

MIT Open Access Articles

Casimir-Polder interaction for gently curved surfaces

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

Citation: Bimonte, Giuseppe, Thorsten Emig, and Mehran Kardar. "Casimir-Polder interaction for gently curved surfaces." *Phys. Rev. D* 90 (2014): 081702(R). © 2014 American Physical Society.

As Published: <http://dx.doi.org/10.1103/PhysRevD.90.081702>

Publisher: American Physical Society

Persistent URL: <http://hdl.handle.net/1721.1/91244>

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

Terms of Use: Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.



Casimir-Polder interaction for gently curved surfacesGiuseppe Bimonte,^{1,2} Thorsten Emig,³ and Mehran Kardar⁴¹*Dipartimento di Scienze Fisiche, Università di Napoli Federico II, Complesso Universitario di Monte S. Angelo, Via Cintia, I-80126 Napoli, Italy*²*INFN Sezione di Napoli, I-80126 Napoli, Italy*³*Laboratoire de Physique Théorique et Modèles Statistiques, CNRS UMR 8626, Bât. 100, Université Paris-Sud, 91405 Orsay cedex, France*⁴*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 24 September 2014; published 28 October 2014)

We use a derivative expansion for gently curved surfaces to compute the leading and the next-to-leading curvature corrections to the Casimir-Polder interaction between a polarizable small particle and a nonplanar surface. While our methods apply to any homogeneous and isotropic surface, explicit results are presented here for perfect conductors. We show that the derivative expansion of the Casimir-Polder potential follows from a resummation of its perturbative series, for small in-plane momenta. We consider the retarded, nonretarded and classical high-temperature limits.

DOI: [10.1103/PhysRevD.90.081702](https://doi.org/10.1103/PhysRevD.90.081702)

PACS numbers: 12.20.-m, 03.70.+k, 42.25.Fx

I. INTRODUCTION

Quantum and thermal vacuum fluctuations of the electromagnetic field are at the cause of so-called dispersion forces between two polarizable bodies. A particular instance of dispersion interaction is the Casimir-Polder force [1] between a small polarizable particle (like an atom or a molecule) and a nearby material surface. Recent advances in nanotechnology and in the field of ultracold atoms have made possible quite precise measurements of the Casimir-Polder interaction. (For recent reviews, see Refs. [2,3].)

There is presently considerable interest in investigating how the Casimir-Polder interaction is affected by the geometrical shape of the surface, and several experiments have been recently carried out [4–7] to probe dispersion forces between atoms and microstructured surfaces. The characteristic nonadditivity of dispersion forces makes it very difficult to compute the Casimir-Polder interaction for nonplanar surfaces in general. Detailed results have been worked out only for a few specific geometries. The example of a uniaxially corrugated surface was studied numerically in Ref. [8] within a toy scalar field theory, while rectangular dielectric gratings were considered in Ref. [9]. In Ref. [10], analytical results were obtained for the case of a perfectly conducting cylinder. A perturbative approach is presented in Ref. [11], where surfaces with smooth corrugations of any shape, but with small amplitude, were studied. The validity of the latter is restricted to particle-surface separations that are much larger than the corrugation amplitude. In this paper we present an alternative approach that becomes exact in the opposite limit of small particle-surface distances. In this limit, the proximity force approximation (PFA) [12] can be used to obtain the leading contribution to the Casimir-Polder potential. Our approach is based on a systematic *derivative expansion* of the potential, extending to the Casimir-Polder interaction an analogous expansion successfully used recently [13–15] to

study the Casimir interaction between two nonplanar surfaces. It has also been applied to other problems involving short-range interactions between surfaces, like radiative heat transfer [16] and stray electrostatic forces between conductors [17]. From this expansion we could obtain the leading and the next-to-leading curvature corrections to the PFA for the Casimir-Polder interaction.

The paper is organized as follows: In Sec. II we present the derivative expansion for the general case of a dielectric surface. Explicit results are presented for the special case of a perfectly conducting surface. In Sec. III the example of a two-state atom is considered, and we present the potential in two limits: the retarded Casimir-Polder limit and the nonretarded London limit. In Section IV we conclude, pointing out some avenues for further exploration. Finally, in the Appendix we show how the derivative expansion of the potential can be obtained by a resummation of the perturbative series to all orders.

II. DERIVATIVE EXPANSION OF THE CASIMIR-POLDER POTENTIAL

Consider a particle (an “atom,” a molecule, or any polarizable microparticle) near a dielectric surface S . We assume that the particle is small enough (compared to the scale of its separation d from the surface), such that it can be considered as pointlike, with its response to the electromagnetic (em) fields fully described by a dynamic electric dipolar polarizability tensor $\alpha_{\mu\nu}(\omega)$. (We assume for simplicity that the particle has a negligible magnetic polarizability, as is usually the case.) Let us denote by Σ_1 the plane through the atom which is orthogonal to the distance vector (which we take to be the $\hat{\mathbf{z}}$ axis) connecting the atom to the point P of S closest to the atom. We assume that the surface S is characterized by a *smooth* profile $z = H(\mathbf{x})$, where $\mathbf{x} = (x, y)$ is the vector spanning Σ_1 , with the origin at the atom’s position (see Fig. 1). In what follows, greek indices μ, ν, \dots label all coordinates (x, y, z) ,

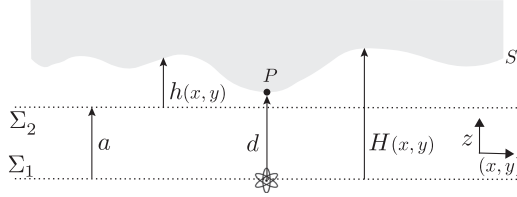


FIG. 1. Coordinates parametrizing a configuration consisting of an atom or nanoparticle near a gently curved surface.

while latin indices $i, j, k \dots$ refer to (x, y) coordinates in the plane Σ_1 . Throughout, we adopt the convention that repeated indices are summed over.

The exact Casimir-Polder potential at finite temperature T is given by the scattering formula [18,19]

$$U = -k_B T \sum_{n=0}^{\infty} \text{Tr}[\mathbb{T}^{(S)} \mathbb{U} \mathbb{T}^{(A)} \mathbb{U}](\kappa_n). \quad (1)$$

Here $\mathbb{T}^{(S)}$ and $\mathbb{T}^{(A)}$ denote, respectively, the T-operators of the plate S and the atom, evaluated at the Matsubara wave numbers $\kappa_n = 2\pi n k_B T / (\hbar c)$, and the primed sum indicates that the $n = 0$ term carries weight $1/2$. In a plane-wave basis $|\mathbf{k}, Q\rangle$ [20], where \mathbf{k} is the in-plane wave vector, and $Q = E, M$ denotes, respectively, the electric (transverse magnetic) and magnetic (transverse electric) modes, the translation operator \mathbb{U} in Eq. (1) is diagonal with matrix elements e^{-dq} where $q = \sqrt{k^2 + \kappa_n^2} \equiv q(k)$, $k = |\mathbf{k}|$. The matrix elements

of the atom T-operator in the dipole approximation are

$$\mathcal{T}_{QQ'}^{(A)}(\mathbf{k}, \mathbf{k}') = -\frac{2\pi\kappa_n^2}{\sqrt{qq'}} e_{Q\mu}^{(+)}(\mathbf{k}) \alpha_{\mu\nu}(ic\kappa_n) e_{Q'\nu}^{(-)}(\mathbf{k}'), \quad (2)$$

where $q' = q(k')$, $\mathbf{e}_M^{(\pm)}(\mathbf{k}) = \hat{\mathbf{z}} \times \hat{\mathbf{k}}$ and $\mathbf{e}_E^{(\pm)}(\mathbf{k}) = -1/\kappa_n(ik\hat{\mathbf{z}} \pm q\hat{\mathbf{k}})$, with $\hat{\mathbf{k}} = \mathbf{k}/k$. There are no analytical formulas for the elements of the T-operator of a curved plate $\mathbb{T}^{(S)}$, and its computation is in general quite challenging, even numerically. We shall demonstrate, however, that for any smooth surface it is possible to compute the leading curvature corrections to the potential in the experimentally relevant limit of small separations. The key idea is that the Casimir-Polder interaction falls off rapidly with separation, and it is thus reasonable to expect that the potential U is mainly determined by the geometry of the surface S in a small neighborhood of the point P of S which is *closest* to the atom. This physically plausible idea suggests that for small separations d the potential U can be expanded as a series expansion in an increasing number of derivatives of the height profile H , evaluated at the atom's position. Up to fourth order, and assuming that the surface is homogeneous and isotropic, the most general expression which is invariant under rotations of the (x, y) coordinates, and that involves at most four derivatives of H [but no first derivatives, since $\nabla H(\mathbf{0}) = 0$] can be expressed [at zero temperature, and up to $\mathcal{O}(d^{-1})$] in the form

$$U = -\frac{\hbar c}{d^4} \int_0^{\infty} \frac{d\xi}{2\pi} \left\{ \beta_1^{(0)} \alpha_{\perp} + \beta_2^{(0)} \alpha_{zz} + d \times \left[(\beta_1^{(2)} \alpha_{\perp} + \beta_2^{(2)} \alpha_{zz}) \nabla^2 H + \beta_3^{(2)} \left(\partial_i \partial_j H - \frac{1}{2} \nabla^2 H \delta_{ij} \right) \alpha_{ij} \right] + d^2 \right. \\ \left. \times \left[\beta^{(3)} \alpha_{zi} \partial_i \nabla^2 H + (\nabla^2 H)^2 (\beta_1^{(4)} \alpha_{\perp} + \beta_2^{(4)} \alpha_{zz}) + (\partial_i \partial_j H)^2 (\beta_3^{(4)} \alpha_{\perp} + \beta_4^{(4)} \alpha_{zz}) + \beta_5^{(4)} \nabla^2 H \left(\partial_i \partial_j H - \frac{1}{2} \nabla^2 H \delta_{ij} \right) \alpha_{ij} \right] \right\}, \quad (3)$$

where the Matsubara sum has been replaced by an integral over $\xi = \kappa d$, $\alpha_{\perp} = \alpha_{xx} + \alpha_{yy}$, and it is understood that all derivatives of $H(\mathbf{x})$ are evaluated at the atom's position, i.e. for $\mathbf{x} = \mathbf{0}$. The coefficients $\beta_p^{(p)}$ are dimensionless functions of ξ , and of any other dimensionless ratio of frequencies characterizing the material of the surface. The derivative expansion in Eq. (3) can be formally obtained by a resummation of the perturbative series for the potential for small in-plane momenta \mathbf{k} (see the Appendix). We note that there are additional terms involving four derivatives of

H which, however, yield contributions $\sim 1/d$ (as do terms involving five derivatives of H) and are hence neglected.

As demonstrated in the Appendix [see Eqs. (A12), (A13)], the coefficients $\beta_q^{(p)}$ in Eq. (3) can be extracted from the perturbative series of the potential U , carried to second order in the deformation $h(\mathbf{x})$, which in turn involves an expansion of the T-operator of the surface S to the same order. The latter expansion was worked out in Ref. [21] for a dielectric surface characterized by a frequency-dependent permittivity $\epsilon(\omega)$. It reads

$$\mathcal{T}_{QQ'}^{(S)}(\mathbf{k}, \mathbf{k}') = (2\pi)^2 \delta^{(2)}(\mathbf{k} - \mathbf{k}') \delta_{QQ'} r_Q^{(S)}(ic\kappa_n, \mathbf{k}) \\ + \sqrt{qq'} \left[-2B_{QQ'}(\mathbf{k}, \mathbf{k}') \tilde{h}(\mathbf{k} - \mathbf{k}') + \int \frac{d^2 \mathbf{k}''}{(2\pi)^2} (B_2)_{QQ'}(\mathbf{k}, \mathbf{k}'; \mathbf{k}'') \tilde{h}(\mathbf{k} - \mathbf{k}'') \tilde{h}(\mathbf{k}'' - \mathbf{k}') + \dots \right], \quad (4)$$

TABLE I. The coefficients $\beta_q^{(p)}$ are obtained by multiplying the third column by $e^{-2\xi}$ and adding the fourth column times $\text{Ei}(2\xi) = -\int_{2\xi}^{\infty} dt \exp(-t)/t$.

p	q	$\times e^{-2\xi}$	$\times \text{Ei}(2\xi)$
0	1	$\frac{1}{8}(1 + 2\xi + 4\xi^2)$	0
	2	$\frac{1}{4}(1 + 2\xi)$	0
2	1	$-\frac{1}{32}(3 + 6\xi + 6\xi^2 + 4\xi^3)$	$-\frac{\xi^4}{4}$
	2	$-\frac{1}{16}(1 + 2\xi - 2\xi^2 + 4\xi^3)$	$\xi^2(1 - \frac{\xi^2}{2})$
	3	$-\frac{1}{32}(3 + 6\xi + 2\xi^2 - 4\xi^3)$	$\frac{\xi^4}{4}$
3	1	$\frac{1}{32}(1 + 2\xi - 2\xi^2 + 4\xi^3)$	$-\frac{\xi^2}{4}(2 - \xi^2)$
	2	$\frac{1}{384}(3 + 6\xi + 15\xi^2 + 22\xi^3 + 2\xi^4 - 4\xi^5)$	$\frac{\xi^4}{48}(6 - \xi^2)$
4	2	$-\frac{1}{960}(15 + 542\xi + 259\xi^2 - 546\xi^3 - 14\xi^4 + 28\xi^5)$	$\frac{120\xi^4}{7}(20 - \xi^2)$
	3	$\frac{1}{192}(15 + 30\xi - 9\xi^2 + 70\xi^3 + 2\xi^4 - 4\xi^5)$	$\frac{\xi^4}{24}(18 - \xi^2)$
	4	$\frac{1}{480}(45 + 218\xi - 59\xi^2 + 146\xi^3 + 14\xi^4 - 28\xi^5)$	$\frac{\xi^4}{60}(40 - 7\xi^2)$
	5	$\frac{1}{96}(9 + 18\xi - 27\xi^2 + 50\xi^3 - 2\xi^4 + 4\xi^5)$	$\xi^4(1 + \frac{\xi^2}{12})$

where $r_Q^{(S)}(i\kappa_n, \mathbf{k})$ denotes the familiar Fresnel reflection coefficient of a flat surface. Explicit expressions for $B_{QQ'}(\mathbf{k}, \mathbf{k}')$ and $(B_2)_{QQ'}(\mathbf{k}', \mathbf{k}'; \mathbf{k}'')$ are given in Ref. [21].

The computation of the coefficients $\beta_q^{(p)}$ involves an integral over \mathbf{k} and \mathbf{k}' [as is apparent from Eq. (1)] that cannot be performed analytically for a dielectric plate, and has to be estimated numerically. In this paper, we shall content ourselves to considering the case of a perfect conductor, in which case the integrals can be performed analytically. For a perfect conductor, the matrix $B_{QQ'}(\mathbf{k}, \mathbf{k}')$ takes the simple form

$$B(\mathbf{k}, \mathbf{k}') = \begin{pmatrix} \frac{\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' \kappa_n + \kappa_n \kappa_n'}{q q'} & \frac{\kappa_n}{q} \hat{\mathbf{z}} \cdot (\hat{\mathbf{k}} \times \hat{\mathbf{k}}') \\ \frac{\kappa_n}{q} \hat{\mathbf{z}} \cdot (\hat{\mathbf{k}} \times \hat{\mathbf{k}}') & -\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' \end{pmatrix}, \quad (5)$$

where the matrix indices 1,2 correspond to E, M , respectively. For perfect conductors, the matrix $(B_2)_{QQ'}(\mathbf{k}, \mathbf{k}'; \mathbf{k}'')$ is simply related to B by

$$(B_2)(\mathbf{k}, \mathbf{k}'; \mathbf{k}'') = 2q'' B(\mathbf{k}, \mathbf{k}'') \sigma_3 B(\mathbf{k}'', \mathbf{k}'), \quad (6)$$

where $\sigma_3 = \text{diag}(1, -1)$. For perfect conductors, the coefficients $\beta_q^{(p)}$ are functions of ξ only, and we list them in Table I.

The geometric significance of the expansion in Eq. (3) becomes more transparent when the x and y axis are chosen to be coincident with the principal directions of S at P , in which case the local expansion of H takes the simple form $H = d + x^2/(2R_1) + y^2/(2R_2) + \dots$, where R_1 and R_2 are the radii of curvature at P . In this coordinate system, the derivative expansion of U reads

$$U = -\frac{\hbar c}{d^4} \int_0^\infty \frac{d\xi}{2\pi} \left\{ \beta_1^{(0)} \alpha_\perp + \beta_2^{(0)} \alpha_{zz} + \left(\frac{d}{R_1} + \frac{d}{R_2} \right) (\beta_1^{(2)} \alpha_\perp + \beta_2^{(2)} \alpha_{zz}) + \frac{\beta_3^{(2)}}{2} \left(\frac{d}{R_1} - \frac{d}{R_2} \right) (\alpha_{xx} - \alpha_{yy}) \right. \\ + d^2 \beta^{(3)} \alpha_{zi} \partial_i \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + \left(\frac{d}{R_1} + \frac{d}{R_2} \right)^2 (\beta_1^{(4)} \alpha_\perp + \beta_2^{(4)} \alpha_{zz}) \\ \left. + \left[\left(\frac{d}{R_1} \right)^2 + \left(\frac{d}{R_2} \right)^2 \right] (\beta_3^{(4)} \alpha_\perp + \beta_4^{(4)} \alpha_{zz}) + \frac{\beta_5^{(4)}}{2} \left[\left(\frac{d}{R_1} \right)^2 - \left(\frac{d}{R_2} \right)^2 \right] (\alpha_{xx} - \alpha_{yy}) \right\}. \quad (7)$$

III. TWO-STATE ‘‘ATOM’’

The β coefficients in Eq. (3) are significantly different from zero only for rescaled frequencies $\xi \lesssim 1$. Therefore, for separations small compared to the radii of surface

curvature but $d \gg c/\omega_r$, where ω_r is the typical atomic resonance frequency, we can replace $\alpha_{\mu\nu}(i\kappa)$ in Eqs. (3) and (7) with its static limit $\alpha_{\mu\nu}(0) \equiv \alpha_{\mu\nu}^0$. Upon performing the ξ integrals, we obtain the *retarded* Casimir-Polder potential

$$U_{\text{CP}} = -\frac{\hbar c}{\pi d^4} \left\{ \frac{\alpha_{\mu\mu}^0}{8} - \left(\frac{d}{R_1} + \frac{d}{R_2} \right) \left(\frac{3\alpha_{\perp}^0}{40} + \frac{\alpha_{zz}^0}{15} \right) - \frac{1}{40} \left(\frac{d}{R_1} - \frac{d}{R_2} \right) (\alpha_{xx}^0 - \alpha_{yy}^0) + \frac{d^2}{30} \alpha_{zi}^0 \partial_i \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right. \\ \left. + \left(\frac{d}{R_1} + \frac{d}{R_2} \right)^2 \left(\frac{3\alpha_{\perp}^0}{280} - \frac{\alpha_{zz}^0}{240} \right) + \left[\left(\frac{d}{R_1} \right)^2 + \left(\frac{d}{R_2} \right)^2 \right] \left(\frac{13\alpha_{\perp}^0}{280} + \frac{3\alpha_{zz}^0}{40} \right) + \frac{9}{560} \left[\left(\frac{d}{R_1} \right)^2 - \left(\frac{d}{R_2} \right)^2 \right] (\alpha_{xx}^0 - \alpha_{yy}^0) \right\}. \quad (8)$$

In the special case of a spherical atom near a cylindrical metallic shell, the leading curvature correction in the above formula reproduces Eq. (30) of Ref. [10]. Before turning to the nonretarded limit, it is instructive to consider the

classical high-temperature limit, where the Casimir free energy is given by the first term of the Matsubara sum in Eq. (1). From the limit $\kappa \rightarrow 0$ of the coefficients $\beta_q^{(p)}$, we obtain the classical free energy as

$$U_{\text{classical}} = -\frac{k_B T}{2} \frac{1}{d^3} \left\{ \frac{1}{8} \alpha_{\perp}^0 + \frac{1}{4} \alpha_{zz}^0 - \frac{3}{64} \left(3 \frac{d}{R_1} + \frac{d}{R_2} \right) \alpha_{xx}^0 - \frac{3}{64} \left(\frac{d}{R_1} + 3 \frac{d}{R_2} \right) \alpha_{yy}^0 - \frac{1}{16} \left(\frac{d}{R_1} + \frac{d}{R_2} \right) \alpha_{zz}^0 \right. \\ \left. + \frac{1}{128} \left(17 \frac{d^2}{R_1^2} + 5 \frac{d^2}{R_2^2} + 2 \frac{d^2}{R_1 R_2} \right) \alpha_{xx}^0 + \frac{1}{128} \left(17 \frac{d^2}{R_2^2} + 5 \frac{d^2}{R_1^2} + 2 \frac{d^2}{R_1 R_2} \right) \alpha_{yy}^0 + \frac{1}{64} \left(5 \frac{d^2}{R_1^2} + 5 \frac{d^2}{R_2^2} - 2 \frac{d^2}{R_1 R_2} \right) \alpha_{zz}^0 \right\}. \quad (9)$$

From the above result we obtain the *nonretarded* London interaction between the surface and a two-state atom for small distances $d \ll d_r = c/\omega_r$ at any finite temperature T . The dynamic dipolar polarizability of an atom or molecule on the imaginary frequency axis is given by

$$\alpha_{\mu\nu}(\kappa) = \frac{\alpha_{\mu\nu}^0}{1 + (d_r \kappa)^2}. \quad (10)$$

Formally, the nonretarded limit is obtained by taking the velocity of light to infinity ($c \rightarrow \infty$). This implies that the coefficients $\beta_q^{(p)}$ are evaluated at $\xi = \kappa_n d \sim 1/c \rightarrow 0$, while the atom's polarizability tends to the finite limit $\alpha_0/[1 + (2\pi n k_B T/(\hbar \omega_r))^2]$ for $c \rightarrow \infty$. Hence, the Matsubara sum over n can be performed easily, leading to the nonretarded London potential at finite temperature T of

$$U_L = \frac{\hbar \omega_r}{2k_B T} \coth \left(\frac{\hbar \omega_r}{2k_B T} \right) U_{\text{classical}}. \quad (11)$$

IV. CONCLUSIONS AND OUTLOOK

We have developed a derivative expansion for the Casimir-Polder potential between a small polarizable particle and a gently curved dielectric surface, which is valid in the limit of small particle-surface distances. We have demonstrated the power of our approach by computing analytically the leading and next-to-leading curvature corrections to the PFA for the potential, in the idealized limit of a perfectly conducting surface at zero temperature. For a two-level atom, we provide explicit formulas for the

potential in the retarded Casimir-Polder limit, and in the nonretarded London limit.

While the explicit results presented in the paper are for idealized situations, the gradient expansion method allows for many interesting extensions: Specific dielectric properties of the surface can be easily incorporated and estimated numerically; resonances and anisotropy of the material can lead to interesting interplay with shape and curvature. On the side of the ‘‘atom,’’ we can include effects from higher multipoles in the particle's polarizability. It is easy to deduce already from Eq. (7) that curvature of the surface can exert a torque, rotating an anisotropic particle into a specific low-energy orientation. Nonequilibrium situations, involving an excited atom, or a surface held at a different temperature, also provide additional avenues for exploration.

ACKNOWLEDGMENTS

We thank R. L. Jaffe for valuable discussions. This research was supported by the National Science Foundation through Grant No. DMR-12-06323.

APPENDIX: RESUMMATION OF THE PERTURBATIVE SERIES

It has been recently shown that the derivative expansion of the Casimir energy between a flat and a curved surface follows from a resummation of the perturbative series, for small in-plane momenta [22]. In this appendix we show that the derivative expansion of the Casimir-Polder potential U in Eq. (3) can be justified by an analogous procedure. It is first convenient to recast the potential U in Eq. (1) in the form

$$U = -\frac{\hbar c}{d^4} \int_0^\infty \frac{d\xi}{2\pi} \alpha_{\mu\nu}(i c \kappa) U_{\mu\nu}(\xi), \quad (\text{A1})$$

where the coefficients $U_{\mu\nu}$ depend linearly on the matrix elements of $\mathbb{T}^{(S)}$. To specify the perturbative series, we introduce an arbitrary reference plane Σ_2 at distance a from Σ_1 (see Fig. 1), and then we set $H(\mathbf{x}) = a + h(\mathbf{x})$. For sufficiently small h , the coefficients $U_{\mu\nu}$ in Eq. (A1) admit the expansion

$$U_{\mu\nu} = G_{\mu\nu}^{(0)}(a) + \sum_{n \geq 1} \frac{1}{n!} \int d^2 \mathbf{x}_1 \dots \int d^2 \mathbf{x}_n \times G_{\mu\nu}^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n; a) h(\mathbf{x}_1) \dots h(\mathbf{x}_n), \quad (\text{A2})$$

where $G_{\mu\nu}^{(0)}(a)$ denotes the coefficient for a planar surface at distance a from the atom, $G_{\mu\nu}^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n; a)$ are symmetric functions of $(\mathbf{x}_1, \dots, \mathbf{x}_n)$, and for brevity we have omitted the dependence of $G_{\mu\nu}^{(n)}$ on ξ . The kernels $G_{\mu\nu}^{(n)}$ satisfy a set of differential relations, which result from invariance of $U_{\mu\nu}$ under a redefinition of a and $h(\mathbf{x})$:

$$a \rightarrow a + \epsilon, \quad h(\mathbf{x}) \rightarrow h(\mathbf{x}) - \epsilon, \quad (\text{A3})$$

where ϵ is an arbitrary number. Independence of $U_{\mu\nu}$ on ϵ is equivalent to demanding $\partial^p U_{\mu\nu} / \partial \epsilon^p |_{\epsilon=0} = 0$ for all non-negative integers p . It is possible to verify that these conditions are satisfied if and only if the kernels $G_{\mu\nu}^{(n)}$ obey the relations

$$U_{\mu\nu} \simeq G_{\mu\nu}^{(0)}(a) + \sum_{n \geq 1} \left\{ \frac{1}{n!} A_{\mu\nu}^{(n)*}(a) h^n(\mathbf{0}) + \frac{h^{n-1}(\mathbf{0})}{(n-1)!} \left[-\frac{1}{2} B_{\mu\nu|ij}^{(n)*}(a) \partial_i \partial_j h(\mathbf{0}) + \frac{i}{3!} B_{\mu\nu|ijk}^{(n)*}(a) \partial_i \partial_j \partial_k h(\mathbf{0}) + \frac{1}{4!} B_{\mu\nu|ijkl}^{(n)*}(a) \partial_i \partial_j \partial_k \partial_l h(\mathbf{0}) \right] \right\} + \sum_{n \geq 2} \frac{h^{n-2}(\mathbf{0})}{8(n-2)!} C_{\mu\nu|ijkl}^{(n)*}(a) \partial_i \partial_j h(\mathbf{0}) \partial_k \partial_l h(\mathbf{0}), \quad (\text{A7})$$

where

$$A_{\mu\nu}^{(n)}(a) = \tilde{G}_{\mu\nu}^{(n)}(\mathbf{0}, \dots, \mathbf{0}; a), \quad (\text{A8})$$

$$B_{\mu\nu|i_1 \dots i_p}^{(n)}(a) = \partial_{k_{i_1}} c \dots \partial_{k_{i_p}} \tilde{G}_{\mu\nu}^{(n)}(\mathbf{k}, \mathbf{0}, \dots, \mathbf{0}; a) |_{\mathbf{k}=\mathbf{0}}, \quad (\text{A9})$$

and

$$C_{\mu\nu|ijkl}^{(n)}(a) = \partial_{k_i} \partial_{k_j} \partial_{k'_k} \partial_{k'_l} \tilde{G}_{\mu\nu}^{(n)}(\mathbf{k}, \mathbf{k}', \mathbf{0}, \dots, \mathbf{0}; a) |_{\mathbf{k}=\mathbf{k}'=\mathbf{0}}, \quad (\text{A10})$$

and we have only displayed the terms that do not vanish identically on account of the condition $\nabla h(\mathbf{0}) = 0$. The

$$\begin{aligned} \frac{\partial^p G_{\mu\nu}^{(n)}}{\partial a^p}(\mathbf{x}_1, \dots, \mathbf{x}_n; a) \\ = \int d^2 \mathbf{x}_{n+1} \dots \int d^2 \mathbf{x}_{n+p} G_{\mu\nu}^{(n+p)}(\mathbf{x}_1, \dots, \mathbf{x}_{n+p}; a). \end{aligned} \quad (\text{A4})$$

In momentum space, the above relations read

$$\frac{\partial^p \tilde{G}_{\mu\nu}^{(n)}}{\partial a^p}(\mathbf{k}_1, \dots, \mathbf{k}_n; a) = \tilde{G}_{\mu\nu}^{(n+p)}(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{0}, \dots, \mathbf{0}; a), \quad (\text{A5})$$

where our Fourier transforms are defined such that $\tilde{f}(\mathbf{k}) = \int d^2 \mathbf{x} f(\mathbf{x}) \exp(-i \mathbf{k} \cdot \mathbf{x})$, and we set $\tilde{G}_{\mu\nu}^{(0)} \equiv G_{\mu\nu}^{(0)}$. Consider now the perturbative expansion of the coefficients $U_{\mu\nu}$ in Fourier space:

$$U_{\mu\nu} = G_{\mu\nu}^{(0)}(a) + \sum_{n \geq 1} \frac{1}{n!} \int \frac{d^2 \mathbf{k}_1}{4\pi^2} \dots \int \frac{d^2 \mathbf{k}_n}{4\pi^2} \times \tilde{G}_{\mu\nu}^{(n)*}(\mathbf{k}_1, \dots, \mathbf{k}_n; a) \tilde{h}(\mathbf{k}_1) \dots \tilde{h}(\mathbf{k}_n). \quad (\text{A6})$$

For profiles of small slopes, $\tilde{h}(\mathbf{k})$ is supported near zero, and then it is legitimate to Taylor-expand $\tilde{G}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n)$ in powers of the in-plane momenta $(\mathbf{k}_1, \dots, \mathbf{k}_n)$. Upon truncating the Taylor expansion to fourth order, and after going back to position space, we find for $U_{\mu\nu}$ the expression

n -sums in Eq. (A7) can be easily done, because by virtue of Eq. (A5), the A, B, C coefficients satisfy the relations

$$A_{\mu\nu}^{(n)}(a) = \frac{\partial^n G_{\mu\nu}^{(0)}}{\partial a^n}, \quad (\text{A11})$$

$$B_{\mu\nu|i_1 \dots i_p}^{(n)}(a) = \frac{\partial^{n-1} B_{\mu\nu|i_1 \dots i_p}^{(1)}(a)}{\partial a^{n-1}}, \quad (\text{A12})$$

and

$$C_{\mu\nu|ijkl}^{(n)}(a) = \frac{\partial^{n-2} C_{\mu\nu|ijkl}^{(2)}(a)}{\partial a^{n-2}}. \quad (\text{A13})$$

After we substitute the above relations into Eq. (A7), and recalling that $d = a + h(\mathbf{0})$, we obtain the desired result:

$$U_{\mu\nu} \simeq G_{\mu\nu}^{(0)}(d) - \frac{1}{2} B_{\mu\nu|ij}^{(1)*}(d) \partial_i \partial_j h(\mathbf{0}) + \frac{i}{3!} B_{\mu\nu|ijk}^{(1)*}(d) \partial_i \partial_j \partial_k h(\mathbf{0}) + \frac{1}{4!} B_{\mu\nu|ijkl}^{(1)*}(d) \partial_i \partial_j \partial_k \partial_l h(\mathbf{0}) + \frac{1}{8} C_{\mu\nu|ijkl}^{(2)*}(d) \partial_i \partial_j h(\mathbf{0}) \partial_k \partial_l h(\mathbf{0}). \quad (\text{A14})$$

We see that the resummed perturbative series involves the coefficients $B_{\mu\nu|i_1 \dots i_p}^{(1)}(d)$, $p = 2, 3, 4$ and $C_{\mu\nu|ijkl}^{(2)}(d)$, evaluated for $a = d$. As is apparent from Eqs. (A9) and (A10), these coefficients can be extracted, respectively, from the first- and second-order kernels $\tilde{G}_{\mu\nu}^{(1)}(\mathbf{k}; d)$ and $\tilde{G}_{\mu\nu}^{(2)}(\mathbf{k}_1, \mathbf{k}_2; d)$ by Taylor-expanding them for small momenta.

-
- [1] H. B. G. Casimir and D. Polder, *Phys. Rev.* **73**, 360 (1948).
- [2] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Rev. Mod. Phys.* **81**, 1827 (2009).
- [3] *Casimir Physics*, edited by D. A. R. Dalvit *et al.*, Lecture Notes in Physics Vol. 834 (Springer, New York, 2011).
- [4] T. A. Pasquini, M. Saba, G.-B. Jo, Y. Shin, W. Ketterle, D. E. Pritchard, T. A. Savas, and N. Mulders, *Phys. Rev. Lett.* **97**, 093201 (2006).
- [5] H. Oberst, D. Kouznetsov, K. Shimizu, J. I. Fujita, and F. Shimizu, *Phys. Rev. Lett.* **94**, 013203 (2005).
- [6] B. S. Zhao, S. A. Schulz, S. A. Meek, G. Meijer, and W. Scho Ilkopf, *Phys. Rev. A* **78**, 010902(R) (2008).
- [7] J. D. Perreault, A. D. Cronin, and T. A. Savas, *Phys. Rev. A* **71**, 053612 (2005); V. P. A. Lonij, W. F. Holmgren, and A. D. Cronin, *Phys. Rev. A* **80**, 062904 (2009).
- [8] B. Dobrich, M. DeKieviet, and H. Gies, *Phys. Rev. D* **78**, 125022 (2008).
- [9] A. M. Contreras-Reyes, R. Guerout, P. A. Maia Neto, D. A. R. Dalvit, A. Lambrecht, and S. Reynaud, *Phys. Rev. A* **82**, 052517 (2010).
- [10] V. B. Bezerra, E. R. Bezerra de Mello, G. L. Klimchitskaya, V. M. Mostepanenko, and A. A. Saharian, *Eur. Phys. J. C* **71**, 1614 (2011).
- [11] R. Messina, D. A. R. Dalvit, P. A. Maia Neto, A. Lambrecht, and S. Reynaud, *Phys. Rev. A* **80**, 022119 (2009).
- [12] B. V. Derjaguin and I. I. Abrikosova, *Sov. Phys. JETP* **3**, 819 (1957); B. V. Derjaguin, *Sci. Am.* **203**, 47 (1960).
- [13] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, *Phys. Rev. D* **84**, 105031 (2011).
- [14] G. Bimonte, T. Emig, R. L. Jaffe, and M. Kardar, *Europhys. Lett.* **97**, 50001 (2012).
- [15] G. Bimonte, T. Emig, and M. Kardar, *Appl. Phys. Lett.* **100**, 074110 (2012).
- [16] V. A. Golyk, M. Kruger, A. P. McCauley, and M. Kardar, *Europhys. Lett.* **101**, 34002 (2013).
- [17] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, *Phys. Rev. A* **88**, 062501 (2013).
- [18] A. Lambrecht, P. A. Maia Neto, and S. Reynaud, *New J. Phys.* **8**, 243 (2006).
- [19] T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, *Phys. Rev. Lett.* **99**, 170403 (2007).
- [20] We normalize the waves $|\mathbf{k}, Q\rangle$ as in Ref. [21]. Note, though, that the choice of normalization is irrelevant for the purpose of evaluating the trace in Eq. (1).
- [21] A. Voronovich, *Waves Random Media* **4**, 337 (1994).
- [22] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, *Phys. Rev. A* **89**, 062120 (2014).