Linear operators of $\mathbb{R}^2$, November 7, 2005

The purpose of this note is to illustrate the notion of linear operators, by looking at some examples of linear transformations of the plane, $T : \mathbb{R}^2 \to \mathbb{R}^2$. We will consider the standard basis of $\mathbb{R}^2$ and all matrices associated to a linear transformation will be written with respect to this basis.

1. **Rotations**: Let $R_\theta$ be the linear transformation given by rotation counterclockwise by the angle $\theta$. The matrix associated to $R_\theta$ is

$$[R_\theta] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. $$

This is clearly an invertible transformation and the inverse is the rotation by $\theta$ clockwise, or $R_{-\theta}$. The composition of two rotations $R_{\theta_1} \circ R_{\theta_2}$ is again a rotation $R_{\theta_1 + \theta_2}$. Note that the composition of rotations is commutative and that the rotations form a group (the unit element is the identity operator, or the rotation by angle 0). This group is called the special orthogonal group of $\mathbb{R}^2$, and it is denoted by $SO(2)$.

2. **Dilations**: For $a, b > 0$, let $D_{a,b}$ denote the linear transformation which takes a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ into the vector $\begin{pmatrix} ax \\ by \end{pmatrix}$. The matrix associated is diagonal, $[D_{a,b}] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Since $a \neq 0$ and $b \neq 0$, $D_{a,b}$ is invertible and the inverse is the dilation $D_{a^{-1},b^{-1}}$.

3. **Shear transformations**: typical examples are the shear transformations parallel with the $x$-axis, that is given by matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Such a transformation moves the tip of a vector $v$ parallel with the $x$-axis (in general parallel to a line) and fixes the vectors in the $x$-axis. They are invertible transformations and the inverses are given by the opposite shear transformation, e.g. in this case $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$.

4. **Projections**: having fixed a line $L$, the map $P_L$ defined by projecting a vector $v$ on the line $L$ is a linear transformation. Denote the projection on the $x$-axis by $P_x$ and similarly, the projection onto the $y$-axis by $P_y$. The projections are not invertible. For example, $[P_x] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $[P_y] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

5. **Reflections**: the map $S_L$ defined by taking the symmetric of a vector $v$ about a fixed line $L$ is a linear transformation. They are invertible transformations, in fact each reflection is its own inverse. For example, $[S_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We should remark that the translations, i.e. $T(v) = v + v_0$, for some fixed $v_0$, are not linear transformations.

**Proposition.** Any nonzero linear transformation of $\mathbb{R}^2$ can be obtained as a composition of linear transformations of types 1-5.

**Proof.** The heart of the proof is the following decomposition of $2 \times 2$ invertible matrices (it’s actually part of a more general result): any invertible $2 \times 2$ matrix $A$
can be written as a product

\[
A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},
\]

for some \( \theta, a, b, n \). If \( A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) has determinant one, set \( \theta = \arctan\left(\frac{x}{z}\right) \), \( a = \sqrt{x^2 + z^2} \), and \( n = \frac{xv + yz}{x^2 + z^2} \) (\( x \) and \( z \) cannot both be zero, since \( A \) is invertible). Verify by a direct calculation that the decomposition above (with \( b = a^{-1} \)) holds.

The case of general invertible \( A \) follows easily, by multiplying the diagonal matrix with \( \det(A) \). We hope to treat this decomposition in a more conceptual way later in the course.

In this decomposition, the first matrix corresponds to a rotation, the diagonal matrix to a dilation \( D_{a,b} \) or a composition of a dilation with a reflection (if \( a \) and \( b \) are not both positive), and the third matrix corresponds to a shear transformation. Therefore the invertible linear transformation with matrix \( A \) can be realized as a composition of a rotation, a reflection, a dilation, and a shear transformation.

Now, consider a general (maybe noninvertible) linear transformation with matrix \( B \neq 0 \). Then \( B = UR \), where \( U \) is an invertible matrix and \( R \) is the row-reduced echelon form. We know the transformation corresponding to \( U \) can be decomposed as above, so it remains to analyze the transformations corresponding to all reduced echelon forms. If \( R \) is the identity (the case when \( B \) is invertible), there is nothing more left to do. Otherwise, \( R \) can only be of three forms \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix} \), with \( s \neq 0 \), or \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). The first form is clearly the projection \( P_x \). The second form can be decomposed as

\[
\begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.
\]

It means that the corresponding transformation is a composition between a projection and a shear transformation.

The third form can be decomposed as

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

so it is the composition of \( P_x \) with the reflection about the line \( x = y \).

This concludes the proof. \( \square \)