Let $V$ and $W$ be two $\mathbb{F}$-vector spaces. Then there is a natural structure of vector space on $V \times W = \{(v, w) : v \in V, w \in W\}$, as follows
\[
(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2),
\]
\[
k(v, w) := (kv, kw),
\]
for any $v_1, v_2 \in V$, $w_1, w_2 \in W$, $k \in \mathbb{F}$. It is easy to check that the axioms in the definition of a vector space (definition 1.2.4.) hold.

$V$ can be viewed as the subspace $\{(v, 0) : v \in V\}$ of $V \times W$ and $W$ as the subspace $\{(0, w) : w \in W\}$.

Assume $V$ is $n$-dimensional, with a basis $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ and $W$ is $m$-dimensional, with a basis $\mathcal{B}_W = \{w_1, \ldots, w_m\}$. Then a basis for $V \times W$ is given by $\mathcal{B}_{V \times W} = \{(v_i, 0) : 1 \leq i \leq n\} \cup \{(0, w_j) : 1 \leq j \leq m\}$, which implies that the dimension of $V \times W$ is $m + n$. Let us check that $\mathcal{B}_{V \times W}$ is indeed a basis.

**linear independence:** Consider the linear relation $\sum_{i=1}^n a_i (v_i, 0) + \sum_{j=1}^m b_j (0, w_j) = (0, 0)$. This implies $(\sum_{i=1}^n a_i v_i, \sum_{j=1}^m b_j w_j) = (0, 0)$ and therefore $\sum_{i=1}^n a_i v_i = 0$ and $\sum_{j=1}^m b_j w_j = 0$. Since $\{v_1, \ldots, v_n\}$ in $V$, respectively $\{w_1, \ldots, w_m\}$ in $W$ are linearly independent, it follows that $a_i = 0$, for all $1 \leq i \leq n$, respectively $b_j = 0$, for all $1 \leq j \leq m$. This proves the linear independence.

**spanning set:** Let $(v, w)$ be an element in $V \times W$. Then $v \in V$ can be written as a linear combination $v = \sum_{i=1}^n v_i$ because $\{v_1, \ldots, v_n\}$ span $V$. Similarly $w$ can be written as $w = \sum_{j=1}^m w_j$. Then $(v, w) = \sum_{i=1}^n a_i (v_i, 0) + \sum_{j=1}^m b_j (0, w_j)$. This implies that $\mathcal{B}_{V \times W}$ is a spanning set for $V \times W$.

Also remark that if $V$ is a subset of some $\mathbb{F}^k$ and $W$ is a subspace of an $\mathbb{F}^p$, then $V \times W$ is a subspace of $\mathbb{F}^{k+p}$.

A homework exercise asked to show that if $V$ and $W$ are subspaces of the same vector space $U$, then $V + W = \{v + w : v \in V, w \in W\}$ and $V \cap W$ are also subspaces of the same vector space $U$. Note that $V \cup W$ is not a subspace in general. For example, if $V$ and $W$ are two lines containing the origin of $\mathbb{R}^3$ (so they are subspaces), $V \cup W$ is the union of the two lines, so it is not a subspace (there are vectors $v \in V$ and $w \in W$ such that $v + w \notin V \cup W$). In this example $V + W$ is the plane containing the two lines, and $V \cap W$ is just a point, the origin.

Let us show that if $V$ and $W$ are finite dimensional (of dimensions $n$, respectively $m$) then
\[
\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V + W).
\]
In the example above, this is $1 + 1 = 2 + 0$.

Since $V \cap W$ is a subspace of $V$ (or $W$), it is finite dimensional. Assume its dimension is $p \leq \min\{n, m\}$ and let $\{v_1, \ldots, v_p\}$ be a basis for $V \cap W$. The set $\{v_1, \ldots, v_p\}$ is linearly independent as a subset of $V$, so it can be extended to a basis of $V$, $\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n\}$. Similarly there is a basis
of \( W, \{v_1, \ldots, v_p, w_{p+1}, \ldots, w_m\} \). The claim would follow if we showed that the set \( \{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n, w_{p+1}, \ldots, w_m\} \) were a basis for \( V + W \).

**linearly independent:** consider the linear relation \( a_1v_1 + \cdots + a_pv_p + \ldots a_nv_n + b_{p+1}w_{p+1} + \ldots b_mw_m = 0 \). This implies that \( a_1v_1 + \ldots a_nv_n = -(b_{p+1}w_{p+1} + \ldots b_mw_m) \). Call this element \( x \). The left hand side implies that \( x \in V \) and the right hand side that \( x \in W \). So \( x \in V \cap W \) and therefore \( x \) can be expressed as a linear combination of the basis vectors of \( V \cap W \): \( x = c_1v_1 + \cdots + c_pv_p \). Then \( a_1v_1 + \cdots + a_pv_p + a_{p+1}v_{p+1} + \ldots a_nv_n = c_1v_1 + \cdots + c_pv_p \), and so \( (a_1-c_1)v_1 + \cdots + (a_p-c_p)v_p + a_{p+1}c_{p+1} + \ldots a nc_n = 0 \).

Since \( \{v_1, \ldots, v_n\} \) is a basis for \( V \), and in particular linearly independent, it follows that \( a_1 = c_1, \ldots, a_p = c_p \) and \( a_{p+1} = \cdots = a_n = 0 \).

Going back to the relation we started with and using \( a_{p+1} = \cdots = a_n = 0 \), it implies that \( a_1v_1 + \cdots + a_pv_p + b_{p+1}w_{p+1} + \ldots b_mw_m = 0 \). Since \( \{v_1, \ldots, v_p, w_{p+1}, \ldots, w_m\} \) is a basis of \( W \), it follows that \( a_1 = \cdots = a_p = 0 \) and \( b_{p+1} = \cdots = b_m = 0 \). In conclusion all scalars in the linear relation must be zero, and therefore \( \{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n, w_{p+1}, \ldots, w_m\} \) is linearly independent.

**spanning set:** exercise.