Permutations. November 5, 2005

The purpose of this note is to summarize some properties of the permutations. Recall that a permutation of $n$ is a string $\sigma = (\sigma_1, \ldots, \sigma_n)$, where $\sigma_i \in \{1, 2, \ldots, n\}$ and $\sigma_i \neq \sigma_j$ is $i \neq j$. A better definition is to say that a permutation $\sigma$ is a bijective function

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}, \quad \sigma(i) = \sigma_i.$$  

Denote the set of all permutations of $n$ by $S_n$. A distinguished element of $S_n$ is the identity permutation, $\sigma = (1, 2, \ldots, n)$.

If $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$, we say that $(\sigma_i, \sigma_j)$ is an inversion if $\sigma_i > \sigma_j$ and $i < j$. Denote by $\ell(\sigma)$ the number of inversions in the permutation $\sigma$. Then the sign of $\sigma$ is $sg(\sigma) = (-1)^{\ell(\sigma)}$, and we say that $\sigma$ is an even (odd) permutation if $sg(\sigma) = 1$ (respectively, $-1$).

**Example.** In $S_6$, consider $\sigma = (5, 6, 4, 2, 3, 1)$. Then $\ell(\sigma) = 10$, and $sg(\sigma) = 1$.

We record some properties of permutations.

1. It is possible to define multiplication of two permutations. This is clear if we think of permutations as functions; then the multiplication is the composition of functions. In the other notation, if $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\tau = (\tau_1, \ldots, \tau_n)$ are two permutation, define the product $\sigma \cdot \tau$ to be the permutation $\sigma \cdot \tau = (\sigma_{\tau_1}, \sigma_{\tau_2}, \ldots, \sigma_{\tau_n})$. For example, if $\sigma = (4, 2, 3, 1)$ and $\tau = (3, 1, 2, 4)$, then $\sigma \cdot \tau = (3, 4, 2, 1)$, and $\tau \cdot \sigma = (4, 1, 2, 3)$. Note that multiplication of permutations is not commutative, but it is associative (because the composition of functions is).

2. Every permutation has an inverse. If $\sigma \in S_n$ is a permutation, then there exists a permutation $\sigma^{-1} \in S_n$ such that $\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = 1_n$, where by $1_n$ we denote the identity permutation in $S_n$. Again, this statement is clear if we think of permutations as bijective functions. For example, in $S_4$, the inverse of the permutation $\sigma = (4, 2, 3, 1)$ is $\sigma^{-1} = (4, 2, 3, 1)$ (so $\sigma \cdot \sigma = 1_4$) and the inverse of $\tau = (3, 1, 2, 4)$ is $\tau^{-1} = (2, 3, 1, 4)$. In general, the inverse of a permutation $\sigma$ has $i$ on the $\sigma_i$ position.

3. From 1 and 2, we see that the set of permutations $S_n$ has an identity element, a multiplication of elements and every element has an inverse. In general, a set $G$, together with an operation $\ast$, which satisfies the axioms:

   (1) $\ast$ is associative;
   (2) there exists a unit element $e \in G$, such that $x \ast e = e \ast x = x$, for all $x \in G$.
   (3) if $x \in G$, there exists $x^{-1} \in G$ (the inverse) such that $x \ast x^{-1} = x^{-1} \ast x = e$

is called a group. So $S_n$ is the group of permutations of $n$ (or the symmetric group).

4. One homework exercise asked you to identify a connection between the permutation of $n$ and $n \times n$ permutation matrices, i.e. matrices which have a 1 on each row and column and zeros everywhere else. If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a permutation of $n$, define the permutation matrix $P_\sigma$ to be the matrix such that

$$P_\sigma(\sigma_j, j) = 1 \quad \text{and} \quad P_\sigma(i, j) = 0, \text{ if } j \neq \sigma_i.$$  

That is, in the column $j$, we put one on the $\sigma_j$ position, and zeros elsewhere. Note that, a priori, one could define the permutation matrix by applying the procedure
to rows instead of columns. But that definition will not behave well with respect to multiplication.

Verify that \( sg(\sigma) = \text{det}(P_\sigma) \) and \( P_{\sigma \cdot \tau} = P_\sigma \cdot P_\tau \). From this, it also follows that \( sg(\sigma \cdot \tau) = sg(\sigma)sg(\tau) \) and that \( sg(\sigma^{-1}) = sg(\sigma) \). In particular, the product of two even permutations is an even permutation, and the inverse of an even permutation is also even. So the even permutations form a subgroup of \( S_n \) (note that the odd permutations don’t).

5. An important class of permutations is the transpositions. A permutation \( \sigma \) is called said to be a transposition \( (ij) \), if \( \sigma_i = j, \sigma_j = i \) and \( \sigma_k = k \), for all \( k \neq i, j \). Note that each transposition is its own inverse. Their importance comes from the fact that every permutation can be written as a product of transpositions. This fact is not hard to prove by induction. The idea is that, given a permutation \( \sigma \) of \( n \), the number \( n \) appears in some position in \( \sigma \), say \( \sigma_j = n \). Then by multiplying \( \sigma \) by transpositions we can move \( n \) to the \( n \)th position and regard the resulting permutation as a permutation of \( n - 1 \). Then apply the induction. It would be a good exercise to try to write down this idea into a formal proof.

For example \( \sigma = (4, 2, 3, 1) = (14) \) and \( \tau = (3, 1, 2, 4) = (23) \cdot (12) \).

There are many other beautiful and important properties of permutations (for example, the cycle decomposition of a permutation), but we will leave them for some other time.