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Adaptive Control of Scalar Plants in the Presence of Unmodeled Dynamics ^{*}

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Abstract: Robust adaptive control of scalar plants in the presence of unmodeled dynamics is established in this paper. It is shown that implementation of a projection algorithm with standard adaptive control of a scalar plant ensures global boundedness of the overall adaptive system for a class of unmodeled dynamics.

Keywords: Adaptive control, Robustness, Stability Criteria, Lyapunov Stability, Benchmark examples

1. INTRODUCTION

Analysis of adaptive systems in the presence of unmodeled dynamics is a mature research discipline that has been studied extensively over the past three decades. In most physical systems, the presence of non-parametric uncertainties is unavoidable. Hence, it is crucial to investigate and understand the stability and robustness properties of adaptive control systems in the presence of non-parametric uncertainties, such as unmodeled dynamics. There is a vast literature investigating the system design of this important problem, spanning from Fomin V. (1981) to more recent results as in Bobtsov and Nikolaev (2009).

Of the written works, perhaps the most well-known example of adaptive systems in the presence of unmodeled dynamics is by Rohrs et al. (1985); Instability of the adaptive control system was observed due to unbounded parameter drift exciting the high frequency unmodeled dynamics present. Following this counterexample, several robust adaptive control solutions were suggested in the '80s and '90s (see, for example, Narendra and Annaswamy (1989) and Ioannou and Sun (1996)), including specific responses to the counterexample (see, for example, Astrom (1983), Kang et al. (1990), Riedle and Kokotovic (1985), Ioannou and Tsakalis (1986), and Naik et al. (1992)). Most of these were qualitative answers to the observed phenomena, or methods based on local stability, and often involved properties of persistent excitation of the reference input. In this paper, we show that for a class of unmodeled dynamics, including the one in Rohrs et al. (1985), adaptive control of a scalar plant with global boundedness can be established for any reference input. Recent results indicate that extensions to higher-order plants where states are accessible are possible.

2. THE PROBLEM STATEMENT

The problem we address in this paper is the adaptive control of a first-order plant

$$\dot{x}_p(t) = a_p x_p(t) + v(t) \quad (1)$$

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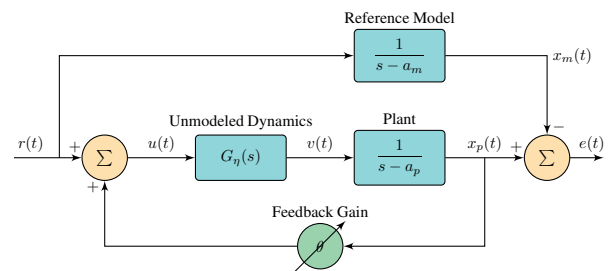


Fig. 1. Adaptive control in the presence of unmodeled dynamics

where a_p is an unknown parameter. It is assumed that $|a_p| \leq \bar{a}$, where \bar{a} is a known positive constant. The unmodeled dynamics are unknown and defined as

$$\begin{aligned} \dot{x}_\eta(t) &= A_\eta x_\eta(t) + b_\eta u(t) \\ v(t) &= c_\eta^T x_\eta(t) \end{aligned} \quad (2)$$

where $A_\eta \in \mathbb{R}^{n \times n}$ is Hurwitz with

$$G_\eta(s) \triangleq c_\eta^T (sI_{n \times n} - A_\eta)^{-1} b_\eta. \quad (3)$$

$x_\eta(t)$ is the state vector, and $u(t)$ is the control input. The goal is to design the control input such that $x_p(t)$ follows $x_m(t)$ which is specified by the reference model

$$\dot{x}_m(t) = a_m x_m(t) + r(t) \quad (4)$$

where $a_m < 0$, and $r(t)$ is the reference input. The adaptive controller we propose is a standard adaptive control input given by (see figure 1)

$$u(t) = \theta(t) x_p(t) + r(t) \quad (5)$$

where the parameter $\theta(t)$ is updated using a projection algorithm given by

$$\dot{\theta}(t) = \gamma \text{Proj}(\theta(t), -x_p(t)e(t)), \quad \gamma > 0 \quad (6)$$

where

$$e(t) = x_p(t) - x_m(t) \quad (7)$$

$$\text{Proj}(\theta, y) = \begin{cases} \frac{\theta^2_{max} - \theta^2}{\theta^2_{max} - \theta^2_{min}} y & [\theta \in \Omega_A, y\theta > 0] \\ y & \text{otherwise} \end{cases} \quad (8)$$

$$\begin{aligned}\Omega_0 &= \{\theta \in \mathbb{R}^1 \mid -\theta'_{max} \leq \theta \leq \theta'_{max}\} \\ \Omega_1 &= \{\theta \in \mathbb{R}^1 \mid -\theta_{max} \leq \theta \leq \theta_{max}\} \\ \Omega_A &= \Omega_1 \setminus \Omega_0\end{aligned}\quad (9)$$

with positive constants θ'_{max} and θ_{max} given by

$$\theta'_{max} > \bar{a} + |a_m| \quad (10)$$

$$\theta_{max} = \theta'_{max} + \varepsilon_0, \quad \varepsilon_0 > 0. \quad (11)$$

Lemma 1. Consider the Adaptive Law in (6) with Projection Algorithm in (8) to (11). Then,

$$\|\theta(t_a)\| \leq \theta_{max} \implies \|\theta(t)\| \leq \theta_{max}, \quad \forall t \geq t_a. \quad (12)$$

Hence, the projection algorithm guarantees the boundedness of the parameter $\theta(t)$ independent of the system dynamics. We refer the reader to Lavretsky and Gibson (2011) for the proof of Lemma 1.

3. CHOICE OF PROJECTION PARAMETERS

The projection algorithm in (8) is specified by two parameters θ'_{max} and θ_{max} . Equation (10) provides the condition for θ'_{max} . To determine ε_0 in (11), the following discussions are needed:

We consider the linear time-invariant system specified by (1), (2), and (5), with the parameter $\theta(t)$ fixed as

$$\theta(t) = -\theta_{max}, \quad \forall t \geq t_a. \quad (13)$$

The closed loop transfer function from $r(t)$ to $x_p(t)$ is given by

$$G_c(s) = \frac{p_\eta(s)}{q_c(s)} \quad (14)$$

where $G_c(s)$ is defined using $G_\eta(s)$ in (3) as

$$G_\eta(s) \triangleq \frac{p_\eta(s)}{q_\eta(s)} \quad (15)$$

$$q_c(s) = q_\eta(s)(s - a_p) + \theta_{max}p_\eta(s). \quad (16)$$

From (10) and (11), it follows that

$$a_p - \theta_{max} < 0, \quad \forall |a_p| \leq \bar{a}. \quad (17)$$

Therefore it follows that there exists a class of unmodeled dynamics (c_η, A_η, b_η) such that $q_c(s)$ has roots in \mathbb{C}^- , the left-half of the complex plane. It is this class that is of interest in this paper.

Definition 1. The triple (c_η, A_η, b_η) is said to belong to $S_\eta(\bar{a}, \theta_{max})$ if $p_\eta(s)$ and $q_\eta(s)$ in (15) are such that the roots of $q_c(s)$ in (16) lie in \mathbb{C}^- for all $|a_p| \leq \bar{a}$.

Let's demonstrate $S_\eta(\bar{a}, \theta_{max})$ with an example. Consider the class of unmodeled dynamics of the form

$$G_\eta(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (18)$$

where $\zeta > 0$ and $\omega_n > 0$. From (14), (16), and (18), the closed-loop dynamics from r to x_p is given by

$$G_c(s) = \frac{\omega_n^2}{q_c(s)} \quad (19)$$

where

$$\begin{aligned}q_c(s) &= s^3 + a_1 s^2 + a_2 s + a_3 \\ a_1 &= (2\zeta\omega_n - a_p) \\ a_2 &= (\omega_n^2 - 2a_p\zeta\omega_n) \\ a_3 &= -a_p\omega_n^2 + \theta_{max}\omega_n^2\end{aligned}\quad (20)$$

For the roots of $q_c(s)$ in (20) to lie in \mathbb{C}^- , the following conditions are necessary and sufficient for all $|a_p| \leq \bar{a}$:

$$(A-i) \quad a_p < \min(2\zeta\omega_n, \frac{\omega_n}{2\zeta})$$

$$(A-ii) \quad \theta_{max} > a_p$$

$$(A-iii) \quad \theta_{max} < \left(-4a_p\zeta^2 + \frac{2\zeta a_p^2}{\omega_n} + 2\zeta\omega_n \right)$$

If

$$a_p < \theta_{max} < a_p + \bar{\theta}^* \quad (21)$$

where

$$\bar{\theta}^* = (2\zeta\omega_n - a_p)\left(1 - \frac{2\zeta a_p}{\omega_n}\right) \quad (22)$$

then conditions (A-ii) and (A-iii) hold.

Hence, any class of unmodeled dynamics (c_η, A_η, b_η) in (18) satisfying condition (A-i) belongs to $S_\eta(\bar{a}, \theta_{max})$. It can be easily shown that the unmodeled dynamics and the plant discussed in the infamous Rohrs counterexample, see Rohrs et al. (1985), satisfies conditions (A-i) to (A-iii) above for some θ_{max} .

We now discuss the choice of ε_0 . Consider the class of unmodeled dynamics $S_\eta(\bar{a}, \theta_{max})$ in Definition 1. Since the closed loop system specified by (1), (2), (5), and (13) is stable, it follows that there exists a Lyapunov function

$$V = \bar{x}^T P \bar{x} \quad (23)$$

with a time derivative

$$\dot{V} = -\bar{x}^T Q \bar{x} \quad (24)$$

where $\bar{x} = [x_p \quad x_\eta^T]^T$. P is the solution to the Lyapunov equation

$$\bar{A}^T P + P \bar{A} = -Q < 0 \quad (25)$$

where

$$\bar{A} = \begin{bmatrix} a_p & c_\eta^T \\ -b_\eta\theta_{max} & A_\eta \end{bmatrix} \quad (26)$$

since \bar{A} is Hurwitz. The latter is true since θ_{max} satisfies (21).

We define two sets $\Omega_u \subset \Omega_A$ and $\Omega_l \subset \Omega_A$ as

$$\Omega_u = \{\theta \in \mathbb{R}^1 \mid -\theta_{max} + \xi_0 \leq \theta < -\theta'_{max}\} \quad (27)$$

$$\Omega_l = \{\theta \in \mathbb{R}^1 \mid -\theta_{max} \leq \theta \leq -\theta_{max} + \xi_0\} \quad (28)$$

where

$$\xi_0 = c\varepsilon_0, \quad c \in (0, 1). \quad (29)$$

We now consider the linear time-varying system specified by (1), (2), and (5), with $\theta(t) \in \Omega_u \cup \Omega_l$. It follows from (11) and (12) that

$$\theta(t) = -\theta_{max} + \varepsilon(t), \quad \forall \theta(t) \in \Omega_u \cup \Omega_l \quad (30)$$

$$\theta(t) = -\theta_{max} + \xi(t), \quad \forall \theta(t) \in \Omega_l \quad (31)$$

where

$$\varepsilon(t) \in [0, \varepsilon_0], \quad \xi(t) \in [0, \xi_0]. \quad (32)$$

Therefore, the closed-loop system is given by

$$\dot{\bar{x}} = \bar{A}\bar{x} + A_\xi(t)\bar{x} + \bar{b}r, \quad \forall \theta(t) \in \Omega_l \quad (33)$$

where

$$A_\xi(t) = \begin{bmatrix} 0 & 0 \\ b_\eta\xi(t) & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ b_\eta \end{bmatrix}. \quad (34)$$

If we choose $V = -\bar{x}^T Q \bar{x}$ with P as in (25), we obtain

$$\dot{V} \leq -\lambda_{Q_{min}} \|\bar{x}\|^2 + 2\lambda_{P_{max}} k \xi_0 \|\bar{x}\|^2 + 2\lambda_{P_{max}} \|\bar{b}\| r_{max} \|\bar{x}\| \quad (35)$$

where

$$\begin{aligned}\lambda_{Q_{min}} &\triangleq \min_i |\Re(\lambda_i(Q))| \\ \lambda_{P_{max}} &\triangleq \max_i |\Re(\lambda_i(P))|\end{aligned}\quad (36)$$

$$\|\bar{b}\| \leq k, \quad r_{max} = \max_{t \geq t_a} |r(t)|. \quad (37)$$

That is,

$$\dot{V} < 0 \quad \text{if } \|\bar{x}\| > x_0 \quad (38)$$

where

$$x_0 = \frac{2\lambda_{P_{\max}} \|\bar{b}\| r_{\max}}{\lambda} \quad (39)$$

$$\bar{\lambda} = \lambda_{Q_{\min}} - 2\lambda_{P_{\max}} k \xi_0. \quad (40)$$

In summary, the closed-loop system has bounded solutions for all $\theta(t) \in \Omega_l$ with $\|x(t)\| \leq x_0$ if (c_η, A_η, b_η) is such that

- (B-i) $q_c(s)$ has roots in \mathbb{C}^- for all $|a_p| \leq \bar{a}$, and
- (B-ii) $\xi_0 < \varepsilon_0$, where
- (B-iii) $\xi_0 < \frac{\lambda_{Q_{\min}}}{2k\lambda_{P_{\max}}}$

We introduce the following definition:

Definition 2. The triple (c_η, A_η, b_η) is said to belong to $S_\eta(\bar{a}, \theta_{\max}, \xi_0)$ if conditions (B-i), (B-ii), and (B-iii) above are satisfied.

4. MAIN RESULT

Theorem 1. Let $z(t) = [e(t) \theta(t)]^T$. The closed-loop adaptive system given by (1)-(11) has globally bounded solutions for all $\theta(t_a) \in \Omega_1$ if $(c_\eta, A_\eta, b_\eta) \in S_\eta(\bar{a}, \theta_{\max}, \xi_0)$.

Definition 3. We define the region A and the boundary regions \bar{B} and \underline{B} as follows

$$\begin{aligned} \bar{B} &= \{z \in \mathbb{R}^2 \mid \theta'_{\max} < \theta \leq \theta_{\max}\} \\ A &= \{z \in \mathbb{R}^2 \mid \theta \in \Omega_0\} \\ \underline{B} &= \{z \in \mathbb{R}^2 \mid -\theta_{\max} \leq \theta < -\theta'_{\max}\} \end{aligned} \quad (41)$$

Definition 4. We divide the boundary region \underline{B} into two regions as follows:

$$\begin{aligned} \underline{B}_U &= \{z \in \mathbb{R}^2 \mid \theta \in \Omega_u\} \\ \underline{B}_L &= \{z \in \mathbb{R}^2 \mid \theta \in \Omega_l\} \end{aligned} \quad (42)$$

with $\underline{B} = \underline{B}_U \cup \underline{B}_L$.

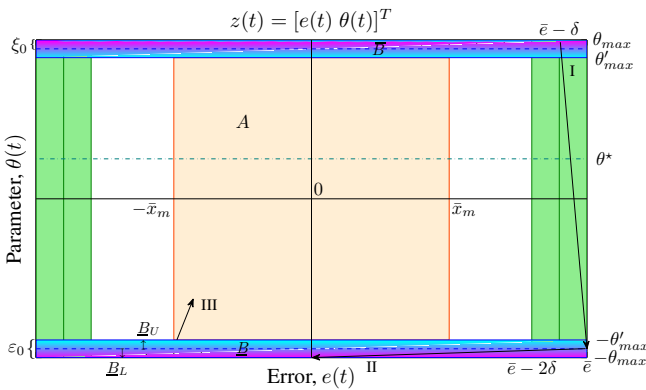


Fig. 2. Definition of regions in (41) and (42), and phases I-III.

Proofs of all propositions are omitted due to page limitations and can be found in Hussain et al. (2013).

Proof. The closed-loop adaptive system has error dynamics in (7) equivalent to¹

$$\dot{e} = a_m e + \tilde{\theta} x_p + \eta \quad (43)$$

where

$$\tilde{\theta} = \theta - \theta^*, \quad \theta^* = a_m - a_p, \quad \eta = v - u. \quad (44)$$

¹ For ease of exposition, we suppress the argument "t" in what follows.

By combining the adaptive law in (6) and (8), and boundary region definitions in (41), we obtain

$$\dot{\theta} = \begin{cases} -\frac{\theta_{\max}^2 - \theta^2}{\theta_{\max}^2 - \theta_{\max}^2} \gamma e x_p & \text{if } z \in (\underline{B} \cup \bar{B}), -e x_p \theta > 0 \\ -\gamma e x_p & \text{otherwise} \end{cases} \quad (45)$$

From Lemma 1, it follows that Theorem 1 is proved if the global boundedness of $e(t)$ is demonstrated. This is achieved in four phases by studying the trajectory of $z(t)$ for all $t \geq t_0$. This methodology was originally proposed in Matsutani et al. (2012) for adaptive control in the presence of time delay.

We begin with suitably chosen finite constants \bar{e} and δ such that $\bar{e} - \delta > 0$. The trajectory then has only two possibilities either (i) $|e(t)| < \bar{e} - \delta$ for all $t \geq t_a$, or (ii) there exists a time t_a at which $|e(t_a)| = \bar{e} - \delta$. The global boundedness of $e(t)$ is immediate in case (i). We therefore assume there exists a t_a where case (ii) holds.

- I) *Entering the Boundary Region:* We start with $|e(t_a)| = \bar{e} - \delta$. We then show that the trajectory enters the boundary region \underline{B} at $t_b \in (t_a, t_a + \Delta T_B)$, and \underline{B}_L at $t_c > t_b$ where ΔT_B and t_c are finite.
- II) *In the Boundary Region, \underline{B}_L :* When the trajectory enters \underline{B} , the parameter is in the boundary of the projection algorithm; e is shown to be bounded in \underline{B}_L by making use of the stability property of the underlying linear time-varying system. For $t > t_c$, the trajectory has only two possibilities: either (i) z stays in \underline{B}_L for all $t \geq t_c$, or (ii) z reenters \underline{B}_U at some $t_d > t_c$ where $|e(t_d)| \leq \bar{x}_m$.
- III) *In the Boundary Region, \underline{B}_U :* For $t > t_d$, the trajectory has three possibilities: either (i) z reenters A at $t = t_e$, (ii) z stays in \underline{B}_U for all $t \geq t_d$, or (iii) z reenters \underline{B}_L at $t_f \in (t_d, t_d + \Delta T_{\underline{B}_L})$ where $\Delta T_{\underline{B}_L}$ is finite.
- IV) *Return to Phase I or Phase II:* If case (i) from Phase III holds, then the trajectory has only two possibilities: either $|e(t)| < \bar{e} - \delta, \forall t > t_e$ which proves Theorem 1, or there exists a $t_g > t_e$ such that $|e(t_g)| = \bar{e} - \delta$ in which case the conditions of Proposition 1 are satisfied with t_a replaced by t_g , and Phases I through III are repeated for $t \geq t_g$. If case (ii) from Phase III holds, then the boundedness of e is established for all $t \geq t_d$. If case (iii) holds, then Phases II and III are repeated for $t \geq t_f$. In all cases, e remains bounded throughout.

4.1 Phase I: Entering the Boundary Region

We start with $|e(t_a)| = \bar{e} - \delta$. From (44), it is easy to see that

$$|\eta| \leq (k_\eta + 1)\theta_{\max}(|e| + \bar{x}_m) + (k_\eta + 1)r_{\max} \quad (46)$$

where $k_\eta = \|G_\eta(s)\|$ and

$$\bar{x}_m = \max_{t \geq t_a} |x_m(t)|. \quad (47)$$

We define \bar{e} as

$$\bar{e} = \max\{e_0, e_1\} \quad (48)$$

where

$$e_0 = |x_p(t_a)| + \bar{x}_m + 2\delta \quad (49)$$

$$e_1 = \frac{1}{2} \left(\bar{c} b_0 + \sqrt{\bar{c}^2 b_0^2 + 4\bar{c} b_1} \right) \quad (50)$$

with b_0 and b_1 defined in (53) and (54), $\delta \in (0, \bar{x}_m)$, $\alpha \in (0, \varepsilon_0]$, c in (29), and

$$\bar{c} = \frac{2\theta'_{max} + \alpha + \frac{\varepsilon_0}{c}}{\delta\gamma}. \quad (51)$$

Phase I is completed by proving the following Proposition:

Proposition 1. Let $z(t_a) \in A$ with $|e(t_a)| = \bar{e} - \delta$ where \bar{e} is given in (48) and $\delta \in (0, \bar{x}_m)$. Then

- (i) $|e(t)| \leq \bar{e}, \quad \forall t \in [t_a, t_a + \Delta T]$
- (ii) $z(t_c) \in \underline{B}_L$ for some $t_c \in (t_a, t_a + \Delta T)$

where

$$\Delta T = \frac{\delta}{b_0\bar{e} + b_1} \quad (52)$$

$$b_0 = |a_m| + (k_\eta + 2)\theta_{max} + |\theta^*| \quad (53)$$

$$b_1 = ((k_\eta + 2)\theta_{max} + |\theta^*|)\bar{x}_m + (k_\eta + 2)r_{max} \quad (54)$$

4.2 Phase II: In the Boundary Region, \underline{B}_L

When the trajectory enters \underline{B}_L , the parameter is in the boundary of the projection algorithm with thickness ξ_0 ; $e(t)$ is shown to be bounded by making use of the underlying linear time-varying system in (33) and (34).

Let $z(t) \in \underline{B}_L$ for $t \in [t_c, t_d)$. That is, $\theta(t) = -\theta_{max} + \xi(t)$ for $t \in [t_c, t_d)$ with $\xi(t)$ satisfying (32) and (29). Since $(c_\eta, A_\eta, b_\eta) \in S_\eta(\bar{a}, \theta_{max}, \xi_0)$, from (38), it follows that

$$\|\bar{x}(t)\| \leq x_0, \quad \forall t \in T_{\underline{B}_L} \quad (55)$$

where $T_{\underline{B}_L}$ is defined as

$$T_{\underline{B}_L} : \{t \mid z(t) \in \underline{B}_L\}. \quad (56)$$

Since $|e(t)| \leq |x_p(t)| + \bar{x}_m$ for all $t \in T_{\underline{B}_L}$ and $\bar{x} = [x_p \ x_\eta^T]^T$, this implies

$$|e(t)| \leq \bar{e}_2, \quad \forall t \in (t_c, t_d) \quad (57)$$

where

$$\bar{e}_2 = x_0 + \bar{x}_m \quad (58)$$

which proves boundedness of e in \underline{B}_L .

We have so far shown that if the trajectory begins in A at $t = t_a$, it will enter the region \underline{B}_L at $t = t_c$, where $t_c < t_a + \Delta T$, and ΔT is finite. For $t > t_c$, there are only two possibilities either (i) z stays in \underline{B}_L for all $t > t_c$, or (ii) z reenters \underline{B}_U at $t = t_d$ for some $t_d > t_c$. If (i) holds, it implies that (57) holds with $t_d = \infty$, proving Theorem 1. The following Proposition addresses case (ii):

Proposition 2. Let $z(t) \in \underline{B}_L$ for $t \in [t_c, t_d)$ and $z(t_d) \in \underline{B}_U$ for some $t_d > t_c$. Then

$$|e(t_d)| \leq \bar{x}_m \quad (59)$$

4.3 Phase III: In the Boundary Region, \underline{B}_U

The boundedness of e has been established thus far for all $t \in [t_a, t_d]$. For $t > t_d$, there are three cases to consider: either (i) z reenters A at $t = t_e$ for some $t_e > t_d$, (ii) z remains in \underline{B}_U for all $t \geq t_d$, or (iii) z reenters \underline{B}_L at $t_f \in (t_d, t_d + \Delta T_{\underline{B}_L})$ where $\Delta T_{\underline{B}_L}$ is finite.

We address case (i) in the following Proposition.

Proposition 3. Let $z(t) \in \underline{B}_U$ for $t \in [t_d, t_e)$ and $z(t_e) \in A$ for some $t_e > t_d$. Then

$$|e(t)| < \bar{x}_m, \quad \forall t \in (t_d, t_e] \quad (60)$$

We now address case (ii) and (iii).

We consider suitably chosen finite constants \bar{e}_3 and δ such that $\bar{e}_3 - \delta > 0$, and

$$\bar{e}_3 = \max\{e_2, e_3\} \quad (61)$$

where

$$e_2 = 2\bar{x}_m + 2\delta \quad (62)$$

$$e_3 = \frac{1}{2} \left(\bar{c}_2 b_0 + \sqrt{\bar{c}_2^2 b_0^2 + 4\bar{c}_2 b_1} \right) \quad (63)$$

and

$$\bar{c}_2 = \frac{(1-c)\varepsilon_0}{\delta\gamma c}. \quad (64)$$

From (59) and the definition of \bar{e}_3 , it follows that

$$|e(t_d)| < \bar{e}_3 - \delta. \quad (65)$$

If $e(t)$ grows without bound, it implies that there exists $t'_d > t_c$ such that

$$|e(t'_d)| = \bar{e}_3 - \delta. \quad (66)$$

Hence,

$$|e(t)| < \bar{e}_3 - \delta, \quad \forall t \in [t_d, t'_d). \quad (67)$$

We show below that if such a t'_d exists, then $z(t)$ must enter \underline{B}_L at $t = t_f$, for some finite $t_f > t'_d$.

Proposition 4. Let $z(t) \in \underline{B}_U$ for all $t \in [t_d, t_f)$, and $\exists t'_d \in (t_d, t_f)$ such that $|e(t'_d)| = \bar{e}_3 - \delta$ where \bar{e}_3 is given in (61) and $\delta \in (0, \bar{x}_m)$. Then

- (i) $|e(t)| \leq \bar{e}_3, \quad \forall t \in [t'_d, t'_d + \Delta T']$
- (ii) $z(t_f) \in \underline{B}_L$ for some $t_f \in (t'_d, t'_d + \Delta T')$

where

$$\Delta T' = \frac{\delta}{b_0\bar{e}_3 + b_1} \quad (68)$$

Proof. We note that Proposition (4) is identical to Proposition 1 with t_a replaced by t'_d , \bar{e} replaced with \bar{e}_3 , and $z(t'_d) \in \underline{B}_U$ which implies $\Delta T_B = 0$. Using an identical procedure, we can prove both Proposition 4(i) and Proposition 4(ii).

We note that if case (ii) holds, it implies that (67) holds for $t'_d = \infty$, which implies that $e(t)$ is globally bounded.

In summary, in Phase III, we conclude that if z enters \underline{B}_U at $t = t_d$,

- (i) z enters A at $t = t_e$ with $|e(t)| < \bar{x}_m$ for all $t \in [t_d, t_e]$,
- (ii) z remains in \underline{B}_U for $t \geq t_d$ with $|e(t)| < \bar{e}_3 - \delta$ for all $t \geq t_d$, or
- (iii) z enters \underline{B}_L at $t = t_f$ for $t_f > t_d$ with $|e(t)| \leq \bar{e}_3$ for all $t \in [t_d, t_f]$.

Therefore, either Phases I and II, or Phases I, II, and III, can be repeated endlessly but with $|e(t)|$ remaining bounded throughout. This is stated in the next section.

4.4 Phase IV: Return to Phase I or Phase II

If Proposition 3 is satisfied, then the trajectory has exited the boundary region and entered Region A . Therefore, $|e(t)| < \bar{e} - \delta$ for all $t \geq t_e$, in which case the boundedness of e is established, proving Theorem 1, or there exists a $t_g > t_e$ such that $|e(t_g)| = \bar{e} - \delta$. The latter implies that the conditions of Proposition 1 are satisfied with t_a replaced by t_g . Therefore, Phases I through III are repeated for $t \geq t_g$.

If Proposition 4 is satisfied instead, then z has reentered \underline{B}_L , in which case Phases II and III are repeated for $t > t_f$.

By combining (48) from Phase I, (57), (58), and (59) from Phase II, and (60) and (61) from Phase III, illustrated in Fig. 3, we obtain

$$|e(t)| \leq \max\{\bar{e}, \bar{e}_2, \bar{e}_3\}, \quad \forall t \geq t_a \quad (69)$$

proving Theorem 1.

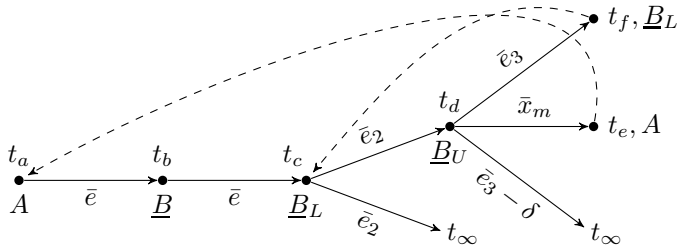


Fig. 3. Trajectory map corresponding to Phases I through IV.

5. NUMERICAL EXAMPLE

In this section we consider the counterexample in Rohrs et al. (1985) with a nominal first order stable plant²

$$x_p(t) = \frac{2}{(s+1)}[u(t)] \quad (70)$$

in the presence of unmodeled dynamics, (18) with

$$\zeta = 0.9912, \quad \omega_n = 15.1327 \quad (71)$$

The plant and reference model differ from (1) and (4) with

$$\dot{x}_p(t) = a_p x_p(t) + k_p v(t) \quad (72)$$

$$\dot{x}_m(t) = a_m x_m(t) + k_m r(t) \quad (73)$$

where

$$a_p = -1, \quad k_p = 2, \quad a_m = -3, \quad k_m = 3. \quad (74)$$

The control input is therefore chosen as

$$u(t) = \theta(t)x_p(t) + k_r r(t) \quad (75)$$

where $k_r = k_m/k_p = 1.5$ so as to match the closed-loop adaptive system when no unmodeled dynamics are present ($G_\eta(s) \equiv 1$).

It can be shown that (18) with (71) corresponds to $S_\eta(a_p, \theta_{max}, \varepsilon_0)$ for $\theta_{max} = 16.7$, $\xi_0 = 4.57 \cdot 10^{-8}$, and $\varepsilon_0 = 0.1\theta_{max}$.

With these choices, the adaptive controller in (45) and (75) guarantees globally bounded solutions for any initial conditions $x_p(0)$ and $\theta(0)$ with $\|\theta(0)\| \leq \theta_{max}$ for the Rohrs unmodeled dynamics in (18) and (71).

5.1 Simulation Studies

In this section, we carry out numerical studies of the adaptive system defined by the plant in (72) in the presence of unmodeled dynamics in (18) and (71) with the reference model in (73), the controller in (75), and the adaptive law in (45) with $\theta_{max} = 16.7$ and $\varepsilon_0 = 1.7$. The resulting plant output, x_p , reference model output, x_m , error, e , and θ are illustrated in Fig. 4 for the reference input

$$r(t) = 0.3 + 1.85 \sin(16.1t) \quad (76)$$

and initial conditions $x_p(0) = 0$ and $\theta(0) = -0.65$. It was observed that all of these quantities became unstable when the projection bound in (11) was removed. It is interesting to note that in this case, only Phases I and II discussed in Section 4 occurred, with Phase I lasting from $t = 0$ to $t = 1377.5s$ and Phase II for all $t \geq 1377.5s$. This clearly validates the main

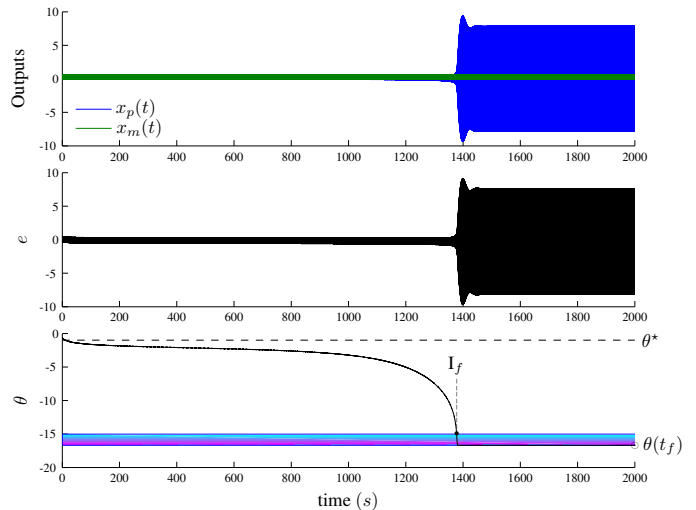


Fig. 4. Simulation of stable plant in the presence of highly damped unmodeled dynamics with adaptive law in (45).

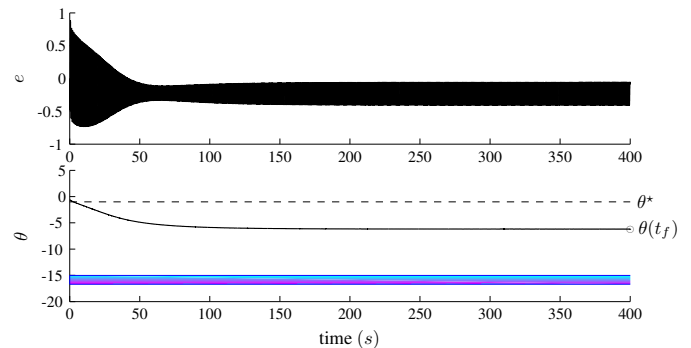


Fig. 5. Simulation results with reference input in (i).

result of this paper reported in Theorem 1. In what follows, we carry out a more detailed study of this adaptive system, by only changing the reference input. As the numerical simulations will show, the behavior of the adaptive system, in terms of which of the four phases reported in Section 4 occur, is directly dependent on the nature of the reference input. Four different choices of the reference input are made, and the corresponding behavior are described.

- (i) $r(t) = 0.3 + 2.0 \sin(8t)$: The error, e , and parameter, θ , corresponding to this reference input are shown in Fig. 5. We observe immediately that $|e(t)| < 1$ for all $t \geq 0$. As a result, the trajectory never enters \underline{B} , eliminating the need for Phases II, III, or IV. Hence, no projection is required in this case.
- (ii) $r(t)$: A pulse for the first one second. That is,

$$r(t) = \begin{cases} 12 & 0 \leq t \leq 1s \\ 0 & t > 1s \end{cases} \quad (77)$$

The corresponding trajectories are shown in Fig. 6, which illustrate that Phase I occurs for $0 \leq t \leq 0.9s$, and Phase II for $0.9s \leq t < 1.0s$. The trajectory exits the boundary region at $t_e = 1.0s$, demonstrating Phase III. Phase I is repeated, and the trajectory reenters \underline{B} at $t_b = 1.3s$, demonstrating Phase IV. The trajectory then settles in \underline{B} for all $t \geq 1.3s$.

- (iii) $r(t) = 10 \forall t$: Fig. 7 illustrates the corresponding limit cycle behavior of the trajectory. We observe that the trajectory first enters \underline{B} at $t_b = 1.80s$. Phase II then occurs

² s in what follows is a differential operator d/dt and not the Laplace variable.

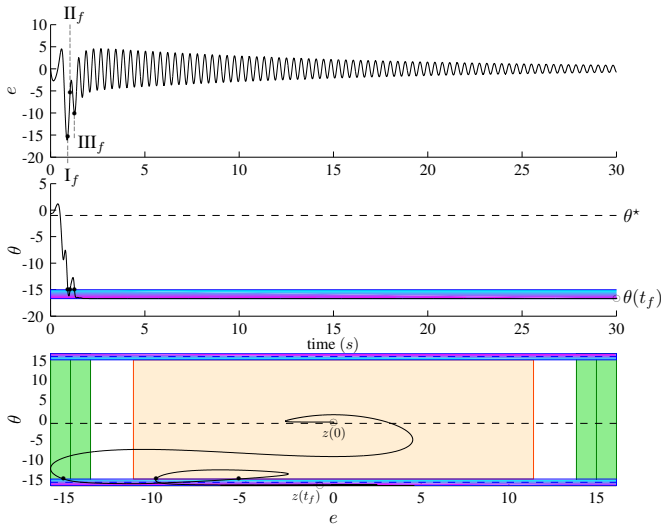


Fig. 6. Simulation results with reference input in (ii).

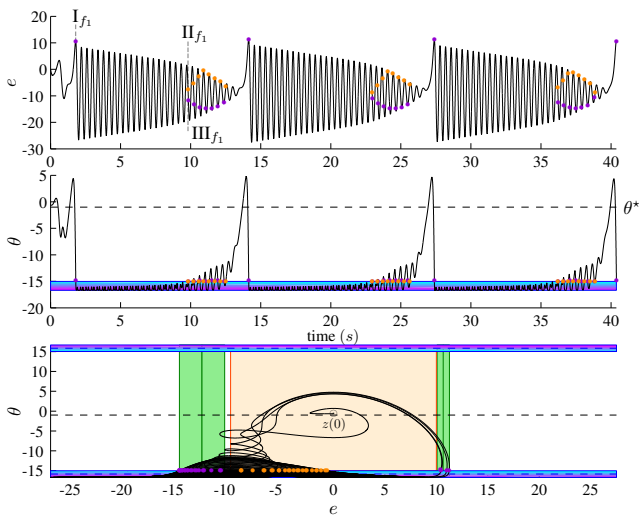


Fig. 7. Simulation results with reference input in (iii).

for $1.80s \leq t < 9.82s$. Phase III occurs for $t_e = 9.82s$, and then Phase I is repeated with the trajectory reentering \underline{B} at $t_b = 9.84s$. Phases I through III are repeated for all $t \geq 9.84$, demonstrating Phase IV, a limit cycle behavior. The points at which the trajectory enters \underline{B} (i.e. Phase II) are shown in orange, and the points at which the trajectory exits \underline{B} (i.e. Phase III) are shown in purple.

- (iv) $r(t) = 2 \sin(\omega_0 t)$, with ω_0 undergoing a continuous sweep from 16.1 rad/s to 2 rad/s over a sixty second interval. The resulting trajectories are shown in phase-plane form in Fig. 8b, for six different initial conditions labeled 1 through 6. It is observed that the trajectory behaves differently for each initial condition. Initial conditions 1 through 3 resulted in trajectories that remained in Phase II for all $t \geq t_b$. Initial condition 4 led to a trajectory with a finite number of occurrences of Phases I through III and finally settled in Region A. Initial conditions 5 and 6 stayed in Region A for all $t \geq 0$. All steady state values are labeled as 1f through 6f.

6. SUMMARY

In this paper, robust adaptive control of scalar plants in the presence of unmodeled dynamics is investigated. It is shown through analytic methods and validated by simulation results

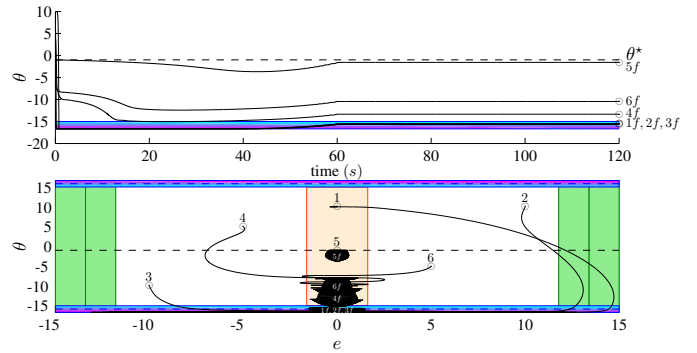


Fig. 8. Simulation results with reference input in (iv).

that a projection algorithm in the standard adaptive control law achieves global boundedness of the overall adaptive system for a class of unmodeled dynamics. The restrictions on the class of unmodeled dynamics and the projection bounds are explicitly calculated and demonstrated using the Rohrs counterexample.

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