this, the force \( \langle F_1^{(2)} \rangle \) in Eq. (2.66) is proportional to \( d^{-6} \), and we can neglect \( \langle F_1^{(1+2)} \rangle \) in this regime, so that for \( d \ll \lambda_{T_1} \),

\[
\lim_{d, \{\delta_\alpha\}, \lambda_{T_1} \gg \{R_\alpha\}} \langle F_1^{(1)} \rangle = - \lim_{d, \{\delta_\alpha\}, \lambda_{T_1} \gg \{R_\alpha\}} \langle F_1^{(2)} \rangle.
\]

We thus observe that at close separations these contributions to the forces on the cylinders are equal and opposite, again because momentum transfer to the environment can be ignored.

In the opposite limit of large distances, \( d \gg \lambda_{T_1} \), one can verify that

\[
\lim_{d, \{\delta_\alpha\}, \lambda_{T_1} \gg \{R_\alpha\}} \frac{\langle F_2^{(1)} \rangle}{L} = \mathcal{O}(d^{-3/2}),
\]  

where the prefactor is a lengthy algebraic function additionally depending on \( d \) in an oscillatory manner via \( \exp(2i\omega d/c) \). Thus, at large separations the self-force is asymptotically less relevant compared to \( \langle F_2^{(1)} \rangle \) (following \( d^{-1} \) in Eq. (2.67)).

**Cylinders from spheres in the dilute limit**

A rarified material can be regarded as a collection of independent molecules; forces between two such materials may then be obtained by pairwise summation of forces between their molecules. Indeed, equilibrium Casimir forces in the optically dilute limit \( (\varepsilon_\alpha(\omega) \to 1) \) can be calculated by integrating the interactions between volume elements. In this subsection we show that even the non-equilibrium force between two cylinders can be obtained from the non-equilibrium force between two spheres [5], by appropriate summation in the optically dilute limit.

We start with \( \langle F_2^{(1)} \rangle \), where we get from Eqs. (2.66) and (2.67) in the limit of
Figure 2-11: Casimir forces between objects can also be obtained by pairwise summation of the forces between all the volume elements.

\( \varepsilon_\alpha \to 1 \), (note that this means that \( \delta_\alpha \) is much larger than the radius of the objects)

\[
\lim_{\lambda \tau_2 \gg d \gg \{R_\alpha\}} \frac{\langle F_2^{(1)} \rangle}{L} = -\hbar \int_0^\infty d\omega n_\omega(T_2) R_1^2 R_2^2 \Re[\varepsilon_1 - 1] \Im[\varepsilon_2] \left( \frac{45}{64d^6} + \frac{3\omega^2}{16c^2d^4} \right),
\]

(2.74)

and

\[
\lim_{d,\lambda \tau_2 \gg \{R_\alpha\}} \frac{\langle F_2^{(1)} \rangle}{L} = \frac{\hbar}{2\pi} \int_0^\infty d\omega n_\omega(T_2) R_1^2 R_2^2 \frac{\omega^5}{c^2d} \Im[\varepsilon_1] \Im[\varepsilon_2].
\]

(2.75)

On the other hand, Eq. (6) from Ref. [5] for the interaction force between two spheres, \( F_2^{(1)}(d) \) can be expanded in the dilute limit to yield the force between two volume elements \( V_1 \) and \( V_2 \).

\[
\lim_{d,\lambda \tau_2 \gg \{R_\alpha\}} F_2^{(1)}(d) = \frac{V_1 V_2 \hbar}{4\pi^3} \int_0^\infty d\omega n_\omega(T_2) \frac{\omega^7}{c^7} \Im[\varepsilon_2] \times \left[ \frac{c^2}{\omega^2 d^2} \Im[\varepsilon_1] - \Re[\varepsilon_1 - 1] \left( \frac{c^3}{\omega^3 d^3} + \frac{2c^5}{\omega^5 d^5} + \frac{9c^7}{\omega^7 d^7} \right) \right],
\]

(2.76)

The force between two cylinders follows then as an integral over the volume of the two cylinders, see Fig. 2-11,

\[
\frac{\langle F_2^{(1)} \rangle}{L} = \pi^2 R_1^2 R_2^2 \int_0^\infty \frac{F_2^{(1)}(\sqrt{d^2 + t^2})}{V_1 V_2} \frac{dt}{\sqrt{d^2 + t^2}}.
\]

(2.77)

\footnote{We have an additional minus sign compared to Eq. (6) from Ref. [5], and we exchanged indices 1 \( \leftrightarrow \) 2.}
Using this and Eq. (2.76), the terms in Eqs. (2.74) and (2.75) are recovered. Since the consistency of the two results is physically expected, it provides a useful check for both calculations. We note, however, that we could not recover the term proportional to $d^{-2}$ in $\langle F_2^{(1)} \rangle$ (corresponding to $d^{-3}$ in Eq. (2.76)) due to mathematical difficulties in the integral over special cylindrical functions.

We note also the corresponding equivalence for the self-force, though somewhat more involved mathematically: in Eq. (7) from Ref. [5], the self force decays as $d^{-2}$ with an oscillating prefactor $\exp(2i\omega d/c)$. Pairwise summation of this result yields exactly a $d^{-3/2}$ separation dependence modulated by $\exp(2i\omega d/c)$, as discussed in relation to Eq. (2.73). Moreover, if we represent a plate of finite thickness by an infinite set of cylinders and place a small sphere in front of this plate, the self-force acting on the sphere scales as $d^{-1}$ with the prefactor of $\exp(2i\omega d/c)$. This matches perfectly with the result for the sphere/plate geometry, in Eq. (19) from Ref. [5], in the limit of large separations. Thus, in the dilute limit one regains the expected connections between different geometries.

### 2.2.2 Numerical Examples

In this section we numerically compute the total Casimir force between two cylinders, in one case composed of dielectrics (SiC) and of conductors (tungsten) in the other. The equilibrium contributions to the force (see Eq. (2.61)) were computed using the methods of Ref. [113] as implemented in the SCUFF-EM code suite [114].

#### Silicon carbide (SiC)

Figure 2-12 depicts the forces on SiC cylinders of radii $R_1 = R_2 = 0.1\mu$m in both cold (0K) and warm (300K) environments; each wire or the environment is at 0K or 300K, for a total of 8 possible combinations. The optical properties of SiC are modeled by the dielectric function given by Eq. (2.42). We evaluated Eqs. (2.64) and (2.71) numerically, restricting $n$ and $m$ to orders $\{-1, 0, 1\}$, and omitting quadratic terms in $\mathbb{T}$, which is justified for $\{R_\alpha\} \ll \{\lambda_{T_\alpha}\}, \{\delta_\alpha\}$, where the forces per unit length will
be proportional to $R_1^2 R_2^2$. We note that $R_\alpha = 0.1\mu m$ is roughly an upper bound for the validity of this asymptotic behavior, as we have checked by including higher order terms. Due to the small thickness of the cylinders, the resulting phenomenology is in close analogy to the case of two spheres [5] (compare also Sec. 2.2.1): in the case when $T_{env} = 0K$, the force starts to deviate strongly from its equilibrium value around $d \approx \lambda_T/2$, where $\lambda_T = \hbar c/k_B T \approx 7.6\mu m$. Cylinder 1 is repelled at large $d$ if $T_2 = 300K$ due to radiation pressure with a force that decays with distance as $d^{-1}$. On the other hand, if additionally $T_1 = 300K$, the oscillating force $\langle F_1^{(1)} \rangle$ is appreciable and in fact dominates the total force for large $d$ if $T_1 = 0K$; the net force now has many zero crossings, where every second one is a stable point of zero force. The wavelength of the oscillations is roughly $6 \mu m$ due to the optical resonance of SiC at wavelength $\lambda_0 \approx 12\mu m$ (the length of the optical path from cylinder 1 to cylinder 2 and back is $2d$, and for a sharp resonance of the dielectric function of cylinder 1 at $\lambda_0$, we have constructive interference at $2d = \lambda_0, 2\lambda_0, \ldots$, for an oscillation wavelength of $\lambda_0/2$). Note that this figure also provides complete information about the force on cylinder 2: e.g., in case $\{T_1 = 0, T_2 = 300K\}$, the blue curve shows the force acting on cylinder 1, while the red one represents the force on cylinder 2. At the crossing of the solid red and dashed blue curves the two cylinders feel equal forces in the same direction, an effect which might have less practical importance for tubes compared to spheres [5].

**Tungsten (W)**

Let us turn to conductors at high temperatures. Figure 2-13 illustrates the total Casimir force for two very thin tungsten cylinders (nanowires) of equal radii $R_\alpha = 0.02\mu m$, and large temperature differences of 2400K. We have chosen this value for $R_\alpha$ as, according to Fig. 2-5(b), it corresponds to the maximum in emissivity of an isolated tungsten cylinder (which is connected to the skin depth being of the same order). As the non-equilibrium force strongly depends on the heat radiation of the objects, we expect it to be also comparatively large for this value. Also, tungsten has a relatively high melting temperature, and the value of 2400K promises large effects. Note that Eqs. (2.66) and (2.67) are not valid here, and the force is not proportional
Figure 2-12: Total force on cylinder 1 per unit length in a system of two SiC cylinders with equal radii $R = 0.1\mu m$ at separation $d$ in a a) cold (0K) b) warm (300K) environment. Dashed lines indicate repulsion. Points of change from repulsive to attractive force with increasing $d$ correspond to stable points of zero force.

to $R_1^2R_2^2$, despite the small thickness of the wires. We use the dielectric function for tungsten given in Eq. (2.53) at $T = 2400K$.

The relative deviation of the total Casimir force from its equilibrium form is not as pronounced as in Fig. 2-12, since equilibrium forces between conductors decay much slower with distance compared to dielectrics. In particular, a Drude dielectric function gives rise to an equilibrium force scaling as $d^{-4}/\log(d/R)$ [102], in good agreement with our numerical results (for dielectrics, our numerical result suggests a $d^{-7}$ law of the equilibrium Casimir force in the small separation retarded regime).

Nevertheless, as in the dielectric case, the force in Fig. 2-13 starts to deviate from the equilibrium curve at approximately $\lambda_T/2$ with a corresponding $\lambda_T \approx 0.95\mu m$. In a cold environment, $T_{\text{env}} = 0K$, the force is attractive at small separations, whereas at large distances it becomes repulsive if $T_2 = 2400K$. When we increase the environment temperature to $T_{\text{env}} = 2400K$, the force shows another feature for $T_1 = T_2 = 0K$, where it is attractive for both small and large distances, but for an intermediate region it becomes repulsive yielding a stable point of zero force in the vicinity of

---

5For the accurate evaluation of the total Casimir force with Eq. (2.61), one must use the cylinder dielectric function at its corresponding temperature. Thus, Eq. (2.53) is strictly only appropriate for $T_1 = T_2 = 2400K$ in Fig. 2-13. However, the dielectric function does not depend significantly on $T$ and the curves for $T_o = 0K$ in Fig. 2-13 are good estimates.
Figure 2-13: Total force on cylinder 1 per unit length in a system of two tungsten (W) cylinders with equal radii $R = 0.02\mu m$ at separation $d$ in a) cold (0K) b) warm (2400K) environment. Dashed lines indicate repulsion. Horizontal line $F_G/L$ denotes the weight of the corresponding tungsten cylinder per unit length.

$d \approx 4\mu m$. This intermediate repulsion is due to the contribution from evanescent waves (which is repulsive because $\langle F_d^{(1)} \rangle$ enters Eq. (2.61) with a minus sign), and then for higher $d$ propagating waves make the force attractive again. In contrast to the dielectric displayed in Fig. 2-12(b), no oscillations are visible in Fig. 2-13. We attribute this to the absence of sharp peaks in the dielectric function of tungsten.

The non-equilibrium force in Fig. 2-13 is much larger compared to Fig. 2-12, again, as a consequence of both replacing the dielectric materials with conductors, as well as choosing the higher temperature $T = 2400K$ instead of $T = 300K$. At the point where the total Casimir force deviates from the equilibrium force the magnitude of the force is roughly $10^6$ times larger for tungsten compared to SiC – approximately $5 \times 10^6$ larger per unit mass. For tungsten nanowires at $T = 300K$ (we do not provide the plot for this case) the non-equilibrium effects are still three orders of magnitude larger when compared to SiC. Thus, the strong enhancement is a result of both material properties and high temperatures. Furthermore, considering wires of length 1$\mu m$ at high temperature $T = 2400K$, the force at the point when non-equilibrium forces start to dominate is approximately 10fN (see Fig. 2-13), significantly larger than the weight $F_G \approx 0.24fN$ (also depicted in the figure) of the nanowires.

There is another force we can use for comparison: if an electric current is used to
heat the wire [17, 18], one requires a current of 17μA to maintain the temperature

\( T = 2400\text{K} \) for tungsten nanowires (obtained by equating heat losses described by Joule’s law to the heat radiation for a single cylinder predicted in Ref. [68]). Such currents correspond to an Ampere’s force of about 0.15fN for \( d = 0.4\mu\text{m} \), which is two orders of magnitude smaller than the nonequilibrium correction to the Casimir force \(^6\). Thus, when switching on an AC or DC current in the wires, the change in force between them due to heating-induced non-equilibrium Casimir force is much larger than that due to Ampere’s force. We thus conclude that hot conducting nanowires are promising candidates for measuring or using non-equilibrium Casimir effects.

\(^6\)This estimate neglects possible couplings between AC or DC currents and thermal current fluctuations [23]
Chapter 3

Proximity Transfer Approximation and Beyond

In this Chapter we reproduce our results, up to some changes in notation, as published in Refs. [70, 71].

Here we study the radiative heat transfer between objects for the experimentally relevant case of small separations exploiting the proximity transfer approximation (PTA), which is significantly more efficient to use in this case when compared to the scattering formalism. PTA works as follows: the heat transfer between two parallel plates (per unit area), $H_{pp}(S(x))$, which is a function of separation $S(x)$, is averaged over the projected surface $\Sigma$. $x = (x, y)$ are Cartesian coordinates in the projected plane $\Sigma$ normal to the $z$ axis (see Fig. 3-1). Thus, PTA yields,

$$H_{PTA} = \int_{\Sigma} d^2 x \ H_{pp}(S(x)).$$  \hspace{1cm} (3.1)

While PTA is generally assumed to hold asymptotically for small separations $d$, there has been no rigorous proof of that before our work [70].

We exploit the above-mentioned definition of PTA in Sec. 3.1 to develop distance expansion for the radiative heat transfer between gently curved objects, in terms of the ratio of distance to radius of curvature. This method, which has been proposed for Casimir forces [115–117], exploits the mapping of coefficients of a perturbative
expansion on one side and a gradient expansion on the other (the detailed derivation of the expansion parameter $\beta$ is provided in App. B). That allows us to go beyond the lowest order PTA and justify it for heat transfer. In Sec. 3.2, we utilize proximity approximation (PA) to study how roughness or modulations modify scaling laws of various interactions, including heat transfer, between closely spaced objects.

### 3.1 Small distance expansion for radiative heat transfer between curved objects

Consider two non-magnetic objects with dielectric permittivities $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$ at temperatures $T_1$ and $T_2$, respectively. The heat transfer $H^1$ between them is symmetric and can be written as an integral over frequency $\omega$, as

$$H = \int_0^\infty d\omega \left[n_\omega(T_1) - n_\omega(T_2)\right] h,$$

where $n_\omega(T)$ is the Bose-Einstein weight given in Eq. (2.5). The spectral density $h$ depends only on the classical scattering amplitudes of the objects and their relative positions [15].

\[^1\text{Note, in this Chapter we omit } \langle \ldots \rangle \text{ to manifest non-equilibrium average, so that, e.g. } \langle H \rangle \equiv H.\]
Figure 3-2: The evanescent electric ($H_{pp}^{ev,N}$), magnetic ($H_{pp}^{ev,M}$) and total propagating ($H_{pp}^{pr}$) contributions to heat transfer between parallel plates per unit area, $H_{pp} = H_{pp}^{ev,N} + H_{pp}^{ev,M} + H_{pp}^{pr}$, as a function of separations for $T_1 = 0K$ and $T_2 = 300K$. Data are shown for two materials, SiC and SiO$_2$, and are normalized by the result for black bodies (2.41).

For the sake of simplicity, let object 1 be a semi-infinite, planar body, filling the space $z < 0$, whereas object 2 has a curved surface with the smooth height profile $z = S(x)$. Note, this is a simplified case compared to Fig. 3-1, where both of the objects are non-planar. PTA is given by Eq. (3.1) and an analogous expression can be written down for the spectral density.

Different contributions [30] to $H_{pp}(S)$ are separately depicted in Fig. 3-2 for two dielectric materials, SiC (see Eq. (2.42)) and SiO$_2$ (computed using optical data). Propagating waves give rise to an almost separation independent contribution, whereas the transfer due to evanescent waves depends strongly on separation. Both parts have in turn two distinct components from electric ($N$) and magnetic ($M$) modes. As mentioned in Chapter 2, in the literature the $N$ and $M$ modes are often referred to as transverse magnetic (TM) and transverse electric (TE), respectively. Notably, if $S$ is the smallest scale, the evanescent $N$-mode diverges as $S^{-2}$, while all other
contributions approach finite values. For the two materials considered, the figure indicates that the $S^{-2}$ contribution is dominant for $S \lesssim 0.03\ \mu\text{m}$ and $S \lesssim 0.3\ \mu\text{m}$, respectively. In general, the range of this regime, sometimes referred to as the non-retarded regime, depends strongly on material properties\footnote{An expansion of the Fresnel coefficients for large evanescent wavevector shows that the (spectral) $S^{-2}$ contribution is dominant if (for two equal materials) $\text{Im}[(\varepsilon - 1)/(\varepsilon + 1)] \gg \text{Im}[(\varepsilon - 1)(\omega d/2c)^2]$ holds.}. Because the behavior of $H_{pp}(S)$ is nontrivial in general, we restrict ourselves to the case $H_{pp}(S) \propto S^{-2}$, such that Eqs. (3.4)-(3.13) below are valid. At the end, we give an outlook to larger separations. The expansion can equally well be performed for the transfer in Eq. (3.2) as for its spectral density, and we choose the latter due to its larger information content. The spectral density for two parallel plates reads accordingly,

$$h_{pp}(S) = \frac{v\omega}{S^2},$$

(3.3)

where the coefficient $v\omega$ uniquely describes the material dependence of the heat transfer in the considered range. Following Refs. [115–117], we assume (justified a posteriori) that $h$ in Eq. (3.2) admits a local expansion in $\nabla S$,

$$h[S(x)] = h_{PTA}[S(x)] + \int_S d^2x \beta_\omega h_{pp}(S) \nabla S \cdot \nabla S + \ldots,$$

(3.4)

where $\beta_\omega$ is a geometry-independent expansion parameter and the dots represent higher order terms in $\nabla \! S$. We will see below that Eq. (3.4) is an expansion in the ratio of separation to radius of curvature of the surface.

The coefficient $\beta_\omega(S)$ is obtained from the perturbative expansion [29] of $h[S]$ for $|s(x)| \ll d$, with $s$ defined with respect to the point of closest proximity $d$, $S(x) = d + s(x)$: If both a perturbative expansion as well as the gradient expansion in Eq. (3.4) exist, the two expansions must have identical coefficients in their common limit, i.e. the expansion of $h$ in Eq. (3.4) has to match the perturbative expansion at
small momenta $k_\perp$, which is given by (with $k_\perp \equiv |k_\perp|$)

$$h[d + s(x)] = \alpha_0(d) + \alpha_1(d)\tilde{s}(0) + \int \frac{d^2k_\perp}{(2\pi)^2} \frac{\alpha_2(k_\perp; d)}{\alpha_2(k_\perp; d)} |\tilde{s}(k_\perp)|^2,$$  \hspace{1cm} (3.5)

where, e.g., $\alpha_0(d)$ is proportional to $h_{pp}(d)$ times the corresponding projected area, and $\tilde{s}(k_\perp)$ is the Fourier transform of $s(x)$ with in-plane wave-vector $k_\perp$. Thus, if it is possible to expand $\alpha_2(k_\perp; d)$ to second order in $k_\perp$, we can match the expansion

$$\alpha_2(k_\perp; d) = \alpha_2^{(0)}(d) + \alpha_2^{(2)}(d)k_\perp^2 + \ldots,$$  \hspace{1cm} (3.6)

to Eq. (3.4) to get

$$\beta_\omega h_{pp}(d) = \alpha_2^{(2)}(d).$$  \hspace{1cm} (3.7)

The gradient expansion of Eq. (3.4) requires knowledge of the coefficient $\beta_\omega$, which, by Eq. (3.7), is connected to $\alpha_2^{(2)}(d)$, in turn computed below. The coefficients $\alpha_0$ and $\alpha_1$ in Eq. (3.5) are hence irrelevant for the remainder of the computation, and we omit their detailed analysis.

In the asymptotically close regime, $\beta_\omega$ is $d$-independent, as both $h_{pp}(d)$ and $\alpha_2^{(2)}(d)$ are proportional to $d^{-2}$. The perturbative calculation for $h[S]$ is done by use of a plane-wave basis with in-plane wave-vector $k_\perp$ (eventually identical to $k_\perp$ in Eq. (3.5)) and modes $Q = N, M$. The reflection matrix of the planar (isotropic) object 1 is diagonal in this basis and given by the Fresnel coefficients, $r_1^Q(k_\perp)$ (see Eq. (2.27)). The corresponding coefficient $R_2^{QQ'}(k_\perp, k'_\perp)$ for the curved object 2 depends on both incoming (primed) and scattered waves, and can be expanded in powers of $\tilde{s}$ [118], as

$$R_2^{QQ'}(k_\perp, k'_\perp) = (2\pi)^2 \delta^{(2)}(k_\perp - k'_\perp)\delta_{QQ'}r_2^Q(k_\perp) + B_2^{QQ'}(k_\perp, k'_\perp)\tilde{s}(k_\perp - k'_\perp)$$

$$+ \int \frac{d^2k_\perp''}{(2\pi)^2} B_2^{QQ'}(k_\perp, k'_\perp; k_\perp'')\tilde{s}(k_\perp - k_\perp'')\tilde{s}(k_\perp'' - k'_\perp) + \ldots$$  \hspace{1cm} (3.8)

The expressions for $B_2^{QQ'}$ and $B_2^{QQ''}$ are given in Ref. [118]. Substituting Eq. (3.8) into the heat transfer expression for corrugated surfaces [35] and expanding to second order in $\tilde{s}(k_\perp)$ yields $\alpha_2(k_\perp; d)$. Expanding the latter for small $k_\perp$ yields $\alpha_2^{(2)}(d)$.
according to Eq. (3.6), and with Eq. (3.7) the desired $\beta_\omega$ is determined (see App. B for more detailed derivation).

For the experimentally most relevant case of a sphere of radius $R$ in front of a plate, Eq. (3.4) is evaluated to yield the following expansion in $d/R$ (where $S(x) = d + R(1 - \sqrt{1 - |x|^2/R^2})$):

$$h(d) = \frac{2\pi R \nu_\omega}{d} \left[ 1 - (2\beta_\omega - 1) \frac{d}{R} \log \frac{d}{d_0} \right] + \mathcal{O}(d^0). \tag{3.9}$$

Here $d_0$ is an unknown constant of integration. PTA in Eq. (3.1) corresponds to the result for $\beta_\omega = 0$ (compare to Eq. (3.4)). Hence in Eq. (3.9) $2\pi \nu_\omega \log (d/d_0)$ represents a trivial correction to the leading $2\pi R \nu_\omega / d$ and appears due to the projection of the sphere’s surface onto the $xy$-plane.

As the term of $\mathcal{O}(d^0)$ in Eq. (3.9) is unknown from the expansion and might not be small compared to the other terms in Eq. (3.9) (we compare it to a logarithmic term), possibly leading to slow convergence in practical cases, a better quantity to consider is its derivative,

$$h'(d) = -\frac{2\pi R \nu_\omega}{d^2} \left[ 1 + (2\beta_\omega - 1) \frac{d}{R} \right] + \ldots, \tag{3.10}$$

which converges faster, as the constant term in Eq. (3.9) drops out.

In order to test these results, we semi-analytically compute the exact heat transfer between a sphere and a plate. This scheme, where the scattering properties of sphere and plate are expanded in cylindrical multipoles [34], directly yields the complete heat transfer between the two objects as a sum over multipole-contributions (in contrast to other methods which e.g. use surface discretization). Here, a maximal multipole order of $l_{\text{max}} = 2500$ is used for the smallest separations $d/R = 0.004$ shown to obtain precise data (including the derivative). We can hence validate Eq. (3.10) (or Eq. (3.9)) by finding $\beta_\omega$ in two independent ways. The main panel of Fig. 3-3 shows $\beta_\omega$ computed by the gradient expansion for SiC, together with five values
Figure 3-3: The gradient expansion parameter $\beta_\omega$ as a function of frequency for SiC, computed via Eq. (3.7), (solid line), and as found by fitting exact data for a sphere in front of a plate to Eq. (3.10) (data points). The dashed line shows $h_{pp}(d = 10 \text{ nm})n_\omega(T = 300K)$, illustrating the dominant range for near-field transfer of SiC. The upper inset provides three examples of fitting the exact data (data points) to Eq. (3.10), where solid (dashed) lines omit (include) the assumed next order term in Eq. (3.10)$^3$. The point styles used are in accordance with the corresponding frequencies in the main figure. In the considered regime (Eq. (3.3)) these curves depend on $R$ only through $d/R$. For $\omega/c = \{0.5934, 0.5967, 0.6050, 0.6075, 0.6100\}\text{rad}/\mu\text{m}$ shown in the main plot, the values computed in the gradient expansion ($\beta_\omega = \{0.774, 0.206, -0.086, -3.026, -2.156\}$) agree well with the fitted ones ($\{0.773, 0.206, -0.105, -3.044, -2.112\}$). The lower inset shows $\beta_\omega$ for SiO$_2$. 


obtained by fitting of exact data to Eq. (3.10). We observe good agreement\(^3\). The plotted frequency region is the relevant interval for SiC, with one dominant resonance around \(\omega/c = 0.6 \text{ rad/} \mu \text{m} \), as indicated by \(h_{pp}(d = 10 \text{ nm}) \) multiplied by the weight \(n_\omega(T = 300K)\) (dashed curve). The upper inset shows the result of Eq. (3.10) using the fitted \(\beta_\omega\) together with the exact data for three frequencies also depicted in the main plot. We have thus demonstrated the validity of the gradient expansion both qualitatively and quantitatively. The lower inset shows \(\beta_\omega\) for SiO\(_2\) computed by the gradient expansion.

The total transfer \(H\) follows from Eqs. (3.2) and (3.9), where we still consider \(H_{pp}(S) \propto S^{-2}\). For materials with \(|\varepsilon|\) not too large (e.g. SiO\(_2\)), this regime requires \(S \ll \{\lambda_{T_1}, \lambda_{T_2}\}\) \((\lambda_T = \hbar c/(k_B T)\) is the thermal wavelength, \(\lambda_T \approx 7.6 \mu \text{m} \) at 300K. It sets the range of wavelengths contributing to the transfer in Eq. (3.2)). Integrating Eq. (3.9) over frequencies \(\omega\), we obtain

\[
H(d) = \frac{2\pi R v}{d} \left[ 1 - (2\beta - 1) \frac{d}{R} \log \frac{d}{d_0} \right] + \mathcal{O}(d^0),
\]

where, \(v = \int_0^\infty d\omega \left[ n_\omega(T_1) - n_\omega(T_2) \right] v_\omega \) and \(\beta = v^{-1} \int_0^\infty d\omega \left[ n_\omega(T_1) - n_\omega(T_2) \right] \beta_\omega v_\omega\).

Figure 3-4 depicts the heat transfer from a sphere to a plate as a function of \(d/R\) for \(T_1 = 300K\) and \(T_2 = 0K\). In order to eliminate the constant term in Eq. (3.11), the main panel depicts a near-field adjusted plot, where we show \(H(d) - H(d = 0.004R)\). Such a plot takes advantage of the faster convergence observed for the derivative in Eq. (3.10), and allows to accurately describe the exact data by use of Eq. (3.11), and in the same manner, experimental data could be compared to Eq. (3.11). We note that the reference separation (here \(d = 0.004R\)) should be chosen in the range where the \(d/R\)-expansion is valid. For both SiC and SiO\(_2\), the exact data points agree remarkably well with Eq. (3.11), over the whole range of separations depicted (up to \(d/R = 0.1\)). We also show PTA from Eq. (3.1), which, again, to the lowest two orders in \(d/R\) is given by Eq. (3.11) with \(\beta = 0\). Its deviation from the exact data is hence quantified by Eq. (3.11) to \(4\pi/\beta v \log[d/0.004R]\). This is shown in the inset, with

\(^3\)For improved fits we assume a next correction to Eq. (3.10) of the form \(\gamma \log(d/R)\) with adjustable \(\gamma\).
convincing agreement to the exact data, which is scattered due to numerical precision (the inset probes differences of $\sim 1\%$ of the total transfer). Based on Fig. 3-2, we expect these curves to describe the heat transfer for separations up to $d \lesssim 0.03$ $\mu$m and $d \lesssim 0.3$ $\mu$m for SiC and SiO$_2$ respectively. E.g., for $R = 1$ $\mu$m ($R = 10$ $\mu$m) for SiC (SiO$_2$), this corresponds to $d/R \lesssim 0.03$.

The subleading term in Eq. (3.11) is a logarithmic correction. Surprisingly, this term is hardly noticeable due to the coincidence that $\beta = 0.5119$ and $\beta = 0.5241$ for SiC and SiO$_2$, respectively (despite very different curves for $\beta_\omega$ for the two materials in Fig. 3-3), and hence the prefactor $(2\beta - 1)$ is very small (0.02 and 0.05). In other words, the second order term in Eq. (3.11) is very small because the term with $\beta$, coming from the second term in Eq. (3.4), almost cancels the one coming from the first term in Eq. (3.4). Due to this coincidence, which is special for the sphere-plate geometry, PTA predicts a logarithmic term which in reality is almost absent.

Other geometries can readily be analyzed with Eq. (3.4). For example, for two spheres of radii $R_1$ and $R_2$, we find,

$$H(d) = \frac{2\pi v}{d} \frac{R_1 R_2}{R_1 + R_2} \left[ 1 + \frac{d}{R_1 + R_2} \log \frac{d}{d_0} - (2\beta - 1) \left( \frac{d}{R_1} + \frac{d}{R_2} \right) \log \frac{d}{d_0} \right] + O(d^0), \tag{3.12}$$

where, in contrast to the sphere-plate case, the logarithmic term is pronounced for $\beta \approx 0.5$, as has been observed in numerical data [51]. Furthermore, for a cylinder of radius $R$ and length $L$ in front of a plate, the expansion converges better than Eq. (3.11), because the second term has no logarithm,

$$\frac{H(d)}{L} = \frac{\pi \sqrt{R v}}{\sqrt{2} d^{3/2}} \left[ 1 + \left( 2\beta - \frac{3}{4} \right) \frac{d}{R} \right] + O(d^0), \tag{3.13}$$

For larger separations, the $d$-dependence of the expansion parameter $\beta_\omega$ in Eq. (3.7) has to be taken into account, and Eq. (3.4) has to be evaluated numerically. This is left for future work.

Thus, the heat transfer between smoothly curved objects can be systematically expanded for small separations. This expansion is well suited for the gradient of
Figure 3-4: Near-field adjusted plot ($H(d) - H(d = 0.004R)$) for a sphere in front of a plate for SiC (circles, left abscissa) and SiO$_2$ (squares, right abscissa), $T_1 = 300$K and $T_2 = 0$K. Exact data (HT here is short for heat transfer), PTA and Eq. (3.11) are shown as data points, solid and dashed lines, respectively. The inset depicts the deviation of PTA from the exact data, where lines show the analytical form $4\pi\beta\nu\log[d/0.004R]$ from Eq. (3.11). In the regime considered (Eq. (3.3)), all curves shown depend on $R$ only through $d/R$. 
the transfer, and a near-field adjusted plot allows accurate prediction of experimentally measurable data. If the separation is the smallest scale, the expansion can be performed analytically, where coefficients are evaluated numerically.

3.2 Interplay of roughness/modulation and curvature at proximity

In this Section we exploit the PA discussed in the previous section to consider the influence of roughness/modulation of the objects’ surfaces on the interactions (e.g. heat transfer) at proximity. We continue to consider two objects with surface-to-surface separation $S(x)$, but in contrast to Sec. 3.1 none of the objects is necessarily planar, as depicted in Fig. 3-5. Additionally, for the sake of generality we, first, consider the unspecified interaction $I$ (e.g. the Casimir force or heat transfer) between the objects, which may depend in a complicated way on the geometry of the objects. However, we assume that at close proximity it is dominated by the local properties of the surfaces near the point of closest approach. In analogy to Eq. (3.1), the proximity approximation $I_{PA}$ for the net interaction is given by

$$I_{PA} = \int_{\Sigma} d^2x I_{pp} (S(x)),$$

(3.14)
where \( I_{pp}(S) \) denotes the interaction between two parallel plates at separation \( S \) (per surface area), which is, measured normal to the projected surface \( \Sigma \). When both objects have arbitrary shapes, there is some ambiguity in the choice of reference plane \( \Sigma \), which may be decided based on symmetries, or through the experimental path of approach.\(^{4}\) It is reasonable to expect that Eq. (3.14) becomes exact for nearly flat surfaces, i.e. in the limit \( d/\rho \to 0 \), where \( \rho \) sets the scale for the local radii of curvatures. Indeed, in certain cases Eq. (3.14) represents the leading term of an expansion of the interaction in powers of the gradient of \( S \) \([70, 115-117]\) (see Eq. (3.4) above and Eq. (3.24) below). This implies that Eq. (3.14) is asymptotically exact in the limit \( d/\rho \to 0 \), the leading curvature correction being of higher order in \( d/\rho \). Equation (3.14) can be alternatively recast in terms of the height distribution function \([119, 120]\) \( f(s) \equiv \int_{\Sigma} d^2x \delta(S(x) - d - s) \), as

\[
I_{PA} = \int ds f(s - d) I_{pp}(s). \tag{3.15}
\]

Note that \( f(s) \) is measured with respect to the distance \( d \) of closest approach, such that \( f(s) = 0 \) for \( s < 0 \). It is thus a function of shape (and orientation) of the objects, and independent of their separation. Equation (3.15) highlights the basic assumption of PA in neglecting the detailed topology of the surface (e.g. gradients), being sensitive only to the projected surface area at separation \( s \). While in general only an approximation, Eq. (3.15) can be used to understand the qualitative effects of modulation or roughness in concrete experimental setups; the function \( f(s) \) can be easily extracted from atomic force microscope (AFM) scans of the surfaces \([119]\), or computed analytically for model surfaces (e.g., below).

In what follows, we assume that in the considered range of distances \( I_{pp}(d) \) can be described by a power law,

\[
I_{pp}(d) = \frac{\chi}{d^\nu}, \tag{3.16}
\]

where \( \nu \) and the dimensional coefficient \( \chi \) depend on the specific interaction under

\(^{4}\)The slowly varying parts of the surfaces mark almost parallel planes in the considered region of closest approach, thus uniquely defining the plane of reference in practice.
study. For example, for heat transfer \( \chi \equiv v \) and \( v = 2 \) according to Eq. (3.3)\(^5\). Similarly, \( v = 2 \) for the critical Casimir energy involving binary fluids \([57]\). In case of quantum Casimir energy between perfectly conducting surfaces, \( v = 3 \) \([10]\). It may also take non-integer values, as e.g sometimes observed for the quantum Casimir energy between real materials \([10]\).

Starting from Eq. (3.16), it is easy to verify that the asymptotic behavior of \( I_{PA} \) at close proximity is determined by the first non-vanishing coefficient \( f^{(n)}(0) \) of the Taylor expansion of \( f \) near \( s = 0 \), as summarized in Table 3.1. Considering for example a sphere of radius \( R \) in front of a plate, we have

\[
f_S(s) = 2\pi(R - s),
\]

for \( 0 \leq s \leq R \) and zero otherwise; \( f_S(s = 0) \) is finite, leading to well known results: \( I_{PA} \propto d^{-2} \) for \( v = 3 \), and \( I_{PA} \propto d^{-1} \) for \( v = 2 \). This geometry is hence case \( C = 1 \) in Table 3.1 (in the following we call \( C \) the case number, defined as the reduction of power compared to two parallel plates). Another example is a square base pyramid of height \( w \) and base length \( l \), with axis perpendicular to a plate, for which

\[
f_P(s) = 2s\frac{l^2}{w^2},
\]

for \( 0 \leq s \leq w \) and zero else. The sharp tip of the pyramid has vanishing surface density (\( C = 2 \) in Table 3.1), such that PA yields a distinctly different behavior than for the sphere-plate configuration: a \( 1/d \)-divergence for \( v = 3 \), or a logarithmic one for \( v = 2 \). Although the PA is not asymptotically exact for singular surfaces with sharp tips, the above conclusion agrees qualitatively with exact computations of the Casimir force and heat transfer in the cone-plate geometry \([34, 121]\), i.e., the resulting power-law seems to be captured correctly by PA. While Eq. (3.18) considers a square base, the height distribution function is linear in \( s \) for any base shape, e.g. including cones.

\(^5\)After subtraction of the far field contribution to heat transfer, for \( \text{SiO}_2 \) at room temperature, this scaling law extends up to \( d \lesssim 300 \) nm (see Fig. 3-2).
Table 3.1: Asymptotic behavior of the interaction between two surfaces, as determined by the order of the first non-vanishing Taylor coefficient $f^{(n)}(0)$ of the height distribution function of the surfaces. The interaction scales as $I_{pp}(d) = \chi/d^\nu$ for two parallel surfaces. The length $d_0$ depends on higher derivatives of $f$.

We have implicitly assumed that the distribution function $f(s)$ admits an analytical expansion close to $s = 0$. This is indeed the generic case as indicated by the examples above. It is in principle possible to envision precisely designed shapes where this is not the case, corresponding to non-integer case numbers. We shall not dwell on such non-generic cases here.

We now turn to the experimentally relevant case of rough or modulated surfaces. The surface-to-surface separation $S$ can be decomposed as $S = S_c + S_r$, where $S_c$ is the separation between two smoothened surfaces with general radius of curvature $\Lambda$, and $S_r$ is the additional shift resulting from roughness or modulation (see Fig. 3-5). If the length scale $\xi$ characterizing the roughness or the modulation is much smaller than the characteristic size $L$ of the interaction area (typically of order $L \sim \sqrt{d\Lambda}$), i.e., $\xi \ll L \ll \Lambda$, we find that the height distribution function $f(s)$ for $S(x)$ is well approximated by the convolution of the corresponding functions $f_c$ of the smoothened
surface, and $f_r$ (normalized by a unit surface area) for roughness/modulation, i.e.

$$f(s) = \int_0^s ds' f_c(s') f_r(s-s'). \quad (3.19)$$

This result also obtains from the commonly-used two-step application of the PA [10], in which the PA is first used to estimate the correction arising from roughness or modulation (per unit area) to the interaction between parallel plates, and then used again to estimate the effect of curvature. It has been noted (see e.g. Ref. [120]) that if the experimental conditions are such that the approximate formula of Eq. (3.19) is not valid for the surfaces of interest, measurements might not be fully repeatable, as they would be sensitive to e.g. uncontrollable relative displacements of the two surfaces in the plane $\Sigma$.

The asymptotic scalings in Table 3.1 also apply to the combined distribution function in Eq. (3.19), where we note an intriguing relation. Restricting to regular, analytic functions near $s = 0$, the case number of the combined function $f$ in Eq. (3.19) is precisely the sum of the case numbers of the individual functions $f_c$ and $f_r$,

$$C(f) = C(f_c) + C(f_r). \quad (3.20)$$

Equation (3.20) can be derived with the convolution theorem for Laplace-transforms, and combined with Table 3.1 provides a means for obtaining the modified scalings.

Figure 3-6 shows $f(s)$ for the case of a sphere with different types of modulation in front a plate$^6$, where the red curve is for the smooth sphere of Eq. (3.17). If the surface of the sphere is covered with (square base) domes of height $w$ ($f_r \equiv f_D = 2(w-s)/w^2$), $f$ is found analytically with the help of Eqs. (3.17) and (3.19), as

$$f_{SD}(s \leq w) = \frac{2\pi s (6wR - 3ws - 3Rs + s^2)}{3w^2}. \quad (3.21)$$

As we see, $f_{SD}(0)$ vanishes, however its first derivative $f_{SD}'(0)$ does not, see the blue curve in Fig. 3-6. Therefore, the dome-like modulation ($C = 1$), yields $C = 2$ when

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$^6$While for simplicity we consider a flat plate, both plates could be modulated as the results only depend on the relative separation.
convoluted with the sphere-plate geometry.

Suppose now that the sphere is covered with pyramids of height \( w \) (again square based for ease of tiling, as in Eq. (3.18)). After convolution with the sphere we now find,

\[
f_{SP}(s \leq w) = \frac{2\pi s^2 (3R - s)}{3w^2}, \tag{3.22}
\]

which is of order \( s^2 \) (green curve in Fig. 3-6). Sharp tips \((C = 2)\), give \( C = 3 \) when convoluted with a curved surface.

We now consider surface roughness, which in general depends on materials and manufacturing [122–124]. While small deformation of amplitude \( w_0 \ll d \) can be treated perturbatively [29, 125], perturbative methods are not applicable when \( d \) is small or comparable to \( w_0 \), and the PA provides a good alternative tool for qualitative understanding \(^7\). For practical ease, the height distribution function \( f_r \) of a rough surface will be approximated by a Gaussian [120],

\[
f_R(s \geq 0) = \frac{1}{\mathcal{N}\sigma\sqrt{2\pi}} \exp \left[ -\frac{(s - s_0)^2}{2\sigma^2} \right], \tag{3.23}
\]

where \( \sigma \) is the standard deviation and \( \mathcal{N} \) is chosen such that \( \int_{0}^{\infty} ds f_R(s) = 1 \). The distribution \( f_R \) has no sharp boundaries (as the peaks and grooves of the surface can in principle have arbitrary extension), and we introduce an additional parameter \( s_0 \) to set \( s = 0 \), i.e., the “touching distance.” This parameter sets the height of the highest peak, which is touched first on approach and acts as a shift on the separation axis. The truncated Gaussian in Eq. (3.23) corresponds to case 1, although with a possibly small value of \( f_R(0) \). After convolution with the curvature of the sphere, we find that the height distribution function of the rough sphere is linear near \( s = 0 \) \((C = 2)\). The magenta curve in Fig. 3-6 starts linearly, but with a small slope (for the chosen value of \( s_0 = 2\sigma \)), such that the rough surface is intermediate between the dome-like and pyramid-like modulations considered earlier.

As a specific example, we consider radiative heat transfer \( \Pi \) between a rough

\(^7\)Reference [120] shows that estimates of the Casimir force obtained by means of the PA are in good agreement with experiments on rough surfaces for \( d \sim w_0 \).
Figure 3-6: Height distribution functions for model surfaces (indicated by icons/text) in the sphere-plate geometry; for modulated surfaces \( w = R/10 \), while \( \sigma = R/40 \) and \( s_0 = 2\sigma \) for the rough surface. The larger range of \( s \) in the inset emphasizes that the impact of deformations is mostly for small \( s \).
Figure 3-7: Heat transfer between a flat plate and deformed model spheres (as shown by icons/text) of radius $R = 50\mu m$. Note, for this case of heat transfer $\chi$ (see Eq. (3.16)) is equivalent to $v$ (see Sec. 3.1). The amplitude $w$ for modulations (dome or pyramid), and the width $\sigma$ for roughness are as indicated, and 4 times smaller in the inset. The two curves for roughness correspond to $s_0 = 2\sigma$ (upper) and $s_0 = 3\sigma$ (lower).
or modulated sphere and a (perfectly flat) plate, demonstrating that roughness or modulation limit the maximum achievable non-contact energy transfer, an issue of much current technological and experimental interest [26, 27, 126]. As shown in Sec. 3.1, PTA approaches the exact result at small separation. Using the concept of height distribution function, we rewrite the asymptotic expansion (3.4) for the heat transfer $H$ in powers of the gradient of $S$,

$$H = H_{PTA} + \int ds \beta g(s - d)H_{pp}(s) + \cdots. \quad (3.24)$$

where $g(s) = \int \sum d^2x \delta(S(x) - d - s) \nabla S \cdot \nabla S$ encodes the mean squared gradient at $s$. As discussed in Sec. 3.1, the dimensionless coefficient $\beta$ is typically of order of unity [70]. The function $g(s)$ for a rough or modulated surface can be found in a manner similar to Eq. (3.19), and then Eq. (3.24) allows us to judge whether the scaling laws obtained from the PTA are asymptotically exact for specific shapes.

For small separations $d$, the radiative transfer between parallel plates (per unit area) diverges as a power law with exponent $\nu = 2$. Figure 3-7 illustrates the transfer as given by PTA for the cases shown in Fig. 3-6. In all cases we subtracted from $H$ the far field contribution (computed at the reference separation $d = 300$ nm). The smooth sphere shows the expected divergence of $1/d$ (red curve); dome-like modulations (with $f(s)$ in Eq. (3.21)) reduce this divergence into a logarithmic form (blue curve), such that

$$\lim_{d \to 0} H_{SD}(d) = -4\pi \frac{R}{w} \log \frac{d}{d_0}. \quad (3.25)$$

For this type of modulation, the gradient correction in Eq. (3.24) is of order $d^0$, and therefore Eq. (3.25) becomes exact for vanishing separation.\(^8\)

A surface with pyramid-like modulations ($f(s)$ as per Eq. (3.22)) shows an even more drastic slow down: according to Table 3.1, the transfer levels off to a constant value (green curve). However, evaluation of the gradient term in Eq. (3.24) shows that this correction is now of the same order as the PTA term. From this we infer

\(^8\)Judging from Ref. [70], we can expect good agreement of Eq. (3.25) to the exact result for $d/r \lesssim 0.1$, where $r$ is the radius of curvature of the dome apex.
that PTA is not exact in this regime, while the predicted saturation to a constant is presumably correct. *For rough surfaces* the height distribution function is somehow intermediate between those for dome-like and pyramid-like modulations. Confirming this expectation, we find that $H$ diverges logarithmically for rough surfaces, but only with a small prefactor. Depending on $\sigma$ and $s_0$ in Eq. (3.23), this divergence may not be visible in practical cases, giving the appearance of saturation as in the case of pyramid-like modulations. Reducing the amplitude of roughness/modulation does not change the asymptotic behavior, but differences from a smooth sphere become visible only at closer separations (inset of Fig. 3-7).

On a practical note, the right axis in Fig. 3-7 shows the ratio to the classical far field ($d \to \infty$) transfer for SiO$_2$, for $T_1 = 0$ and $T_2 = 300$ K (which is $\sim 4.2\mu W$ [19]). With $\chi = 0.2558nW$ (see Fig. 3-4), and taking $R = 50\mu m$ and $\sigma = 10$ nm, we predict that (rather disappointingly) heat transfer can only be approximately doubled by reducing the separation to 1 nm (for the considered range of $s_0$)

We note that this analysis predicts similar behavior as presented in Fig. 3-7 for any interaction with $\nu = 2$, and might shed new light on reported anomalies for measurements of electrostatic forces [93, 127] (where $\nu = 2$ should e.g. apply for

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9For separations of $\sim 1$ nm, microscopic details of the considered materials might become relevant
conducting surfaces kept at constant potential difference).

In principle saturation can also be achieved for Casimir forces with \( \nu \geq 3 \) by appropriate surface fabrication. Figure 3-8 depicts a combination of three height profiles, a sphere with a smooth modulation and an additional sharp modulation. If the corresponding three lengths are well separated, e.g. for \( R = 100\mu m \), and modulations lengths of 1\( \mu m \) and 10 nm, respectively, the convolution in Eq. (3.19) (and also Eq. (3.20)) can be applied two successive times, to yield \( C = 1 + 1 + 2 \). The Casimir force between two such dielectric or metallic objects, for which \( \nu < 4 \) [10], should saturate to a constant value.
Chapter 4

Linear response relations in
fluctuational electrodynamics

In this Chapter we reproduce our results, up to some changes in notation, as published in Refs. [72].

We consider a collection of arbitrary well-separated objects in vacuum, perturbed by changing the temperature or velocity of one object. We derive linear response relations in FE motivated by analogous relations in other areas of Physics [63-67, 73, 74]. For example, the heat conduction coefficient $\kappa$, can be expressed in terms of the heat flux [128] as

$$\kappa = \lim_{v \to \infty} \lim_{v' \to \infty} \frac{1}{k_BT^2V} \int_0^{v'} dt \langle \mathcal{J}(t) \mathcal{J}(0) \rangle^\text{eq},$$

where $V$ and $T$ are the volume and temperature of the system, respectively, and $\mathcal{J}(t)$ is the total heat flux in the direction of the temperature gradient. We again denote averages in equilibrium by $\langle \ldots \rangle^\text{eq}$, while non-equilibrium averages are indicated as $\langle \ldots \rangle$. Note that $\langle \mathcal{J}(t) \rangle^\text{eq} = 0$.

Another example that has proven useful in simulation-analysis [129] is the so-called Kirkwood formula, expressing the friction $\gamma$ of a particle [73-75] (quoting the
result in one dimension) as

\[ \gamma = \frac{1}{k_B T} \int_0^\infty dt \langle \delta F(t) \delta F(0) \rangle \text{eq}, \]  

(4.2)

where \( \delta F(t) = F(t) - \langle F(t) \rangle \text{eq} \) is the fluctuating part of the instantaneous total force \( F(t) \) acting on the particle (this notation is used for all observables in the following).

While Eq. (4.2) is an example of the fluctuation dissipation theorem [130] (position and force are conjugate variables in the Hamiltonian), Eq. (4.1) is obtained from taking the limit of small spacial variation of thermodynamic driving forces (e.g. temperatures gradients) [128]. The case of radiative heat transfer is hence different as we consider disjoint objects.

### 4.1 Perturbing temperature

#### 4.1.1 Radiative heat transfer

Consider an arrangement of \( N \) arbitrary objects such that \( n \) of them are held at one set of conditions (temperature \( T_1 \) and velocity \( v_1 \)), while the remaining \( N - n \) objects are at slightly different conditions \( (T_2, v_2 = 0) \). In the following we will denote the two groups by \( \{\alpha, \beta\} = 1, 2 \), keeping in mind that each entity can be made up of disconnected pieces. This collection is immersed in a vacuum at temperature \( T_{env} \).

Starting from the equilibrium situation with \( T = \{T_\alpha\} = T_{env} \) and \( v_1 = v_2 = 0 \), we first introduce a small perturbation in the temperature of one of the objects (see Fig. 4-1), aiming to connect the corresponding linear heat transfer coefficient to the fluctuations of the heat flux in equilibrium, in analogy to Eq. (4.1). While the former has been derived in Ref. [15], the latter will be found below.

The total radiation energy \( H^{(\beta)} \) absorbed by object(s) \( \beta \) can be written as an integral over the volume(s) \( V_\beta \) of the local work which is the product of the electric
field \( \mathbf{E} \) and current \( \mathbf{J} \) at point \( \mathbf{r} \) and time \( t \), leading to

\[
H^{(\beta)}(t) = \int_{\mathbf{r} \in V_{\beta}} d^3 \mathbf{r} \{ E_i(\mathbf{r}, t), J_i(\mathbf{r}, t) \}_S. \tag{4.3}
\]

This expression can be recast as the surface integral of the Poynting vector (2.20) through the Poynting theorem [15, 97]. In this Chapter we continue to use the Einstein summation convention (which implies summation over the vector index \( i \) in Eq. (4.3)). Note that \( \langle H^{(\beta)}(t) \rangle_{eq} = 0 \).

The correlations between fluctuations of \( H^{(\alpha)}(t) \) in equilibrium can be formally written (note that \( \int_0^\infty dt \langle \{ A(t), B(0) \}_S \rangle_{eq} = \int_0^\infty dt \langle A(t) B(0) \rangle_{eq} \), making symmetrization needless on the left hand side) as

\[
\int_0^\infty dt \langle H^{(\alpha)}(t) H^{(\beta)}(0) \rangle_{eq} = \int_0^\infty dt \int_{\mathbf{r} \in V_{\beta}} d^3 \mathbf{r} d^3 \mathbf{r}'
\times \langle \{ E_i(\mathbf{r}, t), J_i(\mathbf{r}, t) \}_S \{ E_j(\mathbf{r}', 0), J_j(\mathbf{r}', 0) \}_S \rangle_{eq}. \tag{4.4}
\]

The spatial integrals are restricted to the corresponding volumes according to Eq. (4.3). Equation (4.4) contains a four-point correlation function of the electric field (noting the linear relation between \( \mathbf{E} \) and \( \mathbf{J} \) in Eq. (4.10) below). Given the Gaussian distribution of the electric field, Eq. (4.4) can be rewritten in terms of time-ordered two-point correlation functions via Wick’s theorem,

\[
\int_0^\infty dt \langle H^{(\alpha)}(t) H^{(\beta)}(0) \rangle_{eq} = \int_0^\infty dt \int_{\mathbf{r} \in V_{\beta}} d^3 \mathbf{r} d^3 \mathbf{r}'
\times \left[ \langle E_i(\mathbf{r}, t) J_j(\mathbf{r}', 0) \rangle_{eq} \langle J_i(\mathbf{r}, t) J_j(\mathbf{r}', 0) \rangle_{eq} + \langle E_i(\mathbf{r}, t) J_j(\mathbf{r}', 0) \rangle_{eq} \langle J_i(\mathbf{r}, t) E_j(\mathbf{r}', 0) \rangle_{eq} \right], \tag{4.5}
\]

where the term \( \int_0^\infty dt \langle H^{(\alpha)}(t) \rangle_{eq} \langle H^{(\beta)}(0) \rangle_{eq} \) vanishes. We emphasize that in the equation above, the equilibrium expectation values in the integrand are not symmetrized.

We define the corresponding spectral density for non-symmetrized field correlator
(compare to Eq. (2.2)),

\[ \langle E_i(r, t)E_j(r', 0) \rangle^{eq} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \langle E_i(r)E_j^*(r') \rangle^{eq}. \] (4.6)

After Fourier transforming in time, the first integrand in Eq. (4.5) reads (the other one is treated analogously),

\[ \int_0^{\infty} dt \langle E_i(r, t)E_j^*(r', 0) \rangle^{eq} \langle J_i(r, t)J_j^*(r', 0) \rangle^{eq} = \int_0^{\infty} \frac{d\omega}{2\pi} \langle E_i(r)E_j^*(r') \rangle^{eq} \langle J_i(r)J_j^*(r') \rangle^{eq}. \] (4.7)

While the equilibrium spectral density for the symmetrized electric field correlator has been given above in Eq. (2.4), the equilibrium spectral density \( \langle E_i(r)E_j^*(r') \rangle^{eq} \) can be similarly expressed via the dyadic retarded Green’s function \( G_{ij} \) of the system [74, 96, 128],

\[ \langle E_i(r)E_j^*(r') \rangle^{eq} = \frac{8\pi\hbar}{1 - e^{-\hbar\omega/kB T}} \frac{\omega^2}{c^2} \text{Im} G_{ij}(r, r'; \omega). \] (4.8)

This Green’s function is straightforwardly found for a two component system [15] as

\[ G = (1 + G_0 T_2) \frac{1}{1 - G_0 T_1 G_0 T_2} (1 + G_0 T_1) G_0. \] (4.9)

The Green's function of free space, \( G_0 \), relates the total field and the total current, as used in Eq. (4.3), by

\[ E_i(\omega) = 4\pi i \frac{\omega}{c^2} G_{0,ij} J_j(\omega). \] (4.10)

For a single object with operator \( T \), the Green’s function reduces accordingly to \( G = (1 + G_0 T)G_0 \) in agreement with Eq. (2.13).

After some computation steps, we find a closed form for the correlation function in Eq. (4.4) in terms of \( G_0 \) and the T-operators of the entities, see Eq. (C.1). One important step is that the integrals in Eq. (4.4) can eventually be taken over all space (due to the fact that \( T_\alpha = T_\alpha(r, r') \) is only nonzero if both arguments are within \( V_\alpha \) [14, 15]), such that together with the summation over vector index \( i \),
Figure 4-1: The system under consideration consists of two (possibly multi-component) entities (blue and green). In equilibrium with $T_1 = T_2 = T_{env}$, the average heat absorbed by object 1 (illustrated by the blue arrows) is zero, and the net force on it is the equilibrium Casimir force. If $T_1$ or $T_2$ slightly deviate from equilibrium, the finite heat absorption and the non-equilibrium Casimir force are given by Eqs. (4.11) and (4.12) respectively.

an operator trace arises. A comparison to the previously computed radiative heat transfer $\langle H^{(\beta)} \rangle$ [15] (see also Eq. (C.5)), denoting the energy absorbed by object $\beta$ in the non-equilibrium situation with $T_1$, $T_2$ and $T_{env}$ unequal, explicitly shows the following equality

$$\kappa^{(\beta)}_{\alpha} = - \frac{d\langle H^{(\beta)} \rangle}{dT_{\alpha}} \bigg|_{T_{\alpha} = T_{env} = T} = \frac{1}{k_B T^2} \int_0^\infty dt \langle H^{(\alpha)}(t) H^{(\beta)}(0) \rangle_{eq}. \quad (4.11)$$

Here we define the linear radiative heat transport coefficient $\kappa^{(\beta)}_{\alpha}$, as a measure of the change in the heat absorption $\langle H^{(\beta)} \rangle$ by object $\beta$ in response to a small change in temperature of $\alpha$. It is interesting to note that for $\alpha \neq \beta$, Eq. (4.11) relies on a nonlocal correlation between fluctuations in the different objects, in contrast to the purely local character of Eq. (4.1).

As a side note, Eq. (4.11) directly shows the positivity of the linear transport coefficient $\kappa^{(\alpha)}_{\alpha}$, as equilibrium auto-correlation functions have non-negative Fourier transforms [131]. On the other hand, Eq. (4.11) for $\alpha \neq \beta$ does not allow us to make a statement about the sign of $-\kappa^{(\beta)}_{\alpha}$, which however is non-negative as well [15, 37].
Figure 4-2: An object moving with velocity $v_1$ in the presence of a static object, gives rise to the vacuum friction $-\gamma_1^{(1)} v_1 \equiv \frac{dF_1^{(1)}}{dv_1} \big|_{v_1=0} v_1$ acting on the moving object, and the force $-\gamma_1^{(2)} v_1 \equiv \frac{dF_1^{(2)}}{dv_1} \big|_{v_1=0} v_1$ acting on the static one.

### 4.1.2 Casimir force

Now consider the change in the force $F^{(\beta)}$ \[3, 5, 11, 15, 19, 36, 42\] when all objects are at rest, but with one temperature perturbed to out of equilibrium, as in Fig. 4-1. We find that variations in force are related to the equilibrium correlation function of heat flux and force (compare to Eqs. (C.2) and (C.8)) by

$$\frac{d\langle F^{(\beta)} \rangle}{dT_{\alpha}} \bigg|_{T_{\alpha}=T_{\text{env}}=T} = -\frac{1}{k_B T^2} \int_0^\infty dt \langle F^{(\beta)}(t) H^{(\alpha)}(0) \rangle^{eq}.$$

This relation is found by steps analogous to the ones above Eq. (4.11), starting from the Lorentz force acting on $\beta$, given by the volume integral

$$F_i^{(\beta)}(t) = \frac{1}{c} \int_{V_\beta} \epsilon_{ijk} \{ J_j(r, t), B_k(r, t) \}_S,$$  \hspace{1cm} (4.12)

where $B_k$ is the $k$-th component of the magnetic field, and $\epsilon_{ijk}$ is the Levi-Civita symbol. As before, the equality in Eq. (4.12) is established by direct comparison to the result for the Casimir force in the non-equilibrium situation with $T_1, T_2$ and $T_{\text{env}}$ unequal given by Eq. (C.7) \[15\]. (See Eq. (C.2) for the explicit result of the correlation function in Eq. (4.12).)
The relation (4.12) is anticipated from linear response in the density matrix, yielding the time integral containing the energy dissipation [132] (in our case $H$). The expected general relation for observable $\mathcal{O}(t)$

\[
\frac{d\langle \mathcal{O} \rangle}{dT_\alpha} \bigg|_{T_\alpha = T} = -\frac{1}{k_B T^2} \int_0^\infty dt \langle \mathcal{O}(t) H^{(\alpha)}(0) \rangle^{eq},
\]

is however yet unproven in this framework.

\section{4.2 Perturbing Velocity}

\subsection{4.2.1 Casimir force (vacuum friction)}

The equilibrium system can also be perturbed by moving object(s) $\alpha$ with a small velocity $v_\alpha$. The corresponding change in the Casimir force acting on $\beta$, expressed in terms of the linear force coefficient $\gamma_\alpha^{(\beta)} \equiv -\frac{d\langle F_\beta^{(\alpha)} \rangle}{dv_\alpha}|_{v_\alpha = 0}$ (see Fig. 4-2), is related to the auto-correlation function of the Casimir force in equilibrium [75, 130, 133], in analogy to the Kirkwood formula in Eq. (4.2) (the diagonal part $\gamma_\alpha^{(\alpha)}$ is the friction coefficient of $\alpha$). Here, we explicitly confirm this relation for the fluctuating electromagnetic field, thereby providing a closed expression for the vacuum thermal friction. We find, elaborating in analogy to the derivation of Eqs. (4.11) and (4.12), for the fluctuations of the Casimir force,

\[
\langle \gamma_\alpha^{(\beta)} \rangle_{ij} = \frac{1}{k_B T} \int_0^\infty dt \langle \delta F_i^{(\beta)}(t) \delta F_j^{(\alpha)}(0) \rangle^{eq} = -\frac{\hbar^2}{\pi k_B T} \int_0^\infty d\omega \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} \times \text{Im} \text{Tr} \left\{ \partial_i (1 + G_0 T_\alpha) \frac{1}{1 - G_0 T_\alpha G_0 T_\alpha^*} G_0 [i(\partial_j T_\alpha^* - T_\alpha \partial_j) - 2T_\alpha \partial_j \text{Im}[G_0] T_\alpha^*] \right\} (4.14)
\]

\[
\times \frac{1}{1 - G_0^* T_\alpha G_0 T_\alpha^*} (\delta_{\alpha \beta} + \delta_{\alpha \beta} G_0 T_\alpha G_0^* T_\alpha^*)
\]

where $\bar{\alpha} = 1$ if $\alpha = 2$ and vice versa. Note that the matrix $\langle \gamma_\alpha^{(\beta)} \rangle_{ij}$ has in general non-zero off-diagonal elements in $ij$, and the force need not be parallel to the velocity. While Eq. (4.14) contains both the thermal and zero point contributions to the net
Casimir force, at $T = 0K$, the linear force coefficient $\gamma^{(\alpha)}$ vanishes, and there is no linear response in velocity.

There is, however, response related to higher time derivatives of displacement, in accordance with known results (see, e.g. Refs. [75, 130, 134, 135]). The friction coefficient in Eq. (4.14) has been previously computed for the special cases of two parallel plates, and for a small particle in front of a plate [75].

The first equality sign in Eq. (4.14) can be confirmed by deriving the linear force coefficient directly. Then the term $-iG_0[\bar{a}(\partial_j \bar{T}_\alpha - T_{\alpha} \partial_j) - 2T_{\alpha} \partial_j \text{Im}[G_0]T_\alpha^*]G_0^*$ is found as the disturbed field correlator due to the moving object given by Eq. (C.16) [136]. This field then undergoes scattering due to the surrounding objects, and computing the force introduces another gradient, $\partial_i$, in analogy to Eq. (C.8) and Ref. [15].

In case of an isolated object, the friction tensor $\gamma^{(\alpha)}$ simplifies (where we omit the label $\alpha$) to

$$\gamma_{ij} = \frac{1}{k_B T} \int_0^\infty dt \langle \delta F_i(t) \delta F_j(0) \rangle = \frac{2\hbar^2}{\pi k_B T} \int_0^\infty d\omega \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} \times \text{Im Tr} \{ \partial_i (1 + G_0) \partial_j \text{Im}[G_0] \bar{T}_\alpha^* \} . \tag{4.15}$$

This is equivalent to the force acting on the object at rest in a photon gas moving in direction $j$ [137] (see also Eqs. (C.8) and (C.16)). The latter has the electric field correlator $\propto -i\partial_j \text{Im}[G_0]$.

The trace in Eqs. (4.14) and (4.15) can be readily evaluated in any basis, as exemplified in detail in Ref. [15] for heat transfer and the non-equilibrium Casimir force. For example, Eq. (4.15) reads for a sphere,

$$\gamma_{ij} = -\delta_{ij} \frac{2\hbar^2}{3\pi k_B T} \int_0^\infty d\omega \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} \omega^2 \sum_{P,l,m} \text{Re} \left[ T_l^P + 3a(l, m)^2 T_l^P T_l^{P*} + 6b(l, m)^2 T_l^P T_{l+1}^{P*} \right], \tag{4.16}$$

where $T_l^P \equiv T_l^P(\omega, R)$ is the T-matrix element for the scattering of a spherical wave with frequency $\omega$, wave numbers $l, m$ and polarization $P = \{N, M\}$ from a sphere.
with radius $R$ (see e.g. Ref. [15] for the expressions for $T$). $\overline{P} = N$ if $P = M$ and vice versa. The coefficients in Eq. (4.16) are

$$a(l, m) = \frac{m}{l(l+1)}, \quad \text{Eq. (4.17)}$$

$$b(l, m) = \frac{1}{l+1} \sqrt{\frac{l(l+2)(l-m+1)(l+m+1)}{(2l+1)(2l+3)}}, \quad \text{Eq. (4.18)}$$

Expanding Eq. (4.16) to lowest order in $R$ (noting that $\frac{\partial T}{\partial r} \propto R^3$ to lowest order), and by relating the term linear in $\frac{\partial T}{\partial r}$ to the polarizability, we recover the result obtained in Ref. [137]. Interestingly, if the sphere is a perfect mirror (in which case it does not emit heat radiation), Eq. (4.16) gives a finite result. Specifically, in the limit of a small spherical mirror, we find

$$\hat{\gamma}_{ij} = \delta_{ij} \frac{896\pi^7}{135} \frac{hR^6}{\lambda_T^8}, \quad \text{Eq. (4.19)}$$

where $\lambda_T = \frac{hc}{k_B T}$ is the thermal wavelength, indicating that the friction coefficient is proportional to $T^8$.

### 4.2.2 Radiative heat transfer

An additional anticipated general linear response relation that is complimentary to Eq. (4.13) reads,

$$\frac{d\langle O \rangle}{dv_\alpha} \bigg|_{v_\alpha=0} = -\frac{1}{k_B T} \int_0^\infty dt \langle O(t) \delta F^{(\alpha)}(0) \rangle^{eq}. \quad \text{Eq. (4.20)}$$

Consider $O = H^{(\alpha)}$, then by comparing Eq. (4.20) to Eq. (4.12) we can finally provide the Onsager theorem [76] for FE by writing,

$$\frac{d\langle H^{(\alpha)} \rangle}{dv_\beta} \bigg|_{v_\beta=0} = -T \frac{d\langle F^{(\beta)} \rangle}{dT_\alpha} \bigg|_{\{T_\alpha\} = T_{:\\text{eq}}}. \quad \text{Eq. (4.21)}$$

Here we used the symmetry $\int_0^\infty dt \langle H^{(\alpha)}(t) \delta F^{(\beta)}(0) \rangle^{eq} = -\int_0^\infty dt \langle \delta F^{(\beta)}(t) H^{(\alpha)}(0) \rangle^{eq}$, as found explicitly by using the methods outlined above.
4.3 Experimental relevance and summary

Let us finally comment on experimental relevance of the above results. While the friction in Eq. (4.14) is in principle measurable in precision force experiments [138], the fluctuations of $H$ in Eq. (4.11) are harder to access. We propose instead a method for indirect detection based on equilibrium fluctuations of internal energy $E^{(a)}(t)$ from Eq. (4.11). Energy conservation requires (in the absence of other heat sources) that

$$\frac{\partial}{\partial t} \delta E^{(a)}(t) = H^{(a)}(t), \quad (4.22)$$

using which Eq. (4.11) can be recast as

$$\kappa^{(b)} = \frac{1}{k_B T^2} \lim_{t \to 0} \frac{\partial}{\partial t} \langle \delta E^{(a)}(0) \delta E^{(b)}(t) \rangle^{eq}. \quad (4.23)$$

Relations of this type are sometimes referred to as macroscopic fluctuation-dissipation conditions. The spectrum of energy fluctuations of $a$ in the environment of other objects can be related to $\kappa^{(a)}$ and its heat capacity $C^{(a)}$. Omitting the index $a$ for brevity, the equal time correlations of energy are obtained by standard statistical physics arguments as $\langle \delta E(0)^2 \rangle^{eq} = C k_B T^2$. Hence, by integrating Eq. (4.23) we obtain,

$$\langle \delta E(t) \delta E(0) \rangle^{eq} = C k_B T^2 \left[ 1 - \frac{\kappa t}{C} + \ldots \right] \approx C k_B T^2 e^{-t/\tau}. \quad (4.24)$$

The dots imply higher powers in $t$, which we have assumed lead to an overall exponential decay, with $\tau = C/\kappa$. Thus, if the object’s heat coupling to the remainder of the system is dominated by vacuum heat transfer $H$, then its internal energy will fluctuate with timescale $\tau$. The equilibrium Casimir force is a function of temperature. If its fluctuations $\delta F^{(a)}(t)$ can be assumed to depend on $\delta E^{(a)}(t)$, then they should also exhibit a signature of the timescale $\tau$. Without needing to specify the explicit dependence of $F^{(a)}(t)$ on $\delta E^{(a)}(t)$, we can thus claim that a Fourier-analysis of $F^{(a)}(t)$ should reveal $\tau$ (besides other characteristic timescales), and hence provide an equilibrium means of detecting the vacuum heat conductivity. In order to fulfill
Eq. (4.22), any mechanical contact to the object (e.g. by a cantilever) should be thermally insulated. Furthermore, the relative fluctuations of energy are enhanced for smaller $C$ (per Eq. (4.24)) favoring smaller objects. For example, a setup of a silicon sphere of radius 1µm in front of a silicon plate at a separation of 100nm, leads to a timescale of $\tau \approx 50\mu s$, which is large enough for experimental detection.
Chapter 5

Outlook

In this thesis we have utilized and developed various approaches based on scattering theory and Rytov’s FE that allow us to describe and gain insights into the topics of heat radiation, heat transfer, Casimir forces and vacuum friction. In future theoretical research, it would be interesting to consider the following topics:

- In this thesis we have exploited Rytov’s FE that uses a simplification that there is local thermal equilibrium within each object. However, in real systems where temperatures of objects and environment differ substantially (e.g. hot objects in cold environment), the assumption of temperature homogeneity inside objects breaks down. Hence, it would be of both theoretical and practical importance to extend our methods to the case of non-homogeneous temperatures inside objects as that would allow to quantify non-equilibrium experiments.

- In Chapter 2 we numerically studied heat radiation and non-equilibrium Casimir forces for nanowires with radii of tens of nanometres using bulk dielectric function for the corresponding materials. It would be important to justify our approach and find out for which sizes of objects a bulk dielectric function still accurately describes dielectric properties of that object. We expect that in order to accurately describe dielectric properties of objects of particularly small sizes, one will have to consider a dielectric function that depends both on frequency and wave-number.

- The methods exploited in this thesis are restricted to the case of linear media in which the electric displacement field is linear in the electric field. Thus, the next
step would be to generalize our methods to the case of materials with non-linear electromagnetic response. Interactions between such materials could exhibit non-trivial new non-linear effects.

- In Chapter 4 we computed the vacuum friction (see Fig. 4-2) by considering the auto-correlation function of the Casimir force in equilibrium (see Eq. (4.14)). As the next step, it would be interesting to describe the vacuum friction with tools analogous to Oseen tensor in hydrodynamics, which is the fundamental solution for velocity of Stokes equations for an infinite viscous fluid. Translating that to the case of vacuum friction, objects could play roles of particles, whereas vacuum could play the role of a "viscous" medium that mediates interactions.

- In Chapter 4 we expressed non-equilibrium quantities in terms of equilibrium correlation functions. While those expressions hold only up to the linear order in perturbation, it would be important to go beyond the linear order in perturbation and obtain exact relations between non-equilibrium and equilibrium quantities.

- While our work on the gradient expansion in Sec. 3.1 provides an easy tool to compute the heat transfer at proximity, it has also left few questions open. First, using the gradient expansion (see Eq. (3.4)) one can develop a better insight into the heat transfer for geometries for which semi-analytical methods of scattering theory are not possible (e.g. paraboloids or ellipsoids). Secondly, it is important to extend the gradient expansion to conducting materials and to separations at which the power-law, Eq. (3.3), for the heat transfer does not hold (preliminary results indicate an unphysical divergence in the computation of $\beta(d)$ outside the regime where Eq. (3.3) holds). Finally, it is worth understanding in more detail the intriguing coincidence that the gradient expansion parameter $\beta$ roughly equals 0.5 for all dielectric materials at all temperatures investigated.

While our work has been purely theoretical, it provided insights and suggestions for possible measurements. One study has already been carried out [88], where the thermal radiation of a SiO$_2$ nanowire has been measured. It was found that our general formula for heat radiation given in Sec. 2.1 describes those measurements very accurately, in contrast to the classic Planck's law. In terms of future experiments, we
suggest to consider the following ideas:

- We believe it would be very beneficial to measure the heat radiation by a thin metallic wire (e.g. tungsten wire with $R = 20\text{nm}$) as according to our results in Sec. 2.2, the heat radiation by such an object is several times larger than the one by the black-body of the same shape and size. This effect has not been observed experimentally yet to the best of our knowledge. We note that if one considers to perform the suggested measurement, it should be taken into account that there is a possible inaccuracy of our prediction due to the dependence of dielectric function on wave-number for very thin wires (see the second bullet point in this chapter).

- For thin conducting wires, the non-equilibrium part of the Casimir forces can be two orders larger than the gravitational weight of the cylinder, thereby being in the observable range in principle. According to our rough estimate, the Ampere's force between the nanowires (based on the currents necessary to keep their temperatures constant) is also much smaller than the non-equilibrium correction, rendering conducting nanowires potential candidates for experimental studies of non-equilibrium Casimir forces.

- In private conversations with Gang Chen's group at MIT we learned about their measurements of heat transfer between a dielectric sphere and a dielectric plate as a function of separation. Their results deviated from the anticipated $1/d$ separation dependence (see Fig. 3-4) at close proximity. It is worth checking if this deviation can be attributed to the roughness of a plate/sphere as described in Sec. 3.2 (see Fig. 3-7).

- In the end of Chapter 4 we suggest a way to measure vacuum heat conductivity while keeping a system in thermal equilibrium. For example, in case of heat transfer between a sphere and a plate, this can be done by measuring the time-scale of Casimir force fluctuations acting on the sphere. The measured time-scale and the heat capacities of the objects directly yield the vacuum heat conductivity between two objects.
Appendix A

Cylindrical basis

A.1 Cylindrical harmonics and Free Green's function in cylindrical basis

According to Ref. [98], the cylindrical harmonics can be written as,

\[ R_{M}^{reg}(r) = \left[ \frac{in}{qr} J_n(qr)e_\phi - J_n'(qr)e_r \right] e^{ik_z z + in\phi}, \]

\[ R_{N}^{reg}(r) = \frac{c}{\omega} \left[ ik_z J_n'(qr)e_\phi - \frac{nk_z}{qr} J_n(qr)e_\phi + qJ_n(qr)e_z \right] e^{ik_z z + in\phi}, \]

(A.1)

where \( J_n \) is the Bessel function of order \( n \). \( R_{M}^{reg} \) and \( R_{N}^{reg} \) correspond to regular magnetic multipole (TE) and electric multipole (TM) waves respectively. Also, \( k_z \) and \( q \) are the wavevectors parallel and perpendicular to the cylindrical \( z \)-axis respectively satisfying the relation \( q = \sqrt{k^2 - k_z^2} \), \( k = \omega/c \). \( J_n' \) corresponds to the first derivative with respect to the argument. Furthermore, we denote the corresponding outgoing waves by \( M_{n,k_z}^{out} \) and \( N_{n,k_z}^{out} \), which differ from regular ones by replacing \( J_n \) with the Hankel function of the first kind \( H_n^{(1)} \).

The above solutions correspond to transverse waves, i.e. \( \nabla \cdot R_{M}^{reg} = \nabla \cdot R_{N}^{reg} = 0 \). Moreover, they obey useful relations \( R_{M}^{reg} = \frac{c}{\omega} \nabla \times R_{N}^{reg} \), \( R_{N}^{reg} = \frac{c}{\omega} \nabla \times R_{M}^{reg} \). These relations are also valid for outgoing waves.
The free Green's function in cylindrical waves reads \[98\],

\[
\mathcal{G}_0(r, r') = \sum_{p=-M, N} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{idk_z}{8\pi} \left\{ \left[ \mathbf{P}^{\text{reg}}_{n, k_z}(r) \otimes \mathbf{P}^{\text{out}}_{-n, -k_z}(r') \right], \quad r' > r \right\} \left[ \mathbf{P}^{\text{out}}_{n, k_z}(r) \otimes \mathbf{P}^{\text{reg}}_{-n, -k_z}(r') \right], \quad r' < r.
\]

(A.2)

The following relation for propagating cylindrical waves is useful for deriving the Poynting vector,

\[
\text{Re} \left[ ir \int_0^{2\pi} d\phi \left( \mathbf{P}^{\text{reg}}_{n, k_z}(r) \times \nabla \times \mathbf{P}^{\text{out}*}_{n, k_z}(r) \right) T_{n, k_z}^{P' P'} + \mathbf{P}^{\text{out}}_{n, k_z}(r) \times \nabla \times \mathbf{P}^{\text{reg}*}_{n, k_z}(r) T_{n, k_z}^{P P'} \right] \cdot \mathbf{e}_r
\]

\[
= 4\delta_{P, P'} \text{Re} \left[ T_{n, k_z}^{P P'} \right],
\]

\[
\text{Re} \left[ ir \int_0^{2\pi} d\phi \left( \mathbf{P}^{\text{out}}_{n, k_z}(r) \times \nabla \times \mathbf{P}^{\text{out}*}_{n, k_z}(r) \right) \cdot \mathbf{e}_r \right] = 4\delta_{P, P'}. \quad (A.3)
\]

### A.2 Scattering of electromagnetic waves from uniaxial cylindrical objects

Consider a wave propagating in an anisotropic medium, with dielectric permittivity tensor (2.32) and magnetic permeability \(\mu(\omega)\).

Considering the system's uniaxial symmetry, we look for wave-solutions in the form of cylindrical harmonics (A.1),

\[
\mathbf{M}_{n, k_z}^{\text{reg}, \text{in}}(r) = \left[ \frac{ik_z}{q_M r} J_n(q_M r) \mathbf{e}_r - \frac{n}{q_M r} J'_n(q_M r) \mathbf{e}_\phi \right] e^{ik_z z + in\phi},
\]

\[
\mathbf{N}_{n, k_z}^{\text{reg}, \text{in}}(r) = \frac{c}{\omega} \left[ ik_z J'_n(q_N r) \mathbf{e}_r - J_n(q_N r) \mathbf{e}_\phi + \gamma_N q_N J_n(q_N r) \mathbf{e}_z \right] e^{ik_z z + in\phi}, \quad (A.4)
\]

where \(\gamma_N\) is some constant which modifies the \(N\)-polarized cylindrical harmonic due to uniaxial anisotropy. Importantly, we do not care about keeping the harmonics (A.4) normalized as that does not influence calculations of \(T\) matrix elements in which we are interested. Note that the time dependence \(\exp(-i\omega t)\) is omitted here.
The following Maxwell's equations must be satisfied,

\[- \nabla \times \nabla \times \mathbf{M}^\text{reg, in}_{n,k_z}(\mathbf{r}) = \frac{1}{c^2} \left( \varepsilon \mu \frac{\partial^2}{\partial t^2} \mathbf{M}^\text{reg, in}_{n,k_z}(\mathbf{r}) \right), \]

\[\nabla \cdot [\varepsilon \mathbf{M}^\text{reg, in}_{n,k_z}(\mathbf{r})] = 0, \tag{A.5}\]

with analogous relations for \(\mathbf{N}^\text{reg, in}_{n,k_z}(\mathbf{r})\).

Substituting expressions (A.4) into equations (A.5), we obtain the following un-normalized wave solutions,

\[\mathbf{M}^\text{reg, in}_{n,k_z}(\mathbf{r}) = \left[ \frac{in}{q_{M}r} J_n(q_{M}r) \mathbf{e}_r - J'_n(q_{M}r) \mathbf{e}_\phi \right] e^{ik_z z + in\phi}, \]

\[\mathbf{N}^\text{reg, in}_{n,k_z}(\mathbf{r}) = \frac{c}{\omega} \left[ ik_z J'_n(q_{N}r) \mathbf{e}_r - \frac{n k_z}{q_{N}r} J_n(q_{N}r) \mathbf{e}_\phi + \frac{\varepsilon_r}{\varepsilon_z} q_{N} J_n(q_{N}r) \mathbf{e}_z \right] e^{ik_z z + in\phi}, \tag{A.6}\]

where \(q_M\) and \(q_N\) are the wave-vector components perpendicular to the z-axis for the two solutions respectively,

\[q_M = \sqrt{\varepsilon_r \mu k^2 - k_z^2}, \quad q_N = \sqrt{\varepsilon_z/\varepsilon_r \sqrt{\varepsilon_r \mu k^2 - k_z^2}}. \tag{A.7}\]

The first solution is an ordinary wave and is \(M\)-polarized, whereas the second is called an extraordinary wave and possesses \(N\)-polarization \cite{103}.

In order to solve the scattering problem for the cylinder, we expand the electromagnetic field in cylindrical basis (A.1) and (A.6), outside and inside the cylinder respectively. The expansion coefficients for the field inside and outside can be obtained by matching boundary conditions at the cylinder's surface for field components tangential to the surface.

Using the definition of the \(T\) matrix, we describe the scattering process of a regular magnetic wave by the field outside the cylinder, which is

\[\mathbf{E}^M_{n,k_z} = \mathbf{M}^\text{reg}_{n,k_z} + T^MM_{n,k_z} \mathbf{M}^\text{out}_{n,k_z} + T^NM_{n,k_z} \mathbf{N}^\text{out}_{n,k_z} \tag{A.8}\]
and the field inside the cylinder,
\[
E_{n,k_z}^{M,\text{in}} = A_{n,k_z}^{MM} M_{n,k_z}^{\text{reg,in}} + A_{n,k_z}^{NN} N_{n,k_z}^{\text{reg,in}}. \tag{A.9}
\]

Analogously, for an incident electric (TM) multipole field, the field outside the cylinder becomes
\[
E_{n,k_z}^{N,\text{out}} = N_{n,k_z}^{\text{reg}} + T_{n,k_z}^{MM} M_{n,k_z}^{\text{out}} + T_{n,k_z}^{NN} N_{n,k_z}^{\text{out}}. \tag{A.10}
\]
and the field inside the cylinder,
\[
E_{n,k_z}^{N,\text{in}} = A_{n,k_z}^{MN} M_{n,k_z}^{\text{reg,in}} + A_{n,k_z}^{NN} N_{n,k_z}^{\text{reg,in}}. \tag{A.11}
\]

We next derive the specific form of the \( T \) matrix coefficients by matching the boundary conditions for the medium, i.e. the continuity of \( E_\phi, E_z, H_\phi \) and \( H_z \) across the cylindrical surface. Plugging the explicit form of cylindrical harmonics (A.1) and (A.6) into these conditions we obtain two sets of four linear equations for the expansion coefficients. Using \( B = -i(c/\omega) \nabla \times E \) and \( H = B/\mu \) we can write the system of equations for reflection and transmission amplitudes in case of the incident magnetic waves in the form,
\[
M_{n,k_z} \begin{pmatrix} A_{n,k_z}^{MM} \\ T_{n,k_z}^{MM} \\ A_{n,k_z}^{NN} \\ T_{n,k_z}^{NN} \end{pmatrix} = \begin{pmatrix} \frac{c}{\omega} q J_n(q R) \\ J'_n(q R) \\ 0 \\ J_{q n R} \end{pmatrix}, \tag{A.12}
\]
with the matrix
\[
M_{n,k_z} = \begin{pmatrix}
\frac{2mc}{\mu \omega} J_n(q R) & -\frac{qc}{\omega} H_n^{(1)}(q R) & 0 & 0 \\
J'_n(q R) & -H_n^{(1)}(q R) & \frac{nk_c}{q \omega R} J_n(q R) & -\frac{nk_c}{q \omega R} H_n^{(1)}(q R) \\
0 & 0 & \frac{q c}{\epsilon_z} J_n(q R) & -\frac{qc}{\omega} H_n^{(1)}(q R) \\
\frac{nk_c}{q \omega M R \omega} J_n(q M R) & -\frac{nk_c}{q \omega R} H_n^{(1)}(q R) & \epsilon_r J'_n(q R) & -H_n^{(1)}(q R)
\end{pmatrix}. \tag{A.13}
\]
For the incident electric waves the linear equations are

\[
M_{n,k_z} \begin{pmatrix} A^{MN} \\ T^{MN} \\ A^{NN} \\ T^{NN} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{n k_z c}{q R \omega} J_n(q R) \\ \frac{c}{\omega} q_j J_n(q R) \\ J'_n(q R) \end{pmatrix}
\] (A.14)

The solutions to these sets of equations (A.12) and (A.14) are provided in Eqs. (2.33)-(2.40).

### A.3 Small R expansion of the \( \mathbb{T} \) operator of the cylinder

In order to derive Eqs. (2.43), (2.44), (2.66), (2.67), (2.72) and (2.73), we need the expansion of the \( \mathbb{T} \) operator in terms of \( \omega R / c \). For a cylinder made of isotropic material with magnetic permeability \( \mu(\omega) \) and dielectric permittivity \( \varepsilon(\omega) \), we find for the limit \( R \ll \{\delta, c / \omega\} \),

\[
T^{NN}_{0,k_z} = -\frac{i \pi}{4} (\varepsilon - 1)(\tilde{k}_z^2 - 1)(\omega R / c)^2,
\] (A.15)

\[
T^{MM}_{0,k_z} = -\frac{i \pi}{4} (\mu - 1)(\tilde{k}_z^2 - 1)(\omega R / c)^2,
\] (A.16)

\[
T^{NN}_{1,k_z} = T^{NN}_{-1,k_z} = \frac{i \pi \tilde{k}_z^2 (\mu + 1)(\varepsilon - 1) + (\mu - 1)(\varepsilon + 1)}{4(\varepsilon + 1)(\mu + 1)}(\omega R / c)^2,
\] (A.17)

\[
T^{MM}_{1,k_z} = T^{MM}_{-1,k_z} = \frac{i \pi \tilde{k}_z^2 (\mu - 1)(\varepsilon + 1) + (\mu + 1)(\varepsilon - 1)}{4(\varepsilon + 1)(\mu + 1)}(\omega R / c)^2,
\] (A.18)

\[
T^{MN}_{1,k_z} = T^{NM}_{1,k_z} = -T^{MN}_{-1,k_z} = -T^{NM}_{-1,k_z} = \frac{i \pi}{2} \frac{(\varepsilon \mu - 1)\tilde{k}_z}{(\varepsilon + 1)(\mu + 1)}(\omega R / c)^2,
\] (A.19)

where \( \tilde{k}_z = k_z / k \).
A.4 Leading term of \( T \) operator for \( c/\omega \gg R \gg \delta \)

For \( c/\omega \gg R \gg \delta \), the leading term of \( T \) operator is the \( T_{0,k_2}^{NN} \) element which has then the following form,

\[
\lim_{c/\omega \gg R \gg \delta} T_{0,k_2}^{NN} = \frac{-\pi}{\pi + 2i\gamma_E + \frac{2}{(1-k_2^2)(1+\sqrt{\epsilon R/c})}} + 2i \log [\sqrt{1-k_2^2\omega R/2c}],
\]

(A.20)

where \( \gamma_E \approx 0.577 \) is the Euler-Mascheroni constant. It can be numerically shown that other elements of \( T \) matrix are negligible.

A.5 Auxiliary functions

In defining functions below, the superscript emphasizes that we deal with the interaction force, whereas the numerical subscript indicates the force’s power of decay in the axis-to-axis separation between cylinders.

\[
g_6^{in}(\varepsilon_1(\omega), \varepsilon_2(\omega)) = \frac{45}{2048} \operatorname{Im} \left[ \frac{1}{\varepsilon_2 + 1} \right] \frac{1}{|\varepsilon_1 + 1|^2} \left[ (|\varepsilon_1|^2 - 1)(4(33 + 5 \operatorname{Re}[\varepsilon_1])
\right.
\]

\[
+ (7 + 3 \operatorname{Re}[\varepsilon_1])|\varepsilon_2 + 1|^2 + (\operatorname{Re}[\varepsilon_1]^2 - 1)(40 + 6|\varepsilon_2 + 1|^2) \right] ;
\]

(A.21)

\[
g_4^{in}(\varepsilon_1, \varepsilon_2) = \frac{3}{256} \operatorname{Im} \left[ \frac{1}{\varepsilon_2 + 1} \right] \frac{1}{|\varepsilon_1 + 1|^2} \left[ \operatorname{Im}[\varepsilon_1]^2(|\varepsilon_2 + 1|^2(7 - \operatorname{Re}[\varepsilon_1]) + 12 \operatorname{Re}[\varepsilon_1] + 76)
\]

\[
+(\operatorname{Re}[\varepsilon_1]^2 - 1)(|\varepsilon_2 + 1|^2(5 - \operatorname{Re}[\varepsilon_1]) + 12 \operatorname{Re}[\varepsilon_1] + 100) \right] ;
\]

(A.22)

\[
\begin{align*}
g_1^{in}(\varepsilon_1(\omega), \varepsilon_2(\omega)) &= \frac{2}{15\pi} \operatorname{Im} \left[ \frac{1}{\varepsilon_1 + 1} \right] \operatorname{Im} \left[ \frac{1}{\varepsilon_2 + 1} \right] \\
&\times (|\varepsilon_1 + 1|^2|\varepsilon_2 + 1|^2 + |\varepsilon_1 + 1|^2|\varepsilon_2 + 1|^2 + 36) ; \quad \text{(A.23)}
\end{align*}
\]

\[
f_6^{in}(\varepsilon_{01}, \varepsilon_{02}) = \frac{15\pi^2}{4096} \frac{(\varepsilon_{01} - 1)(172 + (13 + 3\varepsilon_{01})(\varepsilon_{02} + 1)^2 + 20\varepsilon_{01})}{(\varepsilon_{01} + 1)(\varepsilon_{02} + 1)^2} ;
\]

(A.24)

\[
f_4^{in}(\varepsilon_{01}, \varepsilon_{02}) = \frac{\pi^4(\varepsilon_{01} - 1)}{1280(\varepsilon_{01} + 1)} \left[ \frac{12\varepsilon_{01} + 100}{(\varepsilon_{02} + 1)^2} - \varepsilon_{01} + 5 \right] ;
\]

(A.25)
\[ f_1^{in}(\varepsilon_{01}, \varepsilon_{02}) = \frac{16\pi^7}{225} \frac{(\varepsilon_{01} + 1)^2 + (\varepsilon_{02} + 1)^2 + (\varepsilon_{01} + 1)^2(\varepsilon_{02} + 1)^2 + 36}{(\varepsilon_{02} + 1)^2(\varepsilon_{01} + 1)^2}. \] (A.26)
Appendix B

Derivation of the gradient expansion parameter $\beta$

Here we provide the derivation of the gradient expansion parameter $\beta$, and $\beta$ accordingly, from Chapter 3. Some of the notation is different from the main text, and in such cases we will comment on the discrepancy. We restrict ourselves to the case when Eq. (3.3) holds, i.e. we consider the radiative heat transfer only by $N$-polarized evanescent electromagnetic waves (see Fig. 3-4).

B.1 Heat transfer between two plates

The heat transfer between two planar objects, at separation $d$, is computed via Eq. (3.2),

$$H = \int_0^\infty d\omega [n_\omega(T_1) - n_\omega(T_2)]h,$$

where \cite{30, 35}

$$h = \frac{\hbar\omega}{2\pi} \int_{|k_\perp| > \omega/c} \frac{d^2 k_\perp}{(2\pi)^2} \langle k_\perp, N \mid U \left( R_2 - R_2^f \right) U^f \left( R_1^f - R_1 \right) \mid k_\perp, N \rangle$$

Here $\langle k_\perp, p \rangle$ is the normalized electromagnetic wave in planar basis, described by polarization $p = \{M, N\}$ and in-plane momentum $k_\perp$. As mentioned above, we
restrict ourselves to $N$-polarized evanescent waves, $|k_\perp, N\rangle$, as they dominate the heat transfer at small separations (see Fig. 3-4). $R_1$ and $R_2$ are scattering matrices of the planar objects that correspond to usual Fresnel coefficients in the case of perfect plates. We also define,

$$U = \frac{1}{1 - R_2 R_1^*}. \quad \text{(B.3)}$$

### B.2 Perturbing a planar surface profile

Consider now that the plate 1 is perfectly planar, so that its scattering matrix, $R_1 \equiv R_1^{(0)}$, is described by the usual Fresnel coefficient,

$$\langle k_\perp, N | R_1 | k_\perp, N \rangle = (2\pi)^2 \delta^{(2)}(k_\perp - k'_\perp) r_1(k_\perp) e^{2ik_\perp d}, \quad \text{(B.4)}$$

where $k_\perp = \sqrt{\omega^2/c^2 - |k_\perp|^2}$ is the component of the momentum perpendicular to the plates and $r_1(k_\perp) \equiv r_N^1(k_\perp)$ is the Fresnel coefficient given by Eq. (2.27).

On the other hand, plate 2 is rough, and we define the separation profile as $S(x) = d + s(x)$, where $s(x)$ is the deformation amplitude (in accordance with the main text). According to Eq. (3.8) from the main text,

$$\langle k_\perp, N | R_2 | k_\perp, N \rangle = R_2^{NN}(k_\perp, k'_\perp). \quad \text{(B.5)}$$

Now we can expand the reflection operator, $R_2$, up to the second order in the Fourier transform, $\tilde{s}(k_\perp) = \int d^2 x e^{-ik_\perp x} s(x)$, of the deformation amplitude $s(x)$, as

$$R_2 = R_2^{(0)} + R_2^{(1)} + R_2^{(2)}, \quad \text{(B.6)}$$

with,

$$\langle k_\perp, N | R_2^{(0)} | k_\perp, N \rangle = (2\pi)^2 \delta^{(2)}(k_\perp - k'_\perp) r_2(k_\perp), \quad \text{(B.7)}$$

where $r_2(k_\perp) \equiv r_2^N(k_\perp)$ is the usual Fresnel coefficient. Furthermore,

$$\langle k_\perp, N | R_2^{(1)} | k_\perp, N \rangle = B_{ij}(k_\perp, k'_\perp) \tilde{s}(k_\perp - k'_\perp), \quad \text{(B.8)}$$

with $B_{ij}(k_\perp, k'_\perp)$ being the Fresnel coefficients for the rough surface.
\begin{align}
\langle k_L, N | \mathcal{R}_2^{(2)} | k_L, N \rangle &= \int_{|k'_L| > \omega_0/c} \frac{d^2 k'_L}{(2\pi)^2} B_{(ii)}(k_L, k'_L) \bar{s}(k_L - k'_L) |^2, \\
\text{where in accordance to Eq. (B.5), } B_{(i)}(k_L, k'_L) &= B_{(ii)}^{NN}(k_L, k'_L) \text{ and } B_{(ii)}(k_L, k'_L) \equiv B_{(ii)}^{NN}(k_L, k'_L), \text{ and the specific expressions for these quantities can be found in Ref. [118].}
\end{align}

Taking into account that \( U(1 - R_2 R_1) = 1 \), we can obtain the expansion of \( U \) up to the second order in perturbation as

\begin{equation}
U = U^{(0)} + U^{(0)} R_2^{(1)} R_1 U^{(0)} + 2 U^{(0)} R_2^{(1)} R_1 U^{(0)} R_2^{(1)} R_1 U^{(0)}, \\
\end{equation}

where the operator \( U^{(0)} = 1/(1 - R_2 R_1) \) has the matrix elements \( U^{(0)}(k_L) \equiv U^{(0)}(k_L) = 1/(1 - r_2(k_L) r_1(k_L) e^{2ik_L d}). \)

Using that \( \bar{s}(k_L - k'_L) = \bar{s}^*(k'_L - k_L) \), we incorporate Eqs. (B.4)-(B.10) into Eq. (B.2) to obtain an expression in accordance with Eq. (3.5),

\begin{equation}
\begin{split}
\alpha_0(d) + \alpha_1(d) \bar{s}(0) + \int_{|k_L| > \omega_0/c} \frac{d^2 k_L}{(2\pi)^2} \alpha_2(k_L; d) |\bar{s}(k_L)|^2,
\end{split}
\end{equation}

where

\begin{equation}
\alpha_2(k_L; d) = \int_{|k_L| > \omega_0/c} \frac{d^2 k_L}{(2\pi)^2} \Delta(k_L, k'_L - k_L),
\end{equation}

with

\begin{equation}
\Delta(k_L, k'_L) = \frac{2\hbar \omega}{\pi} |U^{(0)}(k_L)|^2 e^{-2|k_L| d} \left( 2 e^{-2|k_L| d} \text{Re} \left[ U^{(0)}(k_L) B_{(ii)}(k_L, k'_L) r_1(k_L) \right] \text{Im} r_2(k_L) \text{Im} r_1(k_L) \\
+ \text{Im} B_{(ii)}(k_L, k'_L) \text{Im} r_1(k_L) + \text{Im} r_1(k_L) \text{Im} \left[ U^{(0)}(k'_L) r_1(k'_L) e^{-2|k'_L| d} B_{(ii)}(k_L, k'_L) B_{(ii)}(k'_L, k_L) \right] \\
+ 2 \text{Im} r_2(k_L) \text{Im} r_1(k_L) e^{-2|k_L| d} \text{Re} \left[ U^{(0)}(k'_L) U^{(0)}(k'_L) r_1(k'_L) r_1(k'_L) e^{-2|k'_L| d} B_{(ii)}(k_L, k'_L) B_{(ii)}(k'_L, k_L) \right] \\
- \text{Im} r_1(k_L) e^{-2|k_L| d} \text{Im} \left[ U^{(0)}(k'_L) r_1(k'_L) \right] |B_{(ii)}(k_L, k'_L)|^2 \\
+ |U^{(0)}(k'_L)|^2 |r_1(k'_L)|^2 e^{-4|k'_L| d} \text{Im} r_2(k'_L) \text{Im} r_1(k_L) |B_{(ii)}(k_L, k'_L)|^2) .
\end{equation}

Using the expression for \( \alpha_2(k_L; d) \), one can directly compute \( \alpha_2^{(2)}(d) \) defined by Eq. (3.6). Finally, that allows to compute \( \beta_\omega \) using Eq. (3.7).
Appendix C

Non-equilibrium Trace formulae

C.1 Equilibrium correlations

We consider a system of two arbitrary objects (or two sets of distinct objects) in equilibrium with the environment at temperature $T$. The objects' scattering properties are described by their scattering (T) operators $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Then, following the derivation outlined in Chapter 4, we obtain the following results for the desired correlation functions,

$$\int_0^\infty dt \langle H^{(1,2)}(t) H^{(2)}(0)\rangle_{eq} = \frac{2\hbar^2}{\pi} \int_0^\infty d\omega \frac{\omega^2 e^{\hbar \omega / k_B T}}{(e^{\hbar \omega / k_B T} - 1)^2} \text{Im} \text{Tr} M^{(2)}_{1,2},$$

(C.1)

$$\int_0^\infty dt \langle H^{(1,2)}(t) F^{(2)}(0)\rangle_{eq} = -\frac{2\hbar^2}{\pi} \int_0^\infty d\omega \frac{\omega e^{\hbar \omega / k_B T}}{(e^{\hbar \omega / k_B T} - 1)^2} \text{Re} \text{Tr} \nabla M^{(2)}_{1,2},$$

(C.2)

where we have introduced the operators

$$M^{(2)}_1 = (1 + \mathbb{C}_0 \mathbb{T}_2) \frac{1}{1 - \mathbb{C}_0 \mathbb{T}_1 \mathbb{C}_0 \mathbb{T}_2} \mathbb{C}_0 \text{Im}[\mathbb{T}_1] - \mathbb{T}_1 \text{Im}[\mathbb{C}_0] \mathbb{T}_1^* \mathbb{C}_0^* \frac{1}{1 - \mathbb{T}_2^* \mathbb{C}_0^* \mathbb{T}_1^* \mathbb{C}_0^*} \mathbb{T}_2^*$$

(C.3)

$$M^{(2)}_2 = (1 + \mathbb{C}_0 \mathbb{T}_1) \frac{1}{1 - \mathbb{C}_0 \mathbb{T}_2 \mathbb{C}_0 \mathbb{T}_1} \mathbb{C}_0 \text{Im}[\mathbb{T}_2] - \mathbb{T}_2 \text{Im}[\mathbb{C}_0] \mathbb{T}_2^* \frac{1}{1 - \mathbb{T}_1^* \mathbb{C}_0^* \mathbb{T}_2^* \mathbb{C}_0^*} \mathbb{T}_1^*.$$  

(C.4)
C.2 Heat transfer and Casimir force

We summarize the relevant results for heat transfer and non-equilibrium Casimir forces from Ref. [15]. Consider objects 1 and 2 held at temperatures $T_1$ and $T_2$, and with the environment at temperature $T_{env}$. The heat absorbed by object 2 is given by Eq. (69) from Ref. [15] as

$$\langle H^{(2)}_2 \rangle (T_1, T_2, T_{env}) = \sum_{\alpha=1,2} \langle H^{(2)}_\alpha \rangle (T_\alpha) - \langle H^{(2)}_\alpha \rangle (T_{env}).$$  \hspace{1cm} (C.5)

Here, $\langle H^{(2)}_1 \rangle$ is the heat transfer from object 1 to object 2, and $\langle H^{(2)}_2 \rangle$ is the so-called self-emission by object 2, corresponding to the heat lost by object 2 due to the presence of object 1. These are given by Eqs. (56) and (65) from Ref. [15] respectively,

$$\langle H^{(2)}_1 \rangle (T_{1,2}) = -\frac{2\hbar \pi}{\omega} \int_0^\infty \frac{\omega d\omega}{e^{\hbar \omega/k_B T_{1,2}} - 1} \text{Im Tr} \, M^{(2)}_{1,2}. \hspace{1cm} (C.6)$$

The Casimir force acting on an arbitrary object 2 is given by Eq. (79) in Ref. [15], and can be written as

$$\langle F^{(2)}_2 \rangle (T_1, T_2, T_{env}) = \langle F^{(2)}_2 \rangle^{eq} (T_{env}) + \sum_{\alpha=1,2} \left[ \langle F^{(2)}_\alpha \rangle (T_\alpha) - \langle F^{(2)}_\alpha \rangle (T_{env}) \right]. \hspace{1cm} (C.7)$$

We note, that this equation is equivalent to Eq. (2.61) for $N$ objects. The equilibrium Casimir force $\langle F^{(2)}_2 \rangle^{eq}$ is well-studied [14], and not relevant for our analysis. The non-equilibrium contribution $\langle F^{(2)}_1 \rangle$ acts on object 2 due to the sources in object 1. The other non-equilibrium contribution is the self-force, $\langle F^{(2)}_2 \rangle$, and represents the force that acts on object 2 due to the sources in the object itself. These non-equilibrium contributions to the Casimir force are given by Eqs. (76) and (77) in Ref. [15] as

$$\langle F^{(2)}_1 \rangle (T_{1,2}) = \frac{2\hbar}{\pi} \int_0^\infty \frac{d\omega}{e^{\hbar \omega/k_B T_{1,2}} - 1} \text{Re Tr} \, \nabla M^{(2)}_{1,2}. \hspace{1cm} (C.8)$$

With Eqs. (C.1),(C.2),(C.5)–(C.8), the relations (4.11) and (4.12) of the main text can be confirmed.
C.3 Field correlations sourced by a moving object

Here we compute the spectral density \( C^{\text{obj}}(r, r') = \langle E_i(r) E_j^*(r') \rangle_{S,\omega} \) resulting from an isolated object moving with velocity \( \mathbf{v} \), to linear order in velocity. Without loss of generality, consider the object moving along the \( p \)-axis, so that \( \mathbf{v} = v \hat{e}_p \).

We first consider an arbitrary equilibrium situation viewed in a reference frame moving with velocity \( \mathbf{v} \), which follows from the covariant treatment in Ref. [136]. The spectral density can be expressed in terms of the system’s Green’s function as

\[
C_{ij}(r, r') = \langle E_i(r) E_j^*(r') \rangle_{S,\omega} = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3h}{(2\pi)^3} e^{i(k \cdot r + h \cdot r')} C_{ij}(k, h)
\]

where we use the symmetric version of expression for \( C_{ij}(k, h) \) from Ref. [136],

\[
C_{ij}(k, h) = -\text{sgn}(\omega) \frac{2i\pi\hbar\omega^2}{c^2} \left\{ \coth \left( \frac{\hbar(\omega - k_p v)}{2k_B T} \right) G_{ij}(\omega, k, h) - \coth \left( \frac{\hbar(\omega + k_p v)}{2k_B T} \right) G_{ji}^*(\omega, -h, -k) \right\} .
\]

We have set \( k_p = k \cdot \hat{e}_p \), and \( G_{ij}(\omega, k, h) \) is the spatial/temporal Fourier transform of the Green’s function \( G_{ij}(t, r, r') \) for the system. Note that for \( v = 0 \) the equilibrium correlator in the rest frame is recovered. By expanding the field correlations to linear order in \( v \), we obtain

\[
\left. \frac{dC_{ij}(k, h)}{dv} \right|_{v=0} = -\text{sgn}(\omega) \frac{4i\pi\hbar^2\omega^2}{c^2k_B T} \left( e^{\hbar\omega/k_B T} - 1 \right)^2 \left( k_p G_{ij}(\omega, k, h) + h_p G_{ji}^*(\omega, -h, -k) \right) .
\]

Transforming back to real space we get for the Lorentz transformed field correlator
to linear order in $v$,
\[
\left. \frac{dC_{ij}(\mathbf{r}, \mathbf{r'})}{dv} \right|_{v=0} = - \text{sgn}(\omega) \frac{4\pi \hbar^2 \omega^2}{c^2 k_B T} \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} \left( \partial_{\mathbf{r}} G_{ij}(\omega, \mathbf{r}, \mathbf{r'}) + \partial_{\mathbf{r'}} G_{ji}^*(\omega, \mathbf{r'}, \mathbf{r}) \right).
\]
(C.12)

For an isolated object, Eq. (C.12) enables computing the field correlator $C$ in a frame which is moving with respect to both the object and the environment. It follows from the Green’s function of the system, expressed in terms of the object’s $T$ operator, $G = G_0 + G_0 T G_0$ (see Eq. (2.13)), and reads
\[
\left. \frac{dC}{dv} \right|_{v=0} = - \text{sgn}(\omega) \frac{4\pi \hbar^2 \omega^2}{c^2 k_B T} \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} \left[ 2\partial_{\mathbf{r}} \text{Im}[G_0] - i(G_0 \partial_{\mathbf{r}} T G_0 - G_0^* T^* \partial_{\mathbf{r}} G_0^*) \right].
\]
(C.13)

To linear order in $v$, the result in Eq. (C.13) can alternatively be found by a decomposition into two terms: one arising from the motion of the empty environment (with the static object present) and the other one resulting from the moving object in a static environment,
\[
\left. \frac{dC}{dv} \right|_{v=0} = \left. \frac{d(C_{\text{env}} + C_{\text{obj}})}{dv} \right|_{v=0}.
\]
(C.14)

We are interested in the latter component, $C_{\text{obj}}$. The field sourced by the moving environment in the presence of a static object is computed by first considering Eq. (C.12) for the empty environment (described by $G_0$), and then scattering at the static object [15], to get
\[
\left. \frac{dC_{\text{env}}}{dv} \right|_{v=0} = - \text{sgn}(\omega) \frac{8i\pi \hbar^2 \omega^2}{c^2 k_B T} \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} (1 + G_0 T) \partial_{\mathbf{r}} \text{Im}[G_0](T^* G_0^* + 1).
\]
(C.15)

The desired correlator can now be found by use of Eqs. (C.13), (C.14) and (C.15), and reads
\[
\left. \frac{dC_{\text{obj}}}{dv} \right|_{v=0} = - \text{sgn}(\omega) \frac{4\pi \hbar^2 \omega^2}{c^2 k_B T} \frac{e^{\hbar \omega/k_B T}}{(e^{\hbar \omega/k_B T} - 1)^2} G_0 \left[ i(\partial_{\mathbf{r}} T^* - \mathbf{T} \partial_{\mathbf{r}}) - 2\mathbf{T} \partial_{\mathbf{r}} \text{Im}[G_0] T^* G_0^* \right].
\]
(C.16)
This is precisely the source term in the expression of Eq. (4.14) in the main text, demonstrating the equivalence of the force correlator in Eq. (4.14) and the hereby found linear response result.
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