Essays in Capital Markets

by

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Abstract

The first two chapters of this dissertation study financial asset markets which are not "frictionless." The first chapter focuses on the effects of transaction costs. The second chapter focuses on the interaction between asymmetric information and strategic behavior. The third chapter empirically assesses the informativeness of certain types of price indicators based on technical analysis.

In Chapter 1 (co-authored with Andrew Lo and Jiang Wang) we propose a dynamic equilibrium model of asset pricing and trading volume with heterogeneous investors facing fixed transactions costs. We show that even small fixed costs can give rise to large "no-trade" regions for each investor's optimal trading policy and a significant illiquidity discount in asset prices. We perform a calibration exercise to illustrate the empirical relevance of our model for aggregate data. Our model also has implications for the dynamics of order flow, bid/ask spreads, market depth, the allocation of trading costs between buyer and seller, and other aspects of market microstructure, including a square-root power law between trading volume and fixed costs which we confirm using historical US stock market data from 1993 to 1997.

Chapter 2 develops an equilibrium model of a dynamic asymmetric information economy. The model is solved under two circumstances: where the informed and uninformed sectors are both competitive, and where the informed sector is competitive and the uninformed sector consists of a single, strategic agent. The strategic uninformed agent, when facing the same signals as the uninformed competitive sector, manages to extract different information about the state of the economy. I find that expected returns, return variability, and unexpected trading volume differ between the competitive and the strategic economies. Furthermore, this difference depends on the degree of informational asymmetry between the two sectors. In the strategic economy, less surplus is lost due to informational arbitrage by the informed sector. Interestingly, the presence of asymmetric information allows even the competitive uninformed agents to gain surplus from allocational trade. Finally, I examine the incentives of agents to become better informed, and find that sometimes both competitive and strategic agents are better off under worse information.

Technical analysis, also known as "charting," has been a part of financial practice for many decades, but this discipline has not received the same level of academic scrutiny and acceptance as more traditional approaches such as fundamental analysis. One of the main obstacles is the highly subjective nature of technical analysis—the presence of geometric shapes in historical price charts is often in the eyes of the beholder. In Chapter 3 (co-authored with Andrew Lo and Jiang Wang), we propose a systematic and automatic approach to technical pattern recognition using nonparametric kernel regression, and apply this method to a large number of U.S. stocks from 1962 to 1996 to evaluate the effectiveness of technical analysis. By comparing the unconditional empirical distribution of daily stock returns to the conditional distribution—conditioned on specific technical indicators such as head-and-shoulders or double-bottoms—we find that over the 31-year sample period, several technical indicators do provide incremental information and may have some practical value.

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I dedicate this thesis to my family.

To my family.

Contents

Chapter 1

Asset Prices and Trading Volume under Fixed Transaction Costs

1.1 Introduction

It is well known that transactions costs in asset markets are an important factor in determining investors' trading behavior.[1] Consequently, transactions costs should also affect market liquidity and asset prices in equilibrium.[2] However, the magnitude and direction of their effects on asset prices, trading volume, and other aspects of market microstructure are still subject to considerable controversy and disagreement.

The earlier studies of transactions costs in asset markets are based primarily on partial equilibrium analysis. For example, by comparing exogenously specified returns of two assets—one with transactions costs and another without—that yield the same utility, Constantinides (1986) argues that proportional transactions costs can only have a small impact on asset prices. However, using the present value of transactions costs under a set of candidate trading policies as a measure of the liquidity discount in asset prices, Amihud and Mendelson (1986) conclude that the liquidity discount can be substantial despite relatively small transactions costs.

[1]The literature on optimal trading strategies in the presence of transactions costs is vast. See, for example, Atkinson and Wilmott (1995), Constantinides (1986), Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Eastham and Hastings (1988), Fleming, Grossman, Vila, and Zariphopoulou (1992), Harrison, Sellke, and Taylor (1983), Korn (1998), Morton and Pliska (1995), Schroeder (1998), Shreve and Soner (1992), and Schroeder (1998).

[2]See, for example, Aiyagari and Gertler (1991), Allen and Gale (1994), Amihud and Mendelson (1986), Bensaid et al. (1992), Constantinides (1986), Demsetz (1968), Dumas (1992), Easley and O'Hara (1987), Epps (1976), Foster and Viswanathan (1990), Garman and Ohlson (1981), Grossman and Laroque (1990), Heaton and Lucas (1996), Huang (1997), Jarrow (1992), Kyle (1985, 1989), Tiniç (1972), Tuckman and Vila (1992), Uppal (1993), Vayanos (1998), and Vayanos and Vila (1997).

More recently, several authors have developed equilibrium models to address this issue. For example, Heaton and Lucas (1996) numerically solve a model in which agents trade to share their labor income risk. Their analysis shows that symmetric transactions costs do not affect asset prices significantly.[3] Vayanos (1998) develops a model in which agents trade to smooth life-time consumption, and shows that the price impact of proportional transaction costs is linear in the costs and for realistic magnitudes, their impact is small. Huang (1998) considers investors that are exposed to surprise liquidity shocks and who are able to trade in a liquid and an illiquid financial asset. He finds that for investors faced with borrowing constraints, the stochastic nature of the liquidity shock induces a high liquidity premium. However, for non-constrained investors or for investors who know when their liquidity shock will arrive, the liquidity premium is small.

A common feature of these equilibrium models is the infrequent trading needs that they imply for investors, and calibrations of such models may understate the effect of transactions costs on asset prices given the levels of trading that we observe empirically.[4] After all, it is the high-frequency trading needs that are affected most significantly by transactions costs. Moreover, there is a substantial empirical literature that documents the importance of trading frictions for asset prices and investment management.[5] This suggests the need for a more plausible model of investors' trading behavior to fully capture the economic implications of transactions costs in financial markets.

In this paper, we hope to provide such a model by investigating the impact of a fixed transactions cost on asset prices and trading behavior in a continuous-time general equilibrium model with heterogeneous investors. Investors are endowed with a risky non-tradeable asset, e.g., labor income, and in a frictionless economy, they will wish to trade continuously

[3]In Heaton and Lucas (1996), agents trade two assets, a risky stock and a riskless bond. Transaction costs on the stock alone only have negligible effect on asset prices; agents can use the bond to achieve most of their risk-sharing needs. However, if transactions costs are also imposed on the bond, their effect on the prices become important. In this paper, we assume that the bond market is frictionless.

[4]It may be possible to calibrate the model of Heaton and Lucas (1997) to allow for high-frequency trading needs, and we hope to explore this possibility in a future study.

[5]See, for example, Arnott and Wagner (1990), Berkowitz, Logue, and Noser (1988), Birinyi (1995), Bodurtha and Quinn (1990), Brennan and Copeland (1988), Brinson, Hood, and Beebower (1986), Brinson, Singer, and Beebower (1991), Chan and Lakonishok (1993, 1995), Collins and Fabozzi (1991), Cuneo and Wagner (1975), Demsetz (1968), Gammill and Pérold (1989), Hasbrouck and Schwartz (1988), Huang and Stoll (1995), Keim and Madhavan (1995a-c), Kraus and Stoll (1972), Loeb (1983), Pérold (1988), Schwartz and Whitcomb (1988), Sherrerd (1993), Stoll (1989, 1993), Tiniç (1972), Treynor (1981), Turnbull and White (1995), Wagner (1993), Wagner and Banks (1992), and Wagner and Edwards (1993).

and will trade an unbounded quantity in the securities market to hedge their non-traded risk exposure. But in the presence of a fixed transactions cost, they will seek to trade finite amounts and will trade only infrequently. Indeed, we find that even small fixed costs can give rise to large "no-trade" regions for each investor's optimal trading policy, and the the uncertainty regarding the optimality of the investors' asset positions between trades reduces their asset demand, which leads to a decrease in the equilibrium price. We show that this price decrease—a discount due to illiquidity—satisfies a power law with respect to the fixed cost, i.e., it is approximately proportional to the square root of the fixed cost, which implies that small fixed costs can have a significant impact on asset prices. Moreover, the size of illiquidity discount increases with the investors' trading needs at high frequencies and is very sensitive to their risk aversion.

Our model allows us to examine how transactions costs can influence the level of trading volume, and may serve as a bridge between the market microstructure literature and the broader equilibrium asset-pricing literature. In particular, despite the many market microstructure studies that relate trading behavior to market-making activities and the price-discovery mechanism,[6] the seemingly high level of volume in financial markets have often been considered puzzling from a rational asset-pricing perspective (see, for example, Ross, 1986). Some have even argued that additional trading frictions ought to be introduced in the form of a transactions tax to discourage high-frequency trading.[7] Yet in absence of transactions costs, most dynamic equilibrium models will show that it is quite rational and efficient for trading volume to be *infinite* when the information flow to the market is continuous, i.e., a diffusion. An equilibrium model with fixed transactions costs can reconcile these two disparate views of trading volume. In particular, our analysis shows that while fixed costs do imply less-than-continuous trading and finite trading volume, an increase in such costs has only a slight effect on volume at the margin.

Our model also has significant implications for the dynamics of order flow, the evolution of bid/ask spreads and depths, and other aspects of the market's microstructure. In particular, we endogenize not only the price at which trades are consummated, but also the *times* at which trades occur. The standard market-clearing condition—that investors trade a market-

[6]See, for example, Admati and Pfleiderer (1988) Bagehot (1971), Easley and O'Hara (1987), Foster and Viswanathan (1990), and Kyle (1985).

[7]See, for example, Stiglitz (1989), Summers and Summers (1990a,b), and Tobin (1984).

clearing quantity in each transaction—is obviously inadequate in a dynamic context where investors can choose when to transact. We extend the market-clearing condition as follows: investors must wish to trade the same quantities with each other, and they must want to do so at the same time. This feature distinguishes our model from other existing models of trading behavior in the market microstructure literature, models in which order flow is almost always specified exogenously, e.g., Glosten and Milgrom (1985) and Kyle (1985). We find that the expected time between trades satisfies a power law with respect to the fixed transactions cost—it is proportional to the fourth root of the fixed cost. This implies a square-root power law between trading volume and inter-arrival times, an unexpectedly sharp empirical implication that we investigate and confirm using transactions data.

We develop the basic structure of our model in Section 1.2. Section 1.3 discusses the nature of market equilibrium in the presence of fixed transactions costs. We derive explicit solutions for the equilibrium in Section 1.4, and analyze these solutions in Section 1.5. Section 1.6 reports the results of a calibration exercise using empirically plausible values of the parameters derived from the existing literature. Section 1.7 presents an empirical test of some of the model's implications using historical stock market data from 1993 to 1997, and we conclude in Section 1.8.

1.2 The Model

1.2.1 Economy

We consider an economy defined on a continuous time-horizon $[0, \infty)$. There is a single commodity, which is also used as the numeraire. The economy is further defined as follows.

The underlying uncertainty of the economy is characterized by an n-dimensional standard Brownian motion $B = \{B_t : t \geq 0\}$ defined on its filtered probability space $(\Omega, \mathcal{F}, F, \mathcal{P})$. The filtration $F = \{\mathcal{F}_t : t \geq 0\}$ represents the information revealed by B over time.

There are two traded securities: a risk-free bond and a risky stock. The bond pays a positive, constant interest rate r. Each share of the stock pays a cumulative dividend D_t where

$$D_t = \bar{a}_D t + \int_0^t b_D dB_s = \bar{a}_D t + b_D B_t \tag{1.1}$$

$\bar{a}_D$ is a positive constant, and b_D is a $(1 \times n)$ constant matrix. The securities are traded competitively in a securities market. Let $P = \{P_t : t \geq 0\}$ denote the stock price process, which is progressively measurable with respect to F.

Transactions in the bond market are costless, but transactions in the stock market are costly. For each stock transaction, its two sides have to pay a total fixed cost of κ, which is exogenously specified and independent of the amount transacted. However, the allocation of this fixed cost between buyer and seller, denoted by κ^+ and κ^-, respectively, is determined endogenously in equilibrium. More formally, the transactions cost for a trade δ is given by

$$\kappa(\delta) = \begin{cases} \kappa^+ & \text{for } \delta > 0 \\ 0 & \text{for } \delta = 0 \\ \kappa^- & \text{for } \delta < 0 \end{cases} \tag{1.2}$$

where δ is the signed volume (positive for purchases and negative for sales), κ^+ is the cost for purchases, κ^- is the cost for sales, and the sum $\kappa^+ + \kappa^- = \kappa$.

There are two agents in the economy, denoted by $i = 1, 2$. Each agent is initially endowed with zero amount in bonds and $\bar{\theta}$ shares of the stock. In addition, agent i is endowed with a stream of non-traded income with cumulative cash flow N_t^i, where

$$N_t^i = \int_0^t \left[(-1)^i X_s + Y_s/2\right] b_N dB_s \tag{1.3a}$$

$$X_t = \int_0^t (-a_X X_s ds + b_X dB_s) \tag{1.3b}$$

$$Y_t = \int_0^t (-a_Y Y_s ds + b_Y dB_s) \tag{1.3c}$$

a_X, a_Y are positive constants, b_N, b_X, b_Y are $(1 \times n)$ constant matrices. For future convenience, we let $X_t^i \equiv (-1)^i X_t$. Thus, b_N specifies the non-traded risk, $X_t^i + Y_t/2$ gives agent i's total exposure to the non-traded risk. Since $X_t^1 + X_t^2 = 0 \ \forall\ t$, Y_t defines the aggregate level of non-traded risk and X_t^i defines the idiosyncratic component of agent i's non-traded risk.

Each agent chooses his consumption and trading policy to maximize the expected utility over his life-time consumption. Let C denote the agents' consumption space, which consists of F-adapted consumption processes $c = \{c_t : t \geq 0\}$ satisfying certain technical conditions (see Huang and Pages, 1990). The agents' stock trading policy space consists of only simple policies defined as follows:

Definition 1 *Let θ_t^i be agent i's stock holdings at time t. A simple trading policy is given by the processes (l_t, L_t, U_t, u_t), continuous and progressively measurable, with $l_t \leq u_t$, and $\{L_t, U_t\} \in [l_t, u_t]$, such that*

(1) $\theta_t^i \in [l_t, u_t]$ $\forall t > 0$,

(2) for t such that $\theta_t^i \in (l_t, u_t)$ the agent does not trade,

(3) for t such that $\theta_t^i = l_t$, the agent purchases $\delta_t^+ = L_t - l_t$ shares,

(4) for t such that $\theta_t^i = u_t$, the agent sells $\delta_t^- = u_t - U_t$ shares.

If we define τ_k^i as the kth time that agent i chooses to trade, then τ_k^i will be a stopping time of F. Therefore, agent i's stock holdings evolve according to

$$\theta_t^i = \theta_{0-}^i + \sum_{\{k: \tau_k^i \leq t\}} \delta_k^i \tag{1.4}$$

where θ_{0-}^i is his initial endowment of stock shares, which is assumed to be $\bar{\theta}$.

Let M_t^i denote agent i's bond position at t (in value). M_t^i represents agent i's liquid financial wealth. Then,

$$M_t^i = \int_0^t \left(rM_s^i - c_s\right) ds + \int_0^t \left(\theta_s^i dD_s + dN_s^i\right) - \sum_{\{k: \tau_k^i \leq t\}} \left(P_{\tau_k^i} \delta_k^i + \kappa_k^i\right) \tag{1.5}$$

where $\kappa_k^i = \kappa(\delta_k^i)$ and $\kappa(\cdot)$ is given in (1.2). Equation (1.5) defines agent i's budget constraint. Agent i's consumption/trading policy (c, δ) is budget feasible if the associated M_t process satisfies (1.5). We denote the set of budget feasible policies by Φ.[8]

[8] We can also define his total financial wealth W_t^i, including both his bond position and the market value of his stock holdings: $W_t^i = M_t^i + \theta_t^i P_t$. From (1.4) and (1.5), we have

$$W_t^i = \theta_0^i P_0 + \int_0^t \left(rW_s^i + dN_s^i - c_s\right) ds + \int_0^t \theta_s^i (dD_s + dP_s - rP_s ds) - \sum_{\{k: \tau_k^i \leq t\}} \kappa_k^i$$

which also defines agent i's budget constraint.

Both agents are assumed to maximize expected utility of the form:

$$u(c) = \mathrm{E}\left[-\int_0^\infty e^{-\rho t - \gamma c_t} dt\right] \tag{1.6}$$

subject to the terminal wealth condition that $\lim_{t\to\infty} \mathrm{E}\left[-e^{-\rho t - r\gamma(M_t^i + \theta_t^i P_t)}\right] = 0$.[9] Here, ρ and γ (both positive) are the time-discount coefficient and the risk-aversion coefficient, respectively.

1.2.2 Definition of Equilibrium

Definition 2 *An equilibrium in the stock market is defined by:*

(a) a price process $P = \{P_t : t \geq 0\}$ progressively measurable with respect to F

(b) an allocation of the transaction cost (κ^+, κ^-), where κ^+ is the cost for purchases and κ^- is the cost for sales as defined in (1.2)

(c) agents' trading policies (l_t, L_t, U_t, u_t), $i = 1, 2$, given the price process and the allocation of transaction cost

such that:

(i) each agent's trading policy solves his optimization problem:

$$J^i(M_0^i, \theta_0, \cdot) \equiv \sup_{(c,\delta)\in\Phi} \mathrm{E}\left[-\int_0^\infty e^{-\rho t - \gamma c_t} dt\right] \tag{1.7}$$

subject to the transversality condition that $\lim_{t\to\infty} \mathrm{E}_0[J^i(M_t^i, \theta_t, \cdot)] = 0$, where $\cdot$ denotes the relevant state variables[10]

[9] The terminal wealth condition is imposed to prevent the agents from running a Ponzi scheme. This specification of the objective function can be interpreted as the limit of the same utility function over a finite horizon (0, T] with a bequest function of the form $B(W_T, T) = -e^{-\rho T - r\gamma W_T}$, when $T \to \infty$.

[10] The transversality condition for the value function arises from the terminal wealth condition specified earlier. See, for example, Merton (1990) for more discussion on this point.

(ii) the stock market clears:

$$\forall\, k \in \mathbb{N}_+ : \qquad \tau_k^1 = \tau_k^2 \tag{1.8a}$$

$$\sum_{i=1}^{2} \delta_k^i = 0. \tag{1.8b}$$

In the presence of transaction costs, the market-clearing condition consists of two parts: agents' desired trading times match, which is (1.8a), and their desired trade amount match, which is (1.8b). Thus, "double coincidence of wants" must always be guaranteed in equilibrium, which is a very stringent condition.

It should be pointed out that by assuming a constant interest rate, we are not closing the bond market. This assumption simplifies our analysis, but deserves clarification. Three comments are in order. First, our focus is on how transactions costs affect the trading and pricing of a security when agents want to trade it at high frequencies. Assuming constant interest rate allows us to focus on the stock, which is costly to trade, and restrict to simple risk-sharing motives of trading. Closing the bond market, however, would make the interest rate stochastic and introduce additional trading motives (such as intertemporal hedging). Such a complication is unnecessary for our purpose. Second, allowing the interest rate to adjust endogenously in our model would not fundamentally change the high frequency trading needs from simple risk-sharing motives. The bond is locally risk-free and is not used as an instrument for risk sharing at high frequencies. (This is no longer the case at lower frequencies as shown in Heaton and Lucas (1995)). Third, we could avoid the issue of bond market equilibrium by considering a finite horizon version of our model without intermediate consumption. In this case, the bond becomes a numeraire and the only market clearing condition is for the stock. The qualitative features of the model remain the same. The inconvenience is the additional time dependence of the equilibrium. One can then consider the limit when the horizon goes to infinity. We choose the current setting to avoid such a procedure.

1.2.3 Simplifying Assumptions

For parsimony, we make several simplifying assumptions about the agents' non-traded risks, which is given in (1.3). First, we assume that there is no aggregate non-traded risk, which requires that $Y_t = 0 \forall\, t \geq 0$. In the current model, in absence of differences at the individual level, non-traded risk at the aggregate level does not generate any trading needs. It is the difference between agents in their non-traded income that generates trading. Since we are mainly interested in the impact of transaction costs, it is natural to focus on the difference in non-traded risk across agents. After all, transaction costs matter only when agents want to transact.[11]

The difference in the agents' non-traded risk is fully characterized by X_t. We further assume that $a_X = 0$. From (1.3), X_t now follows a Brownian motion: $X_t = b_X B_t$. Thus, changes in the difference between the agents' non-traded risk are persistent. In addition, we assume that the risk in the non-traded income is instantaneously perfectly correlated with risk in stock payoffs. In particular, we set $b_N = -hb_D$, where h is a positive (scalar) constant.[12] This implies that the non-traded risk is actually marketed. (Despite this, we continue to use the term non-traded risk throughout the paper to reflect the fact that it not be marketed.) These two assumptions ($a_X = 0$ and $b_N = -hb_D$) can potentially increase the agents' needs to trade. However, we do not expect them to qualitatively affect our results. They are made to simplify the model.

1.3 Characterization of Equilibrium

Our derivation of the equilibrium is as follows. We first conjecture a set of candidate stock price processes and a set of candidate trading policies. We then solve for each agent's optimization problem within the conjectured policy set under each candidate price process. This optimal policy is further verified to be the true optimal policy among all feasible policies. Finally, we show that the stock market clears for a particular candidate price process.[13]

[11] The coexistence of aggregate and idiosyncratic risks can lead to interesting interactions between the two (see, for example, Caballero, 1991, 1997 and Caballero and Engle, 1993). We hope to analyze this interaction in our setting in future research.

[12] This assumption can be partially relaxed by allowing an additional component of the non-traded income that is independent from the risk of the stock.

[13] Needless to say, following this procedure we does not address the issue on the uniqueness of equilibrium, which is left for future research.

1.3.1 Candidate Price Processes and Trading Policies

In the absence of transaction costs, our model reduces to (a special version of) the model considered by Huang and Wang (1997). Agents trade continuously in the stock market to hedge their non-traded risk. Since their non-traded risks perfectly cancel with each other, the agents can eliminate their non-traded risk through trading. Thus, the equilibrium price remains constant over time, independent of the idiosyncratic non-traded risk as characterized by X_t. In particular, the equilibrium price has the following form:

$$P_t = \frac{\bar{a}_D}{r} - p_0 \quad \forall\, t \geq 0 \tag{1.9}$$

where $\bar{a}_D/r$ gives the present value of expected future dividends, discounted at the risk-free rate, and p_0 gives the discount in the stock price to adjust for risk. The agents' optimal stock holding is linear in his exposure to non-traded risk:

$$\theta_i = \theta_0 + hX_t^i \tag{1.10}$$

where θ_0 is a constant. It is worth noting that here each agent's stock holding is independent of his wealth.[14]

In the presence of transaction costs, the agents only trade infrequently. However, whenever they trade, we expect them to reach optimal risk-sharing. This implies, as in the case of no transactions costs, that the equilibrium price at all trades should be the same, independent of the idiosyncratic non-traded risk X_t. Thus, we consider the candidate stock price processes of the form (1.9) even in the presence of transaction costs.[15] The discount p_0 now reflects the price adjustment of the stock for both its risk and illiquidity.

The candidate trading policies are restricted to be affine functions of the non-traded risk

[14] The agents' optimal trading policy and the equilibrium stock price under zero transaction costs are given in Section 1.4.1 as a special case of the model.

[15] Given the perfect symmetry between the two agents, the economy is invariant under the following transformation: $X_t \to -X_t$. This implies that the price must be an even function of X_t. A constant is the simplest even function.

X_t, given by (l_t, m_t, u_t),[16] where for agent i

$$l_t = z_l + hX_t^i \quad m_t = z_m + hX_t^i \quad u_t = z_u + hX_t^i \tag{1.11}$$

for some constants (z_l, z_m, z_u).[17] The motivation for this choice of policy will be made clear in the next section. Agent i is assumed to maintain his current stock position θ_t between a continuously changing lower bound of l_t and a continuously changing upper bound of u_t. When the lower bound is hit, the agent purchases $\delta_t^+ = m_t - l_t$ shares of the stock. When the upper bound is hit, he sells $\delta_t^- = u_t - m_t$ shares. In both cases, his stock position becomes m_t after the trade. In particular, we assume that $\theta_{0^-} \in (l_0, u_0)$, where θ_{0^-} is the agent's initial stock position. There is no loss of generality by this latter assumption in the model defined in Section 1.2.

We define the stopping time τ_k to be the first time the stock position hit the boundary (l_t, u_t) given the agent's stock position θ_{k-1} (at τ_{k-1}):

$$\tau_k = \inf\{t \geq \tau_{k-1} : \theta_{k-1} \notin (l_t, u_t)\} \quad \forall\, k = 1, 2, \ldots \tag{1.12}$$

where $\tau_0 = 0$. $\{\tau_k : k \in \mathbb{N}_+\}$ then gives the sequence of trading times. The amount of trading at τ_k is given by $\delta_k^+ = m_{\tau_k} - l_{\tau_k}$ or $\delta_k^- = u_{\tau_k} - m_{\tau_k}$, depending on whether l or u is hit.

1.3.2 Optimal Policy within the Candidate Set

Given the candidate stock price process and trading policies, we now examine an agent's optimization problem. We start by conjecturing that each agent's value function is of the form:

$$J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M + \theta \bar{a}_D / r) - V(\theta, X)} \tag{1.13}$$

[16] Note that in the class of affine policies, it can be shown that (l, L, U, u) type policies collapse to (l, m, u) type policies given our set-up.

[17] Cadenillas and Zapatero (1999,2000) study fixed cost control problems similar to our own and show that, under some technical conditions, in the class of impulse control policies, the optimal policy is indeed an affine simple one. See also Vial (1972) and Constantinides (1976).

and $V(\theta, X)$ is twice-differentiable. For simplicity in exposition, the index i is omitted here. Since the agent only trades at discrete times $\{\tau_k : k \in \mathbb{N}_+\}$, his stock position is constant between trades. Thus, for $t \in (\tau_{k-1}, \tau_k)$, the Bellman equation takes the form:

$$0 = \sup_c \left\{ -e^{-\rho t - \gamma c} + \mathcal{D}[J] \right\} \tag{1.14}$$

where $\mathcal{D}[\cdot]$ is the standard Itô operator.[18] The optimal consumption is given by

$$c = -\frac{1}{\gamma} \left[\ln r - r\gamma M - \gamma\theta\bar{a}_D - V(\theta, X) \right]. \tag{1.15}$$

The Bellman equation then yields the following PDE for V:

$$0 = r\left(V - \bar{v}\right) + \tfrac{1}{2}\sigma_X^2 \left(V_X^2 - V_{XX}\right) + \tfrac{1}{2} r^2\gamma^2\sigma_D^2 \left(\theta - hX\right)^2 \tag{1.16}$$

where $\bar{v} = (\rho - r + r\ln r)/r$, $\sigma_D^2 = b_D b_D'$, $\sigma_X^2 = b_X b_X'$, and $\sigma_N^2 = b_N b_N' = h^2\sigma_D^2$. A more complete derivation is given in the Appendix.

Let $z_i \equiv \theta_i - hX_i$ for $i = 1, 2$ and $V(X, \theta) = v(z) + \bar{v}$. Then equation (1.16) reduces to a second order non-linear ordinary differential equation (ODE):

$$\sigma_z^2 v'' = \sigma_z^2 v'^2 + 2rv + (r\gamma)^2 \sigma_D^2 z^2 \tag{1.17}$$

where $\sigma_z^2 = h^2\sigma_X^2$. Furthermore, we can rewrite the value function as follows:

$$J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M + \theta\bar{a}_D/r) - v(z) - \bar{v}}. \tag{1.18}$$

In general, each agent is concerned with two state variables (in addition to his bond position M_t^i), his exposure to non-traded risk X_t^i and his current stock position θ_t^i. Under the assumptions that the non-traded risk is permanent ($a_X = 0$) and marketed ($b_N = -hb_D$), the dimensionality of the state space is reduced. In particular, agent i only needs to be concerned with z_t^i as the state variable of interest, which characterizes his net risk exposure.

[18] Suppose that $dx_i = a_i dt + b_i dB$, where $i = 1, 2, \ldots, m$, and $f = f(x_1, \ldots, x_m)$ is twice differentiable. Let $f_i = (\partial f)/(\partial x_i)$, $f_{ij} = (\partial^2 f)/(\partial x_i \partial x_j)$, and $()'$ denotes the transpose. Then, $\mathcal{D}[f] \equiv \sum_{i=1}^m a_i f_i + \frac{1}{2}\sum_{i,j=1}^m b_i (b_j)' f_{ij}$. Note that in our case $dx_1 = dt$.

We now examine the optimal trading policy within the given candidate set. From the above discussion, agent i's trading policy reduces to the control of z_t^i. From the previous section, we see that solving for the optimal candidate policy (l_t, m_t, u_t) in (1.11) is equivalent to solving for the constants (z_l, z_m, z_u). This is to be understood as follows: when the state variable z_t hits the lower boundary z_l, the agent buys an amount of shares $\delta^+ = z_m - z_l$, and when z_t hits the upper barrier, the agent sells an amount of shares $\delta^- = z_u - z_m$. In both cases, the agent trades in order to bring his state variable z_t to the value z_m immediately after the trade. The optimal trading times are given by $\tau_k = \inf\{t \geq \tau_{k-1} : z_t \notin (z_l, z_u)\}$. The agent's control problem now reduces to finding the optimal (z_l, z_m, z_u) given the transaction costs κ^+, κ^- $(\kappa^+ + \kappa^- = \kappa)$ and price coefficient p_0.

If the trading policy (z_l, z_m, z_u) is optimal, at the trading boundaries (z_l and z_u) and with the optimal trade amounts (δ^+ and δ^-, respectively), the agent must be indifferent between trading and not trading. This leads to the "value-matching" condition:

$$v(z_l) = v(z_m) - r\gamma\left[\kappa^+ - p_0(z_m - z_l)\right] \tag{1.19a}$$

$$v(z_u) = v(z_m) - r\gamma\left[\kappa^- + p_0(z_u - z_m)\right]. \tag{1.19b}$$

In addition, the optimality of the trading boundaries requires the "smooth-pasting" condition:

$$v'(z_l) = v'(z_m) = v'(z_u) = -r\gamma p_0. \tag{1.20}$$

The value-matching condition (1.19) and the smooth-pasting condition (1.20) provide the boundary conditions to solve for the value function and the optimal trading policy (within the candidate set).

1.3.3 Equilibrium Price

An equilibrium price process is given by (1.9) with a particular choice of transaction costs, κ^+ and κ^- $(\kappa^+ + \kappa^- = \kappa)$, and price coefficients, p_0, such that the stock market clears. Given the agents' trading policies, the market-clearing condition (1.8) becomes

$$\delta^+ = \delta^- \tag{1.21a}$$

$$z_m = \bar{\theta}. \tag{1.21b}$$

Equation (1.21a) implies $z_u - z_m = z_m - z_l$. The symmetry between the two agents in their exposure to non-traded risk gives $z_t^1 - z_m = -(z_t^2 - z_m)$. Thus, their optimal trading times perfectly match when (1.21a) is satisfied. Furthermore, at the time of trade, the buyer wants to buy exactly the amount which the seller wants to sell. This trade amount is $\delta = \delta^+ = \delta^-$. Equation (1.21b) requires that both agents trade to the point where their total holdings of the stock equals the supply.

1.4 Solutions to Equilibrium

Solution to the equilibrium of the conjectured form consists of two steps. The first step is to solve for each agent's value function and optimal trading policy, given κ^+ and p_0, which is to solve (1.17) with boundary condition (1.19)–(1.20). This is a free-boundary problem of a non-linear ODE. The second step is to solve for κ^+ and p_0 that the market-clearing condition (1.21) is satisfied. A general solution to the problem in closed form is not readily available. We approach the problem in two ways. We first solve the special case when transaction costs are small and approximate analytical results are obtained. We then solve the general case numerically.

1.4.1 Zero Transactions Costs

It is useful to start with the case when the transactions costs are zero. When $\kappa = 0$, $\delta^+ = \delta^- = 0$ and the agents trade continuously.[19] We have the following theorem:

Theorem 1.1 *For $\kappa = 0$, agent i's optimal trading policy under a constant stock price $P_t = \bar{a}_D/r - p_0$ is*

$$\theta_t^i = \bar{z}_m + hX_t^i$$

[19] We can think of continuous trading in this case as the limit of the progressively measurable simple trading strategies given in Definition 1.

where $\bar{z}_m = p_0/(\gamma\sigma_D^2)$*, and his value function is*

$$J(M_t^i, X_t^i, t) = -e^{-\rho t - r\gamma\left[M_t^i + \theta_t^i(\bar{a}_D/r - p_0) + p_0 h X_t^i\right] - \frac{1}{2}r\gamma^2\sigma_D^2\bar{z}_m^2\left(1-\gamma^2\sigma_N^2\sigma_X^2\right) - \bar{v}}. \tag{1.22}$$

Moreover, in equilibrium, $p_0 = \bar{p}_0 \equiv \gamma\sigma_D^2\bar{\theta}$ *and* $\bar{z}_m = \bar{\theta}$*. Here* $\bar{v} = (\rho - r + r\ln r)/r$*.*

Agent i's stock holding has two components. The first component $\bar{z}$, which is constant, gives his unconditional stock position. For $P_t = (\bar{a}_D/r) - p_0$, the expected excess return on one share of stock is rp_0 and the return variance is σ_D^2. Hence, rp_0/σ_D^2 gives the price of per unit risk (of the stock). Moreover, agent i's risk-aversion (toward uncertainty in his wealth) is $r\gamma$. Thus, his unconditional stock position, $\bar{z}_m = (1/r\gamma)(rp_0/\sigma_D^2) = p_0/(\gamma\sigma_D^2)$, is proportional to his risk tolerance and the price of risk. The second component of agent i's stock position is proportional to X_t^i, his exposure to the non-traded risk. This component reflects his hedging position against non-traded risk and the proportionality coefficient, $h = \sigma_N/\sigma_D$, gives the hedge ratio.

In equilibrium, market clearing requires that $\bar{z}_m = \bar{\theta}$. Thus, $p_0 = \bar{p}_0 \equiv \gamma\sigma_D^2\bar{\theta}$. As mentioned earlier, p_0 gives the discount in the price of the stock for its risk and illiquidity. In absence of transaction costs, the stock is liquid and $p_0 = \bar{p}_0$. Thus, $\bar{p}_0$ can be interpreted as the risk discount of the stock. In the presence of transactions costs, we define the difference between p_0 and $\bar{p}_0$, denoted by π:

$$\pi \equiv p_0 - \bar{p}_0 \tag{1.23}$$

to be the illiquidity discount of the stock.

1.4.2 Infinite Transactions Costs

In order to develop an intuition about the illiquidity discount and to put a bound on its magnitude, we now consider the case when the transaction costs are prohibitively high except at $t = 0$. That is, $\kappa = \widehat{\kappa}\,1_{\{t>0\}}$ where $\widehat{\kappa} \to \infty$. Agents can trade at zero cost at $t = 0$ but cannot trade after all afterwards.[20] We have the following result:

[20] This situation has been considered by Hong and Wang (2000) when they analyze the effect of market closures on asset prices. Closures of the market is equivalent to imposing prohibitive transaction costs.

Theorem 1.2 *For $\kappa = \widehat{\kappa}\, 1_{\{t>0\}}$ where $\widehat{\kappa} \to \infty$ and $4\gamma^2\sigma_N^2\sigma_X^2 < 1$, agent i's stock position is $\bar{\theta}$ and the equilibrium stock price at $t = 0$ is $P_0 = \bar{a}_D/r - p_0$ where*

$$p_0 = \bar{p}_0 \left[1 + \frac{4\gamma^2\sigma_N^2\sigma_X^2}{\left(1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}\right)^2} \right]$$

and $\bar{p}_0$ is given in Theorem 1.1.[21]

In this case, the stock becomes completely illiquid after the initial trade. An illiquidity discount is required in its equilibrium price:

$$\widehat{\pi} \equiv \bar{p}_0 \frac{4\gamma^2\sigma_N^2\sigma_X^2}{\left(1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}\right)^2}.$$

For σ_X^2 small, we have $\widehat{\pi} \approx \gamma^2\sigma_N^2\sigma_X^2\bar{p}_0$.

This extreme case illustrates three points. First, the agents' inability to trade in the future reduces their current demand of the stock. As a result, its price carries an additional discount in equilibrium to compensate for the illiquidity (also see Hong and Wang, 2000). Second, this illiquidity discount is proportional to agents' high frequency trading needs, which is characterized by the (instantaneous) volatility of their non-traded risk, σ_X^2. Third, the liquidity discount also increases with the risk of the stock, which is measured by σ_D^2.

When the transaction costs are finite, agents can trade after the initial date (at a cost) and the stock becomes more liquid. We expect the magnitude of the illiquidity discount to be smaller than the extreme case above. However, the qualitative nature of the results remains the same as we show later.

1.4.3 Small Transactions Costs: An Approximate Solution

We now turn to the case when the transaction costs are small. We seek the solution to each agent's value function, optimal trading policy, the equilibrium cost allocation and stock price

[21] When agents cannot trade (after the initial point), a parameter condition, $4\gamma^2\sigma_N^2\sigma_X^2 < 1$, is required. This condition limits an agent's endowment risk. Unable to unload the risk to the market, the agent's consumption is forced absorb the risk of his endowment. Conditions of the above type is needed to guarantee that his expected utility (over an infinite horizon) is well defined given his endowment. This condition is not needed when agents can trade (even infrequently) to control the risk of his consumption.

that can be approximated by powers of $\varepsilon \equiv \kappa^{\alpha}$ where α is a positive constant. In particular, v takes the form $v(z,\varepsilon)$ and $\kappa^{\pm}$ takes the form:

$$\kappa^{\pm} = \kappa\left(\tfrac{1}{2} \pm \sum_{n=1}^{\infty} k^{(n)}\varepsilon^{n}\right). \tag{1.24}$$

The following theorem summarizes our results on optimal trading policies:

Theorem 1.3 *Let $\varepsilon \equiv \kappa^{\frac{1}{4}}$. For (a) κ small and $\kappa^{\pm}$ in the form of (1.24), and (b) $v(z,\varepsilon)$ analytic for small z and ε, an agent's optimal trading policy is given by*

$$\delta^{\pm} = \phi\kappa^{\frac{1}{4}} \pm \tfrac{6}{11}\left(k^{(1)} - \tfrac{2}{15}r\gamma p_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}}) \tag{1.25a}$$

$$z_m = \frac{p_0}{\gamma\sigma_D^2} + \tfrac{4}{11}\left(k^{(1)} - \tfrac{71}{120}r\gamma p_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}}) \tag{1.25b}$$

where $\phi = \left(\frac{6\sigma_z^2}{r\gamma\sigma_D^2}\right)^{\frac{1}{4}}$.

Here, δ^{+} and δ^{-} are the same to the first order of $\varepsilon = \kappa^{\frac{1}{4}}$, but differ in higher orders of ε.

The stock market equilibrium is obtained by choosing $\kappa^{\pm}$ and p_0 such that the market-clearing condition (1.21) is satisfied. We have the following theorem:

Theorem 1.4 *For (a) κ small and $\kappa^{\pm}$ in the form of (1.24), and (b) $v(z,\varepsilon)$ analytic for small z and ε, the equilibrium price and transaction cost allocation are given by*

$$p_0 = \gamma\sigma_D^2\bar{\theta}\left(1 + \tfrac{1}{6}r\gamma^2\sigma_D^2\phi^2\kappa^{\frac{1}{2}}\right) + o(\kappa^{\frac{1}{2}}) \tag{1.26a}$$

$$\kappa^{\pm} = \kappa\left[\tfrac{1}{2} \pm \tfrac{2}{15}r\gamma p_0\phi\kappa^{\frac{1}{4}} + o(\kappa^{\frac{1}{4}})\right] \tag{1.26b}$$

and the equilibrium trading policies are given by (1.25) with the equilibrium value of p_0 and $\kappa^{\pm}$.

1.4.4 General Transactions Costs: A Numerical Solution

In the general case when κ can take arbitrary values, we have to solve both the optimal trading policy and the equilibrium stock price numerically.

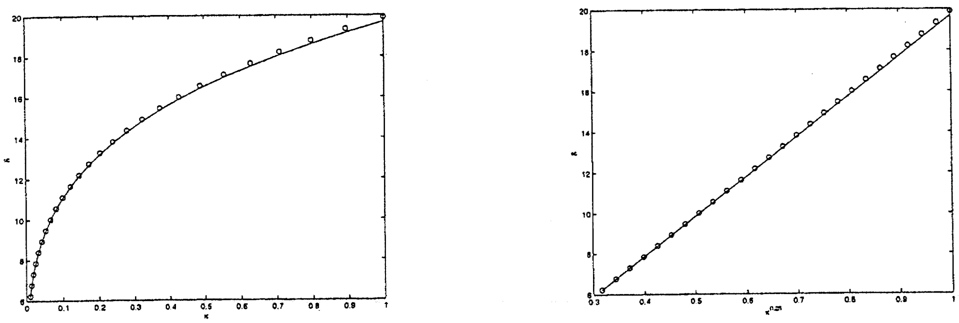


Figure 1.1: Trade amount δ plotted against transaction cost κ and $\kappa^{\frac{1}{4}}$. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 8.8362$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.5$, $\bar{\theta} = 12.8225$, $\bar{a}_D = 0.05$, and $P_t = 0.6486$.

Given p_0 and $\kappa^{\pm}$, we can solve (1.17) and (1.19-1.20) for each agent's optimal trading policy. Figure 1.1 shows the numerical solution for the trade amount for various values of transaction cost. The parameter values in the figures throughout the paper are obtained from a calibration exercise, which is discussed in Section 1.6. Here, we have chosen $\kappa^{\pm}$ such that $\delta^+ = \delta^- \equiv \delta$. Each circle represents the value of δ for a particular value of κ. In the left panel, δ is plotted against the value of κ. In the right panel, δ is plotted against the value of $\kappa^{\frac{1}{4}}$. This transformation is suggested by the approximate solution when κ is small. For comparison, we have also plotted the analytical approximation obtained for small κ as the solid lines.

Given the solution to the agents' optimal trading policies, we can further search for the p_0 and $\kappa^{\pm}$ such that the market-clearing condition (1.21) is satisfied. Figure 1.2 plots the numerical solution (circles) and the analytical approximation (solid line) for the illiquidity discount π ($\pi = p_0 - \bar{p}_0$) in the stock price for various values of the transaction cost. In the left panel, π is plotted against the value of κ. In the right panel, π is plotted against the value of $\kappa^{\frac{1}{2}}$. It is interesting to note that the analytic approximation obtained for small values of transaction costs still fits quite well for fairly large values of κ.

1.5 Analysis of Equilibrium

We now discuss in more detail the impact of transaction costs on agents' trading policies, the equilibrium stock price and trading volume. We focus on the case when κ is small. For convenience, we only maintain the terms up to the lowest appropriate order of κ in our

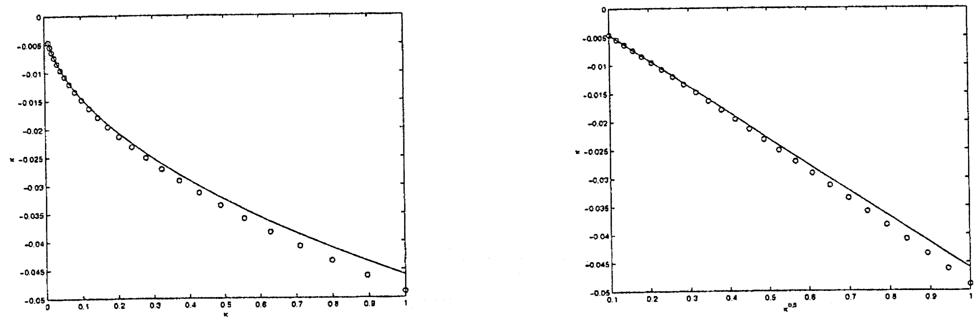


Figure 1.2: Illiquidity discount π plotted against κ and $\kappa^{\frac{1}{2}}$. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 8.8362$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.5$, $\bar{\theta} = 12.8225$, $\bar{a}_D = 0.05$, and $P_t = 0.6486$.

discussion.

1.5.1 Trading Policy

When the transaction costs are zero ($\kappa = 0$), agent i trades continuously in the stock in response to changes in his exposure to non-traded risk, which is characterized by X_t^i ($i = 1, 2$). As stated in Theorem 1.1, the stock position is constantly adjusted such that $\theta_t^i = p_0/(\gamma\sigma_D^2) + hX_t^i$ and $z_t^i = \bar{z}_m = p_0/(\gamma\sigma_D^2)$.

When the transaction costs are positive, it becomes costly to maintain $z_t^i = \bar{z}_m$ at all times. In response, agent i adopts the following policy: He does not trade when z_t^i is within a no-trade region, given by $(z_l, z_u) = (z_m - \delta^+,\ z_m + \delta^-)$. When z_t^i hits the boundary of the no-trade region, agent i trades the necessary amount (δ^+ or δ^-) to bring z_t^i back to the optimal level z_m. Two sets of parameters characterize the agent's optimal trading policy: the bandwidths of the no-trade region, δ^+ and δ^-, and the base level he trades to, z_m, when he does trade. Note that z_m is in general different from $\bar{z}_m$, the position he would trade to in absence of transaction costs. We now discuss these two sets of parameters separately.

To the lowest order of κ, $\delta^+ = \delta^- = \phi\kappa^{\frac{1}{4}}$ as shown in Theorem 1.3. In other words, the bandwidth of the no-trade region exhibits a quartic-root "law" for small transaction costs. We argue that this quartic-root law arises from the boundary conditions, reflecting mainly the nature of the transaction costs. In order to see this, let us consider the simple case when $p_0 = 0$ and $\kappa^+ = \kappa^- = \kappa/2$. Then, $z_m = 0$. We can re-express the boundary conditions in

(1.19) and (1.20) as follows:

$$v(-\delta^-) - v(0) = -r\gamma\kappa = v(\delta^+) - v(0) \tag{1.27a}$$

$$v'(-\delta^-) = v'(0) = v'(\delta^+) = 0. \tag{1.27b}$$

The symmetry between the boundary conditions for the upper and lower no-trade band implies that the band should be symmetric around z_m to the lowest order of κ. That is, $\delta^+ \approx \delta^- \equiv \delta$. Hence, to the zero-th order of κ, $v_1(0) \approx 0 \approx v_3(0)$, where $v_k(0)$ denotes the k-th derivative of v at 0, and furthermore, $v_2(0) \approx 0$ by (1.27b). It follows that for small z, $v(z) \approx \frac{1}{4!}v_4(0)z^4$. The value matching condition (1.27a) then implies that $\delta \propto \kappa^{\frac{1}{4}}$ (if $v_4(0) \neq 0$).[22] The above argument suggests that the quartic-root relation between δ and κ for small κ is determined by the boundary conditions, especially (1.27a), which in turn reflects the form of the transaction cost. For this reason, the quartic-root relation between the width of no-trade region and the fixed transaction cost seems to be a more general result for optimal trading policies.

In the above argument, the quartic-root relation between δ and κ is closely related to the fact that $v(z)$ is quartic in z for small z. The economic intuition behind this property of the value function is as follows. The optimality of the point $z = 0$ requires that $v_1(0) = 0$. If the agent was not allowed to trade, we would have $v(z) \propto z^2$ for z small. However, since the agent always trades back to the optimal position when the trading boundaries are hit, the quadratic term vanishes (to the zero-th order of κ). The symmetry of the boundary conditions further requires that the cubic term vanishes. Thus, $v(z)$ is quartic in z. Intuitively, under fixed transaction costs, the agent always trades back to the optimal stock position. Thus, he can minimize his utility loss without trading too frequently.[23]

[22] More precisely, (1.27b) leads to $v_2(0) + \frac{1}{6}v_4(0)\delta^2 \approx 0$, or $v_2(0) = -\frac{1}{6}v_4(0)\delta^2$. From (1.27a), we have $\frac{1}{12}v_4(0)\delta^4 \approx r\gamma\kappa$, or $\delta \propto \kappa^{\frac{1}{4}}$ if $v_4(0) \neq 0$. In fact, (1.25) gives that $v_4(0) = 2(r\gamma)^2\sigma_D^2/\sigma_z^2$. See the appendix for more details.

[23] The above result on optimal trading policies under fixed transaction costs are closely related to the results of Morton and Pliska (1995) and Atkinson and Wilmott (1995) (see also Schroeder, 1997). Morton and Pliska solve for the optimal trading policy when the agent maximizes his asymptotic growth rate of wealth and pays a cost as a fixed fraction of his total wealth for each transaction. The optimization problem reduces to a free-boundary ODE, which has a closed form solution up to a set of coefficients to be determined by the boundary conditions. They numerically solve for these coefficients. Atkinson and Wilmott (1995), using perturbation techniques, derive an analytic approximations for the solution to the Morton and Pliska model when the transaction cost is small. Interestingly, they also find that the no-trade region is proportional

Having established that the width of no-trade region should be proportional to the quartic root of κ (i.e., $\delta = \phi\kappa^{\frac{1}{4}}$), we now examine the proportionality coefficient ϕ. From Theorem 1.3, we have $\phi = \left(\frac{6\sigma_z^2}{r\gamma\sigma_D^2}\right)^{\frac{1}{4}}$. Note that $r\gamma\sigma_D^2$ corresponds to the certainty equivalence of the (per unit time) expected utility loss of bearing the risk of one stock share. It is then not surprising that ϕ (and δ) is negatively related to $r\gamma\sigma_D^2$. Moreover, σ_z^2 gives the variability of the agent's non-traded risk. For larger σ_z^2, the agent's hedging need is changing more quickly. Given the cost of changing his hedging position, the agent is more cautious in trading on immediate changes in his hedging need. Thus, ϕ (and δ) is positively related to σ_z^2.

Under the optimal trading policy, agents trade only infrequently. Define $\Delta\tau \equiv \mathrm{E}\left[\tau_{k+1} - \tau_k\right]$ to be the average time between two neighboring trades. It is easy to show that

$$\Delta\tau \quad = \quad \delta^2/\sigma_z^2 \approx \left(\phi^2/\sigma_z^2\right)\kappa^{\frac{1}{2}} \qquad (1.28)$$

(see, e.g., Harrison, 1990). Not surprisingly, the average waiting time between trades is inversely related to σ_z, the volatility in the agent's hedging need, and $r\gamma\sigma_D^2$, the cost of bearing the risk of one stock share. Moreover, it is proportional to the square root of the transaction cost. Figure 1.3 plots the average trading interval $\Delta\tau$ versus different values of transaction cost κ as well as the appropriate power law for small κ's.

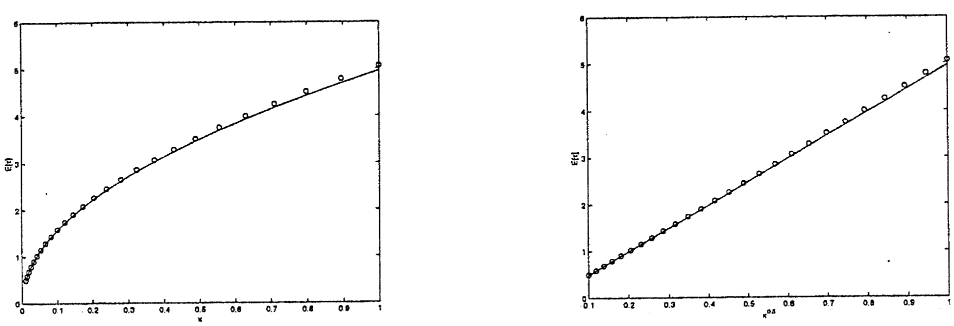


Figure 1.3: Trading Interval. The two panels show the expected inter-arrival times plotted against κ and its square root, respectively. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 8.8362$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.5$, $\bar{\theta} = 12.8225$, $\bar{a}_D = 0.05$, and $P_t = 0.6486$.

When each agent chooses to trade, he trades to a base position z_m. In absence of transaction costs, each agent trades to a position ($\bar{z}_m$) that is most desirable given his current

in size to the fourth root of the transaction cost. (Note that in their model, the transaction cost is a fixed fraction of the total wealth.)

non-traded risk. As his non-traded risk changes, he maintains this desirable position by constantly trading. In the presence of transaction costs, however, an agent only trades infrequently. A position desirable now becomes less desirable later. But he has to stay in this position until the next trade when the gain from trading exceeds the transaction cost. As a result, the agent chooses a position that takes into account the deterioration of its desirability over time and the inability to revise it immediately.

From Theorems 1.3 and 1.4, the shift in the base position is given by $\Delta z_m \equiv z_m - \bar{z}_m = -\frac{1}{6}r\gamma p_0(\sigma_z^2 \Delta\tau)$. It is not surprising that Δz_m is proportional to the total volatility of an agent's non-traded risk over the no-trade period, which is $\sigma_z^2\Delta\tau$. Moreover, Δz_m is proportional to p_0, the risk discount on the stock. In order to further understand this result, let us consider the following heuristic argument. Suppose that the current level of the agent's non-traded asset is zero. The uncertainty in its level over the next no-trade period, denoted by $\tilde{z}$, gives rise to an additional uncertainty in his wealth: $-\tilde{z}(-p_0 + \tilde{d})$, where $\tilde{d}$ denotes the stock dividend over the period. (Here, we set $h = 1$ for simplicity.) Although $\tilde{z}$ has a zero mean, its impact on the overall uncertainty in wealth is not zero. Averaging over $\tilde{z}$ (assumed to be normally distributed with variance σ_z^2), the agent's utility over his future wealth is proportional to $\mathrm{E}_{\tilde{z}}\left[-e^{-r\gamma(\theta-\tilde{z})(-p_0+\tilde{d})}\right] \propto -e^{-r\gamma(\theta+\frac{1}{2}r\gamma p_0\sigma_z^2\Delta\tau)(-p_0+\tilde{d})}$, where $\mathrm{E}_{\tilde{z}}$ denotes the average over $\tilde{z}$, θ is the agent's stock position and $\Delta\tau$ is the length of no-trade period. In other words, the uncertainty in $\tilde{z}$ leads to an effective risk in the agent's wealth that is equivalent to a stock position of size $\frac{1}{2}r\gamma p_0(\sigma_z^2\Delta\tau)$. The size is proportional to p_0 because the uncertainty in wealth generated by uncertainty in $\tilde{z}$ is proportional to p_0. Consequently, the agent reduces his base stock position by the same amount. This shift in the agent's base position reflects the decrease in his demand of the stock in response to its illiquidity.

1.5.2 Stock Price and Illiquidity Discount

In equilibrium, the stock price has to adjust in response to the negative effect of illiquidity on agents' stock demand, giving rise to an illiquidity discount π. For small transaction costs, the illiquidity discount is proportional to the square root of κ. Figure 1.2 further shows that this square-root relation provides a reasonable approximation even for fairly large transaction costs. From Theorem 1.4, we have

$$\pi \approx \gamma \sigma_D^2 \Delta z_m \approx \tfrac{1}{6} r \gamma^3 \sigma_D^4 \phi^2 \bar{\theta} \kappa^{\frac{1}{2}} = \tfrac{1}{6}\gamma \left(r\gamma\sigma_D^2\right) \bar{p}_0 (\sigma_z^2 \Delta\tau). \tag{1.29}$$

As we have shown, fluctuations in his non-traded risk and the cost of adjusting stock positions to hedge this risk reduce an agent's stock demand by Δz_m. Given the linear relation between the agents' stock demand and the stock price, the price has to decrease proportionally to the decrease in demand to clear the market, which gives the illiquidity discount in the first expression of (1.29). Moreover, the decrease in agents' stock demand is proportional to the total risk discount of the stock (p_0) and the volatility of their non-traded risk between trades ($\sigma_z^2 \Delta\tau$), which leads to the second expression.

We thus conclude that the illiquidity discount of the stock is proportional to the product of the unit price of risk, total risk discount of the stock, the variability of agents' desired positions between trades. The proportionality constant is the risk-aversion coefficient γ.

We can also rewrite the illiquidity discount as follows: $\pi = \frac{1}{6}\gamma^2\sigma_D^2(r\gamma\sigma_D^2\bar{\theta})\,(\sigma_z^2\Delta\tau)$. Note that the illiquidity discount is proportional to the cubic power of γ. Comparing with the risk discount which is proportional to γ, we infer that the illiquidity discount is highly sensitive to the agents' risk aversion.

Using a model similar to ours but with proportional transaction costs and deterministic trading needs, Vayanos (1998) finds that the illiquidity discount on the stock is linear in the transaction costs (when they are small). Our result shows that small fixed transaction costs can give rise to a non-trivial illiquidity discount when agents have high frequency trading needs. Given the difference in the nature of transaction costs between our model and Vayanos', our result is not directly comparable to his. But, our result does suggest that the presence of high frequency trading needs is important in analyzing the effect of transaction costs on asset prices.

In order to further confirm this, we consider a special variation of our model, in which $X_t = \bar{a}_X t$. In this case, the agents' non-traded risk evolves deterministically. This gives rise to deterministic needs to trade among agents since they differ in their non-traded risk. We have the following result:

Theorem 1.5 *Let* $\varepsilon = \kappa^{\frac{1}{3}}$. *For (a)* $X_t = \bar{a}_X t$ $(\bar{a}_X \geq 0)$, *(b)* $\kappa^{\pm} = \kappa/2$, *and (c)* $v(z, \varepsilon)$

analytic for small z and ε, agents' optimal trading policies are given by

$$\delta^{1+} = \lambda\kappa^{\frac{1}{3}} + o(\kappa^{\frac{2}{3}}), \quad \delta^{1-} = 0, \quad \delta^{2+} = 0, \quad \delta^{2-} = \delta^{1+} \tag{1.30a}$$

$$z^1_m = \frac{p_0}{\gamma\sigma_D^2} + \tfrac{1}{2}\lambda\kappa^{\frac{1}{3}} - \tfrac{1}{2}\left(\lambda\gamma\sigma_D^2\right)^{-1}\kappa^{\frac{2}{3}} + o(\kappa^{\frac{2}{3}}), \quad z^2_m = \bar{\theta} - (z^1_m - \bar{\theta}) \tag{1.30b}$$

where $\lambda = \left(\frac{6\sigma_N}{r\gamma\sigma_D^3}\right)^{\frac{1}{3}}(\bar{a}_X)^{\frac{1}{3}}$. *The equilibrium stock price is given by* $p_0 = \bar{p}_0 + o(\kappa)$.

It is indeed the case that in absence of high frequency trading needs, the transactions cost does not lead to significant liquidity discount on the stock. Also the power law for the trade amount has now become $\frac{1}{3}$, rather than $\frac{1}{4}$. This is a result of the fact that each agent has only 1 trade boundary (each agent either always sells or always buys), because of the deterministic nature of the endowment process.

1.5.3 Trading Volume

It has long been recognized that continuous-time models have the drawback of implying infinite trading volume in all assets (in the sense that agents' portfolio holdings are of unbounded variation). It has also been long recognized that the introduction of either fixed or proportional transaction costs serves to remedy this problem. In the case of only proportional transaction costs, portfolio holdings become local time processes, and hence of bounded variation. In the case of only a fixed cost, an agent trades only a finite number of times in any finite time interval.

Intuitively, an increase in transaction costs must reduce the volume of trade. Our model suggests a specific form for this relation. In particular, the equilibrium trade size is a constant. From our solution to equilibrium, the volume of trade between time interval t and $t+1$ is given by:

$$\nu_{t+1} = \sum_{\{k:\, t<\tau_k\leq t+1\}} \left|\delta^i_k\right| \tag{1.31}$$

where $i = 1$ or 2. The average trading volume per unit of time is

$$\mathrm{E}\left[\nu_{t+1}\right] = \mathrm{E}\left[\sum_k 1_{\{\tau_k\in(t,\,t+1]\}}\right]\delta \equiv \omega\delta$$

where ω is the frequency of trade (i.e., the number of trades per unit of time). For convenience, we define another measure of average trading volume as the number of shares traded per average trading time, or

$$\nu = \frac{\delta}{\Delta\tau} = \sigma_z^2/\delta \tag{1.32}$$

where $\Delta\tau \equiv \mathrm{E}[\tau_{k+1} - \tau_k] \approx \delta^2/\sigma_z^2$ is the average time between trades.[24] From (1.25), we have

$$\nu = \sigma_z^2 \phi^{-1} \kappa^{-\frac{1}{4}} \left[1 + O\left(\kappa^{\frac{1}{4}}\right)\right].$$

Clearly, as κ goes to zero, trading volume goes to infinity. However, we also have

$$\frac{\Delta\nu}{\nu} \approx -\tfrac{1}{4}\frac{\Delta\kappa}{k}.$$

In other words, (for positive transaction costs) one percentage increase in the transaction cost only decreases trading volume by a quarter of a percent. In this sense, within the range of positive transaction costs, an increase in the cost only reduce the volume mildly at the margin.

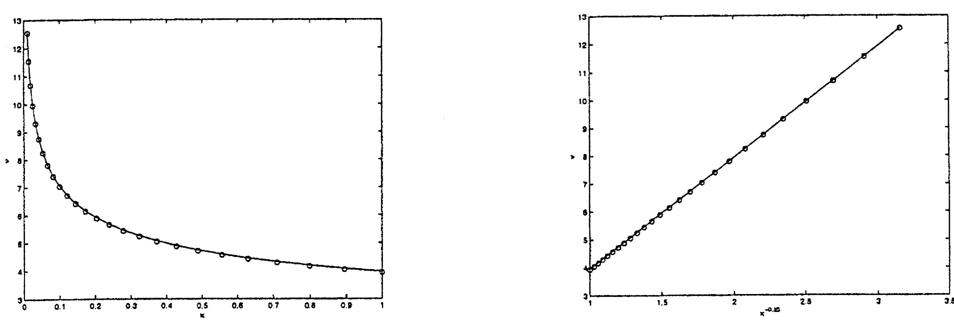


Figure 1.4: Trading volume. The two panels show the volume measure ν plotted against κ (left) and $\kappa^{-\frac{1}{4}}$ (right). The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 8.8362$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.5$, $\bar{\theta} = 12.8225$, $\bar{a}_D = 0.05$, and $P_t = 0.6486$.

Figure 1.4 plots the average volume measure ν versus different values of transaction cost κ as well as the appropriate power laws. Clearly, as κ approaches zero, volume diverges.

[24] Of course, ν is different from $\mathrm{E}[\nu_{t+1}]$ by Jensen's inequality.

1.5.4 Bid/Ask Prices and Depths

Even in the presence of a transaction cost, each agent is willing to transact at the right prices: he becomes willing to buy at a low enough price, P_t^B, and sell at a high enough price, P_t^A. For prices in between these two extremes, the agent prefers not to transact. We define these two critical prices as the agent's bid and ask prices. As it turns out under fixed transaction costs, an agent is willing to buy/sell a finite amount at his bid/ask prices. We define the amount that an agent is willing to transact at his bid and ask prices as the depth of the bid and ask, denoted by δ_t^B and δ_t^A, respectively. Clearly, an agent's bid and ask prices and their depth depend on his current stock position, the current state of his non-traded risk, as well as on the transaction costs.

In Theorem 1.3, we have shown the explicit dependence of the agent's control policy on the stock prices and the allocation of transaction costs. In particular, we have expressed his transaction boundaries (z_l and z_u) as a function of his current risk state (z_t) and the stock price. For a particular allocation of transaction costs (κ^+, κ^-), an agent's bid and ask prices are those prices that put him right at the lower and upper transaction boundaries, respectively. That is

$$z_l(P_t^B) = z_t \tag{1.33a}$$

$$z_u(P_t^A) = z_t. \tag{1.33b}$$

At the bid price P_t^B, the agent is willing to buy only δ_t^B shares and at the ask price P_t^A, he is willing to sell only δ_t^A shares. From Theorem 1.3, the depth of the bid and ask are given by

$$\delta_t^B = z_m(P_t^B) - z_l(P_t^B) \tag{1.34a}$$

$$\delta_t^A = z_u(P_t^A) - z_m(P_t^A). \tag{1.34b}$$

The following theorem characterizes the bid and ask prices, as well as their depth:

Theorem 1.6 *Let $\varepsilon \equiv \kappa^{\frac{1}{4}}$. For (a) κ small and $\kappa^{\pm}$ has the form in (1.24), and (b) $v(z, \varepsilon)$*

analytic for small z and ε, each agent's bid and ask prices are

$$P_t^B = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2 \left[\bar{\theta} + \phi\kappa^{\frac{1}{4}} + (z_t - \bar{\theta}) + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right] \tag{1.35a}$$

$$P_t^A = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2 \left[\bar{\theta} - \phi\kappa^{\frac{1}{4}} + (z_t - \bar{\theta}) + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right]. \tag{1.35b}$$

The corresponding depth δ^B and δ^A are given by

$$\begin{aligned}\delta_t^B &= \phi\kappa^{\frac{1}{4}} + \tfrac{6}{11}k^{(1)}\phi\kappa^{\frac{1}{2}} + \tfrac{4}{55}r\gamma\phi^2\kappa^{\frac{1}{2}}\left(P_t^B - \bar{a}_D/r\right) + o(\kappa^{\frac{1}{2}}) \\ &= \phi\kappa^{\frac{1}{4}} + \tfrac{6}{11}k^{(1)}\phi\kappa^{\frac{1}{2}} - \tfrac{4}{55}r\gamma\bar{p}_0\phi^2\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\end{aligned} \tag{1.36a}$$

$$\begin{aligned}\delta_t^A &= \phi\kappa^{\frac{1}{4}} - \tfrac{6}{11}k^{(1)}\phi\kappa^{\frac{1}{2}} - \tfrac{4}{55}r\gamma\phi^2\kappa^{\frac{1}{2}}\left(P_t^B - \bar{a}_D/r\right) + o(\kappa^{\frac{1}{2}}) \\ &= \phi\kappa^{\frac{1}{4}} - \tfrac{6}{11}k^{(1)}\phi\kappa^{\frac{1}{2}} + \tfrac{4}{55}r\gamma\bar{p}_0\phi^2\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\end{aligned} \tag{1.36b}$$

where ϕ is given in Theorem 1.3.

The second step in (1.36a) follows from the observation that fluctuation in bid/ask prices only have a higher order impact on their depth. Thus, to a "low order approximation" (i.e., to the order of $\kappa^{\frac{1}{2}}$), the depth of the bid and ask are constant, independent of the agent's risk state. This contrasts sharply with the behavior of bid/ask prices themselves, which tend to vary linearly with the agent's risk state z_t. Moreover, with arbitrary allocation of transaction cost between buy and sell (as characterized by $k^{(1)}$), the depth at the bid differs from the depth at the ask.

Given the bid/ask prices of individual agents, we define the bid/ask prices of the market as the best bid/ask prices currently available across all agents in the market. They are denoted by P_t^{MB} and P_t^{MA}, respectively. Thus, $P_t^{MB} = \max[P_t^{1B}, P_t^{2B}]$ and $P_t^{MB} = \min[P_t^{1A}, P_t^{2A}]$. For convenience, we define $\tilde{z}_t^i \equiv z_t^i - \bar{\theta}$. Obviously, $\tilde{z}_t^1 = -\tilde{z}_t^2$. Let $\tilde{z}_t \equiv |\tilde{z}_t^1| = |\tilde{z}_t^2|$. Then, $\max[\tilde{z}_t^1, \tilde{z}_t^2] = |\tilde{z}_t|$ and $\min[\tilde{z}_t^1, \tilde{z}_t^2] = -|\tilde{z}_t|$. We have the following expression for the bid/ask prices of the market:

$$P_t^{MB} = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2 \left[\bar{\theta} + \phi\kappa^{\frac{1}{4}} + \tilde{z}_t + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right] \tag{1.37a}$$

$$P_t^{MA} = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2 \left[\bar{\theta} - \phi\kappa^{\frac{1}{4}} + \tilde{z}_t + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right]. \tag{1.37b}$$

The depth of the market bid and ask prices can be determined from the depth of individual bid/ask prices given in Theorem 1.6. As we mentioned earlier, to a low order approximation, the depth of bid/ask prices are the same across agents. Consequently, the depth of the market bid and ask prices are (approximately) constant, given by the second equation in (1.36a).

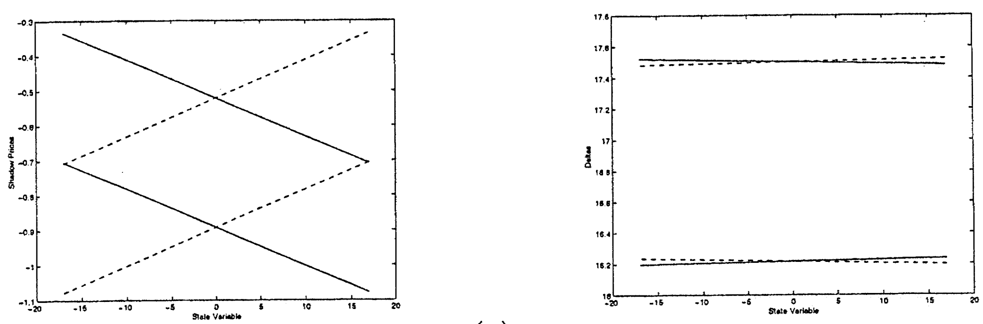
1.5.5 Allocation of Transactions Costs

Given each agent's bid/ask prices, we can now examine the trading process. In the presence of transaction costs, agents do not trade most of the time because one agent's bid price sits below the other agent's ask price. Trading occurs when two things happen at the same time: the bid price of one agent coincides with the ask price of another agent, *and* at this price the two agents want to buy/sell the same amount. In other words, trading occurs when the market bid/ask spread shrinks to zero *and* the depth at the bid equals the depth at the ask.

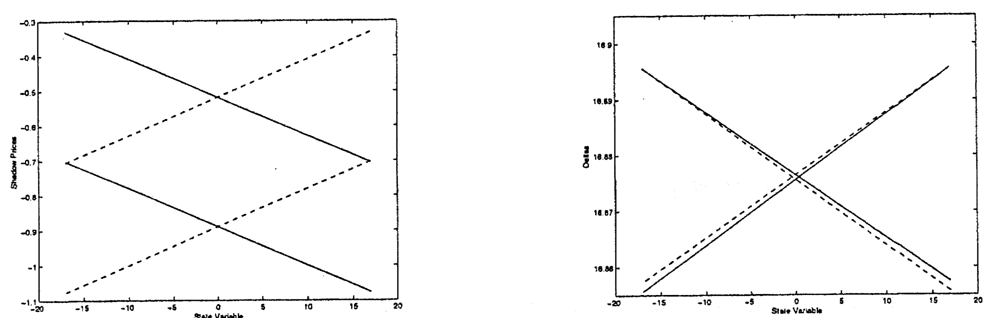
The market bid and ask prices and their depth given above indicate that it can be difficult to meet both of these conditions simultaneously for an arbitrary allocation of the transaction cost (i.e., $\kappa^{\pm}$). In particular, when $\tilde{z}_t = \phi\kappa^{\frac{1}{4}}$, $P_t^{MB} = P_t^{MA}$ and the agents can agree on a transaction price. However, they cannot agree on the amount to transact because in general, $\delta^B \neq \delta^A$. This situation should not be surprising. Under fixed transaction costs, agents always transact a finite amount when they trade. In general, there is no reason to expect any symmetry between the amount they choose to buy and the amount they choose to sell when they decide to trade. This is different from the situation when they face zero transaction costs, in which case only infinitesimal amount is transacted (hence, the symmetry is guaranteed). The lack of symmetry between the depth at the bid and ask prices would prevent the existence of an equilibrium.

In order to allow trading to occur effectively, we need to choose a particular allocation of the fixed cost such that the depth at the bid and ask prices always match when they two prices coincide. From the expressions for the bid/ask depth, this is achievable by setting $k^{(1)} = \frac{2}{15} r\gamma\bar{p}_0\phi$ (to the order of $\kappa^{\frac{1}{2}}$). In this case, we have $\delta_t^{MB} = \delta_t^{MA} = \phi\kappa^{\frac{1}{4}}$. Trading occurs whenever the market bid-ask spread shrinks to zero and the amount $\delta = \phi\kappa^{\frac{1}{4}}$ is transacted. Thus, an equilibrium exists.

The discussion above can be illustrated by looking at agents' bid/ask prices/depth graphically. Figure 1.5 shows the bid and ask prices and their depth of both agents for various values of $z_t^i = \theta_{t-}^i - (\sigma_N/\sigma_D)X_t^i$, within the no-trade region. Since the agents' endowment of non-traded income is opposite to each other, the agents' prices and demands are mirror images of each other around the point $z_t^i = 0$.



(a)



(b)

Figure 1.5: Agents' bid-ask prices and depth. The x-axis represents the level of each agent's state variable $z_t^i = \theta_t^i - (\sigma_N/\sigma_D)X_t^i$ with $X_t^1 = -X_t^2$. Each agent's prices and demands are mirror images around the point $z_t^i = 0$. The dashed and solid lines represent the shadow prices and demands of agents' 1 and 2, respectively. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 41.1808$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.1$, $\bar{\theta} = 64.1127$, $\bar{a}_D = 0.05$, $P_t = 0.6486$, $\kappa_{\%} = 0.5\%$, and $\kappa = 0.00324$. In (a), $\kappa = 0.05$, $\kappa^+ = 1.1\kappa_e^+$, where κ_e^+ is the equilibrium allocation for the buy side transaction cost. In (b), κ^+ and κ^- are assigned their equilibrium values.

Figure 1.5(a) describes the case when $\kappa^+ = \kappa^- = \kappa/2 = 0.025$. The left panel plots the bid/ask prices of the two agents and the right panel plots the depth of the bid and ask prices, respectively. Notice that as deviations in the risk exposure, which has the opposite sign for the two agents, approaches the boundary of no-trade region, the bid price of one agent approaches the ask price of the other agent. At the boundary, the two prices coincide and the two agents would agree on the price to transact. However, their desirable trade amount is different. As shown in the right panel of Figure 1.5(a), at the boundary of no-trade region, the depth of the selling price is lower than the depth of the buying price. This implies that trade would not occur, even though both agents can agree on a price.

The above situation can be avoided if we adjust the allocation of transaction cost. In particular, if we choose κ^+ and κ^- such that the depth of bid and ask prices also coincide at the boundary of no-trade region, trade would occur at the boundary because the agents agree on both the price and the amount of the transaction. Figure 1.5(b) illustrates this case. In this case, an equilibrium exists.

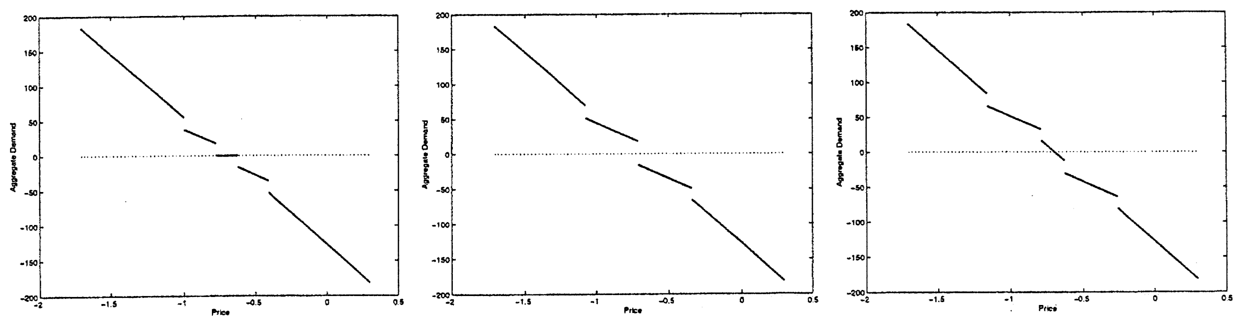


Figure 1.6: Aggregate demand curves for different values of X_t. Respectively, the panels correspond to $X_t = 10, 15, 16.8955, 24$. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 41.1808$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.1$, $\bar{\theta} = 64.1127$, $\bar{a}_D = 0.05$, $P_t = 0.6486$, $\kappa_{\%} = 0.5\%$, and $\kappa = 0.00324$. Here $\kappa^+ = 1.1\kappa_e^+$, where κ_e^+ is the equilibrium allocation for the buy side transaction cost.

Another way to see that an equilibrium may not exist for arbitrary cost allocations is to examine the corresponding aggregate demand curve. It can be seen from Figure 1.6 that the aggregate demand curve exhibits a discontinuity through 0 for some values of X_t. For small values of X_t, both agents have demands of 0 and the market could clear for a range of prices, as can be seen in the first panel of Figure 1.6. For values of X_t which bring both agents outside of their optimal control region (z_l, z_m) (if for example this was the initial endowment in the economy), both agents would like to trade immediately to their optimal allocation, and the trade amount would be $\delta^{\pm} = |z_i - z_m|$. In this case it follows from (1.25) that the market clearing price is

$$p_0 = \gamma\sigma_D^2 \left[\bar{\theta} - \tfrac{4}{11}\left(k^{(1)} - \tfrac{71}{120}\bar{\theta}r\gamma^2\sigma_D^2\phi\right)\phi\kappa^{\frac{1}{2}}\right] + o(\kappa^{\frac{1}{2}}). \tag{1.38}$$

The last panel of Figure 1.6 illustrates this situation. Only for values of X_t such that one agent's state variable is in the vicinity of z_l, the market does not clear for an arbitrary transaction cost allocation (see the middle panel of Figure 1.6). But because X_t evolves continuously, an equilibrium in the economy does not exist almost surely.[25]

For the equilibrium allocation of transaction costs, κ_e^+, the aggregate demand curve remains discontinuous. However, it always passes through 0 for all values of X_t. Figure 1.7 shows the aggregate demand for $\kappa^+ = \kappa_e^+$ at various values of X_t.

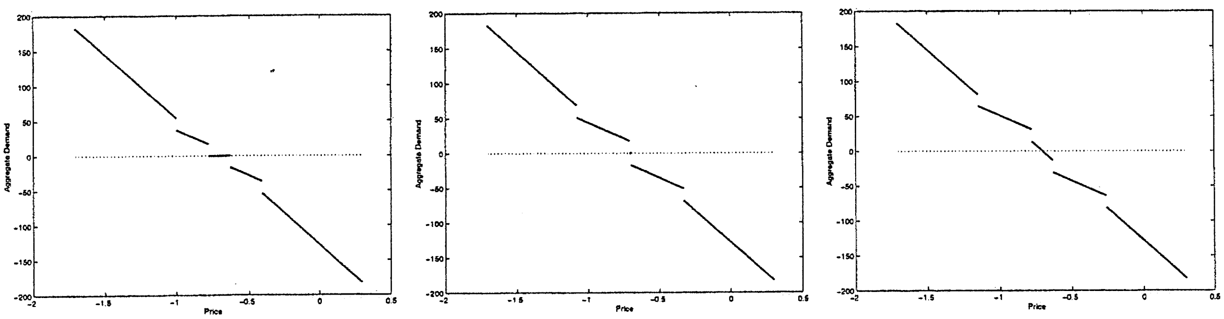


Figure 1.7: Aggregate demand curves for different values of X_t. Respectively, the panels correspond to $X_t = 10, 15, 16.8955, 24$. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 41.1808$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D \sigma_N$, $\gamma = 0.1$, $\bar{\theta} = 64.1127$, $\bar{a}_D = 0.05$, $P_t = 0.6486$, $\kappa_{\%} = 0.5\%$, and $\kappa = 0.00324$. Here κ_e^+ is the equilibrium allocation for the buy side transaction cost.

It is well known that the existence of an equilibrium in the presence of fixed transaction costs is not automatic. In our case, a particular allocation of the cost between the two trading parties is needed to reach an equilibrium. From a practical point of view, one may ask if such an allocation can be implemented through an actual trading process. The answer is affirmative. Let us imagine an electronic trading system through which agents can post their limit orders. Whenever a transaction occurs, the buyer pays κ^+ in addition to the dollar amount of his purchase and the seller receives κ^- less than the dollar amount of his sale. The sum of the charges, $\kappa^+ + \kappa^- = \kappa$ is used to cover the total fixed cost. Such a mechanism can then support the trading process as we discussed.

1.5.6 Trading Process

Let us now examine the actual trading process. As the risk exposure of each agent changes over time, their bid/ask prices and the respective depth also change. A transaction occurs

[25] If the state variable has a jump component this would not be the case: an equilibrium would exist with positive probability.

when the market's bid and ask prices as well as their depth coincide. Figures 1.8 and 1.9 show the realization of a single realization of the economy.

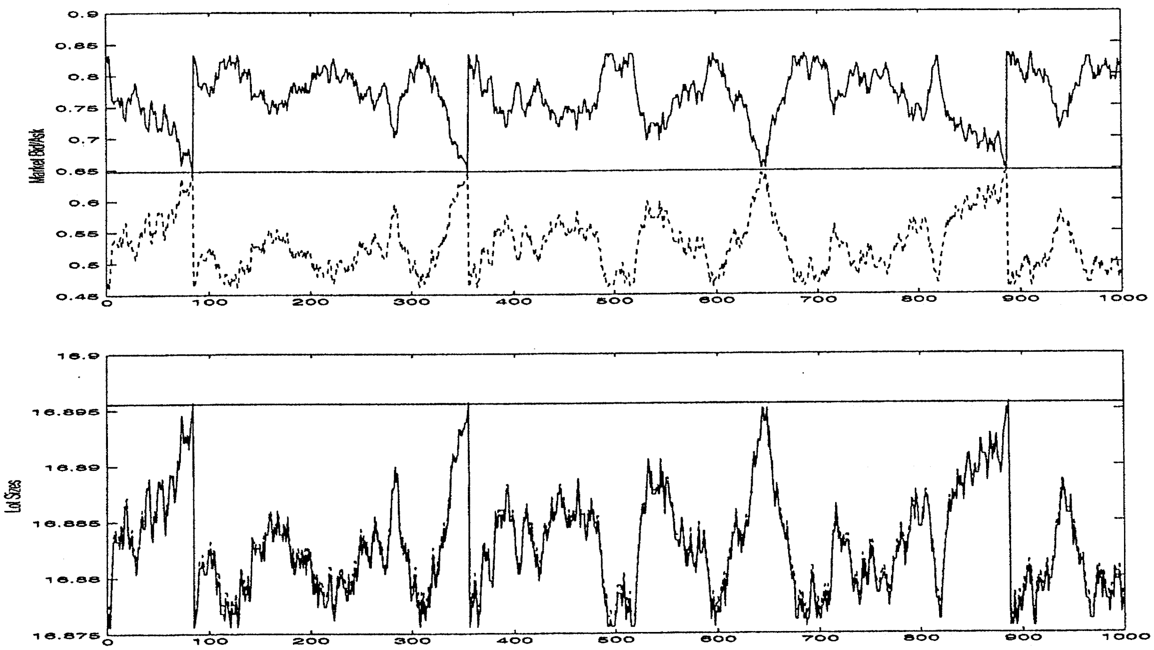


Figure 1.8: A single realization of the economy. In the top panel, the dashed line represents the market bid price, and the solid line represents the market ask price. In the bottom panel, the dashed line represents the depth at the market bid price, and the solid line represents the depth at the market ask price. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 41.1808$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D\sigma_N$, $\gamma = 0.1$, $\bar{\theta} = 64.1127$, $\bar{a}_D = 0.05$, $P_t = 0.6486$, $\kappa_{\%} = 0.5\%$, and $\kappa = 0.00324$, $T = 1/2$, $N = 1000$. Here T is the number of years in the simulation and N is the number of points in the simulated Brownian motion.

Figure 1.8 shows the time evolution of the market bid/ask prices and of the number of shares offered and sought at the ask and bid (their depth), respectively. Note that the depth of the bid/ask prices is not constant over time, but its variation is much smaller than that in the bid/ask prices. We observe that the bid-ask spread approaches zero as a trade occurs and widens discretely right after the trade. This is intuitive because right after a trade, the desire for another trade is minimized. We also observe that the difference in depth between the bid and ask prices exhibits the same pattern, diminishing to zero as a trade occurs and widening discretely after the trade.

Figure 1.9 plots each agent's bid-ask spread and buy/sell amounts. Immediately after a trade, all of these variables revert discontinuously back to their level when each agent's endowment z_t is equal to $\bar{\theta}$ (i.e. as it is immediately after a trade). Interestingly the ultimate trade price is always the half-way point between the market bid and ask (the solid line in the top panel of Figure 1.8 is the mean of the current bid-ask prices).

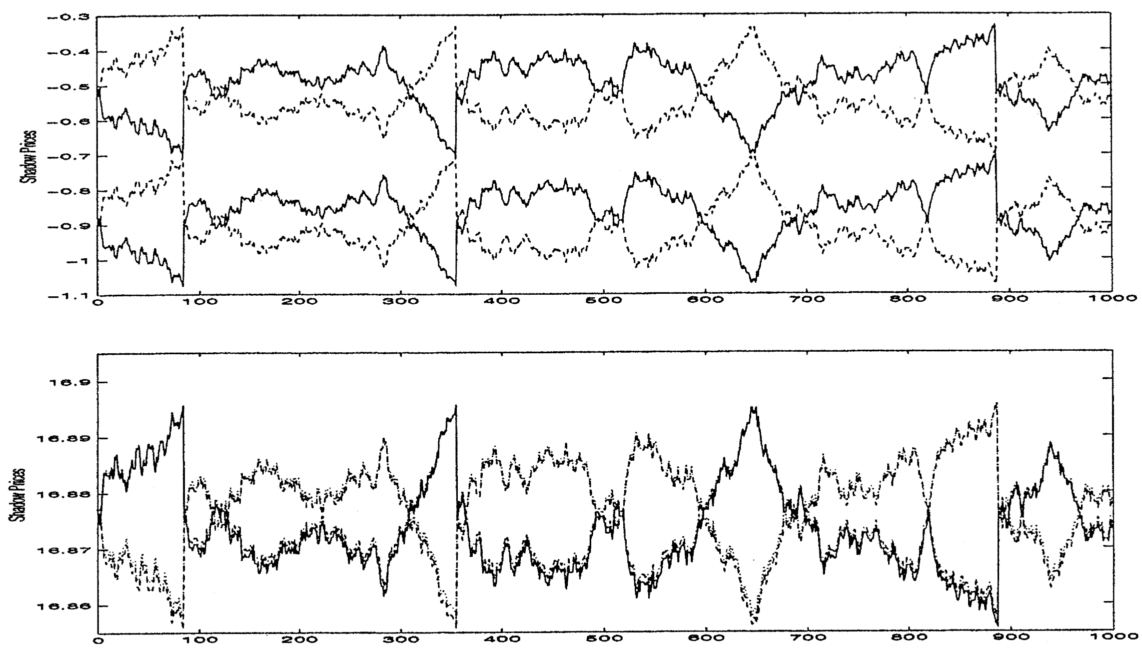


Figure 1.9: A single realization of the economy. In the top panel, the dashed lines represents one agent's bid and ask prices, and the solid lines represent the other's. In the bottom panel, the solid and dotted lines represent one agent's ask and bid amounts respectively. The dashed and dot-dashed lines represent the other agent's ask and bid amounts respectively. The parameter values are $r = 0.037$, $\rho = 0.07$, $\sigma_X = 41.1808$, $\sigma_D = 0.3311$, $\sigma_N = 0.3311$, $\sigma_{DN} = -\sigma_D \sigma_N$, $\gamma = 0.1$, $\bar{\theta} = 64.1127$, $\bar{a}_D = 0.05$, $P_t = 0.6486$, $\kappa_{\%} = 0.5\%$, and $\kappa = 0.00324$, $T = 1/2$, $N = 1000$. Here T is the number of years in the simulation and N is the number of points in the simulated Brownian motion.

1.6 A Calibration Exercise

Our model shows that even small fixed transactions costs imply a significant reduction in trading volume and an illiquidity discount in asset prices. To further examine the impact of fixed costs in equilibrium, we calibrate our model using historical data and derive numerical implications for the illiquidity discount, trading frequency, and trading volume. From (1.29), for small fixed costs κ we can re-express the illiquidity premium π as:

$$\pi = \frac{1}{\sqrt{6}} r^{-\frac{1}{2}} \gamma^{\frac{3}{2}} \sigma_N \sigma_X p_0 \kappa^{\frac{1}{2}} \tag{1.39}$$

Without loss of generality, we set $\sigma_N = 1$, hence the remaining parameters to be calibrated are: the interest rate r, the risk discount p_0, the volatility of the idiosyncratic non-traded risk σ_X, the agents' coefficient of absolute risk aversion γ, and the fixed transaction cost κ.

The starting point for our calibration exercise is a study by Campbell and Kyle (1993). In particular, they propose and estimate a detrended stock-price process of the following form:[26]

$$P_t = V_t - \frac{\lambda}{r} - Y_t \tag{1.40}$$

where V_t (the present value of future dividends discounted at the risk-free rate) is assumed to follow a Gaussian process, Y_t (fluctuations in stock demand) is assumed to follow an AR(1) Gaussian process, r is the risk-free rate, and λ/r is the risk discount. In the Appendix, we show that in the absence of transactions costs, the general non-traded income process (1.3) of our model yields the same price process as (1.40). Moreover, in our model λ/r is denoted by p_0 and Y_t is the aggregate exposure of non-traded risk, which generates changes in stock demand. Therefore, we can obtain values for r, p_0, γ, and σ_Y (the instantaneous volatility of Y_t) from their parameter estimates.

Campbell and Kyle based their estimates on annual time series of US real stock prices and dividends from 1871 to 1986. The real stock price of each year is defined by the Standard & Poors Composite Stock Price Index in January, normalized by the Producer Price Index in the same month. The real dividend each year is taken to be the annual dividend per-

[26] See Campbell and Kyle (1993, equation (2.3), p. 3).

share normalized by the Producer Price Index (over this sample period, the average annual dividend growth rate is 0.013). The price and dividend series are then detrended by an exponential detrending factor $\exp(-0.013\,t)$ and the detrended series are used to estimate (1.40) via maximum likelihood estimation. In particular, they obtain the following estimates for the risk-free rate:[27]

$$r = 0.037 \quad , \quad \lambda = 0.0210$$

Using these estimates, we are able to compute values for the following parameters:

$$\bar{a}_D = 0.050 \quad , \quad p_0 = 0.5676 \; .$$

Our model also contains the parameter σ_X, the volatility of idiosyncratic non-traded risk. Because it is the *aggregate* non-traded risk that affects prices, Campbell and Kyle (1993) only provides an estimate for the volatility σ_Y of *aggregate* non-traded risk as a function of the coefficient of absolute risk aversion γ.[28] Obtaining an estimate for the magnitude of σ_X requires data at a more disaggregated level, which has been performed by Heaton and Lucas (1996) using PSID data. Their analysis shows that the residual variability in the growth rate of individual income—the variability of the component that is uncorrelated with aggregate income—is 8 to 13 times larger than the variability in the growth rate of aggregate income. Based on this result, we use values for σ_X that are 1, 4, 8, and 16 times the value of σ_Y.

The two remaining parameters to be calibrated are the coefficient of absolute risk aversion γ and the fixed cost κ. Since there is little agreement as to what the natural choices are for these two parameters, we calibrate our model for a range of values for both.

Tables 1.1–1.4 report the results of our calibrations. Each of the four tables corresponds to a separate value for the variability σ_X of the idiosyncratic component of income: Table 1.1 sets $\sigma_X = \sigma_Y$, Table 1.2 sets $\sigma_X = 4\sigma_Y$, Table 1.3 sets $\sigma_X = 8\sigma_Y$, and Table 1.4 sets $\sigma_X = 16\sigma_Y$. Within each table, there are four sub-panels. The first sub-panel reports the share price P_t (in dollars) of the risky asset as a function of the absolute risk aversion coefficient γ and the

[27]See Campbell and Kyle's (1993, p. 20) estimates for "Model B".

[28]This was not an oversight on their part, but is merely due to the fact that the idiosyncratic component does not affect prices, and was irrelevant for their purposes. The functional relation between σ_Y and γ implied by their estimates is given in the Appendix.

standard deviation σ_I of annual income. The second sub-panel reports expected trade inter-arrival times τ (in years), the third sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} \equiv \bar{a}_D/r - \bar{p}_0$ in a frictionless economy), and the fourth sub-panel reports the trade size δ (in shares of the risky asset), all as functions of the transactions cost κ, which ranges from 1 basis point to 5 percent of $\bar{P}$,[29] and the absolute risk aversion coefficient γ, which ranges from 0.001 to 5.000. Our motivation for selecting the latter range for γ was derived from inspecting the relation between γ and the certainty equivalent of two specific gambles under constant-absolute-risk-aversion preferences, displayed in Figure 1.10. For Gamble A, a 50/50 gamble to win either \$1 or nothing, Figure 1.10 shows that risk aversion parameters between 0.000 and 5.000 yield certainty equivalents between 50 and 15 cents. However, for Gamble B, a 50/50 gamble to win either \$1,000 or nothing, Figure 1.10 shows that relevant range for the risk aversion parameter lies somewhere between 0.000 and 0.010 since in this range, the certainty equivalents fall between \$500 and \$100.

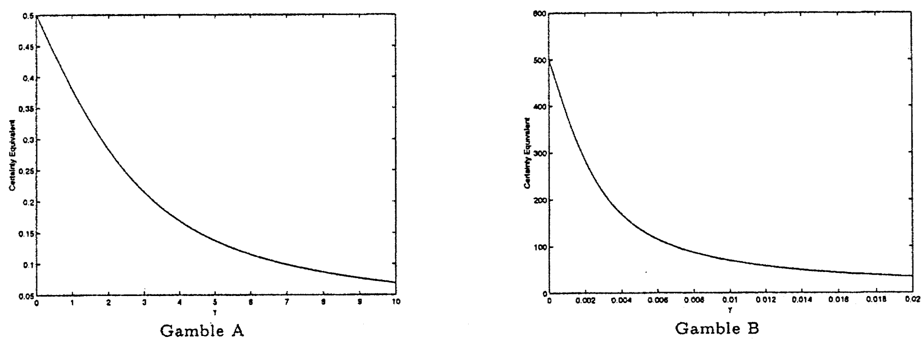


Figure 1.10: Certainty equivalents of two gambles for constant absolute risk aversion preferences, as a function of the risk aversion parameter γ. Gamble A: \$0 with probability 0.5 and \$1 with probability 0.5. Gamble B: \$0 with probability 0.5 and \$1,000 with probability 0.5.

The entries in Tables 1.1–1.4 show that our model is capable of yielding empirically plausible values for trading frequency, trading volume, and the illiquidity discount. In contrast to much of the existing literature, e.g., Huang (1990), Schroeder (1998), Vayanos (1998), we find that transactions costs can have very large impact on both the trading frequency as well as the illiquidity discount in the stock price. For example, Schroeder (1998) finds that when faced with a fixed transactions cost of 0.1%, individuals in his model will trade once every 10 years! In Table 1.1, we see that for a 0.1% fixed cost, individuals in our model will trade

[29] We display transactions costs as a percentage of $\bar{P}$ simply to provide a less scale-dependent measure of their magnitudes. Since κ is a fixed cost, its value is, by definition, scale-dependent and must therefore be considered in the complete context of the calibration exercise.

anywhere between once every 4.3 years and once every 0.003 years (or 333 times per year) as the risk aversion parameter varies from 0.001 to 5.000, respectively. This striking difference between our results and those of the existing literature stems from the fact that our agents have a strong need to trade frequently. The high-frequency nature of their endowment level implies that not trading can be very costly. Furthermore, not trading means that the risk exposure from holding market-clearing levels of the stock is much greater. Other transactions cost models fail to account for a high-frequency component in trading needs.[30]

To get a feel for the level of proportional costs implied by our choice of fixed transaction cost, we report in the bottom panel of Tables 1.1–1.4 the cost as a percentage of the total transaction amount (that is $100 \times \kappa/\delta P$). The proportional cost depends on the choice of fixed cost and the risk-aversion parameter. From Table 1.1, for example, we see that the proportional cost ranges from 0 to 2.6% of the total transaction amount, clearly an empirically plausible range.

As the risk aversion parameter increases, and therefore the endowment volatility σ_X decreases (recall that in the calibration $\gamma\sigma_X$ is a constant), while holding κ fixed, trading becomes less frequent, the illiquidity discount increases, and the turnover also declines.[31] For example, a risk aversion parameter of 5.00 and a fixed cost of 1% of $\bar{P}$ implies that the investor will trade approximately once every two years, with an annual turnover of 24.7%, an illiquidity discount of 1.547% of $\bar{P}$, and a proportional cost of 77 basis point. Of course, Figure 1.10 shows that a risk aversion coefficient of 5.00 is quite extreme for large gambles, hence this case is not particularly compelling from a practical point of view.

As the fixed cost increases while holding γ fixed, the average inter-arrival time and the trade size increase, while the turnover decreases. For risk aversion parameters less than 0.500, the illiquidity discount is never greater than 1.088% over the entire range of fixed costs, but the trading frequency and turnover are considerably more sensitive. For example, for a risk aversion parameter of 0.010, the trading profile varies from $1/.008 = 125.0$ trades per year with an annual turnover of 370%, to $1/0.187 = 5.3$ trades per year with an annual

[30]While some partial equilibrium models, such as Constantinides (1986) and Amihud and Mendelson (1986), do contain a high-frequency component in the uncertainty faced by their investors they still miss these effects because they do not take into account the unwillingness of investors to hold large amounts of the risky asset in the presence of transactions costs.

[31]Turnover is defined in our case as the trade amount per average unit time (in years) expressed as a percentage of the total supply of shares of the risky asset.

turnover of 78%, as the fixed costs vary from 1 basis point to 5 percent of $\bar{P}$, despite the fact that the illiquidity discount increases by only to 14.5 basis points over this range.

As the volatility σ_X of the idiosyncratic component of income increases, Tables 1.2–1.4 show that for each risk-aversion-parameter/fixed-cost combination, trading frequency increases, the turnover increases, and the illiquidity discount increases. For example, in Table 1.3 where σ_X is set at $8\sigma_Y$ (which is in the range that Heaton and Lucas (1996) estimate using PSID data), a risk aversion parameter of 0.50 and a fixed cost of 5.0% of $\bar{P}$ imply a trading frequency of $1/0.166 \approx 6$ trades per year, an annual turnover of 664%, an illiquidity discount of 9.7 percent, and a proportional cost of 16 basis points. These results suggest that transaction costs may, in fact, have substantial effects on asset prices. Furthermore, existing levels of trading frequency and volume in financial markets may not be as unusual or as irrational as many have thought. Although static equilibrium asset-pricing models may not be able to justify significant amounts of trading activity, the calibration results in Tables 1.1–1.4 show that our dynamic equilibrium model is clearly capable of generating empirically plausible implications.

1.7 An Empirical Test

Fixed costs have a number of empirical implications for asset prices, trading volume, trading frequency, trade size, and bid/ask spreads and depths, as Sections 1.4–1.6 demonstrate. Perhaps the most direct implications are the power laws for trade sizes δ and inter-arrival times $\Delta\tau$ implied by Theorem 1.3:

$$\delta \approx \phi\kappa^{\frac{1}{4}} \quad , \quad \Delta\tau \approx \left(\phi^2/\sigma_z^2\right)\kappa^{\frac{1}{2}} . \tag{1.41}$$

A direct test of (1.41) can be readily performed by regressing $\log\delta$ and $\log\Delta\tau$, respectively, on $\log\kappa$ and testing the null hypotheses that the slope coefficients are $\frac{1}{4}$ and $\frac{1}{2}$, respectively. However, κ is generally not observable, hence the direct approach is difficult to implement.[32]

An indirect test of (1.41) can be performed by combining the two equations to yield the

[32] While certain components of fixed costs for stock trading are observable, e.g., ticket charges, there are other unobservable components that may be considerably larger, such as the opportunity cost of the time and effort spent on information acquisition and processing as well as the decision making and implementation involved in the trading process.

rather unexpected relation:

$$\delta \;\approx\; \sigma_z(\Delta\tau)^{\frac{1}{2}} \tag{1.42}$$

which can be tested by regressing the logarithm of trade size on the logarithm of inter-arrival times and testing the null hypothesis that the slope coefficient is $\frac{1}{2}$. This is a less-than-satisfying test of the impact of fixed costs on δ and $\Delta\tau$ because κ does not appear in (1.42).

A more compelling test of our model of fixed costs can be developed by applying (1.41) and (1.42) to the case of stock splits. The typical motivation for stock splits is a method for enhancing liquidity in the face of indivisibilities associated with round-lot trading conventions, high share prices, or exchange-mandated minimum price variation rules.[33] For example, if round lots are cheaper to trade than odd lots, then a 2:1 stock split will reduce the cost of trading 50 pre-split shares. Such arguments for increased post-split liquidity are based on a decrease in fixed costs.[34] If we denote by κ_b and κ_a the fixed cost of trading before and after an $s:1$ split, respectively, and denote by δ_b and δ_a the optimal number of shares traded before and after an $s:1$ split, respectively, then we have:

$$\delta_b \;\approx\; \phi\kappa_b^{\frac{1}{4}} \quad , \quad \frac{\delta_a}{s} \;\approx\; \phi\kappa_a^{\frac{1}{4}} \tag{1.43}$$

where δ_a is renormalized by the split factor s because the split should have no impact on the optimal trade size (other than through its impact on κ). This yields the relation

$$\xi_\delta \;\equiv\; \log\left(\frac{\delta_a/s}{\delta_b}\right) \;=\; \tfrac{1}{4}\log\left(\frac{\kappa_a}{\kappa_b}\right) . \tag{1.44}$$

[33] See, for example, Angel (1997). Another motivation for stock splits is a signaling mechanism for revealing private information to investors; see Brennan and Copeland (1988), McNichols and Dravid (1990), and Pilotte and Manuel (1996). Muscarella and Vetsuypens (1996) attempt to differentiate between these two motives empirically using ADR "solo splits" and conclude that the liquidity effect dominates.

[34] Proportional costs are also affected by a split, but the evidence seems to suggest that these costs *increase*. For example, Conroy, Harris, and Benet (1990) conclude that the percentage bid/ask spreads of NYSE-listed companies typically increase after splits. Therefore, if liquidity enhancement is indeed and outcome of a stock split, it must be accomplished through a reduction in fixed costs. An indirect indication that splits reduce fixed costs is the fact that the number of shareholders tends to increase after a split, documented by Barker (1956) and Lamoureux and Poon (1987).

A similar relation for $\Delta\tau$ follows from (1.41):

$$\xi_{\Delta\tau} \equiv \log\left(\frac{\Delta\tau_a}{\Delta\tau_b}\right) = \tfrac{1}{2}\log\left(\frac{\kappa_a}{\kappa_b}\right) \tag{1.45}$$

and combining (1.44) with (1.45) yields

$$\zeta \equiv \frac{\xi_\delta}{\xi_{\Delta\tau}} \approx \frac{1}{2} \tag{1.46}$$

which is an empirically testable implication that has the advantage of involving a clear and significant change in fixed costs (otherwise companies would not go to the expense of a split) without the need to observe the magnitudes of those costs. Moreover, (1.44) and (1.45) provide two theoretically independent estimates of the change in fixed costs after a split. We examine these implications in Sections 1.7.1–1.7.3.

1.7.1 The Data

To empirically test (1.46), we begin by identifying all stock splits that occurred during the period from January 1, 1993 to December 31, 1997 using the Center for Research in Security Prices (CRSP) event file. To ensure that our sample consists only of splits, we select only those stocks whose "share factor" changes match their "price factor" changes, and we eliminate all split events in which the split factor "`facshr(i)`" is not an integer when multiplied by 1, 2, 3, 4, or 5. This yields 2,842 split events during the five-year period.

For each of these split events, we use the New York Stock Exchange's Trades and Quotes (TAQ) database to obtain the trades and time stamps for these stocks over a 14-day windowed centered symmetrically around the split date. Some of the stocks identified in the CRSP database were not present in the TAQ database, hence we dropped the events of such stocks from our sample. For the remaining events, we collect all trades from the TAQ database for the 14-day window surrounding each event, dropping TAQ observations with correction codes 7–12 (see the TAQ User's Guide for more information), or observations missing a time stamp, trade size, or price. This leaves a total of 2,169 split events and 6,495,403 trades. Table 2 summarizes the number of split events in our sample according to split factor and year. Note that the more extreme split events, 2:1 and 3:2, dominate the sample in all years, accounting for at least 80% of all the split events in each year.

For each stock and each event, we eliminate the lowest and highest 5% of the trade sizes and inter-arrival times during the 14-day window to reduce the impact of outliers, and use the remaining trade sizes and inter-arrival times to perform our empirical analysis. If a stock had no data for trade size or inter-arrival times either before or after the split, we eliminate that event from our sample.

Table 3 reports means and standard deviations for trade size δ and inter-arrival times τ over 1-day, 2-day, 3-day, and 7-day intervals before and after splits. For a 1-day window and the entire sample of split events, the pre-split average trade size and inter-arrival time are 1139 shares and 728 seconds, respectively; the post-split average trade size and inter-arrival time are 740 shares and 503 seconds, respectively. Using a longer window yields similar results as the rest of Table 3 shows—splits do enhance liquidity in the sense that average trade sizes and inter-arrival times always decline after splits, i.e., more frequent trading of smaller lots. Therefore, it is likely that fixed costs have declined after the split date.

1.7.2 The Empirical Results

To compute the ratio ζ in (1.46), we first construct the quantities

$$\log(\overline{\delta}^a/(s\overline{\delta}^b)) \qquad \text{and} \qquad \log(\overline{\Delta\tau}^a/(s\overline{\Delta\tau}^b))$$

for each split event using the pre- and post-split average trade sizes and inter-arrival times for 1-day to 7-day windows. We eliminate the lowest and highest 5% of these log-ratios from our sample to reduce the impact of outliers, and with the remaining sample we compute the ratio ζ for each event and summarize the sampling distributions of these ratios in Table 4.

The entries in the '1-Day' sub-panel show that the average *zeta* using the entire sample of split events is 0.482, which is remarkably consistent with the theoretical value of $\frac{1}{2}$ given in (1.46). Similar averages are obtained for 2:1 and 3:2 splits. However, 4:3 and 5:4 splits yield an average ζ of -0.583 and -0.150, respectively, for the 1-day window. The same patterns emerge from 2-day and 3-day windows: the average ζ is approximately $\frac{1}{2}$ when the entire sample of split events is used, but deviates significantly from $\frac{1}{2}$ for 4:3 and 5:4 splits. Not surprisingly, the 7-day window results are the farthest from (1.46)—over longer periods, factors other than fixed costs will influence trade size and inter-arrival times, adding noise to the power laws on which (1.46) is based. But overall, the relation (1.46) seems to be well supported by the majority of splits in our sample, especially those that involve more extreme split factors, which are precisely the cases in which the reduction in fixed costs are expected to be the greatest.

1.7.3 A Control

A natural control to our empirical analysis in Section 1.7.2 is to consider the implications of (1.41) for non-split dates. In particular, let $(\delta_b, \Delta\tau_b)$ and $(\delta_a, \Delta\tau_a)$ denote the optimal trade size and inter-arrival time before and after an arbitrary non-split date, respectively. Then $\delta_b = \delta_a$, $\Delta\tau_b = \Delta\tau_a$, and the split factor $s = 1$, which implies

$$\xi_\delta \;=\; \xi_{\Delta\tau} \;=\; 0\,. \tag{1.47}$$

Table 5 reports estimates of ξ_δ and $\xi_{\Delta\tau}$ for the same data set used in Table 4, but where the "before" and "after" windows are centered either before the split date or after the split date, and where a 1-day window is used to compute average trade sizes and inter-arrival times. For example, the first sub-panel labeled '*Dates* -5 *and* -4' contains estimates for ξ_δ and $\xi_{\Delta\tau}$ where the "before" period is the fifth day before the split and the "after period is the fourth day before the split. In contrast to the entries in Table 4, the estimates of ξ_δ and $\xi_{\Delta\tau}$ are considerably smaller in magnitude and fluctuate around 0.000 without any discernible pattern. These results, and those of Table 4, suggest that our model of fixed costs may be a reasonable approximation for US equity markets.

1.8 Conclusions

We have developed a continuous-time equilibrium model of asset prices and trading volume with heterogeneous investors and fixed transactions costs. With prices, trading volume, and inter-arrival times determined endogenously, we show that even a small fixed cost of trading can have a substantial impact on the frequency of trade. Investors follow an optimal policy of not trading until their risk level reaches either a lower or upper boundary, at which point they incur the fixed cost and trade back to an optimal level of risk exposure. As the investors' endowment uncertainty increases, their "no-trade" region increases as well, despite the fact that the expected time between trades declines. Investors optimally balance their desire to hedge their endowment risk exposure against the fixed cost of transacting.

We also show that small fixed costs can induce a relatively large premium in asset prices. The magnitude of this illiquidity premium is more sensitive to the risk aversion of agents than is the risk premium. Because investors must incur a transactions cost with every trade, they do not rebalance very often. In between trades, they face some uncertainty as to the level of their holdings of the risky asset. This increases the effective risk faced by the investor for holding the risky asset, which reduces his demand for the risky asset at any given price, and to clear the market, the equilibrium price must compensate investors for the illiquidity of the shares that they hold. The price effect, then, relies heavily on the market-clearing motive, hence partial equilibrium models are likely to underestimate the effect of transactions costs on asset returns because they ignore this mechanism.

Because our model is dynamic, the market-clearing condition we propose has an auxiliary requirement: agents must want to trade at the same time. Imposing this double "double coincidence of wants" endogenizes the market's order flow, and inter-arrival times between trades are determined in equilibrium as well as the quantities traded. Despite the fact that every buyer must have a seller and vice versa, we allow the fixed cost to be divided endogenously between the buyer and seller so that one agent can bear a larger share of the cost to induce the other agent to trade earlier than he otherwise would. This division of the fixed cost between buyer and seller is a means of representing the compensation for the provision of "immediacy" that typically accrues to market makers, and provides a natural bridge between the asset-pricing literature (in which risk sharing is the prime motive for trading) and the market microstructure literature (in which the facilitation of trade through market making activities is the main focus).

Although our model has many interesting theoretical and empirical implications, it is admittedly a rather simple parameterization of a considerably more complex set of phenomena. In particular, our assumption of perfect correlation between the dividend and endowment flows is likely to exaggerate the hedging motive in our economy. If a perfect hedging vehicle were not available, then individuals would certainly trade less often. However, as the amount of risk that cannot be shared increases at the economy-wide level, the equilibrium price-effect may actually increase because individuals must bear an ever increasing amount of uncertainty at market-clearing levels of asset holdings. The persistence of the endowment shocks in our economy does increase both the illiquidity discount and the desire to trade. Moreover, we do not allow for an aggregate endowment component (indeed our aggregate endowment is exactly zero), which certainly does exist in reality. All of these are interesting and important extensions of our model.

Another set of questions has to do with the effects of investor and security heterogeneity. For example, Vayanos and Vila (1999) and Huang (1998) consider the implications of transactions costs that are asymmetric across different securities. Also, fixed costs may differ across individuals. Who, then, is the marginal, or price-setting investor? It is unclear what effect transactions costs may have in the presence of many small heterogeneous agents. It is a fundamental question as to whether a CAPM-type result holds in the presence of multiple securities and an entry fee (i.e. the fixed cost is only paid to enter the market, but then agents may transact in any security). A more complete understanding of transactions costs will involve a resolution of some of these outstanding issues.

1.9 Appendix

A.1 Derivation of the Bellman Equation (1.16)

Given the conjectured value function in (1.13) and the equations for X_t and M_t in (1.3b) and (1.5), respectively, for $t \in (\tau_k,\ \tau_{k+1})$ (i.e., between trades) the Bellman equation (1.14) leads to:

$$0 = \sup_{c} \left\{ -e^{-\rho t - \gamma c} + J\left[-\rho - r\gamma(rM + \theta\bar{a}_D - c) + \tfrac{1}{2}(r\gamma)^2\sigma_D^2(\theta - hX)^2 + \tfrac{1}{2}\sigma_X^2(V_X^2 - V_{XX}) \right] \right\}. \tag{A.1}$$

The first order condition with respect to c is $e^{-\rho t - \gamma c} = -rJ$, which gives the optimal consumption in (1.15) (the second order condition is always satisfied given the concavity of the utility function). Substituting the optimal consumption back into (A.1), we obtain the differential equation in (1.16). The boundary conditions needed for its solution are discussed in the text.

At the optimum, the Bellman equation requires that $\mathcal{D}[J] = e^{-\rho t - \gamma c} = -rJ$. It follows that $\mathrm{E}_0[J(M_T, \cdot, T)] \to 0$ as T gets large. Thus, the transversality condition (1.7) is satisfied.

A.2 Proof of Theorem 1.1

When $\kappa = 0$, the conjectured price process is a constant. The agents' conjectured value function has the form: $J(W, X, t) = -e^{-\rho t - r\gamma W - v(X) - \bar{v}}$, where $\bar{v} = (\rho - r + r\ln r)/r$. The Bellman equation for each agent's optimization problem has the same form as in (1.14), except that the agent trades continuously to choose the optimal θ. His budget constraint can be expressed in terms of his wealth, given in Footnote 8 without the terms associated with transactions costs. In particular, we have for $t \geq 0$:

$$\begin{aligned} 0 &= \sup_{c,\,\theta} \left\{ -e^{-\rho t - \gamma c} + \mathcal{D}[J] \right\} \\ &= \sup_{c,\,\theta} \left\{ -e^{-\rho t - \gamma c} + J\left[-\rho - r\gamma(rW + \theta r p_0 - c) + \tfrac{1}{2}(r\gamma)^2\sigma_D^2(\theta - hX)^2 + \tfrac{1}{2}\sigma_X^2\left(v'^2 - v''\right) \right] \right\}. \end{aligned} \tag{A.2}$$

The optimal policies are given by

$$c = -\frac{1}{\gamma}[\ln r - \gamma r W - v(X)] \quad \text{and} \quad \theta_t = \frac{p_0}{\gamma\sigma_D^2} + hX_t.$$

Substituting the optimal policies into (A.2), we find the solution for the value function: $v(X) = v_0 + r\gamma p_0 hX$, where $v_0 = \frac{1}{2}r\gamma^2\sigma_D^2\bar{z}_m^2(1-\gamma^2\sigma_N^2\sigma_X^2)$ and $\bar{z}_m = p_0/(\gamma\sigma_D^2)$. The same argument as in the $\kappa > 0$ case shows that the transversality condition is satisfied.

Since for agents $i = 1, 2$, $X_t^1 = -X_t^2$. Market clearing only requires that $p_0 = \gamma\bar{\theta}\sigma_D^2$ and the equilibrium price is indeed constant.

Proof of Theorem 1.2

When $k = \widehat{\kappa}\,1_{\{t>0\}}$ with $\widehat{\kappa} \to \infty$, agents do not trade for $t > 0$. We conjecture that for $t > 0$, $J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M - \theta a_D/r) - V(\theta, X)}$. The corresponding Bellman equation is identical to (A.1). Again the optimal choice of c is given by (1.15). The Bellman equation has the following solution: $V(\theta, X) = v_0 - v_2(\theta - hX)^2$ where

$$v_0 = \bar{v} - 1 - \tfrac{1}{4}r\left(1 - \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}\right) \quad \text{and} \quad v_2 = \frac{r\gamma^2\sigma_D^2}{1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}}.$$

Apparently, we must require that $4\gamma^2\sigma_N^2\sigma_X^2 < 1$.

At $t = 0$, agents are free to trade costlessly. They choose the optimal θ to maximize their expected utility:

$$\theta = \arg\sup\ J(M - \theta P, \theta, X, 0) = \frac{r\gamma p_0}{2v_2} + hX.$$

where $P = \bar{a}_D/r - p_0$. Since for agents $i = 1, 2$ we have $X_t^1 = -X_t^2$, the market clearing condition, $\theta_1 + \theta_2 = \bar{\theta}$, requires that $\bar{\theta} = 2r\gamma p_0/v_2$. Thus, the risk discount is

$$p_0 = \frac{2\bar{\theta}v_2}{r\gamma} = \bar{p}_0\frac{2}{1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}} = \bar{p}_0\left(1 + \frac{1 - \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}}{1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}}\right)$$

where $\bar{p}_0 = \gamma\sigma_D^2\bar{\theta}$.

Proof of Theorem 1.3

When the value function is analytic in the interval (z_l, z_u), we can express it in the form of a Taylor series:

$$v(z) = \sum_{k=0}^{\infty} \tfrac{1}{k!} v_k (z - z_m)^k. \tag{A.3}$$

Substituting this into (1.17), we obtain

$$\begin{aligned} \sigma_z^2 \sum_{n=0}^{\infty} \tfrac{1}{n!} v_{n+2} (z - z_m)^n &= \sigma_z^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \tfrac{1}{(n-m)!m!} v_{n-m+1} v_{m+1} (z - z_m)^n \\ &+ 2r \sum_{n=0}^{\infty} \tfrac{1}{n!} v_n (z - z_m)^n + (r\gamma)^2 \sigma_D^2 z^2. \end{aligned} \tag{A.4}$$

Since the expansion is around z_m, we can write z^2 as $z^2 = (z - z_m)^2 + 2z_m(z - z_m) + z_m^2$. Matching powers on both sides of equation (A.4), we have the following conditions for the coefficients in the Taylor series:

$$0 = \sigma_z^2 \left(v_1^2 - v_2 \right) + 2rv_0 + (r\gamma)^2 \sigma_D^2 z_m^2 \tag{A.5a}$$

$$0 = \sigma_z^2 \left(2v_1 v_2 - v_3 \right) + 2rv_1 + 2(r\gamma)^2 \sigma_D^2 z_m \tag{A.5b}$$

$$0 = \sigma_z^2 \left(v_2^2 + v_1 v_3 - \tfrac{1}{2} v_4 \right) + rv_2 + (r\gamma)^2 \sigma_D^2 \tag{A.5c}$$

$$0 = \sigma_z^2 \left[\sum_{l=0}^{k} \tfrac{1}{l!(k-l)!} v_{l+1} v_{k-l+1} - \tfrac{1}{k!} v_{k+2} \right] + \tfrac{2}{k!} r v_k \quad \forall\, k > 2. \tag{A.5d}$$

It is obvious that the other coefficients in the Taylor series can be expressed as polynomials of only two coefficients, v_1 and v_2. Note that v_0 does not enter into any of the higher order coefficients in (A.5d). Solving the value function (and the optimal trading policy) now reduces to solving v_1 and v_2. It is immediate that the smooth-pasting condition gives

$$v_1 = -r\gamma p_0. \tag{A.6}$$

The remaining conditions determine v_2 and the policy parameters, z_l, z_m, and z_u, which depend on κ^+, κ^-, p_0 and the other parameters of the model.

When κ^+ and κ^- are small, we consider the solution to v_k, z_l, z_m, and z_u of the form:

$$v_k = \sum_{n=0}^{\infty} v_k^{(n)} \varepsilon^n \tag{A.7a}$$

$$z_m - z_l \equiv \delta^+ = \sum_{n=0}^{\infty} b^{(n)} \varepsilon^n \tag{A.7b}$$

$$z_u - z_m \equiv \delta^- = \sum_{n=0}^{\infty} s^{(n)} \varepsilon^n \tag{A.7c}$$

where $\varepsilon \equiv \kappa^{\frac{1}{4}}$.

The two value-matching conditions are

$$\tfrac{1}{2}v_2\delta^{-2} + \tfrac{1}{6}v_3\delta^{-3} + \tfrac{1}{24}v_4\delta^{-4} + \tfrac{1}{120}v_5\delta^{-5} + \cdots = -r\gamma\kappa^- \tag{A.8a}$$

$$\tfrac{1}{2}v_2\delta^{+2} - \tfrac{1}{6}v_3\delta^{+3} + \tfrac{1}{24}v_4\delta^{+4} - \tfrac{1}{120}v_5\delta^{+5} + \cdots = -r\gamma\kappa^+ \tag{A.8b}$$

One of the smooth pasting conditions is satisfied by $v_1 = -r\gamma p_0$. We write the remaining two as follows

$$v_2\delta^- + \tfrac{1}{2}v_3\delta^{-2} + \tfrac{1}{6}v_4\delta^{-3} + \tfrac{1}{24}v_5\delta^{-4} + \cdots = 0 \tag{A.9a}$$

$$v_2\delta^+ - \tfrac{1}{2}v_3\delta^{+2} + \tfrac{1}{6}v_4\delta^{+3} - \tfrac{1}{24}v_5\delta^{+4} + \cdots = 0 \tag{A.9b}$$

We have four equations and four unknowns: $v_2, z_m, \delta^-, \delta^+$ (the dependence on z_m enters through v_3 by (A.5b)). Using the equations for the coefficients given in (A.5), and the expansions of $v_k, z_m, \delta^-, \delta^+$ given in (A.7), we match powers of ε in (A.9) and (A.8). That is, for every $n = \{0, 1, 2, 3, \dots\}$, we write the system of equations involving ε^n. Each system is linear in the i-th order coefficients. Proceeding in this way we obtain

$$v_2 = -2r\gamma\phi^{-2}\varepsilon^2 + o(\varepsilon^2) \tag{A.10a}$$

$$v_3 = \frac{r\gamma}{\phi^3}\left(\tfrac{240}{55}k^{(1)} + \tfrac{78}{55}r\gamma p_0\phi\right)\varepsilon^2 + o(\varepsilon^2). \tag{A.10b}$$

With more work, we can compute higher order approximations for all the coefficients.

Proof of Theorem 1.4

First we set $\delta^+ = \delta^-$ in equation (1.25). This gives us the value of κ^+ in equation (1.26a). Then we set $z_m = \bar{\theta}$ in equation(1.25). This gives us p_0 in (1.26a).

The Numerical Solution

To solve the boundary value ODE problem, we use a first-order expansion finite difference scheme set out in Press, et. al. (1992). The general idea is to convert our equation into a set of two coupled first-order finite difference equations of the form

$$\begin{aligned} y_2' &= y_1 \\ \sigma_z^2 y_1' &= \sigma_z^2 y_1^2 + 2ry_2 + (r\gamma)^2\sigma_D^2 z^2 \end{aligned}$$

where $y_2(\cdot) = v(\cdot)$ and the derivatives are understood to mean $y'(z) = (y(z+\Delta) - y(z))/\Delta$ for a grid spacing Δ. The system is then iterated using a first-order Taylor approximation until convergence. The free boundaries are found by using a numerical root finder in Matlab to find values of (z_l, z_m, z_u) such that the value matching and smooth pasting conditions are satisfied.

We can solve for an equilibrium by finding values of (p_0, κ^+) such that the optimal policy (z_l, z_m, z_u) satisfies the market clearing conditions. However, this requires a nested iteration: to solve for the equilibrium price and transactions cost allocation we need to solve the free-boundary problem for each candidate (p_0, κ^+). A faster approach is to find values of (p_0, κ^+, δ) such that the solution to a free boundary problem with boundaries $(\bar{\theta} - \delta, \bar{\theta} + \delta)$ satisfies the optimality conditions for the policy $(\bar{\theta} - \delta, \bar{\theta}, \bar{\theta} + \delta)$. This avoids the nested iterations required by the first approach.

Proof of Theorem 1.5

Consider the case where the endowment level is deterministic $X_t = \bar{a}_X t$ $(\bar{a}_X \geq 0)$ and $\kappa^\pm = \kappa/2$. We conjecture that $J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M - \theta \bar{a}_D/r) - V(\theta, X)}$. The Bellman equation reduces to a differential equation for V:

$$0 = \tfrac{1}{2}(r\gamma)^2\sigma_D^2(\theta - hX)^2 + r(V - \bar{v}) - \bar{a}_X V_X \tag{A.11}$$

where $\bar{v} = (\rho - r + r\ln r)/r$. This is essentially the same equation as (1.16) (with a similar derivation), except that here $\sigma_X = 0$ and there is an additional term $-\bar{a}_X V_X$ due to the deterministic drift in X. Because the σ_X term drops out in this case, the differential equation for V is linear. Letting $z = \theta - hX$ and $V(\theta, X) = v(z) + \bar{v}$, we can reduce the PDE above to the following first-order linear free boundary problem

$$0 = \tfrac{1}{2}(r\gamma)^2\sigma_D^2 z + rv + \bar{a}_X h v'. \tag{A.12}$$

The boundary conditions are the same smooth-pasting and value matching conditions we had before, with the exception that the optimal policy consists of only two points: for the agent with endowment X_t, the optimal policy is (z_l, z_m) and for the agent with endowment $-X_t$, the optimal policy if (z_m, z_u) (z_m in both policies is the same). The reason for this type of policies is that the risk state z_t for one agent only decreases between trades, and for the other agent the risk state only increase. Thus for the agent with endowment X_t, z_t can only decrease and when it deviates sufficiently from the optimal point z_m, the agent rebalances back to the optimal point.

The solution to (A.12) is given by

$$v(z) = v_0 + 2v_1 z + v_2 z^2 + \beta e^{\alpha t}$$

where β is determined using the boundary conditions, and the other constants are given by

$$v_0 = -\bar{a}_X^2\gamma^2\sigma_N^2/r, \quad v_1 = \tfrac{1}{2}\bar{a}_X\gamma^2\sigma_N\sigma_D, \quad v_2 = -\tfrac{1}{2}r\gamma^2\sigma_D^2, \quad \alpha = -r\sigma_D/(\bar{a}_X\sigma_N).$$

For $P_t = \bar{a}_D/r - p_0$ and $\kappa^+ = \kappa^- = \kappa/2$, the boundary conditions for the X_t agent are

$$\begin{aligned} v(z_l) &= v(z_m) - r\gamma(\kappa/2 - p_0\delta^+) \\ v'(z_l) &= v'(z_m) = -r\gamma p_0 \end{aligned}$$

and the boundary conditions for the $-X_t$ agent are

$$v(z_u) = v(z_m) - r\gamma(\kappa/2 + p_0\delta^-)$$
$$v'(z_u) = v'(z_m) = -r\gamma p_0.$$

Given the solution for v, the boundary conditions for the $X_t = \bar{a}_X t$ agent are

$$-r\gamma p_0 = 2v_2 z_m + 2v_1 + \beta\alpha e^{\alpha z_m} \tag{A.13a}$$
$$0 = -2v_2\delta + \beta\alpha e^{\alpha z_m}(e^{-\alpha\delta} - 1) \tag{A.13b}$$
$$0 = v_2\delta^2 - (2v_2 z_m + r\gamma p_0)\delta + r\gamma\kappa/2 \tag{A.13c}$$

where $\delta^+ = \delta$. Note that for the $-X_t$ agent, if we replace $\bar{a}_X$ in (A.13) with $-\bar{a}_X$ and let $\delta^- = -\delta$, then the algebraic form of the boundary conditions remains exactly the same. Hence solving (A.13) for $\bar{a}_X$ and for $-\bar{a}_X$ gives us solutions for both agents' control problems. The unknown variables are (β, z_m, δ). We are unable to solve these non-linear algebraic equations in closed form. We expand the unknowns as follows

$$z_m = \sum_{i=0}^{\infty} z_m^{(i)}\varepsilon^i, \quad \delta = \sum_{i=1}^{\infty} \delta^{(i)}\varepsilon^i, \quad \text{and} \quad \beta = \sum_{i=0}^{\infty} \beta^{(i)}\varepsilon^i$$

where the appropriate power law is given by $\varepsilon = \kappa^{\frac{1}{3}}$. Substituting the expansions into (A.13), and collecting terms for successive powers of ε, we are left with a series of linear equations for the coefficients in the above expansions. Hence, we are able to solve for (β, z_m, δ) in the approximate form.

Proof of Theorem 1.6

We first show how to compute the agent's bid price and bid amount. The ask price and ask amount are handled in the analogous way. For an agent with risk level z_t, we find a price $P_t^B = \bar{a}/r - p_0^B$, such that $z_t + \delta^+(P_t^B) = z_m(P_t^B)$. Using the values of $\delta^{\pm}$ and z_m in Theorem 1.3 and doing a little algebra, we can solve for P_t^B, the agent's bid price, and for $\delta^+(P_t^B)$, the agent's bid amount.

Calibration

Here, we establish the equivalence between the model estimated by Campbell and Kyle (1993), with a price process given in (1.40), and our model in absence of transactions costs. Our model without transactions costs is analyzed in detail in Huang and Wang (1997). Given the dividend process (1.1), the agents' non-traded income (1.3), and their preferences, we have the following result:

Theorem 1.7 *In the economy defined in Section 1.2 with $\kappa = 0$, the equilibrium stock price is*

$$P_t = \frac{\bar{a}_D}{r} - p_0 - p_Y Y_t \tag{A.14}$$

where $p_0 = \gamma\bar{\theta}\,(\sigma_D^2 + 2p_Y\sigma_{DY} + p_Y^2\sigma_Y^2)$, $p_Y = [r\gamma\sigma_{DN} + (\sigma_{DY}/\sigma_Y^2)(r/2 + a_Y + r\gamma\sigma_{NY} - u)]\,/(r + u)$, *and* $u = \sqrt{-r^2\gamma^2\sigma_Y^2 + (r/2 + a_Y + r\gamma\sigma_{NY})^2}$.

(A.14) has exactly the same form as (1.40), with $V_t = \bar{a}_D/r$ and an additional scaling constant p_Y for Y_t. In order to match with the correlation structure in Campbell and Kyle (1993), we require $\sigma_{NY}/(\sigma_N\sigma_Y) = \sigma_{DY}/(\sigma_D\sigma_Y) \equiv \rho_Y$ and $\sigma_{DN} = -\sigma_D\sigma_N$ with $\sigma_N = 1$.

Let $Q_t \equiv \int_0^t (dP_t + dD_t - rP_t dt)$ denote the excess dollar return on one share of the stock and $M_t \equiv (r + a_Y)p_Y Y_t$. Then we have

$$dQ_t = (rp_0 + M_t)\,dt + b_Q dB_t \tag{A.15a}$$

$$dM_t = -a_M M_t dt + b_M dB_t \tag{A.15b}$$

where $a_M = a_Y$, $b_Q = b_D - p_Y b_Y$ and $b_M = (r + a_Y)p_Y b_Y$. Equation (A.15) is identical to the equations (3.3) and (3.4) in Campbell and Kyle (1993) (p. 10) to be estimated, except that they use M_t for the excess share return on the stock and N_t for the varying mean-return variable while we use Q_t and M_t, respectively. Following our notation, we have $\sigma_Q^2 = b_Q{}'b_Q = \sigma_D^2 + 2p_Y\sigma_{DY} + p_Y^2\sigma_Y^2$, $\sigma_M^2 = b_M{}'b_M = (r + a_Y)^2 p_Y^2\sigma_Y^2$, and $\sigma_{QM}^2 = b_Q{}'b_M = (r + a_Y)^2 p_Y^2(\sigma_{DY} + p_Y\sigma_Y^2)^2$.

Campbell and Kyle gave the following estimates $\sigma_Q = 0.3311$, $\sigma_M = 0.0173$ and $\sigma_{QM}/(\sigma_Q\sigma_M) = -0.5176$. Together with their estimates for $r = 0.0370$, $a_D = 0.050$ (the unconditional mean of V_t in equation (1.40)), $\lambda = 0.0210 = rp_0$ and $a_Y = 0.0890$, we have $\sigma_D = 0.2853$, $\rho_Y = -0.1194$, and $\gamma\sigma_Y = 1.347$.

References

Admati, A. and P. Pfleiderer (1988), "A Theory of Intraday Patterns: Volume and Price Variability", *Review of Financial Studies* 1, 3–40.

Aiyagari, R. and M. Gertler (1991), "Asset Returns with Transaction Costs and Uninsured Individual Risk," *Journal of Monetary Economy* , 27, 311-331.

Allen, F. and D. Gale (1994), *Financial Innovation and Risk Sharing*, MIT Press, Cambridge, MA.

Amihud, Y. and H. Mendelson (1986a), "Asset Pricing and the Bid-Asked Spread," *Journal of Financial Economics* 17, 223–249.

Amihud, Y. and H. Mendelson (1986b), "Liquidity And Stock Returns," *Financial Analysts Journal* 42, 43–48.

Angel, J. (1997), "Tick Size, Share Prices, and Stock Splits", *Journal of Finance* 52, 655–681.

Arnott, R. and W. Wagner (1990), "The Measurement And Control of Trading Costs", *Financial Analyst Journal* 46, 73–80.

Bagehot, W. (a.k.a. Jack Treynor) (1971), "The Only Game in Town", *Financial Analysts Journal* 22, 12–14.

Bensaid, B., Lesne, J., Pages, H. and J. Scheinkman (1992), "Derivative Asset Pricing With Transaction Costs", *Mathematical Finance* 2, 63–86.

Berkowitz, S., D. Logue and E. Noser, Jr. (1988), "The Total Cost Of Transactions On The NYSE", *Journal of Finance* 43, 97–112.

Birinyi, L. (1995), *What Does Institutional Trading Cost?* Greenwich, CT: Birinyi Associates, Inc.

Bodurtha, S. and T. Quinn (1990), "Does Patient Program Trading Really Pay?" *Financial Analysts Journal* 46, 35–42.

Brennan, M. (1975), "The optimal number of securities in a risky portfolio when there are fixed costs of transacting: theory and some empirical results," *Journal of Financial and Quantitative Analysis*, 483–496.

Brennan, M. and T. Copeland (1988), "Stock Splits, Stock Prices, and Transaction Costs", *Journal of Financial Economics* 22, 83–101.

Brinson, G., Hood, R. and G. Beebower (1986), "Determinants of Portfolio Performance", *Financial Analysts Journal* 42, 39–44.

Brinson, G., Singer, B. and G. Beebower (1991), "Determinants of Portfolio Performance II: An Update", *Financial Analysts Journal* 47, 40–48.

Cadenillas, A. and F. Zapatero (1999), "Optimal central bank intervention in the foreign exchange market," *Journal of Economic Theory*, 87, 218–242.

Cadenillas, A. and F. Zapatero (2000), "Classical and impulse stochastic control of the exchange rate using interest rates and reserves," *Mathematical Finance*, 10 (2), 141–156.

Campbell, J., S. Grossman and J. Wang (1993), "Trading Volume and Serial Correlation in Stock Returns", *Quarterly Journal of Economics* 108, 905–939.

Campbell, J. and A. Kyle (1993), "Smart Money, Noise Trading, and Stock Price Behavior", *Review of Economic Studies* 60, 1–34.

Chan, L, and J. Lakonishok (1993), "Institutional Trades and Intra-Day Stock Price Behavior", *Journal of Financial Economics* 33, 173–199.

Chan, L. and J. Lakonishok (1995), "The Behavior of Stock Prices Around Institutional Trades", *Journal of Finance* 50, 1147–74.

Collins, B. and F. Fabozzi (1991), "A Methodology for Measuring Transaction Costs", *Financial Analysts Journal* 47, 27–36.

Conroy, R., Harris, R. and B. Benet (1990), "The Effects of Stock Splits on Bid-Ask Spreads", *Journal of Finance* 45, 1285–1295.

Constantinides, G.M. (1976), "Stochastic cash management with fixed and proportional transaction costs," *Management Science*, 22 (12), 1320–1331.

Constantinides, G.M. (1986), "Capital Market Equilibrium with Transaction Costs," *Journal of Political Economy* , Vol.94 (4), 842-862.

Cuneo, L. and W. Wagner (1975), "Reducing the Cost of Stock Trading", *Financial Analysts Journal* 26, 35–44.

Demsetz, H. (1968), "The Cost of Transacting", *Quarterly Journal of Economics* 82, 33–53.

Dumas, B. (1992), "Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World", *Review of Financial Studies* 5, 153–180.

Easley, D. and M. O'Hara (1987), "Price, Trade Size, and Information in Securities Markets", *Journal of Financial Economics* 19, 69–90.

Eastham, J.F. and K.J. Hastings (1988), "Optimal Impulse Control of Portfolios," *Mathematics of Operations Research*, Vol. 13 (4), 588-605.

Epps, T. W. (1976), "The Demand For Brokers' Services: The Relation Between Security Trading Volume And Transaction Cost", *Bell Journal of Economics* 7, 163–196.

Foster, D. and S. Viswanathan (1990), "A Theory of the Interday Variations In Volume, Variance, and Trading Costs In Securities Markets", *Review of Financial Studies* 3, 593–624.

Gammill, J. and A. Perold (1989), "The Changing Character Of Stock Market Liquidity", *Journal of Portfolio Management* 15, 13–18.

Garman, M. and J. Ohlson (1981), "Valuation Of Risky Assets In Arbitrage-Free Economies With Transactions Costs", *Journal of Financial Economics* 9, 271–280.

Grossman, S.J. and G. Laroque (1990), "Asset Pricing and optimal portfolio choice in the presence of illiquid durable consumption goods," *Econometrica* , 58, 25-52.

Harrison, J.M. (1990), *Brownian Motion and Stochastic Flow Systems*, Krieger, Florida.

Harrison, J.M., T.M. Sellke, and A.J. Taylor (1983), "Impulse Control of Brownian Motion," *Mathematics of Operations Research*, Vol. 8 (3), 454-466.

Hasbrouck, J. and R. Schwartz (1988), "Liquidity and Execution Costs in Equity Markets", *Journal of Portfolio Management* 14, 10–16.

Heaton, John and Deborah J. Lucas (1996), "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," *Journal of Political Economy* , Vol.104 (3), 443-487.

Huang, Chi-fu and Henri Pages (1990), "Optimal consumption and portfolio policies with an infinite horizon: Existence and convergence," mimeo, *MIT*.

Huang, Jennifer, and Jiang Wang (1997), "Market Structure, Security Prices, and Informational Efficiency," *Macroeconomic Dynamics*, 1, 169-205.

Huang, Ming (1998), "Liquidity Shocks and Equilibrium Liquidity Premia," mimeo, *Stanford GSB.*

Huang, R. and H. Stoll (1995), "Dealer versus Auction Markets: A Paired Comparison of Execution Costs on NASDAQ and the NYSE", Working Paper 95-16, Financial Markets Research Center, Owen Graduate School of Management, Vanderbilt University.

Jarrow, R. (1992), "Market Manipulation, Bubbles, Corners, And Short Squeezes", *Journal of Financial and Quantitative Analysis* 27, 311–336.

Keim, D. and A. Madhavan (1995a), "The Anatomy of the Trading Process", *Journal of Financial Economics* 37, 371–398.

Keim, D. and A. Madhavan (1995b), "The Upstairs Market for Large-Block Transactions: Analysis and Measurement of Price Effects", to appear in *Review of Financial Studies.*

Keim, D. and A. Madhavan (1995c), "Execution Costs and Investment Performance: An Empirical Analysis of Institutional Equity Trades", working paper, School of Business Administration, University of Southern California.

Korn, R. (1998), "Portfolio optimization with strictly positive transaction costs and impulse control," *Finance & Stochastics*, 2, 85-114.

Kraus, A. and H. Stoll (1972), "Price Impacts of Block Trading on the New York Stock Exchange", *Journal of Finance* 27, 569–588.

Kyle, A. (1985), "Continuous Auctions And Insider Trading", *Econometrica* 53, 1315–1336.

Kyle, A. (1989), "Informed Speculation With Imperfect Competition", *Review of Economic Studies* 56, 317–356.

Lo, A. and J. Wang (1999a), "Trading Volume: Definitions, Data Analysis, and Implications of Portfolio Theory," *Review of Financial Studies* , forthcoming.

Lo, A. and J. Wang (1998b), "Trading Volume: Implications of An Intertemporal Equilibrium Model," work in progress, MIT.

Loeb, T. (1983), "Trading Cost: The Critical Link Between Investment Information and Results", *Financial Analysts Journal* 39, 39–44.

McNichols, M. and A. Dravid (1990), "Stock Dividends, Stock Splits, and Signaling", *Journal of Finance* 45, 857–879.

Merton, R. (1973), "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41, 867–887.

Muscarella, C. and M. Vetsuypens (1996), "Stock Splits: Signaling or Liquidity? The Case of ADR 'Solo-Splits"', *Journal of Financial Economics* 42, 3–26.

Pérold, A. (1988), "The Implementation Shortfall: Paper Versus Reality", *Journal of Portfolio Management* 14, 4–9.

Pilotte, E. and M. Timothy (1996), "The Market's Response to Recurring Events: The Case of Stock Splits", *Journal of Financial Economics* 41, 111–127.

Press, W.H., S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery (1992), *Numerical Recipes in C*, Cambridge University Press.

Schroeder, Mark (1998), "Optimal Portfolio Selection with Fixed Transaction Costs," working paper, Northwestern University.

Schwartz, R. and D. Whitcomb (1988), *Transaction Costs and Institutional Investor Trading Strategies.* Monograph Series in Finance and Economics 1988–2/3, New York: Salomon Brothers Center for the Study of Financial Institutions, New York University.

Sherrerd, K., ed. (1993), *Execution Techniques, True Trading Costs, and the Microstructure of Markets.* Charlottesville, VA: Association for Investment Management and Research.

Stiglitz, J. (1989), "Using Tax Policy to Curb Speculative Short-Term Trading", *Journal of Financial Services Research* 3, 101–115.

Stoll, H. (1989), "Inferring the Components of the Bid-Ask Spread: Theory and Empirical Tests", *Journal of Finance* 44, 115–134.

Stoll, H. (1993), *Equity Trading Costs.* Charlottesville, VA: Association for Investment Management and Research.

Summers, L. and V. Summers (1990a), "The Case for a Securities Transactions Excise Tax", *Tax Notes* (August 13), 879–884.

Summers, L. and V. Summers (1990b), "When Financial Markets Work Too Well: A Cautious Case for a Securities Transactions Tax", *Journal of Financial Services Research* 3, 261–286.

Tiniç, S. (1972), "The Economics of Liquidity Services", *Quarterly Journal of Economics* 86, 79–93.

Tobin, J. (1984), "On the Efficiency of the Financial Market System", *Lloyds Bank Review* 153, 1–15.

Treynor, J. (1981), "What Does It Take to Win the Trading Game?", *Financial Analysts Journal* 37, 55–60.

Tuckman, B. and J. Vila (1992), "Arbitrage With Holding Costs: A Utility-Based Approach", *Journal of Finance* 47, 1283–1302.

Turnbull, A. and R. White (1995), "Trade Type and the Costs of Making Markets", working paper, Western Business School University of Western Ontario.

Uppal, R. (1993), "A General Equilibrium Model of International Portfolio Choice", *Journal of Finance* 48, 529–553.

Vayanos, Dimitri (1998), "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model," *Review of Financial Studies* , Vol.11(1), 1-58.

Vayanos, D. and J.L. Vila (1999), "Equilibrium Interest Rate and Liquidity Premium With Transaction Costs," *Economic Theory*, 13, 509–539.

Vial, J.-P. (1972), "A continuous time model for the cash balance problem," in Szego/Shell (eds.), *Mathematical Methods in Investment and Finance*, North Holland, 244–291.

Wagner, W. (1993), "Defining and Measuring Trading Costs", in K. Sherrerd, ed.: *Execution Techniques, True Trading Costs, and the Microstructure of Markets.* Charlottesville, VA: Association for Investment Management and Research.

Wagner, W. and M. Banks (1992), "Increasing Portfolio Effectiveness Via Transaction Cost Management", *Journal of Portfolio Management* 19, 6–11.

Wagner, W. and M. Edwards (1993), "Best Execution", *Financial Analyst Journal* 49, 65–71.

Wang, J. (1994), "A Model of Competitive Stock Trading Volume", *Journal of Political Economy* 102, 127–168.

Table 1.1: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 1 (i.e. $\sigma_X = 1 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transaction cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\bar{\theta}\tau}$), and the fifth sub-panel reports the transaction cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta \bar{P}}$). These quantities are reported as functions of the transaction cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	1347.026	134.703	13.470	2.694	1.347	0.898	0.674	0.269
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.003	0.008	0.026	0.059	0.084	0.103	0.118	0.187
0.050	0.006	0.019	0.059	0.132	0.187	0.229	0.265	0.419
0.100	0.008	0.026	0.084	0.187	0.265	0.325	0.375	0.593
0.300	0.015	0.046	0.145	0.325	0.459	0.563	0.650	1.029
0.500	0.019	0.059	0.187	0.419	0.593	0.727	0.840	1.331
1.000	0.026	0.084	0.265	0.593	0.840	1.029	1.190	1.886
5.000	0.059	0.187	0.593	1.331	1.886	2.314	2.676	4.257
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.002	0.007	0.021	0.048	0.068	0.083	0.096	0.152
0.050	0.005	0.015	0.048	0.107	0.152	0.186	0.215	0.341
0.100	0.007	0.021	0.068	0.152	0.215	0.264	0.304	0.483
0.300	0.012	0.037	0.118	0.264	0.373	0.458	0.529	0.840
0.500	0.015	0.048	0.152	0.341	0.483	0.592	0.684	1.088
1.000	0.021	0.068	0.215	0.483	0.684	0.840	0.971	1.547
5.000	0.048	0.152	0.483	1.088	1.547	1.903	2.206	3.546
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.000	0.000	0.001	0.003	0.004	0.005	0.006	0.010
0.050	0.000	0.001	0.003	0.007	0.010	0.012	0.014	0.022
0.100	0.000	0.001	0.004	0.010	0.014	0.017	0.019	0.031
0.300	0.001	0.002	0.008	0.017	0.024	0.029	0.034	0.054
0.500	0.001	0.003	0.010	0.022	0.031	0.038	0.044	0.070
1.000	0.001	0.004	0.014	0.031	0.044	0.054	0.063	0.100
5.000	0.003	0.010	0.031	0.070	0.100	0.124	0.144	0.235
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	658.08	370.06	208.09	139.15	117.01	105.72	98.38	78.23
0.050	440.13	247.47	139.15	93.04	78.23	70.68	65.77	52.29
0.100	370.06	208.09	117.01	78.23	65.77	59.43	55.30	43.96
0.300	281.18	158.11	88.89	59.43	49.96	45.13	41.99	33.37
0.500	247.47	139.15	78.23	52.29	43.96	39.71	36.94	29.35
1.000	208.09	117.01	65.77	43.96	36.94	33.37	31.04	24.65
5.000	139.15	78.23	43.96	29.35	24.65	22.26	20.70	16.41
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.001	0.004	0.007	0.010	0.012	0.024
0.050	0.000	0.001	0.004	0.015	0.024	0.033	0.041	0.082
0.100	0.000	0.001	0.007	0.024	0.041	0.056	0.069	0.137
0.300	0.001	0.003	0.017	0.056	0.094	0.127	0.158	0.313
0.500	0.001	0.004	0.024	0.082	0.137	0.186	0.231	0.459
1.000	0.001	0.007	0.041	0.137	0.231	0.313	0.388	0.771
5.000	0.004	0.024	0.137	0.459	0.771	1.044	1.295	2.567

Table 1.2: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 4 (i.e. $\sigma_X = 4 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transaction cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\,\bar{\theta}\,\tau}$), and the fifth sub-panel reports the transaction cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transaction cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	5388.103	538.810	53.881	10.776	5.388	3.592	2.694	1.078
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.001	0.002	0.007	0.015	0.021	0.026	0.030	0.047
0.050	0.001	0.005	0.015	0.033	0.047	0.057	0.066	0.105
0.100	0.002	0.007	0.021	0.047	0.066	0.081	0.094	0.148
0.300	0.004	0.011	0.036	0.081	0.115	0.141	0.162	0.257
0.500	0.005	0.015	0.047	0.105	0.148	0.182	0.210	0.332
1.000	0.007	0.021	0.066	0.148	0.210	0.257	0.297	0.470
5.000	0.015	0.047	0.148	0.332	0.470	0.576	0.666	1.059
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.009	0.027	0.086	0.192	0.272	0.334	0.386	0.611
0.050	0.019	0.061	0.192	0.431	0.611	0.750	0.868	1.381
0.100	0.027	0.086	0.272	0.611	0.868	1.065	1.233	1.968
0.300	0.047	0.149	0.473	1.065	1.516	1.865	2.161	3.476
0.500	0.061	0.192	0.611	1.381	1.968	2.425	2.814	4.548
1.000	0.086	0.272	0.868	1.968	2.814	3.476	4.042	6.596
5.000	0.192	0.611	1.968	4.548	6.596	8.239	9.675	16.495
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.001	0.002	0.005	0.012	0.017	0.021	0.025	0.039
0.050	0.001	0.004	0.012	0.028	0.039	0.048	0.056	0.089
0.100	0.002	0.005	0.017	0.039	0.056	0.069	0.080	0.128
0.300	0.003	0.010	0.030	0.069	0.098	0.121	0.141	0.230
0.500	0.004	0.012	0.039	0.089	0.128	0.159	0.185	0.304
1.000	0.005	0.017	0.056	0.128	0.185	0.230	0.269	0.450
5.000	0.012	0.039	0.128	0.304	0.450	0.573	0.683	1.260
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	5264.66	2960.52	1664.79	1113.27	936.12	845.86	787.15	625.95
0.050	3520.68	1979.79	1113.27	744.43	625.95	565.58	526.31	418.48
0.100	2960.52	1664.79	936.12	625.95	526.31	475.54	442.52	351.83
0.300	2249.49	1264.94	711.25	475.54	399.82	361.23	336.13	267.19
0.500	1979.79	1113.27	625.95	418.48	351.83	317.86	295.77	235.07
1.000	1664.79	936.12	526.31	351.83	295.77	267.19	248.60	197.53
5.000	1113.27	625.95	351.83	235.07	197.53	178.38	165.92	131.61
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.001	0.002	0.004	0.005	0.006	0.012
0.050	0.000	0.000	0.002	0.007	0.012	0.017	0.021	0.041
0.100	0.000	0.001	0.004	0.012	0.021	0.028	0.035	0.069
0.300	0.000	0.001	0.008	0.028	0.047	0.064	0.079	0.157
0.500	0.000	0.002	0.012	0.041	0.069	0.093	0.116	0.230
1.000	0.001	0.004	0.021	0.069	0.116	0.157	0.194	0.386
5.000	0.002	0.012	0.069	0.230	0.386	0.523	0.649	1.287

Table 1.3: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 8 (i.e. $\sigma_X = 8 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transaction cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\,\theta\,\tau}$), and the fifth sub-panel reports the transaction cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transaction cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ σ_X	0.001 10776.2	0.010 1077.62	0.100 107.762	0.500 21.552	1.000 10.776	1.500 7.184	2.000 5.388	5.000 2.155
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.000	0.001	0.003	0.007	0.010	0.013	0.015	0.023
0.050	0.001	0.002	0.007	0.017	0.023	0.029	0.033	0.052
0.100	0.001	0.003	0.010	0.023	0.033	0.041	0.047	0.074
0.300	0.002	0.006	0.018	0.041	0.057	0.070	0.081	0.129
0.500	0.002	0.007	0.023	0.052	0.074	0.091	0.105	0.166
1.000	0.003	0.010	0.033	0.074	0.105	0.129	0.149	0.236
5.000	0.007	0.023	0.074	0.166	0.236	0.290	0.335	0.538
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.017	0.054	0.172	0.386	0.546	0.670	0.775	1.233
0.050	0.038	0.121	0.386	0.868	1.233	1.516	1.756	2.814
0.100	0.054	0.172	0.546	1.233	1.756	2.161	2.507	4.042
0.300	0.094	0.298	0.951	2.161	3.094	3.824	4.451	7.287
0.500	0.121	0.386	1.233	2.814	4.042	5.011	5.847	9.678
1.000	0.172	0.546	1.756	4.042	5.847	7.287	8.542	14.443
5.000	0.575	1.233	4.042	9.678	14.443	18.462	22.123	41.509
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.001	0.003	0.011	0.025	0.035	0.043	0.050	0.080
0.050	0.002	0.008	0.025	0.056	0.080	0.098	0.114	0.185
0.100	0.003	0.011	0.035	0.080	0.114	0.141	0.164	0.269
0.300	0.006	0.019	0.061	0.141	0.204	0.254	0.297	0.501
0.500	0.008	0.025	0.080	0.185	0.269	0.337	0.396	0.684
1.000	0.011	0.035	0.114	0.269	0.396	0.501	0.596	1.077
5.000	0.037	0.080	0.269	0.684	1.077	1.444	1.812	4.527
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	14890.69	8374.52	4708.68	3148.72	2647.65	2392.35	2226.27	1770.29
0.050	9959.04	5599.66	3148.72	2105.43	1770.29	1599.53	1488.44	1183.40
0.100	8374.52	4708.68	2647.65	1770.29	1488.44	1344.83	1251.39	994.81
0.300	6363.26	3577.71	2011.58	1344.83	1130.60	1021.42	950.39	755.25
0.500	5599.66	3148.72	1770.29	1183.40	994.81	898.69	836.14	664.26
1.000	4708.68	2647.65	1488.44	994.81	836.14	755.25	702.59	557.79
5.000	3149.32	1770.29	994.81	664.26	557.79	503.38	467.87	369.24
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.000	0.002	0.003	0.004	0.004	0.009
0.050	0.000	0.000	0.002	0.005	0.009	0.012	0.015	0.029
0.100	0.000	0.000	0.003	0.009	0.015	0.020	0.024	0.049
0.300	0.000	0.001	0.006	0.020	0.033	0.045	0.056	0.111
0.500	0.000	0.002	0.009	0.029	0.049	0.066	0.082	0.162
1.000	0.000	0.003	0.015	0.049	0.082	0.111	0.137	0.273
5.000	0.002	0.009	0.049	0.162	0.273	0.369	0.457	0.902

Table 1.4: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 16 (i.e. $\sigma_X = 16 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transaction cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\bar{\theta}\tau}$), and the fifth sub-panel reports the transaction cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transaction cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	21552.410	2155.241	215.524	43.105	21.552	14.368	10.776	4.310
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.000	0.001	0.002	0.004	0.005	0.006	0.007	0.012
0.050	0.000	0.001	0.004	0.008	0.012	0.014	0.017	0.026
0.100	0.001	0.002	0.005	0.012	0.017	0.020	0.023	0.037
0.300	0.001	0.003	0.009	0.020	0.029	0.035	0.041	0.065
0.500	0.001	0.004	0.012	0.026	0.037	0.046	0.053	0.084
1.000	0.002	0.005	0.017	0.037	0.053	0.065	0.075	0.120
5.000	0.004	0.012	0.037	0.084	0.120	0.148	0.173	0.304
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.034	0.109	0.345	0.775	1.101	1.353	1.566	2.507
0.050	0.077	0.243	0.775	1.756	2.507	3.094	3.595	5.847
0.100	0.108	0.345	1.101	2.507	3.595	4.451	5.187	8.542
0.300	0.188	0.599	1.927	4.451	6.453	8.057	9.459	16.113
0.500	0.233	0.775	2.507	5.847	8.542	10.733	12.669	22.132
1.000	0.345	1.101	3.595	8.542	12.669	16.113	19.222	35.322
5.000	0.775	2.507	8.542	22.132	35.322	47.713	59.958	131.419
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.002	0.007	0.022	0.050	0.071	0.087	0.102	0.164
0.050	0.005	0.016	0.050	0.114	0.164	0.204	0.238	0.396
0.100	0.007	0.022	0.071	0.164	0.238	0.297	0.349	0.596
0.300	0.012	0.038	0.125	0.297	0.440	0.559	0.666	1.225
0.500	0.015	0.050	0.164	0.396	0.596	0.767	0.925	1.813
1.000	0.022	0.071	0.238	0.596	0.925	1.225	1.518	3.484
5.000	0.050	0.164	0.596	1.813	3.484	5.821	9.552	-26.683
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	42117.08	23683.81	13317.72	8905.28	7487.89	6765.70	6295.90	5005.90
0.050	28168.42	15837.87	8905.28	5954.05	5005.90	4522.74	4208.40	3345.04
0.100	23686.72	13317.72	7487.89	5005.90	4208.40	3801.97	3537.51	2810.94
0.300	17995.50	10118.73	5688.53	3801.97	3195.52	2886.32	2685.06	2131.37
0.500	15837.71	8905.28	5005.90	3345.04	2810.94	2538.51	2361.10	1872.30
1.000	13317.72	7487.89	4208.40	2810.94	2361.10	2131.37	1981.58	1566.79
5.000	8905.28	5005.90	2810.94	1872.30	1566.79	1408.08	1302.25	982.77
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.000	0.001	0.002	0.002	0.003	0.006
0.050	0.000	0.000	0.001	0.004	0.006	0.008	0.010	0.020
0.100	0.000	0.000	0.002	0.006	0.010	0.014	0.017	0.034
0.300	0.000	0.001	0.004	0.014	0.023	0.032	0.039	0.078
0.500	0.000	0.001	0.006	0.020	0.034	0.047	0.058	0.114
1.000	0.000	0.002	0.010	0.034	0.058	0.078	0.097	0.191
5.000	0.001	0.006	0.034	0.114	0.191	0.258	0.318	0.600

Table 2

Number of split events of NYSE/AMEX/NASDAQ stocks in our sample after filtering for errors and other irregularities, from January 1, 1993 to December 31, 1997.

Split Factor	1993	1994	1995	1996	1997	Total
2:1	174	140	189	223	276	1002
3:2	158	108	140	170	235	811
4:3	11	13	9	13	14	60
5:4	37	31	39	38	45	190
Other	12	19	13	26	37	107
Total	392	311	390	470	607	2170

Table 3

Summary statistics for trade sizes δ and inter-arrival times $\Delta\tau$ before and after stock splits, from January 1, 1993 to December 31, 1997.

Window	Split Factor	Before or After	Sample Size	δ (Shares) Mean	δ (Shares) S.D.	$\Delta\tau$ (Seconds) Mean	$\Delta\tau$ (Seconds) S.D.
1-Day	All	Before	1626	1139	930	728	807
	All	After	1621	740	657	503	535
	2:1	Before	791	1116	817	519	609
	2:1	After	791	641	490	364	428
	3:2	Before	598	1174	960	921	923
	3:2	After	599	836	777	635	586
	4:3	Before	37	961	727	650	516
	4:3	After	37	754	406	520	432
	5:4	Before	118	1272	1297	1286	1178
	5:4	After	118	944	730	1031	1093
2-Day	All	Before	1749	1149	896	871	950
	All	After	1749	739	579	604	660
	2:1	Before	827	1122	712	599	718
	2:1	After	827	652	431	400	473
	3:2	Before	654	1219	1057	1095	1072
	3:2	After	654	841	693	763	751
	4:3	Before	45	1136	1260	761	613
	4:3	After	45	813	592	695	561
	5:4	Before	143	1130	901	1617	1516
	5:4	After	143	869	602	1388	1386
3-Day	All	Before	1805	1141	800	932	1010
	All	After	1805	757	615	662	730
	2:1	Before	846	1118	670	655	768
	2:1	After	846	667	418	446	543
	3:2	Before	677	1213	899	1144	1112
	3:2	After	677	843	634	816	803
	4:3	Before	49	1077	936	762	646
	4:3	After	49	1007	1754	799	574
	5:4	Before	148	1122	831	1628	1490
	5:4	After	148	910	621	1355	1228
7-Day	All	Before	1869	1144	680	972	997
	All	After	1869	793	548	747	799
	2:1	Before	870	1122	602	724	851
	2:1	After	870	701	422	525	649
	3:2	Before	699	1194	698	1159	1049
	3:2	After	699	886	601	899	859
	4:3	Before	53	1100	806	816	563
	4:3	After	53	835	584	766	559
	5:4	Before	160	1153	870	1610	1378
	5:4	After	160	921	595	1438	1183

Table 4

Summary statistics for the log-ratios $\xi_\delta \equiv \log(\overline{\delta}^a/(s\overline{\delta}^b))$ and $\xi_{\Delta\tau} \equiv \log(\overline{\Delta\tau}^a/\overline{\Delta\tau}^b)$, the implied ratios of fixed costs $(\kappa^a/\kappa^b)_\delta$ and $(\kappa^a/\kappa^b)_{\Delta\tau}$ based on trade sizes and inter-arrival times, respectively, and ratios $\zeta \equiv \xi_\delta/\xi_{\Delta\tau}$, across stock splits from January 1, 1993 to December 31, 1997, where $\overline{\delta}^b$ and $\overline{\delta}^a$ are the average trade size (in shares) before and after a split, respectively, s is the split factor, and $\overline{\Delta\tau}^b$ and $\overline{\Delta\tau}^a$ are the average inter-arrival times before and after a split, respectively.

Window	Split Factor	Sample Size	ξ_δ		$\xi_{\Delta\tau}$		$(\kappa_i^a/\kappa_i^b)_\delta$		$(\kappa_i^a/\kappa_i^b)_\tau$		ζ	
			Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1-Day	All	1619	−0.418	0.018	−0.396	0.018	0.984	0.061	0.739	0.018	0.482	0.053
	2:1	791	−0.524	0.023	−0.411	0.024	0.456	0.036	0.695	0.023	0.494	0.085
	3:2	598	−0.298	0.028	−0.368	0.030	1.759	0.185	0.810	0.034	0.558	0.085
	4:3	36	−0.089	0.128	−0.291	0.153	6.401	3.360	1.216	0.199	−0.583	0.251
	5:4	118	−0.264	0.080	−0.296	0.065	3.062	0.664	0.821	0.077	−0.150	0.226
2-Day	All	1746	−0.416	0.015	−0.392	0.018	0.661	0.036	0.672	0.015	0.419	0.047
	2:1	827	−0.531	0.019	−0.424	0.024	0.329	0.022	0.620	0.018	0.503	0.075
	3:2	654	−0.300	0.024	−0.378	0.027	1.324	0.153	0.714	0.027	0.361	0.072
	4:3	44	−0.146	0.130	−0.109	0.110	1.947	0.561	1.264	0.170	−0.337	0.520
	5:4	142	−0.224	0.059	−0.119	0.112	2.736	0.676	1.305	0.217	−0.036	0.170
3-Day	All	1801	−0.405	0.013	−0.387	0.017	0.582	0.026	0.664	0.013	0.484	0.046
	2:1	846	−0.502	0.018	−0.440	0.021	0.351	0.023	0.602	0.016	0.582	0.075
	3:2	677	−0.319	0.022	−0.352	0.027	0.822	0.062	0.699	0.024	0.329	0.070
	4:3	47	−0.174	0.101	0.037	0.100	2.615	1.435	1.661	0.269	−0.278	0.372
	5:4	145	−0.164	0.054	−0.142	0.086	1.690	0.274	1.131	0.148	0.280	0.166
7-Day	All	1864	−0.383	0.011	−0.323	0.014	0.454	0.017	0.703	0.012	0.597	0.045
	2:1	870	−0.458	0.015	−0.380	0.020	0.278	0.013	0.624	0.015	0.820	0.065
	3:2	698	−0.305	0.018	−0.285	0.022	0.590	0.035	0.747	0.022	0.406	0.070
	4:3	51	−0.251	0.066	−0.043	0.059	1.173	0.276	1.147	0.114	−0.252	0.483
	5:4	159	−0.198	0.046	−0.092	0.059	1.305	0.165	1.144	0.090	0.021	0.264

Table 5

Summary statistics for the log-ratios $\xi_\delta \equiv \log(\overline{\delta}^a/\overline{\delta}^b)$ and $\xi_{\Delta\tau} \equiv \log(\overline{\Delta\tau}^a/\overline{\Delta\tau}^b)$ across dates prior to and after stock splits stock splits, from January 1, 1993 to December 31, 1997, where $\overline{\delta}^b$ and $\overline{\delta}^a$ are the average trade size (in shares) in the "before" and "after" periods, respectively, and $\overline{\Delta\tau}^b$ and $\overline{\Delta\tau}^a$ are the average inter-arrival times in the "before" and "after" periods, respectively. '*Dates* -5 *and* -4' indicates that the "before" period is the fifth day prior to a split event and the "after" period is the fourth day prior to the same event.

Split Factor	Sample Size	ξ_δ		$\xi_{\Delta\tau}$	
		Mean	S.D.	Mean	S.D.
		Dates -5 *and* -4			
All	1513	0.010	0.019	−0.021	0.018
2:1	741	0.013	0.024	−0.006	0.023
3:2	562	0.039	0.034	−0.030	0.034
4:3	36	0.059	0.147	−0.047	0.166
5:4	108	−0.138	0.080	−0.079	0.073
		Dates $+4$ *and* $+5$			
All	1646	0.003	0.016	0.014	0.017
2:1	803	0.019	0.022	0.020	0.020
3:2	609	−0.011	0.028	0.000	0.032
4:3	40	0.034	0.152	−0.026	0.143
5:4	117	−0.190	0.067	0.046	0.072
		Dates -7 *and* -6			
All	1526	0.008	0.018	0.024	0.017
2:1	749	−0.031	0.022	0.014	0.023
3:2	569	0.015	0.035	0.037	0.030
4:3	36	0.073	0.124	−0.041	0.134
5:4	108	0.113	0.075	0.005	0.080
		Dates $+6$ *and* $+7$			
All	1621	0.028	0.019	0.005	0.017
2:1	792	0.024	0.024	0.006	0.022
3:2	599	0.017	0.033	−0.024	0.029
4:3	37	−0.159	0.143	−0.043	0.138
5:4	119	0.179	0.088	0.099	0.066

Chapter 2

Market Power and Imperfect Information

2.1 Introduction

Today's financial markets are populated by many large traders such as mutual funds, pension and endowment funds, and securities dealers. The behavior of large traders may therefore have important implications for the behavior of markets. There is strong empirical evidence that large trades do have a significant impact on prices. Furthermore it seems that institutions trade to their desired portfolio positions slowly, thereby showing evidence of strategic trading behavior. Despite their access to resources and their obvious concern with the actual mechanism of the trading process, institutional investors as a whole do not provide excess, risk-adjusted returns. Yet, the (typically small) trades of corporate insiders seem to do just that.[1] The following statement from O'Hara (1995) seems representative of a prevalent point of view.

> (E)lectronic clearing networks such as Instinet and POSIT can attract large uninformed institutional traders if the structure of their trading mechanisms provides a "better deal" for executing the large orders such traders enter.

Hence we are faced with a world view where certain large institutions are clearly concerned about the way in which their trades are executed, and yet are relatively poorly informed

[1] See Holthausen, Leftwich, and Mayers (1990) and Keim and Madhavan (1996) for evidence about the price impact of large trades. Chan and Lakonishok (1995) find that large investors execute inventory shifts over several days. See Carhart (1997) for evidence about the poor risk-adjusted performance of investment funds. Lakonishok and Lee (1998) suggest that insider trades are informative to a degree. And Jeng, Metrick, and Zeckhauser (2000) (using the methodology in Carhart (1997)) argue that portfolios consisting of insider trades do provide risk-adjusted excess returns.

about the prospects of the securities in which they trade. It would seem then that certain securities (the exact number of which is an empirical question) have clienteles which consist of large, yet poorly informed, investors. This raises some important theoretical and empirical questions. How are returns and trading patterns of such securities different from securities whose uninformed clienteles are smaller? Are large, poorly informed agents, by virtue of their potential market power, able to extract different types of information from the market than small, poorly informed traders? And finally, how do the incentives to gather information differ between large and small poorly informed agents — that is, why do the uninformed remain uninformed? The purpose of this paper is to provide a unified and self-consistent theoretical framework in which these questions can be addressed, as well as to point to promising directions for future empirical work.

In order to understand the dynamic trading behavior of an economy with asymmetrically informed agents, we must endow these agents with motives to trade. We can classify trading motives into two general categories: informational and allocational.[2] For our purposes, informational trade occurs when agents trade in order to take advantage of information which is superior to that of the rest of the market. Allocational trade occurs when agents need to rebalance their portfolios to reach their optimal level of asset holdings. In this paper, we will focus on the interaction of market power and information in an economy whose agents trade for both of the above reasons. In particular, we will construct two comparable economies which will differ only in the strategic behavior of their uninformed agents. The competitive economy will then provide a natural benchmark to study the effects of market power, and how these are affected by the degree informational asymmetry. While most previous studies have focused on the behavior of large, informed traders, a relevant scenario, and perhaps the more prevalent one, has been left unexplored. This paper, then, has a dual purpose: (1) to study the asset pricing and trading volume implications of the presence of large, poorly informed agents, and (2) to study the optimal behavior of such agents in an equilibrium

[2]The seminal works on the effects of informational asymmetry are by Grossman (1976), Grossman and Stiglitz (1980), and Hellwig (1980). More recent papers which have extended their analysis to a richer setting include Wang (1993,1994), He and Wang (1995), Hong (1999), Zhou (1997). The papers by Kyle (1985,1989) were the first to analyze the interaction between strategic behavior and asymmetric information. Extensions of the Kyle model to richer settings can be found in Vayanos(1999a), Chau (1999), Back (1992), and Back, Cao, and Willard (1999). Related models involving informational asymmetry and strategic behavior are Glosten and Milgrom (1985), Glosten (1988), Vayanos (2000), Madhavan and Smidt (1993), and Spiegel and Subrahmanyam (1992). See O'Hara (1995) for a detailed overview of much of the related literature.

setting.

In a seminal and related paper, Kyle (1985) studies the behavior of a large, well informed agent (about the asset payoff), and finds that the agent balances off the desire to trade gradually to minimize price impact versus the desire to act quickly in order to exploit a (short lived) informational advantage. Related papers (Vayanos (2000), Chau (1999)) study the behavior of large agents with private endowment shocks and with identical or superior asset payoff information, and reach a similar conclusion. There is ample empirical evidence that specialists have some market power in the securities which they trade, and that they are at an informational disadvantage relative to certain agents in the market. Also relevant, then, is the market microstructure literature on the behavior of specialists (see Amihud and Mendelson (1980), Ho and Stoll (1981), Glosten and Milgrom (1985), Easley and O'Hara (1987), Glosten (1988), and Madhavan and Smidt (1993)). The general findings of this literature are: (1) specialists exploit their market power in order to extract surplus from allocational trade, and (2) the liquidity of the market decreases (e.g. the bid-ask spread widens) as the informational disadvantage of the specialist increases.

In this paper, I will consider an infinite horizon, discrete time, stationary economy. The assets in the economy consist of a riskless bond paying a fixed return and a stock paying a dividend flow in every period. The stock price is determined endogenously in equilibrium. The economy consists of two sectors: one which has perfect information about the state variables in the economy (the informed sector), and one which has imperfect information about the dividend growth rate and the asset holdings of the informed sector (the uninformed sector). Trade takes place through a series of auctions in which the informed agents submit their demand curves, and in which the uninformed agents choose a trade amount in either a strategic or a competitive manner (this distinction will be made clear later). The uninformed agents receive a direct signal about the dividend level in every period, and can also make inferences from the price and dividend history, which are both public information. The remaining agents in the economy are perfectly informed about the state of the economy: in particular, they know the dividend level and the asset holdings of the large uninformed agent. Both types of agents are risk averse and both receive exogenously specified endowment shocks.[3] These assumptions induce the two sectors to engage in risk sharing with each other.

The model predicts that, holding constant the quality their signals, an uninformed strategic agent will learn different things about the state of the economy than an uninformed competitive sector. In particular, the strategic agent will have a more precise estimate of the asset holdings of the informed sector, and a less precise estimate of the growth rate of the dividend. The intuition for this result follows from the fact that it is costlier for a strategic agent to trade out of an undesirable position than it is for a competitive agent to do so, since the strategic agent takes into account the price impact of his or her trades. This causes equilibrium prices in the strategic economy to be relatively less informative about dividend prospects, and to be more informative about asset holdings, than in the competitive economy.

The framework of the model allows us to make some predictions about the behavior of asset returns. In particular, we find that returns are increasing in the information asymmetry present in the economy. Furthermore, returns are typically lower in the strategic economy. The difference between expected returns in the strategic and competitive economy increases with informational asymmetry. This result has to do with the long-run asset holdings in the strategic versus the competitive economies.

I find that return variability first decreases and then increases with informational asymmetry. The decrease is due to a negative correlation between innovations to future cash flows and innovations to the uninformed sector's misestimate of the dividend growth rate. Returns are more volatile in the strategic economy. And the difference between returns in the strategic and competitive economies as a function of informational asymmetry exhibits an inverted U-shaped pattern. This arises from the fact that return variability in the strategic economy is less sensitive to changes in the information structure than it is in the competitive economy.

Another implication of the model has to do with the size of unexpected trading volume. I find that unexpected trading volume decreases and then increases with informational asymmetry. The former effect is due to a negative correlation between endowment innovations and innovations to the uninformed sector's misestimate of the dividend growth rate.

[3] We can think of the endowment shocks as originating from a position in a security with similar cash flows or from non-traded labor income.

Furthermore, because of strategic considerations, unexpected trading volume is lower in the strategic economy. And similarly to the case of return variability, the difference between trading volume in the two economies as a function of the informational asymmetry exhibits an inverted U-shaped pattern. The reason again is similar: unexpected volume is less sensitive to changes in the information structure in the strategic economy.

The last two sets of results, therefore, suggest that the presence of a strategic uninformed agent dampens, to a degree, the effects informational asymmetries on return variability and turnover. The fact that trading out of undesirable positions carries an additional price impact cost in the strategic economy forces the uninformed strategic agent to be more cautious in his or her responses to changes in estimated state variables. This effect becomes evident when we consider the dynamic response of the economy to an innovation in the exogenous signal of the uninformed sector. In the strategic economy the trading response of the uninformed agent is more muted over time than in the competitive economy. Furthermore, the surplus lost to the informed sector is smaller in the strategic economy.

An interesting consequence of trade in this setting is that informational asymmetry allows the competitive uninformed sector to extract surplus from allocational trade with the informed agents. The reason for this effect is that in equilibrium the misestimate of the uninformed agents enters the price. Then an endowment shock in the informed sector causes the a misestimate of future cash flows by the uninformed agents. This leads the uninformed to buy the stock exactly when its price is artificially low due to misinformation. Note that this effect does not depend on strategic behavior. Strategic behavior introduces an additional effect. The uninformed strategic investor extracts surplus from allocational trade even under symmetric information. Though the amount of surplus obtained from this channel can increase or decrease with informational asymmetry, it is always positive, and therefore strategic uninformed agents extract more surplus from allocational trade under asymmetric information than do competitive uninformed traders.

The above arguments suggest that it may be possible for uninformed agents to be better off in some circumstances under a worse information set. Of course, having a worse information set makes the uninformed subject to more information arbitrage by the informed agents. However, in situations where the allocational trade by the informed sector is intense enough, the former effect may dominate, and uninformed agents (both strategic and competitive) may be better off from a welfare point of view under worse information. Because of the additional trade surplus channel to which the strategic agents have access, their incentives to become better informed can differ significantly from those of the competitive uninformed agents. Nonetheless, all of this suggests that the presence of poorly informed agents can be an endogenous outcome in a fully rational economy.

The remainder of the paper is organized as follows. Section 2.2 formally defines the model. Section 2.3 analyzes the control problems of the agents in the economy, and solves for an equilibrium. Section 2.4 analyzes the equilibrium of the economy. Section 2.5 discusses the incentives of the uninformed agents to improve their information set. Section 2.6 discusses some empirical implications of the model. Section 2.7 concludes. All proofs are in the Appendix.

2.2 The Model

The economy consists of an uninformed sector and of a continuum of measure 1 of competitive informed agents. There is a single consumption good, which is also the numeraire. There is a dividend paying stock and a riskless bond in the economy. Trading takes place every h units of time. The t^{th} trade takes place at time $t \times h$, where $t \in \{0, 1, 2, ...\}$.

A. Uncertainty

The uncertainty in the economy is characterized by the shocks $\boldsymbol{\epsilon}_t \equiv [\ \epsilon_U \quad \epsilon_I \quad \epsilon_F \quad \epsilon_\theta \quad \epsilon_D\ ]'$, which are independent through time. The covariance matrix, $\boldsymbol{\Sigma}$, is constant through time. I will assume that the error terms are also independent contemporaneously, although this simplification can be relaxed.

B. Securities Markets

Agents can trade in two securities, a perfectly liquid bond and a stock in fixed supply. The bond pays a return of $e^{rh} - 1$ per period. Here r is fixed. The stock pays a dividend $D_t h$ per period, where

$$D_{t+1} - D_t = (F_{t+1} - k_D D_t)h + \epsilon_{D,t+1} \tag{2.1}$$

$$F_{t+1} - F_t = -k_F F_t h + \epsilon_{F,t+1} \tag{2.2}$$

F_t/k_D is the short term level[4] of the dividend. Since F_t is mean reverting to 0, the long term level of the dividend is zero. The disturbance terms, ϵ_D and ϵ_F are Gaussian, with means zero and variances $\sigma_D^2 h$ and $\sigma_F^2 h$, respectively. This specification of the dividend process allows for a well defined limit of the model as h goes to 0. It will be convenient to write the dividend process as

$$D_{t+1} = A_D D_t + F_{t+1}h + \epsilon_{D,t+1} \tag{2.3}$$

$$F_{t+1} = A_F F_t + \epsilon_{F,t+1} \tag{2.4}$$

where A_D and A_F have the obvious definitions. It will be shown that the equilibrium share price in the economy will have the following form

$$P_t = P_F F_t + P_D D_t + \tilde{P}(\mathbf{\Psi}_t) \tag{2.5}$$

where P_F and P_D are constants, $\tilde{P}(\cdot)$ is a linear function, which I will refer to as the *risk discount*, and $\mathbf{\Psi}_t$[5] are the period t state variables in the economy.

C. Information Structure

The uninformed sector in the economy does not have perfect information about either the dividend level of the stock, or about the endowments of the informed traders. On the other hand, the uninformed agents observe in each period the paid dividend and the transaction price. Both of these contain signals about the level, F_t/k_D, of the dividend. Furthermore, the uninformed agents receive directly a signal about the dividend level; this signal is a modeling device which will allow us to change the amount of informational asymmetry present in the economy. The uninformed sector rationally updates its beliefs in each period to reflect new information. A precise statement of the inference problem of the uninformed agents will be made later in this paper.

D. Agents

There are 2 classes of agents in the economy: the informed competitive sector and the large uninformed sector. In the case of the strategic economy, we will think of the uninformed sector as consisting of a single strategic agent. In the competitive case, we will think of

[4] Going forward, I will also refer (incorrectly) to F_t as the dividend level. This will simplify exposition.
[5] Bold letters refer to vectors and matrixes in what follows. Regular letters refer to scalar quantities.

the uninformed sector as containing a measure 1 set of identical (in all respects, including information sets) agents. In the strategic case, trading proceeds by the informed agents submitting their limit orders in each period, and the large agent choosing a trade amount (and therefore the price). Each informed agent is small relative to the market, and hence has no price impact. In the competitive case, the uninformed agents submit their orders believing that these induce no price impact.

The uninformed sector's endowment is given by U_t and the informed sector's endowment is given by I_t. Each agent receives an endowment and a cash account shock in every period. The period $t+1$ endowment shock is $\epsilon_{a,t+1}$ shares for $a \in \{_U, _I\}$.[6] The shock to the cash account is $-\epsilon_{a,t+1}P_t$.[7] At the end of each period, agents submit market orders, $\delta_{a,t}$. Hence endowments evolve according to

$$A_{t+1} = A_t + \delta_{a,t} + \epsilon_{a,t+1} \tag{2.6}$$

here $A_t \in \{U_t, I_t\}$.

In each period t events occur in the following sequence

- traders observe the period t values of the state variables which are in their information set,
- each informed trader submits a demand curve $\delta_{I,t}(P_t)$ and chooses a consumption amount $c_{I,t}(P_t) \times h$,
- the strategic uninformed agent observe the price function and its dependence on their trade amount, and then choose a trade, $\delta_{U,t}$ and a consumption amount $c_{U,t} \times h$; the competitive uninformed agents observe the equilibrium price, believe that they do not have price impact, and optimally choose to submit their own trade and consumption amounts,
- the trade and consumption are both charged against agents' holdings of the consumption good,

[6] All quantities subscripted with U are associated with the uninformed agents, and quantities subscripted with I are associated with either a single informed agent, or with the informed sector (the meaning will always be clear from the context).

[7] The cash account shock makes the uncertainty faced by agents independent of the dividend level and dividend growth rate. The cash account and the endowment shocks can be thought of as exogenously imposed asset purchases or liquidations.

- the consumption good is invested until period $t+1$ and gives a return $e^r h - 1$,

- at the end of period t, agents receive dividends from their holdings of the risky asset in the amount $(A_t + \delta_{a,t} + \epsilon_{a,t+1})D_{t+1}h$

- finally, period $t+1$ shocks are realized and state variables evolve according to their laws of motion.

The dynamics of each agent's holdings of the numeraire good (cash account) are thus given by

$$M_{t+1} = e^{rh}(M_t - c_t h - (\delta_t + \epsilon_{a,t+1})P_t) + (U_t + \delta_t + \epsilon_{a,t+1})D_{t+1}h \tag{2.7}$$

All agents are assumed to maximize a time separable CARA utility of the form

$$E_t\left[\sum_{s=t}^{\infty} -e^{-\rho s h - \gamma_A c} h\right] \tag{2.8}$$

subject to the dynamics of the money account and of the holdings of the risky asset given above. The expectation above is taken with respect to the agents' information sets at time $t \times h$. Also a Merton type transversality condition is imposed: $\lim_{t\to\infty} E_s[J(M_{a,t},\ldots,t)] = 0$. Here, ρ and γ, both positive, are respectively the time-discount factor and the risk-aversion coefficient. Notice that γ is allowed to vary across the two sectors in the economy.

The state variables of interest to the continuum of informed agents are

$$\{M_{I,t}, D_t, F_t, \Psi_{I,t}, P_t\}$$

where $\Psi_{I,t} \equiv \{U_t, I_t, \hat{F}_t - F_t\}$. Here $\hat{F}_t$ is the uninformed agent's time $t \times h$ estimate of the dividend level; hence $\hat{F}_t - F_t$ is U's misestimate of the dividend level. P_t is the price of the risky asset. The state variables of interest to the uninformed sector are

$$\{M_{U,t}, D_t, \hat{F}_t, \Psi_{U,t}\}$$

where $\Psi_{U,t} \equiv \{U_t, \hat{I}_t\}$.[8] Here $\hat{I}_t$ is the uninformed agent's estimate of the asset holdings of the informed agents. This is not a state variable for the informed agents because the

[8]The distributional assumptions of the model will render the uninformed agents' inference and control problems separable. Hence the uninformed sector is justified in replacing unobserved state variables with its estimates of them, as long as the estimation error is accounted for in the state variable dynamics. The inference problem will be discussed in detail later in the paper.

misestimate $\hat{I}_t - I_t$ will be seen to be perfectly correlated with $\hat{F}_t - F_t$; hence the informed agent is always able to deduce one from the other.

E. Strategic Equilibrium Concept

I look for a Nash equilibrium of the economy (see Kyle (1985,1989)). A Nash equilibrium of this economy is defined as

- the trading mechanism given above,
- a set of agents' strategies $\{(c^*_{a,t}, \delta^*_{a,t})\}$ for $a \in \{\upsilon, \imath\}$,

such that

- each agent maximizes (2.8) over his consumption and trading policies, given his beliefs about the trading strategies $\{(c^*_{a,t}, \delta^*_{a,t})\}$ of the other agents, [9]
- there exists a price which clears the market in the risky security in each trading period, that is

$$\delta_{\upsilon,t} + \mu(\mathcal{I}) \times \delta_{\imath,t}(P_t) = 0 \tag{2.9}$$

  where $\mathcal{I}$ is the set of informed agents, and μ is the appropriate measure on this set (in particular $\mu(\mathcal{I}) = 1$),
- agents have rational beliefs about the price process.

Equation (2.9) simply states that the sum of buy orders from the large agent and from the measure 1 continuum of small informed traders must add up to zero in every trading period. This can be thought of as a *flow* market clearing condition.

Following Kyle (1985), the large agent can not pre-commit to a particular strategy in order to affect the decisions of the informed sector. Instead, the large agent must take as given the behavior of the informed sector, and solve for his optimal response to this behavior.

F. Competitive Equilibrium Concept

[9] The large, uninformed assumes that all the informed agents will follow their equilibrium strategies, and then solves his investment–consumption problem. Each informed agent assumes that the other measure 1 of informed agents and the uninformed agent all follow their equilibrium strategies. Then the atomic informed agent solves for his own optimal strategy. In equilibrium this strategy must be the same as the conjectured strategy of the other informed agents.

I look for a competitive equilibrium of the economy. This equilibrium is defined as

- a price function, $P_t = P(D_t, F_t, U_t, I_t, \Omega_t)$
- a trade function in each period $\delta_t = \delta(U_t, I_t, \Omega_t)$
- a set of agents' strategies $\{(c^*_{a,t}, \delta^*_{a,t})\}$ for $a \in \{\upsilon, \imath\}$,

such that

- each agent maximizes (2.8) over his consumption and trading policies, given his beliefs about the price and trade functions,
- market clearing takes place

$$\mu(\mathcal{U})\delta_{U,t}(P_t) + \mu(\mathcal{I}) \times \delta_{I,t}(P_t) = 0 \tag{2.10}$$

  where $\mathcal{U}$ is the set of uninformed agents ($\mu(\mathcal{U}) = 1$), and $\mathcal{I}$ is as given above,

- actual trade is equal to agents' beliefs about trade,

$$\delta_{U,t} = -\delta_{I,t} = \delta_t \tag{2.11}$$

This economy differs from the strategic one only because uninformed and informed agents all behave in a price taking manner. The information sets of the agents stay the same. Hence any differences between the two economies will be due to strategic versus competitive behavior by the uninformed sector. One feature of this model is to maintain the asset holdings of the two sectors as separate state variables. This is necessary in order to support an information structure equivalent to that of the strategic economy.

G. Discussion of the Model

With no strategic behavior, models very similar to this one have been analyzed by Wang (1993, 1994). Under the competitive equilibrium concept, the model differs from the Wang models because of its endowment structure. The particular endowment structure chosen here facilitates the analysis of the inference problem of the strategic uninformed agent. Dynamic models with strategic behavior and asymmetric information have been analyzed by Vayanos (1999a,2000) and Chau (1999). The structure of the Vayanos and Chau papers differs from this one in that their strategic agent is also informed. Furthermore, the information asymmetry in Vayanos (1999a,2000) only applies to agents' endowments. Chau considers a large, informed trader where both his endowment and the dividend level are private information.

2.3 Determination of Equilibrium

In this section I discuss the solution of the strategic and competitive economies described in Section 2.2.

2.3.1 Strategies and Price

I consider a linear Nash equilibrium of the economy. I will state the result about the equilibrium of the economy as a proposition.

Theorem 2.1 *Assuming that a linear Nash equilibrium of the strategic and competitive economy exists, the aggregate demand (i.e. of the measure 1 continuum of informed agents) of the informed sector is given by*

$$\delta_{I,t} = \boldsymbol{A}_I \boldsymbol{\Psi}_{I,t} + \pi P_F F_t + \pi P_D D_t - \pi P_t \tag{2.12}$$

The demand of the uninformed agents is given by

$$\delta_{U,t} = \boldsymbol{F}_U \boldsymbol{\Psi}_{U,t} \tag{2.13}$$

in the strategic and competitive case. Finally the price is given by

$$P_t = P_F F_t + P_D D_t + P_U U_t + P_I I_t + P_\Omega(\hat{F}_t - F_t) \tag{2.14}$$

All the constants satisfy a set of non-linear algebraic equations given in the Appendix. P_F and P_D are known in closed form. All constants, except P_F and P_D, depend on whether the economy is strategic or competitive.

To see how this result obtains in the strategic case, let us simply assume that the aggregate demand of the small informed investors is given by (2.12). Let us decompose the price into its dividend and non-dividend components.

$$P_t = \tilde{P}_t(\delta_{U,t}) + P_F F_t + P_D D_t \tag{2.15}$$

The market clearing condition (2.9) must hold. Plugging in the demands of the agents, we find that

$$\boldsymbol{A}_I \boldsymbol{\Psi}_{I,t} - \pi \tilde{P}_t + \delta_{U,t} = 0$$

Hence

$$\tilde{P}_t(\delta_{U,t}) = \frac{1}{\pi}\left(\boldsymbol{A}_I \boldsymbol{\Psi}_{I,t} + \delta_{U,t}\right) \tag{2.16}$$

and

$$P_t = \frac{1}{\pi}\boldsymbol{A}_I \boldsymbol{\Psi}_{I,t} + P_F F_t + P_D D_t + \frac{1}{\pi}\delta_{U,t} \tag{2.17}$$

This is the sense in which the price depends on the demands of the strategic uninformed agent. Given this price impact function, the uninformed agent will be shown to have demands linear in $\boldsymbol{\Psi}_{U,t}$, as proposed in Theorem 2.1. In particular, his demand for the risky asset will not depend on $(M_{U,t}, \hat{F}_t, D_t)$, as will be shown in Section 2.3.3. It then becomes clear that the non-dividend component of the price $\tilde{P}_t$ will also be linear in $\boldsymbol{\Psi}_{I,t}$.[10]

In the competitive case, the agents take as given the price process and the trade amount process. Using these, they compute optimal demands. The requirement is that their optimal demands be equal to the conjectured demand process, and that the market clears.

2.3.2 Filtering Problem of the Uninformed Agents

An uninformed agent does not directly observe the dividend level F_t or the risky asset holdings of the informed investors, I_t. In each period t he receives a signal given by

$$S_t^{(\theta)} = hF_t + \epsilon_{\theta,t} \tag{2.18}$$

This signal is also observed by the informed agents (i.e. the informed agents know the information set of the uninformed). The uninformed agent must form predictions for these unobserved variables using publically available information and his own signal. The information set of the uninformed agent at time th is given by $\mathcal{F}_\tau^U = \{D_\tau, P_\tau, S_\tau^{(\theta)}, \delta_{U,\tau} : \tau \leq t\}$. From his knowledge of the dividend history, in each period t the uninformed agent observes

$$S_t^{(D)} = hF_t + \epsilon_{D,t} \tag{2.19}$$

[10] The uninformed agent's state variables are $(U_t, \hat{I}_t)$. His demand only puts weights on these 2 variables. However as will become clear in Section 2.3.2, $\hat{I}_t = i_1 I_t + i_2(\hat{F}_t - F_t)$, for some constants i_1 and i_2. Hence U's linear demands in $\boldsymbol{\Psi}_{U,t}$ translate into linear demands in $\boldsymbol{\Psi}_{I,t}$.

In the strategic case, from his observation of the price function in (2.17), the uninformed agent is able to extract

$$S_t^{(P)} = \left(P_F - \frac{1}{\pi}A_{I,\Omega}\right) F_t + \frac{1}{\pi}A_{I,I}I_t \tag{2.20}$$

The competitive uninformed agents, on the other hand, observe

$$S_t^{(P)} = (P_F - P_\Omega)\, F_t + P_I I_t \tag{2.21}$$

Let us note that in the strategic economy, the signal in (2.21) is simply some multiple of the signal in (2.20), and hence contains exactly the same information. This result is proved in the Appendix. Hence any difference in the resultant information sets of the uninformed agents across the two economies is not a result of different signals, but is instead a result of the different behavioral assumptions about the uninformed sector.

From the dynamics of I_t given in (2.6), and from the market clearing condition $\delta_{U,t}+\delta_{I,t} = 0$, we can write

$$I_{t+1} = I_t - \delta_{U,t} + \epsilon_{I,t+1} \tag{2.22}$$

which we can further decompose into

$$\begin{aligned} I_{t+1} &= I_{-\infty} - \sum_{s=-\infty}^{t} \delta_{U,s} + \sum_{s=-\infty}^{t+1} \epsilon_{I,s} \\ &= I_{-\infty} + I_{D,t} + I_{\epsilon,t+1} \end{aligned} \tag{2.23}$$

We interpret this decomposition as follows: $I_{D,t}$ is the part of I_t which has come from the trades of the uninformed agent, and which must therefore be know to the uninformed agent; and $I_{\epsilon,t+1}$ is the unknown and stochastic part of the informed agents' endowments. The dynamics of these components of I_t are

$$I_{D,t} = I_{D,t-1} - \delta_{U,t} \tag{2.24}$$

$$I_{\epsilon,t+1} = I_{\epsilon,t} + \epsilon_{I,t+1} \tag{2.25}$$

Then we see that the inference problem about I_t reduces to an inference problem about $I_{\epsilon,t}$. And we can re-write the signal received from price as follows

$$S_t^{(P)} = \left(P_F - \frac{1}{\pi}A_{I,\Omega}\right) F_t + \frac{1}{\pi}A_{I,I}I_{\epsilon,t} \tag{2.26}$$

since the uninformed agent knows $I_{D,t-1}$ in (2.20).

Given this formulation, the uninformed agents are faced with a standard Kalman filtering problem. Let us define the vectors $\boldsymbol{z}_t = [F_t \;\; I_{\epsilon,t}]'$ and $\boldsymbol{S}_t = [S_t^{(P)} \;\; S_t^{(D)} \;\; S_t^{(\theta)}]'$. Furthermore, let $\hat{\boldsymbol{z}}_t = E[\boldsymbol{z}_t|\mathcal{F}_t^U]$. The following theorem characterizes the law of motion of $\hat{\boldsymbol{z}}_t$.

Theorem 2.2 *Given that $\mathcal{F}_t^U = \{D_s, P_s, \delta_{U,s} : s \leq t\}$, the conditional expectation $\hat{Z}_t$ is given by*

$$\begin{bmatrix} \hat{F}_t \\ \hat{I}_{\epsilon,t} \end{bmatrix} = \begin{bmatrix} A_F & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{F}_{t-1} \\ \hat{I}_{\epsilon,t-1} \end{bmatrix} + \boldsymbol{K} \left(\begin{bmatrix} S_t^{(P)} \\ S_t^{(D)} \\ S_t^{(\theta)} \end{bmatrix} - E\left(\begin{bmatrix} S_t^{(P)} \\ S_t^{(D)} \\ S_t^{(\theta)} \end{bmatrix} \middle| \mathcal{F}_{t-1}^U \right) \right) \tag{2.27}$$

where $\boldsymbol{K}$ is a 2×3 matrix defined in Appendix A.

Let us define $\Omega_t \equiv \hat{F}_t - F_t$ to be the uninformed investor's misestimate of the dividend level. In Appendix A, I show that Ω_t follows an AR(1) process given by

$$\Omega_{t+1} = A_\Omega \, \Omega_t + \boldsymbol{B}_\Omega \, \epsilon_{t+1} \tag{2.28}$$

where $A_\Omega < 1$ and $\boldsymbol{B}_\Omega$ is a 1×5 vector. Hence the misestimate of the dividend level is mean reverting to 0, which suggests that any errors made in estimating F_t tend to be corrected over time. The magnitude of A_Ω is a measure of the accuracy of the estimate. As A_Ω approaches 0, any errors in the estimation of $\hat{F}_t$ tend to be corrected faster.

From $S^{(P)}$ we see that the uninformed investor observes a sum of the current dividend level and the holdings of the informed investors. Hence, the following must be true of his estimates of $I_{\epsilon,t}$ and F_t

$$\left(P_F - \frac{1}{\pi} A_{I,\Omega}\right) \hat{F}_t + \frac{1}{\pi} A_{I,I} \hat{I}_{\epsilon,t} = \left(P_F - \frac{1}{\pi} A_{I,\Omega}\right) F_t + \frac{1}{\pi} A_{I,I} I_{\epsilon,t} \tag{2.29}$$

Let us define

$$\zeta \equiv \frac{\pi P_F - A_{I,\Omega}}{A_{I,I}} \tag{2.30}$$

which measures the informativeness of the price about the dividend level relative to the informativeness of the price about the asset holdings of the informed sector. Adding back in $I_{D,t}$, we can then rewrite (2.29) as

$$\hat{I}_t - I_t = -\zeta \Omega_t \tag{2.31}$$

It turns out that $\zeta < 0$. Hence misestimates of the asset holdings of the informed sector and of the dividend growth rate, F, are perfectly correlated. Furthermore, $\Omega_t > 0$ if and only if $\hat{I}_t - I_t > 0$. The intuition for this is straightforward. If the uninformed agents underestimate the dividend growth rate, then they must also underestimate the asset holdings of the informed sector (and hence the discount in the stock price) in order for their perceived price to be equal to the actual price (which, of course, they observe).

From the point of view of the uninformed agent, Ω_t is actually a Gaussian random variable with mean 0 and with variance given by the upper left entry in the covariance matrix $\boldsymbol{O}$ given in Appendix A when conditioning on $\mathcal{F}_t^U$. In fact, it is shown in Appendix A that investor U believes that $\hat{F}_t$ follows an AR(1) process given by

$$\hat{F}_{t+1} = A_F \hat{F}_t - C_F \Omega_t + \boldsymbol{B}_F \boldsymbol{\epsilon}_{t+1} \tag{2.32}$$

where Ω_t is the random variable described above, and the constant 1×5 matrix $\boldsymbol{B}_F$ and the constant C_F are given in Appendix A. Similarly, we have for $\hat{I}_{\epsilon,t}$ that

$$\hat{I}_{\epsilon,t+1} = \hat{I}_{\epsilon,t} - C_I \Omega_t + \boldsymbol{B}_I \boldsymbol{\epsilon}_{t+1} \tag{2.33}$$

Since $\hat{I}_{t+1} = \hat{I}_{\epsilon,t+1} + I_{D,t}$ and $I_{D,t} = I_{D,t-1} - \delta_{U,t}$, we have

$$\hat{I}_{t+1} = \hat{I}_t - \delta_{U,t} - C_I \Omega_t + \boldsymbol{B}_I \boldsymbol{\epsilon}_{t+1} \tag{2.34}$$

Equations (2.32) and (2.34) describe the dynamics of U's beliefs about the current dividend level and the current inventory of the informed sector, respectively.

We should note at this point the role which the exogenous signal, $S^{(\theta)}$, plays in the model. The variance of ϵ_θ controls the degree of information asymmetry in the model. As $\epsilon_\theta \to 0$, the information sets of the two classes of agents become the same. As σ_θ increases the informational asymmetry increases in the sense that the uninformed sector's estimates of the dividend growth rate and of the informed asset holdings become less precise. However as $\sigma_\theta \to \infty$, it is not the case that the information set of the uninformed agents becomes progressively worse. At some point the uninformed agents simply assign zero weight to the exogenous signal in the updating equation (2.27). The economy then behaves as if the exogenous signal did not exist.

Given the distributional assumptions of this model, the inference problem of the uninformed investor and his control problem are separable once we account for the estimation error of F_t (i.e. $-\Omega_t$) in the evolution of D_t (see Dothan and Feldman (1986), Detemple (1986), and Gennotte (1986)). That is we know from (2.3–2.4), and the fact that $\Omega_t = \hat{F}_t - F_t$, that we can write the dividend dynamics as

$$\begin{aligned} D_{t+1} &= A_D D_t + (A_F F_t + \epsilon_{F,t+1})h + \epsilon_{D,t+1} \\ &= A_D D_t + (A_F \hat{F}_t + \epsilon_{F,t+1})h - A_F \Omega_t h + \epsilon_{D,t+1} \end{aligned}$$

where Ω_t is a Gaussian random variable when conditioning on $\mathcal{F}_t^U$.

A potentially complicating issue in the strategic economy, is that the uniformed investor controls the evolution of I_t by his trading decisions in period $t-1$. However, as I have just shown, we can separate I_t into a controlled and an exogenous component (from the point of view of the uninformed investor), and then the inference problem only involves the exogenously specified component, $I_{\epsilon,t}$, of I_t. Also, the actions of the uninformed do not affect the information content of the dividend. Hence the separation principle holds in the strategic economy. This greatly simplifies the analysis of the control problem of the uninformed strategic agent.

2.3.3 Control Problem of the Uninformed Agents

In this section we will consider the control problems of the strategic and the competitive uninformed agents. In both cases, the separation principle implies that the uninformed investor can simply replace all unobserved state variable with his estimates of them as discussed above. The theorem stated below applies to both the strategic and the competitive cases, with distinctions between the two explicitly noted.

The state variables of interest to the uninformed investor are $M_{U,t}, D_t, \hat{F}_t$, and $\Psi_{U,t} = \{U_t, \hat{I}_t\}$ in the strategic case, and $\Psi_{U,t} = \{U_t, \tilde{U}_t, \hat{I}_t\}$ in the competitive case. Here $\tilde{U}_t$ is the asset holdings of an atomic uninformed agent. The uninformed investor solves the following problem ($\mathcal{U}$):

$$J(M_{U,t}, D_t, \hat{F}_t, \Psi_{U,t}, t) = \sup_{c,\delta} E_t \left[\sum_{s=t}^{\infty} -e^{-\rho s h - \gamma c} h \right]$$

where the expectation is taken with respect to $\mathcal{F}_t^U = \{D_s, P_s, \delta_{U,s} : s \leq t\}$. The dynamics of the state variables are given by

$$\begin{aligned} M_{U,t+1} &= e^{rh}(M_{U,t} - c_{U,t}h - (\delta_{U,t} + \epsilon_{U,t+1})P_t) + (U_t + \delta_{U,t} + \epsilon_{U,t+1})D_{U,t+1}h \\ U_{t+1} &= U_t + \delta_{U,t} + \epsilon_{U,t+1} \\ \hat{I}_{t+1} &= \hat{I}_t - \delta_{U,t} - C_I\Omega_t + \boldsymbol{B}_I\boldsymbol{\epsilon}_{t+1} \\ \hat{P}_t &= P_F\hat{F}_t + P_D D_t + \tilde{P}_t \end{aligned}$$

where Ω_t is a Gaussian mean 0 random variable in period t. In the strategic economy the risk discount is given by $\tilde{P}_t = \frac{1}{\pi}\left(\tilde{\boldsymbol{A}}_I\boldsymbol{\Psi}_{U,t} + \delta_{U,t}\right)$, where $\tilde{A}_I = [A_{I,U} \;\; A_{I,I}]$. In the competitive economy, $\tilde{P}_t = \boldsymbol{A}_P\boldsymbol{\Psi}_{U,t}$, where the price vector $\boldsymbol{A}_P$ is taken as given. Notice that $\delta_{U,t}$ above is a choice variable in the strategic economy. In the competitive economy, the atomic uninformed agent simply assumes some $\delta_{U,t}$ as a linear function of $\boldsymbol{\Psi}_{U,t}$ (of course not depending on $\tilde{U}_t$). Then the evolution of the atomic agent's asset holdings is given by

$$U_{t+1}^C = U_t^C + \delta_{U,t}^C + \epsilon_{U,t+1}$$

where $\delta_{U,t}^C$ is the optimal trade amount of the atomic agent. In equilibrium, of course, we must have that $\delta_{U,t}^A = \delta_{U,t}$.

For succinctness we can write

$$\boldsymbol{\Psi}_{U,t+1} = \boldsymbol{A}_{\Psi,U}\boldsymbol{\Psi}_{U,t} + \boldsymbol{i}\delta_{U,t}^A + \boldsymbol{B}_{\Psi,U}\boldsymbol{\epsilon}_{U,t+1} \tag{2.35}$$

where $\boldsymbol{i} \equiv [1 \;\; -1]'$ in the strategic case, where $\boldsymbol{i} \equiv [\; 0 \;\; 1 \;\; 0 \;]'$ in the competitive case, and $\boldsymbol{\epsilon}_{U,t} = [\epsilon_{U,t} \; \epsilon_{I,t} \; \epsilon_{F,t} \; \epsilon_{\theta,t} \; \epsilon_{D,t} \; \Omega_{t-1}]'$. Here $\delta_{U,t}^A$ is the chosen trade amount of the uninformed agent in either economy (so in the strategic case $\delta_{U,t}^A \equiv \delta_{U,t}$).

The following theorem states the relevant result.

Theorem 2.3 *The solution of $(\mathcal{U})$ in both the strategic and in the competitive case is given by*

$$J(M_{U,t}, D_t, \hat{F}_t, \boldsymbol{\Psi}_{U,t}, t) = -\kappa e^{-\rho th - \lambda(M + UP_F\hat{F} + UP_D D) - \frac{1}{2}\boldsymbol{\Psi}'\boldsymbol{V}\boldsymbol{\Psi}} \tag{2.36}$$

where

$$\lambda = \gamma\frac{1 - e^{-rh}}{h}$$

and where κ is a constant, $\boldsymbol{V}$ is a symmetric 2×2 matrix in the strategic economy (which is solved by a system of 3 non-linear algebraic equations), and $\boldsymbol{V}$ is a symmetric 3×3 matrix in the strategic economy (which is solved by a system of 6 non-linear algebraic equations). Furthermore, consumption c_t will be linear in the state variables. The trade amount, $\delta_{U,t}$, depends only on $\boldsymbol{\Psi}_{U,t}$ and can therefore be written

$$\delta_{U,t} = \boldsymbol{F}_U \boldsymbol{\Psi}_{U,t}$$

The coefficients $\boldsymbol{F}_U$ will differ across the two economies. Also it is easy to show that in the competitive case the second entry in $\boldsymbol{F}_U$ (i.e. the loading on U_t^C) will be -1.

The Appendix gives the exact expression for c_t and $\delta_{U,t}$.

Because of the form of the price function, the trade of the agent does not depend on either of the dividend components, D_t or F_t. Furthermore, because of the CARA utility assumption, the agent's trade does not depend on his wealth. These two facts greatly simplify the control problem, as well as the equilibrium in the model. Also note that part of the optimal trade of the competitive informed agent is to first sell his or her entire asset holdings. Hence the post-trade optimal position of the competitive agent does not depend on the pre-trade position. This is course is not the case for a strategic agent, whose post-trade optimal position clearly should depend on his or her pre-trade position.

2.3.4 Control Problem of the Informed

The control problem faced by each investor in the informed competitive sector is similar to the uninformed agent's control problem. The state variables of interest to the informed investors are $M_{I,t}, D_t, F_t, \tilde{I}_t$, and $\boldsymbol{\Psi}_{I,t} = \{U_t, I_t, \Omega_t, \tilde{P}_t\}$.[11] Here I_t is the aggregate holdings of the risky asset by the entire informed sector, and $\tilde{I}_t$ is the individual holdings of each informed agent. The control problem of the informed agent takes as given the behavior of all of the other informed and uninformed agents; hence, each small informed agent finds an optimal response to the behavior of all of the other agents in the informed sector and to the behavior of the uninformed strategic trader. The atomic informed agent takes as given the endowment behavior given in (2.6), as well as a candidate price process of the form in (2.17). This form of the price is implied by the aggregate demand curve of the informed sector.

[11] Recall that $\Omega_t = \hat{F}_t - F_t$ and the $\tilde{P}_t$ is the non-dividend part of prices.

The competitive agents use this form of the price process in order to compute the expectation of their one period ahead value function in the Bellman equation. However, the agents solve for their demands at a given point in time for all possible prices P_t, even those which are not consistent with their belief about the form of the price process. This is a complication which is unnecessary in models of competitive trading. However, in the presence of a strategic agent, we need to know the demand elasticity of the informed sector even for out of equilibrium prices. That is, in making his trade decision, the strategic agent will consider trade amounts which are out of equilibrium before settling on the optimal (equilibrium) choice. In a competitive economy, the slope of agents' demand curves can still be determined (given by the weighting in their demand function on the price). However, this slope is computed only for equilibrium prices, and is therefore inadequate for our purposes. In order to identify the slope of the informed sector's demand curve (for out of equilibrium prices), we need to consider the period t price as a state variable.

Let us say that the demand of the uninformed strategic agent is given by $\delta_{U,t} = \boldsymbol{F}_U \boldsymbol{\Psi}_{U,t}$. From (2.31), we find that

$$\hat{I}_t = I_t - \frac{(\pi P_F - A_{I,\Omega})}{A_{I,I}} \Omega_t \tag{2.37}$$

Expressed in terms of $\boldsymbol{\Psi}_{I,t}$, we can write $\delta_{U,t} = \tilde{F}_U \boldsymbol{\Psi}_{I,t}$, where $\tilde{F}_{U,U} = F_{U,U}$, $\tilde{F}_{U,I} = F_{U,I}$, $\tilde{F}_{U,\hat{I}} = 0$, $\tilde{F}_{U,\Omega} = -\frac{(\pi P_F - A_{I,\Omega})}{A_{I,I}} F_{U,I}$, and $\tilde{F}_{U,P} = 0$.

The informed investor solves the following problem ($\mathcal{I}$):

$$\sup_{c,\delta} E_t \left[\sum_{s=t}^{\infty} -e^{-\rho sh - \gamma c} h \right]$$

where the expectation is taken with respect to the informed agent's information set in period

t. The dynamics of the state variables are given by

$$\begin{aligned}
M_{I,t+1} &= e^{rh}(M_{I,t} - c_{I,t}h - \delta_{I,t}P_t - \epsilon_{I,t+1}(P_F F_t + P_D D_t)) + \\
&\quad (I_t + \delta_{I,t} + \epsilon_{I,t+1})D_{I,t+1}h \\
U_{t+1} &= U_t + \delta_{U,t} + \epsilon_{U,t+1} \\
I_{t+1} &= I_t - \delta_{U,t} + \epsilon_{I,t+1} \\
\tilde{I}_{t+1} &= \tilde{I}_t + \delta_{I,t} + \epsilon_{I,t+1} \\
\Omega_{t+1} &= A_\Omega \Omega_t + \boldsymbol{B}_\Omega \epsilon_{t+1} \\
\tilde{P}_{t+1} &= \frac{1}{\pi}\left(\tilde{\boldsymbol{A}}_I + \tilde{\boldsymbol{F}}_U\right)\boldsymbol{\Psi}_{I,t+1} \quad \text{or} \quad \tilde{P}_{t+1} = \boldsymbol{A}_P \boldsymbol{\Psi}_{I,t+1}
\end{aligned}$$

where $\tilde{\boldsymbol{A}}_I$ has the same loadings on U_t, I_t, Ω_t as $\boldsymbol{A}_I$, and a zero loading on $\tilde{P}_t$. The vector $\boldsymbol{A}_P$ gives the price in the competitive economy. Notice that the informed agent's trade decision only affects $\tilde{I}_t$, and does not affect the endowments of any of the other agents in the economy. Also the endowment of the atomic informed agent is not affected by the trade of the strategic investor. Finally, the atomic informed agent takes the price as given. In the Appendix, I show that the dynamics of the informed agent's state variables are given by

$$\boldsymbol{\Psi}_{I,t+1} = \boldsymbol{A}_{\Psi,I}\boldsymbol{\Psi}_{I,t} + \boldsymbol{B}_{\Psi,I}\epsilon_{t+1} \tag{2.38}$$

The following theorem states the relevant result.

Theorem 2.4 *The solution of $(\mathcal{I})$ is given by*

$$J(M_{I,t}, D_t, F_t, \tilde{I}_t, \boldsymbol{\Psi}_{I,t}, t) = -\kappa e^{-\rho th - \lambda(M + \tilde{I}_t P_t) - \frac{1}{2}\boldsymbol{\Psi}'\boldsymbol{V}\boldsymbol{\Psi}} \tag{2.39}$$

where

$$\lambda = \gamma \frac{1 - e^{-rh}}{h}$$

and where κ is a constant, and $\boldsymbol{V}$ is a symmetric 4×4 matrix which is solved by a system of 10 non-linear algebraic equations (see Appendix). Furthermore, consumption c_t will be linear in the state variables. And the trade, $\delta_{I,t}$ depends only on $\boldsymbol{\Psi}_{I,t}$ and $\tilde{I}_t$, and can therefore be written

$$\delta_{I,t} = \boldsymbol{F}_I \boldsymbol{\Psi}_{I,t} - \tilde{I}_t$$

The Appendix gives the exact expression for c_t and $\delta_{I,t}$.

The fact that $\tilde{I}_t$ enters into the agent's demands with a coefficient of -1 says that the agent's optimal position in the risky asset does not depend on his current level of inventory, which is the standard result for competitive investors. Let us define

$$\boldsymbol{\Gamma}_I \equiv \lambda\left((h + P_D)\boldsymbol{B}'_D + P_F\boldsymbol{B}'_F + \boldsymbol{B}'_P\right)$$

We can then write the optimal post trade position of the informed agent as

$$\tilde{I}_t + \delta_{I,t} = (\boldsymbol{\Gamma}'_I\Xi\boldsymbol{\Gamma}_I)^{-1}\left(\lambda(\boldsymbol{A}_{EP} - e^{rh}\boldsymbol{A}_P) - \boldsymbol{\Gamma}'_I\Xi[\lambda\boldsymbol{B}'_I\boldsymbol{A}_{EP} + \boldsymbol{B}'_\Psi\boldsymbol{V}\boldsymbol{A}_\Psi]\right)\boldsymbol{\Psi}_{I,t} \qquad (2.40)$$

where $\boldsymbol{A}_P = [0\ 0\ 0\ 0\ 1]$, and where Ξ, $\boldsymbol{A}_{EP}$, and $\boldsymbol{B}_P$ are given in the Appendix. The first term in this expression gives the excess return per share divided by the riskiness of the asset, and reflects the agent's risk-return tradeoff. The second term reflects the hedging demand which arises because changes in the state variables $\boldsymbol{\Psi}_{I,t}$ change the investment opportunities available to the investor. Notice that this term is just the optimal (in the mean squared error sense) projection of the investment uncertainty faced by the agent onto the space spanned by the agent's investment opportunity set $\boldsymbol{\Gamma}_I$.

2.3.5 Equilibrium of the Economy

A. Strategic Case

Determination of the Nash equilibrium of this economy proceeds as follows. First I conjecture the aggregate demand curve of the informed sector, specified by $\boldsymbol{A}_I$ and π. Then the uninformed agent solves his control problem, taking $\boldsymbol{A}_I$ and π as given. This leads to an optimal demand of the uninformed agent, given by $\boldsymbol{F}_U\boldsymbol{\Psi}_{U,t} = \tilde{\boldsymbol{F}}_U\boldsymbol{\Psi}_{I,t}$. For an atomic informed trader I then solve his control problem, taking $\boldsymbol{A}_I$, π, and $\tilde{\boldsymbol{F}}_U$ as given.

Taking $\delta_{I,t} = \boldsymbol{F}_I\ \boldsymbol{\Psi}_{I,t}$ as the demand of an atomic informed agent, a Nash equilibrium is defined as the pair $\{\boldsymbol{A}_I, \pi\}$ such that

$$[\ F_{I,U}\ \ F_{I,I}\ \ F_{I,\Omega}\ ] \;=\; \boldsymbol{A}_I \qquad (2.41)$$

$$F_{I,P} \;=\; -\pi \qquad (2.42)$$

In other words, the actual demand of an atomic informed agent is equal to the conjectured demand for the informed sector. And the actual price elasticity of demand of an atomic informed agent is equivalent to the conjectured price elasticity.

B. Competitive Case

Here I proceed in much the same way as in the strategic case. First I conjecture the aggregate demand curve of the informed sector, and the optimal trade amount (though not a demand curve) for the strategic sector. These are given by $\boldsymbol{A}_I$, π, and $\boldsymbol{A}_U$. Then, using (2.17), I solve for the resultant equilibrium price. Using this price, I first solve for the optimal demand of an atomic informed agent, which can be expressed as $\tilde{\boldsymbol{F}}_U \ \boldsymbol{\Psi}_{I,t}$. Then I solve for the optimal demand of an atomic informed agent, given by $\delta_{I,t} = \boldsymbol{F}_I \ \boldsymbol{\Psi}_{I,t}$.

An equilibrium is defined as the vector $\{\boldsymbol{A}_I, \pi, \boldsymbol{A}_U\}$ such that

$$[\ F_{I,U} \ \ F_{I,I} \ \ F_{I,\Omega}\ ] = \boldsymbol{A}_I \tag{2.43}$$

$$F_{I,P} = -\pi \tag{2.44}$$

$$[\ \tilde{F}_{U,U} \ \ \tilde{F}_{U,I}\ ] = \boldsymbol{A}_U \tag{2.45}$$

In other words, the demands of the atomic informed and strategic agents are exactly equal to the conjectured sector demands.

This procedure is identical to conjecturing a price function and a trade function directly. The conjectured price function will be the same as the price given by (2.17), and the conjectured trade amount (expressed without loss of generality in terms of the uninformed sector's demand) will be given by $\boldsymbol{A}_U$.

2.4 Analysis of Equilibrium

In this section I analyze the equilibrium of the economy. In particular I focus on how strategic behavior on the part of the uninformed agents affects prices and trading patterns. First I will discuss some basic features of the equilibrium. Then I will examine the effect that market power has on the information set which results from the inference performed by the uninformed agents. I will then consider the effects of information asymmetry on expected returns in the competitive and in the strategic economies; and in particular, I will discuss how and why expected returns differ across the two. Finally, I will repeat this same analysis for the variability of returns and for trading volume. Note that in the discussion which follows, all variables with a c superscript refer to the competitive economy; variables with no superscript refer to the strategic economy.

2.4.1 Properties of Equilibrium

Recall from (2.14) that the price in this economy is given by

$$P_t = P_F F_t + P_D D_t + P_U U_t + P_I I_t + P_\Omega(\hat{F}_t - F_t) \tag{2.46}$$

As is shown in the appendix, the coefficients on the dividend levels, D_t and F_t are given by

$$P_D = \frac{A_D h}{e^{rh} - A_D} \tag{2.47}$$

$$P_F = \frac{A_F e^{rh} h^2}{(e^{rh} - A_F)(e^{rh} - A_D)} \tag{2.48}$$

where from (2.3–2.4) we see that $A_D = 1 - k_D h$ and $A_F = 1 - k_F h$. To simplify these somewhat, we note that a simple application of l'Hopital's rule shows that in the limit as $h \to 0$ we have

$$P_D = \frac{1}{r + k_D}$$

$$P_F = \frac{1}{(r + k_D)(r + k_F)}$$

These coefficients simply give the present value of expected future dividends. In fact at any time t we have that

$$P_D D_t + P_F F_t = \mathrm{E}_t\left[\sum_{s=t+1}^{\infty} e^{-rhs} D_s h\right] \tag{2.49}$$

where the expectation is with respect to the information set at time t. In the limit as either of the agents in the economy become risk neutral or as the noise in the economy goes to zero, the actual price will converge to $P_t^0 \equiv P_D D_t + P_F F_t$. Hence any deviation of the price from P_t^0 reflects a risk loading on the state variables in the economy. Although I have not solved for them in closed form, P_I and P_U are negative for all parameterizations of the economy which I have considered. The sign of the coefficients induces risk sharing in the economy. Again, though I have not solved for it in closed form, P_Ω is positive for all parameterizations of the economy which I have considered. It serves as a deterrent to excessive informational arbitrage by the informed sector. In response to a large positive value of Ω_t, the informed agents will, at a given price, buy a large position in the risky asset, with the intent of selling it over time to the misinformed large agent at high prices. The positive loading on Ω_t in the price function serves to mitigate this behavior.

As the mean-reversion of the dividend growth rate increases, knowledge of F_t becomes less valuable. In fact when $k_F \to 1/h$ (next period's growth rate approaches white noise), the informational advantage of the informed agents over the uninformed agent disappears. Also, the contribution of the current growth rate to the price disappears (that is $P_F = 0$).

Recall that the risk discount in the price is given by $\tilde{P}_t = P_U U_t + P_I I_t + P_\Omega \Omega_t$. Let us define

$$\tilde{P}_\infty = \lim_{T \to \infty} E[\tilde{P}_T | \mathcal{F}_t] \tag{2.50}$$

as the long run limit of the risk discount in the stock price. From (2.28), we know that in expectation the estimation error Ω_t tends to go to 0 in the long run. Let us define U_∞ and I_∞ as the expected long run levels of the asset holdings in the economy. From market clearing, we can write $U_t = U_\infty + \Delta_t$ and $I_t = I_\infty - \Delta_t$ (and therefore $E_t[\Delta_T] \to 0$ as T gets large). Hence we can re-write the risk discount as

$$\begin{aligned}
\tilde{P}_t &= P_U U_\infty + P_I I_\infty + (P_U - P_I)\Delta_t + P_\Omega \Omega_t \\
\tilde{P}_t &= \tilde{P}_\infty + (P_U - P_I)\Delta_t + P_\Omega \Omega_t
\end{aligned} \tag{2.51}$$

This equation makes clear the respective roles of information and market power to the agents in the economy. For all parameterizations of the economy which I have considered, $P_U - P_I > 0$ and $P_\Omega > 0$. The term $P_U - P_I$ is a measure of the market power of the large agent. When the large agent's inventory is above its long-run levels, that is $\Delta_t > 0$, the large agent sells shares of the risky asset at a price above its long term level, and hence earns surplus from the rest of the economy. When $\Delta_t < 0$, the large agent buys at prices below the long-run level. In the competitive economy, we clearly should have that $P_U = P_I$, since the current levels of the agents' asset holdings do not affect their optimal post-trade positions.

The coefficient P_Ω determines the extent to which prices reflect the uninformed sector's misestimate of the dividend growth rate. One of the roles this coefficient plays is to govern the amount of surplus lost due to misinformed trade by the uninformed agents. Consider what happens when the uninformed receive a shock in their exogenous signal. They overestimate both F_t and I_t (that is $\Omega_t > 0$), and thinking that the current expected return of the risky asset is high begin to buy at prices above the long-run price. In this sense, a high value of P_Ω allows for substantial informational arbitrage by the informed agents. However, there is another effect which works in favor of the uninformed agents. This will be discussed in section 2.4.7.

Since the trade amount of the uninformed agent is given by $\delta_{U,t} = F_{U,U}U_t + F_{U,I}I_t + F_{U,\Omega}\Omega_t$, it is easy to see that the ratio of long run asset holdings in the economy can be written as

$$\frac{U_\infty}{I_\infty} = -\frac{F_{U,I}}{F_{U,U}} \tag{2.52}$$

From this we find that

$$\frac{U_\infty}{U_\infty + I_\infty} = \frac{-F_{U,I}}{F_{U,U} - F_{U,I}} \qquad \frac{I_\infty}{U_\infty + I_\infty} = \frac{F_{U,U}}{F_{U,U} - F_{U,I}} \tag{2.53}$$

Hence the long run price discount (per unit of the risky asset) can be written as

$$\frac{\tilde{P}_\infty}{S_t} = P_I \frac{F_{U,U}}{F_{U,U} - F_{U,I}} - P_U \frac{F_{U,I}}{F_{U,U} - F_{U,I}} \tag{2.54}$$

where $S_t \equiv I_t + U_t$ is the total number of shares of the risky asset held by the two sectors.

2.4.2 Price Impact

From (2.17), we see that π measures the depth of the market faced by the strategic investor. Decreasing marginal utility implies that $\pi > 0$. Furthermore π is decreasing in the risk aversion of the competitive agents in the economy (i.e. γ_I), and in the noisiness of the shocks in the economy (i.e. γ_I, γ_U, γ_F, and γ_D). The limiting cases are easiest to consider. As $\gamma_I \to 0$, we will have $\pi \to \infty$, as the risk neutral demand curve becomes infinitely elastic. Similarly as they become infinitely risk averse, no change in price will be able to induce agents to hold any shares, and hence $\pi \to 0$. Increasing the variability of shocks will increase the variability of share returns. As returns become more volatile the demand elasticity of risk averse agents will again decrease in magnitude (i.e. π will fall).

2.4.3 Inference Problem

Let us consider the effect that strategic behavior has on the inference problem of the uninformed agents in the economy. Figure 2.1 shows the variances of the uninformed sector's estimates of the inventory of the informed agents ($\sigma_{\hat{I}}$) and of the level of the dividend ($\sigma_{\hat{F}}$), as functions of the quality of the uninformed agents' information set (σ_θ). The variances $\sigma_{\hat{F}}$

and $\sigma_{\hat{I}}$ are respectively the upper left hand and the lower right hand entries in the matrix $\boldsymbol{O}$ given in the Appendix.

As the precision of the external signal received by the uninformed improves (that is as $\sigma_\theta \to 0$), the information sets of the informed and uninformed become symmetric, and in both the competitive and in the strategic economies, the variances of the uninformed agents' estimates of the state variables go to 0. As σ_θ increases, $\sigma_{\hat{I}}$ and $\sigma_{\hat{F}}$ increase as well. As can

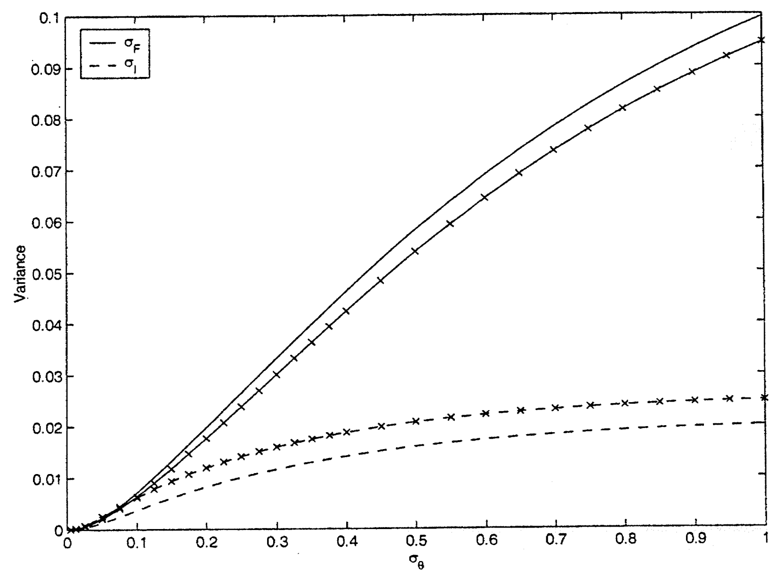


Figure 2.1: $\sigma^2_{\hat{I}}$ and $\sigma^2_{\hat{F}}$ as functions of σ_θ. The dashed lines show $\sigma^2_{\hat{I}}$, and the solid lines shows $\sigma^2_{\hat{F}}$. The marked lines represent the competitive economy. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

be seen from Figure 2.1, the rate of increase of the variance depends on whether we are in the competitive or the strategic economy. We can see that when $\sigma_\theta > 0$, the uninformed agents in the competitive economy will have a more precise estimate of F_t and a less precise estimate of I_t (i.e. $\sigma^C_{\hat{F}} < \sigma_{\hat{F}}$ and $\sigma^C_{\hat{I}} > \sigma_{\hat{I}}$).

To see the intuition for this result, consider what happens when the uninformed agents receive a shock in their external signal. The strategic uninformed agent will realize that his trades imply a price impact: trading into a position based on incorrect information entails a price concession, as does trading out of such a position. These are considerations which a competitive uninformed agent does not make. Not wanting to bear this additional cost of misinformed trade, the uninformed strategic trader reduces the exposure of his demand to his misestimate of the state of the economy (i.e. $F_{U,\Omega} < F^C_{U,\Omega}$). In turn, this reduces the incentives of the informed agent to speculate on their superior information. Hence a larger fraction of trade originating from the informed sector will be due to allocational motives. The equilibrium price, reflecting equilibrium order flow, will therefore contain more information about the asset holdings of the informed sector, and less information about the dividend growth rate. Hence in the strategic economy the asset holdings of the informed sector will be better estimated than in the competitive economy ($\sigma^C_{\hat{I}} > \sigma_{\hat{I}}$), and the dividend growth rate will be more poorly estimated ($\sigma^C_{\hat{F}} < \sigma_{\hat{F}}$). As will become clear in the next section, the strategic uninformed agent will have a better estimate of expected returns than will the competitive uninformed sector.

2.4.4 Expected Returns

In order to get a sense for the effect that strategic behavior has on the properties of prices in the presence of asymmetric information, it will be convenient to work with returns. The natural concept of returns in the current framework is that of *share returns.* Cumulative excess share returns are defined as the profit from a zero-cost investment (from the perspective of a competitive agent) which is rolled over in every period. Hence we have

$$Q_t \equiv \sum_{s=1}^{t} \left(D_s h + P_s - P_{s-1} - (e^{rh} - 1)P_{s-1}\right) \tag{2.55}$$

The per period excess returns are simply $dQ_t \equiv Q_t - Q_{t-1} = D_t h + P_t - e^{rh} P_{t-1}$. The quantity dQ_t/P_{t-1} gives the traditional excess return expressed as a rate. For $\boldsymbol{\Psi}_t = [\ U_t \ \ I_t \ \ \Omega_t\ ]'$, it is easy to show that

$$dQ_t = \boldsymbol{A}_Q \boldsymbol{\Psi}_{t-1} + \boldsymbol{B}_Q \boldsymbol{\epsilon}_t \tag{2.56}$$

for some constant matrices $\boldsymbol{A}_Q$ and $\boldsymbol{B}_Q$ (see Appendix). In particular, note that the excess returns do not depend on either the current dividend level D_t, or on the dividend growth rate F_t. Therefore, a more precise estimate of I_t leads to a more precise estimate of the expected return of the risky asset. Excess returns, however, *do* depend on dividend innovations. This follows because the financing term $(e^{rh} - 1)P_{t-1}$ fully accounts for the expected, but not the unexpected, dividend payments at time t.

Since returns depend on the current state variables, we consider long-term returns as a state independent measure of the expected profits from holding the risky asset. Recalling the definitions of U_∞ and I_∞ from (2.53), we find that expected long term (per single share) returns $E[dQ]$ are given by

$$E[dQ] = \boldsymbol{A}_Q \times \left[\begin{array}{ccc} \dfrac{-F_{U,I}}{F_{U,U} - F_{U,I}} & \dfrac{F_{U,U}}{F_{U,U} - F_{U,I}} & 0 \end{array} \right]'$$

The actual excess returns are then given by $S_t\, E[dQ]$, where S_t is the current supply of the risky asset. Figure 2.2 shows $E[dQ]$ as a function of the informational asymmetry in the competitive and the strategic economies. We find that in both economies expected returns

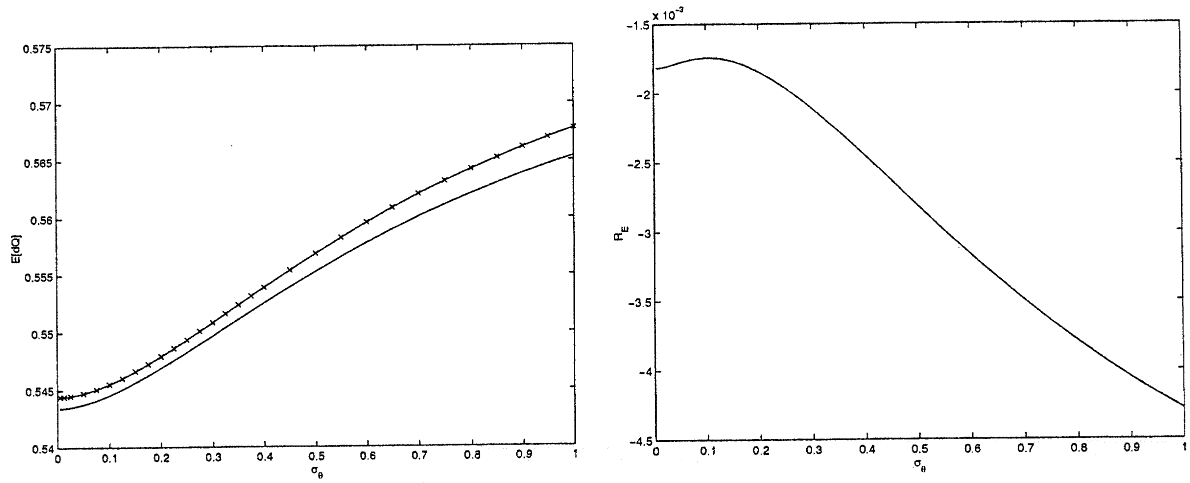


Figure 2.2: The left pannel shows $E[dQ]$ as a function of σ_θ. The marked lines represent the competitive economy. The right panel shows the quantity $(E[dQ] - E[dQ^c])/E[dQ^c]$. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

are increasing with informational asymmetry (see Wang (1993) for a similar result). The intuition for the result is that investors need to be compensated for bearing risk. Since increasing σ_θ increases the riskiness of the stock payout from the point of view of the uninformed investors, the excess return needs to be higher to induce them to hold market clearing levels of the risky asset.

We also see that the return in the strategic economy is lower than it is in the competitive one. To understand this result, consider a situation where the informed sector receives an endowment shock. In order for the uninformed sector to engage in risk sharing, the price of the risky asset needs to fall. Relative to the initial trade amount in the competitive case, the strategic agent can either choose to buy more or fewer shares in response to the endowment shock. Buying more shares will imply a higher price; this strategy can not be optimal (otherwise in the competitive economy, the uninformed agents would have bought more than the equilibrium number of shares at the lower equilibrium price). Hence the strategic agent will choose to initially buy fewer shares, but at a lower price, than in the competitive economy. Hence the equilibrium price discount on the holdings of the competitive sector must be higher in the strategic economy (i.e. $|P_I| > |P_I^C|$).

Consider what happens when the uninformed investor receives a positive endowment shock. By selling more than in the competitive case, he will lower the price and hence sell more shares at a lower price. This can not be optimal (for the same reason as in the case above). But by selling less, the strategic agent will be able to sell at a higher price. This will be the equilibrium outcome. Hence it must be the case that $|P_U| < |P_U^C|$. Since in the competitive case, $P_I^C = P_U^C$, and since both coefficients are negative, we must have in the strategic economy that $P_U - P_I > 0$.

The increase in $|P_I|$ is coupled with the decrease in $|P_U|$. The magnitudes of these changes are related to the relative risk aversions of the two sectors. For example, when the uninformed agents are less risk averse, an endowment shock to the informed sector induces a large amount of trade. Therefore, the scope for price improvement is larger when we move to the strategic economy. In this case, the increase in $|P_I|$ will dominate the decrease in $|P_U|$. However, it will also be the case that in the long-run the uninformed agents will hold a larger number of shares relative to the informed agents. The net result of these two effects is that the price will be higher in the strategic economy since the P_U coefficient receives a higher weight in the long-run price. The expected return will therefore be lower in the strategic economy.

The right panel in Figure 2.2 shows the difference in expected returns between the two economies as a fraction of the expected return in the competitive economy as a function of σ_θ.

$$R_E \equiv \frac{E[dQ] - E[dQ^C]}{E[dQ^C]}$$

Consider again the above example in which the informed sector receives an endowment shock. As the informational asymmetry increases, the decreases in $|P_U|$ relative to the competitive economy increases as well (we will return to why $P_U - P_U^C$ increases with informational asymmetry in Section 2.5). The long-run asset holdings of the uninformed decrease with informational asymmetry because holding the stock becomes progressively riskier for the uninformed agents as their information set deteriorates. However, the decrease in $|P_U|$ is the dominant effect causing the price to be higher (relative to the competitive economy) in the strategic economy with informational asymmetry. Hence as information sets of the uninformed become worse, the difference in expected returns between the two economies becomes progressively larger.

2.4.5 Return Variability

Figure 2.3 shows the conditional variance of share returns, $\text{Var}_{t-1}(dQ_t) = \boldsymbol{B}_Q \Sigma \boldsymbol{B}_Q'$, as a function of the informational asymmetry in the competitive and in the strategic economies. From the Appendix, we see that

$$\boldsymbol{B}_Q = [\, P_U \;\; P_I \;\; P_F + P_D h + h^2 \;\; 0 \;\; P_D + h \,] + P_\Omega \boldsymbol{B}_\Omega \tag{2.57}$$

Here $P_\Omega \boldsymbol{B}_\Omega$ is the contribution to return innovations which comes from the uninformed sector's misestimates of the state variables in the economy. We note that in both economies

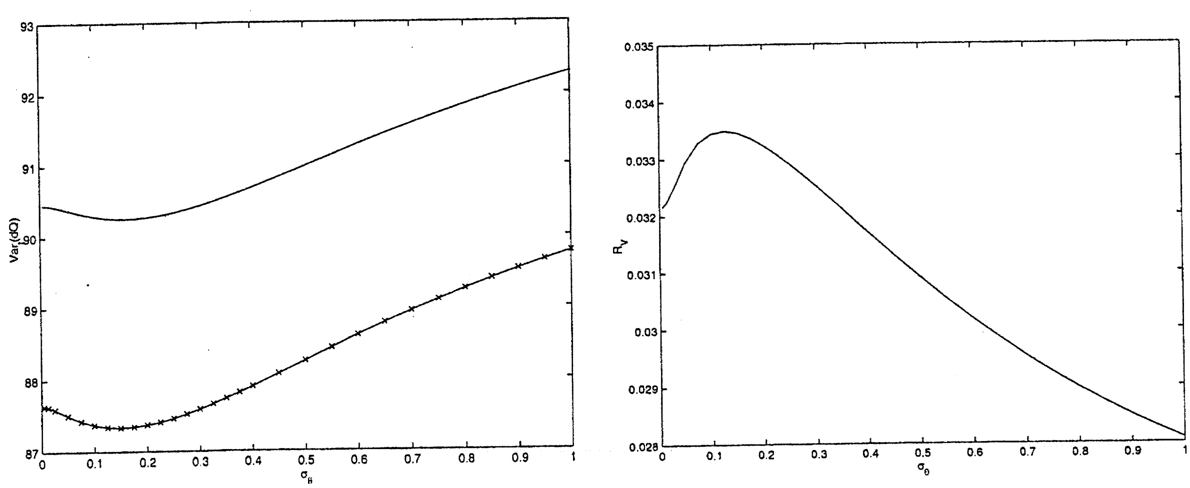


Figure 2.3: The left panel shows $\text{Var}_t(dQ_{t+1})$ as a function of σ_θ. The marked lines represent the competitive economy. The right panel shows the quantity $(\text{Var}(dQ) - \text{Var}(dQ^C))/\text{Var}(dQ^C)$. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

the conditional return variance can decrease or increase with informational asymmetry. As

σ_θ increases expected returns increase (i.e. $|P_U|$ and $|P_I|$ increase), and hence endowment innovations lead to more pronounced changes in actual returns (see Wang (1993) for a related result regarding price innovations). This causes return variance to increase. On the other hand, consider what happens when a dividend growth shock arrives (ϵ_F). This has the effect of making the current dividend higher. It also causes the uninformed agents to underestimate the current dividend growth rate; this negative shock to Ω_t then lowers the price (since $P_\Omega > 0$). The negative correlation between dividend growth innovations and dividend misestimate innovations acts to reduce return variability (that is $\boldsymbol{B}_\Omega$ in (2.57) has a negative weighting on ϵ_F). As Figure 2.3 shows the latter effect dominates for small σ_θ, and the former effect dominates for larger σ_θ. We have the slightly paradoxical result that since, in equilibrium, the misestimates of state variables effect asset prices, that negative correlation between these and innovations to actual state variables may serve to lessen return variability.

The right panel in Figure 2.3 shows the difference in conditional return variances between the two economies as a fraction of the conditional return variance in the competitive economy as a function of σ_θ.

$$R_V \equiv \frac{\mathrm{Var}_t(dQ_{t+1}) - \mathrm{Var}_t(dQ^C_{t+1})}{\mathrm{Var}_t(dQ^C_{t+1})}$$

We see that the return variance in the strategic economy is higher than in the competitive economy. The intuition is straightforward. When the informed sector receives a positive endowment shock, strategic behavior leads to lower initial prices than in the competitive, and this results in higher return variability. Interestingly the difference in return variability between the two economies first increases, with small σ_θ, and then begins to decrease. Recall that in the strategic economy, the uninformed sector reduces (relative to the competitive case) its exposure to trading on misinformation because of the increased cost of trading out of misinformed positions. Reducing the dependence of demand on misinformation ($F_{U,\Omega} < F^C_{U,\Omega}$) reduces the dependence of the price on misinformation ($P_\Omega < P^C_\Omega$). This decreases the sensitivity of the price in the strategic economy to innovations in Ω_t. Since increased variability in Ω_t is the main contributor to the two channels discussed above, we find that in the strategic economy changes in informational asymmetry have less of an effect on return variability. Hence the inverted U-shaped pattern in R_V is caused by the fact that the U-shaped pattern in $\mathrm{Var}_t(dQ^C_{t+1})$ is more pronounced than the one in $\mathrm{Var}_t(dQ_{t+1})$.

2.4.6 Trading Volume

We would like to understand the impact of strategic behavior on trading volume. Recall that from the point of view of the informed competitive sector, trades are given by $\delta_{I,t} = \boldsymbol{F}_I \boldsymbol{\Psi}_t$. We can then write the trade amount as

$$\delta_{I,t+1} = (1 + F_{I,I} - F_{I,U})\delta_{I,t} + F_{I,\Omega}\,(A_\Omega - 1)\Omega_t + F_{I,I}\,\epsilon_{I,t+1} + F_{I,U}\,\epsilon_{U,t+1} + F_{I,\Omega}\,\epsilon_{\Omega,t+1} \quad (2.58)$$

where $\epsilon_{\Omega,t+1} \equiv \boldsymbol{B}_\Omega \boldsymbol{\epsilon}_{t+1}$. In a competitive economy, past trades do not affect future trades, which is equivalent to the requirement that $1 + F_{I,I} - F_{I,U} = 0$. However, this is not the case in a strategic economy, where $1 + F_{I,I} - F_{I,U}$ is typically positive. Hence the dynamics of trade are substantially different in the two economies. In the strategic economy, past trade amounts (turnover) affect current trade amounts (turnover). However, this dependence is not present in the competitive case. Furthermore, $1 + F_{I,I} - F_{I,U}$ can increase or decrease with informational asymmetry, although the effect is small in either case.

In order to have a comparable concept of trading volume in both economies, we will consider deviations of trade amounts from their expectation (conditional on the information set of the informed agents, $\mathcal{F}_t^I$). Let us define the unanticipated trading volume at time $t+1$ as

$$\hat{\delta}_{t+1} \equiv |\, \delta_{I,t+1} - E_t[\delta_{I,t+1}]\,| \quad (2.59)$$

Conditional on $\mathcal{F}_t^I$, $\delta_{I,t+1} - E_t[\delta_{I,t+1}]$ is a mean zero Gaussian random variable, and hence

$$E_t[\hat{\delta}_{t+1}] = \sqrt{\frac{2}{\pi}\mathrm{Var}_t(\delta_{I,t+1} - E_t[\delta_{I,t+1}])} \quad (2.60)$$

Figure 2.4 shows $E_t[\hat{\delta}_{t+1}]$ as a function of the informational asymmetry in the competitive and in the strategic economies. From (2.58), we can write the variance of unanticipated trade as

$$\mathrm{Var}_t(\delta_{I,t+1} - E_t[\delta_{I,t+1}]) = F_{I,I}^2\,\sigma_I^2 + F_{I,U}^2\,\sigma_U^2 + F_{I,\Omega}^2\,\sigma_\Omega^2 + 2\,F_{I,I}\,F_{I,\Omega}\,\mathrm{Cov}(\epsilon_I, \epsilon_\Omega) \quad (2.61)$$

since innovations to uninformed asset holdings are uncorrelated with innovations to informed asset holdings and with innovations to dividend misestimates (ϵ_Ω). We see that in both economies expected unanticipated trading volume first falls with increasing informational

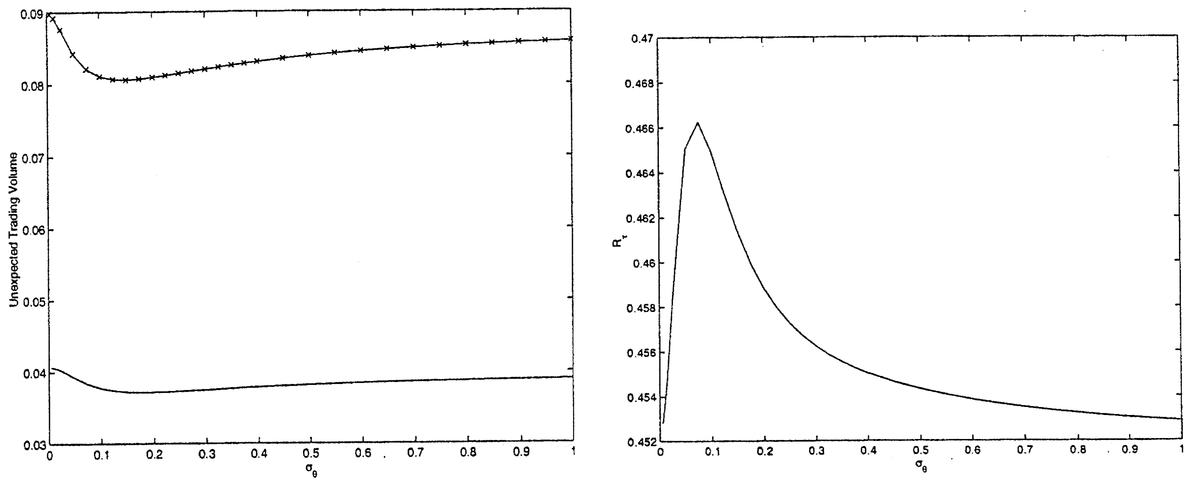


Figure 2.4: The left panel shows $E_t[\hat{\delta}_{t+1}]$ as a function of σ_θ. The marked lines represent the competitive economy. The right panel shows the quantity $E_t[\hat{\delta}_{t+1}]/E_t[\hat{\delta}^C_{t+1}]$. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

asymmetry, and then begins to increase. The first effect of increasing σ_θ is to increase σ_Ω, the variance of innovations to Ω_t. Because the exogenous signal becomes progressively less informative, the uninformed sector attributes more weight to the price and dividend signals in their inference problem. Hence innovations which affect these signals lead to larger changes in the misestimates of the uninformed agents, which serves to increase trading volume.

However, consider the $\text{Cov}(\epsilon_I, \epsilon_\Omega)$ term. When a positive shock to the informed endowment arrives, the uninformed agents will underestimate the informed asset holdings. As we have already discussed this will cause them to also underestimate the dividend growth rate. Hence Ω_t will fall. This means that the covariance between informed endowment innovations and innovations to the uninformed sector's misestimate is negative. Hence a positive informed endowment shock will on the one hand cause the uninformed sector to increase demand due to risk sharing motives, and on the other hand the uninformed demand will fall due to a decrease in Ω_t. The two offsetting effects, picked up by $\text{Cov}(\epsilon_I, \epsilon_\Omega)$ tend to reduce unanticipated trading volume. As we see from Figure 2.4, the latter effect dominates at first, causing unanticipated trading volume to fall. The former effect dominates for larger σ_θ.

The right panel in Figure 2.4 shows the ratio of unanticipated trade volume in the two economies as a function of σ_θ. Because we are considering ratios of trade volume, we may also think of this as a ratio of turnover (since turnover is just some constant times volume).

$$R_\tau \equiv \frac{E_t[\hat{\delta}_{t+1}]}{E_t[\hat{\delta}^C_{t+1}]}$$

From the right panel of Figure 2.4 we see that unanticipated trading volume in the strategic economy is substantially smaller than in the competitive one. The reason for this effect is that trade in the strategic economy is more gradual because of price impact considerations. Hence the coefficients $F_{I,I}$, $F_{I,U}$, and $F_{I,\Omega}$ in (2.61) are all smaller in magnitude in the strategic economy. This reduces the trade variance and also reduces the unanticipated trading volume. Furthermore, the smaller magnitude of the coefficients diminishes the dependence of unanticipated volume in the strategic economy on σ_Ω and on $\mathrm{Cov}(\epsilon_I, \epsilon_\Omega)$. Hence the U-shaped pattern in trading volume is more pronounced in the competitive economy. This effects the inverse U-shaped pattern in R_τ.

2.4.7 Dynamics

In this section we will study the differences in the dynamic responses of the competitive and the strategic economies to different types of shocks. We have previously argued that the additional cost of trading out of misinformed positions in the strategic economy lessens the dependence of uninformed demand on misestimates of state variables. A clear illustration of this can be seen by considering the impulse response of the two economies to a shock in the exogenous signal received by the uninformed sector.[12] A positive realization of ϵ_θ causes the uninformed agent to overestimate the current dividend growth rate, and hence to overestimate the current asset holdings of the informed sector. The return from a share of the risky asset then seems to increase, causing the uninformed to buy shares of the risky asset. As we see from the left panel of Figure 2.5, the competitive uninformed sector trades immediately to their optimal holdings given their new belief about share returns. They then slowly sell the risky asset as their belief about its return converges to the true return over time. In the strategic economy, the uninformed agent realizes that trading immediately to the competitive point entails additional price impact. In addition to this trading in response to changes in a state variable which could be misestimated (i.e. the asset holdings of the informed sector) bears the additional cost of potentially having to trade out of that position,

[12]See the Appendix for a derivation of the impulse response function.

and therefore having to incur additional price impact. Hence the strategic agent buys fewer shares than the competitive uninformed sector. Furthermore, at time 3 (that is three trading periods after the initial shock), the initial misestimate has had some time to correct itself, making the perceived share return lower, and causing the strategic agent to buy fewer shares. By time 4, the strategic agent begins selling off shares of the risky asset. As is clear from the figure, the over-reaction of the strategic uninformed agent is much smaller than the over-reaction of the competitive uninformed sector.

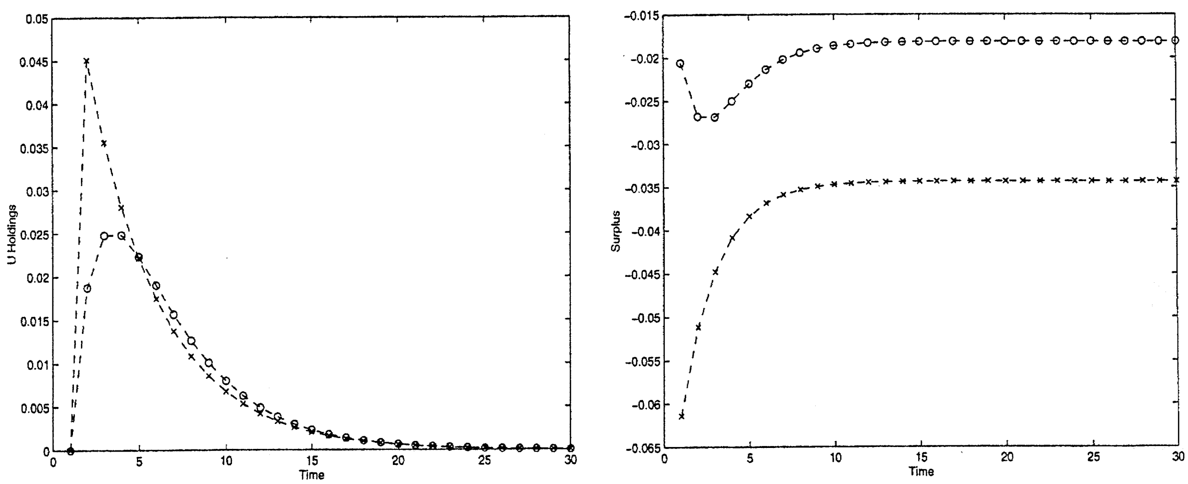


Figure 2.5: The left panel shows the uninformed asset holdings following a shock in the exogenous signal, ϵ_θ. The right panel shows the evolution of the trade surplus. The o's mark the strategic economy, and the x's mark the competitive one. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $\sigma_\theta = 1.0$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

In order to understand the welfare implications of misinformed trade in the two economies let us formulate one concept of trade surplus. From (2.51), we see that the risk discount in the price can be decomposed into a long-run term, and two other terms reflecting market power and the misinformation of the uninformed sector. These latter two terms go to zero in expectation over time. In the case where the dividend level and growth rate are both zero (i.e. the price is equal to the risk discount), we may define the *trade surplus* from a time 0 shock as the trade weighted deviations of the long-run price from the current trade price. That is

$$C_t \equiv \sum_{s=0}^{t} \delta_{U,s} \left(P_\infty - P_s \right) \tag{2.62}$$

Since over time, trade amounts go to zero, C_t approaches a well defined limit C_∞. From the right panel of Figure 2.5 we see that the trade surplus in both economies is negative, reflecting a transfer of wealth from the uninformed to the informed sector. Because of the reduced over-reaction in the strategic economy, we see that the uninformed loss due to the exogenous shock is smaller in the strategic economy than it is in the competitive economy. Hence strategic behavior in the presence of poor information mitigates the effects of misinformed trade.

Let us consider now what happens to trade surplus in response to a positive endowment shock experienced by the informed sector. The left panel of Figure 2.6 shows the evolution of trade surplus in an economy under almost symmetric information. We see that the

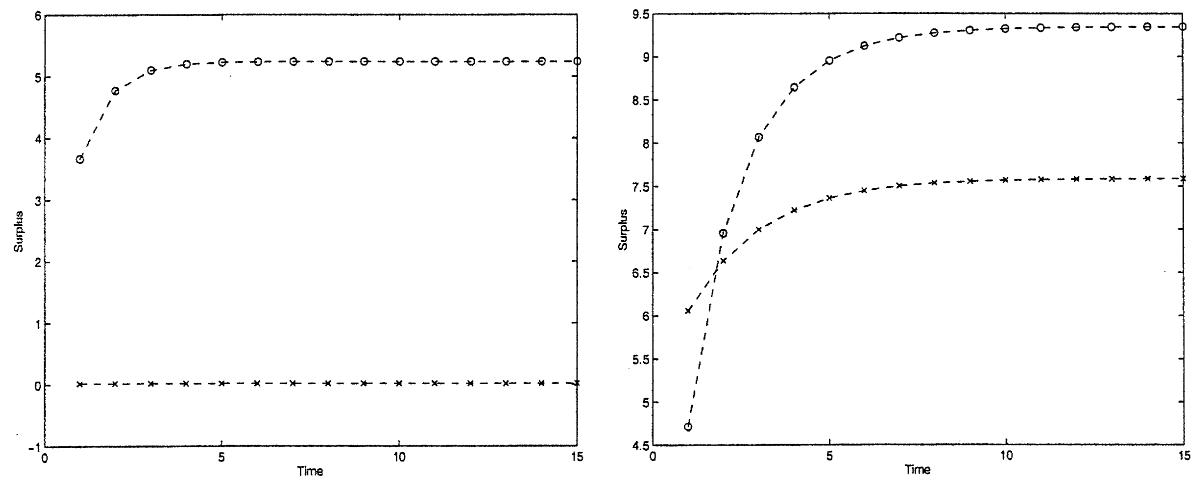


Figure 2.6: The left panel shows the surplus from a unit shock in the informed endowment, $\epsilon_\theta = 1.0$, under almost symmetric information: $\sigma_\theta = 0.005$. The right panel shows the evolution of surplus under asymmetric information: $\sigma_\theta = 1.0$. The o's mark the strategic economy, and the x's mark the competitive one. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

strategic agent, now (almost) symmetrically informed, is able to extract surplus from trade by lessening initial trade amounts relative to the competitive case, and thereby buying the initial shares at prices below the long-run price. In the competitive economy, we see that the trade surplus is (almost) zero. This follows from (2.51) and the fact that $P_U - P_I = 0$ in the competitive economy. However in the asymmetric information case, as we see from the right panel of Figure 2.6, both the strategic and competitive uninformed sectors accumulate trade surplus in response to an informed sector endowment shock. The reason has to do with asymmetric information. In the competitive, asymmetrically informed economy, we have already seen that the arrival of a positive informed sector endowment shock causes the uninformed agents to underestimate informed asset holdings, and therefore the dividend growth rate as well. Hence Ω_t falls in response to a positive ϵ_I. However, the uninformed agents have still revised upward their belief about the informed asset holdings. Hence they believe returns to have gone up, and will begin to buy shares. The share price, reflecting the now negative Ω_t and the fact that $P_\Omega > 0$, will be lower than the long-run price. Hence by engaging in risk sharing, the competitive poorly informed agents are actually able (unwittingly) to extract surplus from allocational trade. Of course, in the strategic economy, this effect is reinforced by strategic trading on the part of the uninformed sector, and the trade surplus is even higher. We see, therefore, that sometimes competitive and strategic agents can be better off by having less information. This is the topic to which we now turn.

2.5 Acquisition of Information

In order to assess the incentive for the poorly informed agents to acquire information, let us conduct the following thought experiment. At time 0, the a poorly informed agent can choose to improve the precision of his exogenous signal $S^{(\theta)}$ at some fixed cost by lowering σ_θ to $\sigma_\theta - \eta$. What will be the effects of such an improvement in his information set? In the current model, the uninformed agents trades in response to three stimuli: own endowment shocks, and perceived and actual endowment shocks of the informed sector. Undertaking this project will increase the precision of the uninformed sector's estimates of asset returns, and *ceteris paribus*, it will make the uninformed agents better off. The adverse selection problem stems largely from the fact that the uninformed trader has to enter the market in order to hedge own endowment shocks. Without this trading motive, a risk-averse uninformed agent could simply refuse to trade with the informed sector, and would therefore not be subject to informational arbitrage. Hence the adverse selection problem will be most severe when either the uninformed sector has a highly volatile endowment or is very risk averse. The value of an incremental improvement in its information set will be most valuable in exactly these circumstances. Figure 2.7 bears this out. It plots the uninformed sector's reservation value ΔM of an η reduction in σ_θ.[13] We find that ΔM is increasing with σ_U. The difference between the competitive economy and the strategic one is negligible, suggesting similar considerations in both settings. Notice that ΔM does not go to zero as σ_U gets small. Even when own hedging demand is not a consideration, the uninformed sector will trade with the rest of the market because some of the latter's trade will be allocational in nature.

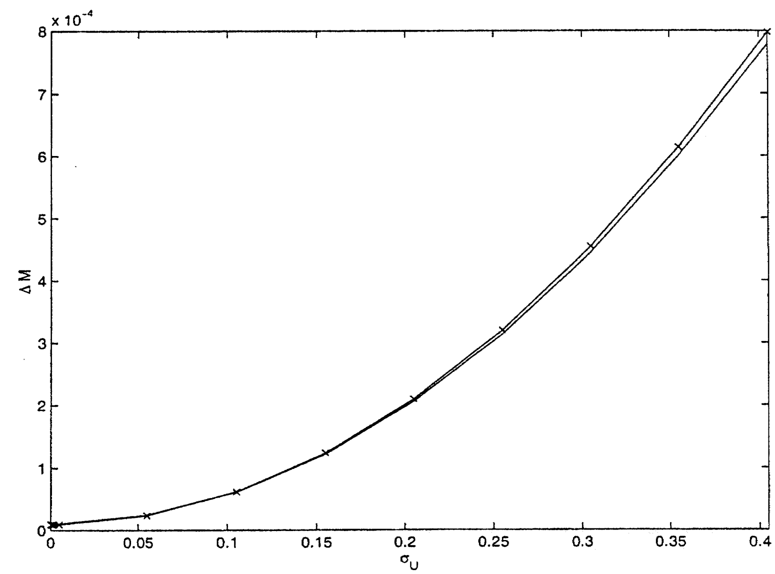


Figure 2.7: The reservation value ΔM of an incremental improvement in the signal received by the uninformed sector as a function of that sector's endowment volatility σ_U. The marked line shows the competitive economy. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_\theta = 0.40$, $\eta = 0.01$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_I = 0.1$, $\gamma_U = 0.1$, and $\rho = 0.09$.

Changing the information set of the uninformed sector will affect two aspects of trade in the economy: the ability of the informed sector to trade against the misinformation of the uninformed, and the mechanism via which allocational trade takes place. Clearly an

[13]The reservation value ΔM solves the following equation

$$J(M_t - \Delta M, D_t, F_t, \Psi_t, t; \sigma_\theta - \eta) = J(M_t, D_T, F_t, \Psi_t, t; \sigma_\theta)$$

This is the cost of acquiring information that makes the uninformed agent indifferent between undertaking an improvement to his information set and doing nothing. I normalize ΔM by assuming that D_t, F_t, and Ψ_t are all zero; this is the only steady-state of the economy (i.e. from which the agents would not trade) for all parameter values of the model. Using (2.36), we find that

$$\Delta M = -\frac{1}{\lambda} \ln \left(\frac{\kappa_\eta}{\kappa} \right)$$

where the η subscript indicates variables under the improved information set.

improvement in the precision of their exogenous signal reduces the misinformed trade by the uninformed sector. Hence the trade surplus which accrues to the informed agents because of misinformed trade will decrease. This will make the uninformed sector better off. On the other hand, we have already seen that under greater informational asymmetry, more trade surplus accrues to the uninformed sector when the informed receive endowment shocks. Hence to the degree that improved information reduces this trade surplus, it may make the uninformed agents worse off. Figure 2.8 shows that depending on the hedging needs of the informed sector, either of these effects may dominate. Not surprisingly, improved information

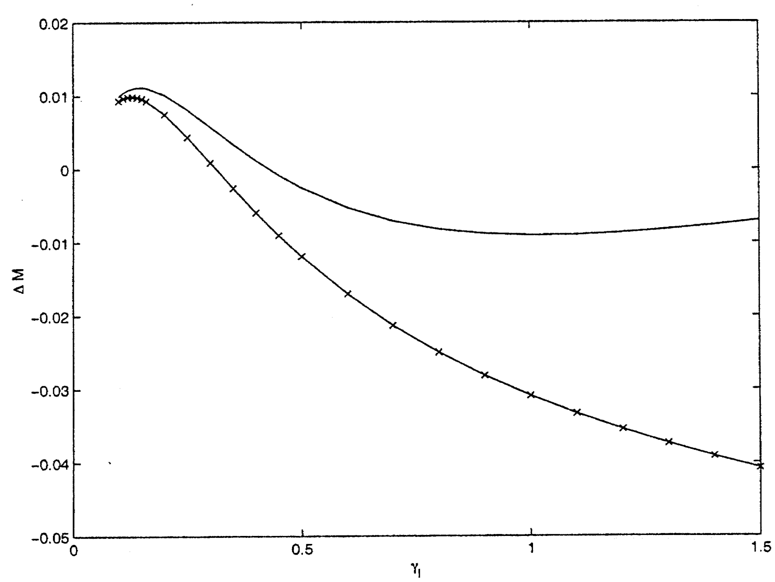


Figure 2.8: The reservation value ΔM of an incremental improvement in the signal received by the large agent as a function of the risk aversion γ_I of the informed sector. The marked line shows the competitive economy. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 1.30$, $\sigma_F = 0.25$, $\sigma_\theta = 0.4$, $\eta = 0.01$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_I = 0.1$, $\gamma_U = 0.1$, and $\rho = 0.09$.

leads to a negative ΔM in those cases where the hedging needs of the informed sector, and hence the trade surplus accruing to the uninformed from allocational trade, are highest. Furthermore, we see that for higher values of γ_I, the improved information set makes the uninformed strategic agents better off than the uninformed competitive agents.

To gain insight into the mechanism underlying this effect, let us consider again equation (2.51), reproduced here for convenience

$$\tilde{P}_t = \tilde{P}_\infty + (P_U - P_I)\Delta_t + P_\Omega \Omega_t$$

We see that, from the point of view of trade surplus, changing the information set will have two effects: it will change $P_U - P_I$ and P_Ω. Figure 2.9 shows how these are affected by an improved information set in the two economies. Let us consider first the competitive

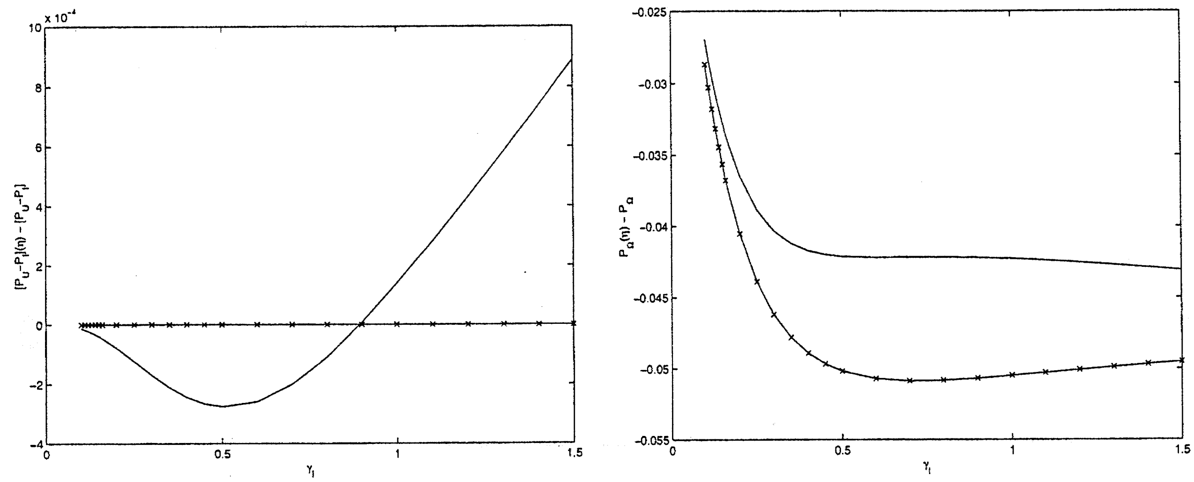


Figure 2.9: The left panel shows the $(P_U - P_I)(\eta) - (P_U - P_I)$ as a function of the risk aversion of the informed sector, γ_I. The right panel shows $P_\Omega(\eta) - P_\Omega$ as a function of γ_I. The marked lines show the competitive economy. The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 1.30$, $\sigma_F = 0.25$, $\sigma_\theta = 0.4$, $\eta = 0.01$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_I = 0.1$, $\gamma_U = 0.1$, and $\rho = 0.09$.

economy, where $P_U - P_I = 0$. Hence the only effect in this setting is through changes in P_Ω. We see that decreasing σ_θ also decreases P_Ω. When the informed sector is very risk averse, and hence when the demand for allocational trade is high, a reduction in P_Ω substantially reduces the trade surplus accruing to the uninformed agent when the informed sector receives an endowment shock (see the discussion in Section 2.4.7). This effect begins to dominate the reduction in misinformed trade resulting from the improved information set. Hence ΔM in the competitive economy falls with γ_I.

In the strategic economy, an additional effect enters into play. We recall from Section 2.4.4 that $P_U - P_I > 0$. This allows the uninformed strategic agent to extract surplus from allocational trade, in addition to the effect discussed above. As we see from Figure 2.9 this quantity can increase or decrease in σ_θ. Hence by improving his information set, the uninformed strategic trading lowers P_Ω, but also affects $P_U - P_I$. When $P_U - P_I$ falls with η (i.e. with improved information), the uninformed strategic agent will gain less trade surplus from the market power channel under improved information. However, when $P_U - P_I$ increases with improved information, the uninformed strategic agent will benefit from an added ability to extract surplus from allocational trade. The fact that this latter effect dominates for larger values of γ_I means that the uninformed strategic agent has added incentives to become informed (relative to the competitive sector). However, we see that there is still a region of the economy in which the strategic agent is worse off under better information.

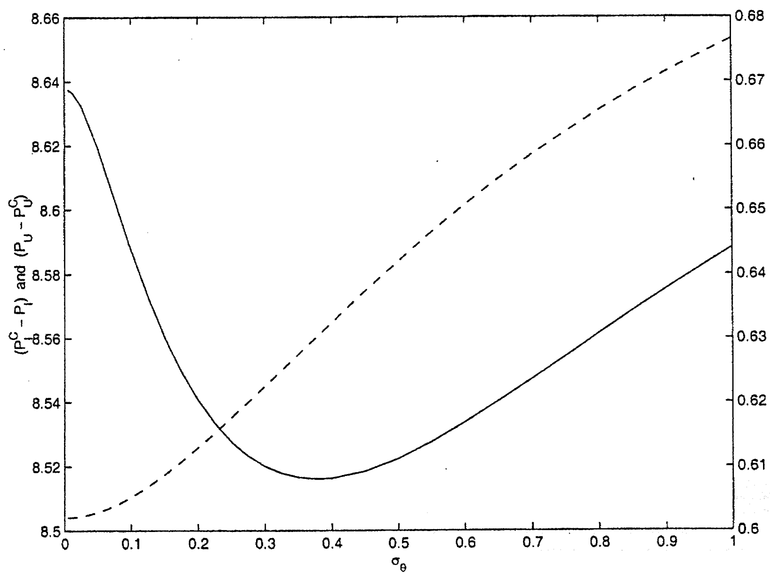


Figure 2.10: The graph shows the difference between price coefficients in the strategic and competitive economies: $P_I^C - P_I$ (solid line, left axis) and $P_U - P_U^C$ (dashed line, right axis). The parameters values are $h = 1$, $k_D = 0.15$, $k_F = 0.15$, $\sigma_U = 0.12$, $\sigma_I = 0.12$, $\sigma_F = 0.25$, $\sigma_D = 1.5$, $r = 0.07$, $\gamma_U = 0.1$, $\gamma_I = 1.5$, and $\rho = 0.09$.

In order to understand the behavior of $P_U - P_I$ as a function of the information structure of the economy, let us examine the behavior of the individual price coefficients P_U and P_I across the two economies. Figure 2.10 shows the differences between the coefficients in the two economies. We see first that the quantity $P_U - P_U^C$ is increasing with σ_θ. The reason is quite intuitive. As its information set deteriorates, the uninformed competitive sector sells more and more aggressively in response to a unit endowment shock. Hence, the scope for price improvement is larger for the strategic agent (see the discussion in Section 2.4.4); the difference between optimal trade amounts will increase progressively with the decrease in informativeness of the signal. And hence the discount in the price due to uninformed asset holdings will experience larger decreases with increased informational asymmetry.

On the other hand, consider $P_I^C - P_I$. At first this quantity begins to decline with information asymmetry. The reason for this is quite interesting. Recall from our earlier discussion that in the strategic economy, the uninformed agent has a better estimate of the asset holdings of the informed sector. Hence as we introduce informational asymmetry into the model, when the informed sector experiences a unit endowment shock, the competitive uninformed sector will have a larger misestimate of it than will the strategic uninformed agent. The strategic agent, thinking that the return on the security is higher than do the competitive uninformed agents, will still buy fewer shares than the competitive agents; however, this decrease will be smaller as σ_θ increases because the strategic agent's perception of the returns of the risky asset will be higher. However, as the information set becomes progressively worse, the strategic agent's price impact will begin to increase (i.e. π will fall). Because trading out of misinformed positions becomes costlier, the strategic uninformed agent will face a progressively less liquid market, and hence will choose to reduce his trade amount by progressively more as his or her information set deteriorates. This causes the initial price to be lower relative to the strategic case as σ_θ becomes very large.

Recalling that $P_I^C = P_U^C$, we see then that the behavior discussed above will imply that $P_U - P_I$ will first begin to decrease with σ_θ and will then begin to increase with σ_θ. Hence we see the behavior in Figure 2.9. For the less risk-averse economies, the improvement in the information set of the strategic uninformed agent will decrease $P_U - P_I$; however, for the more risk-averse economies (where the effects of misinformation are larger) we will have $P_U - P_I$ increase with η, as the information effect discussed above begins to dominate (that is we move to the downward sloping, as a function of σ_θ, part of the $P_U - P_I$ curve). In essence, for high γ_I, introducing better information into the economy (i.e. $\eta > 0$) has the effect of bringing the information set of the uninformed strategic agent closer to that of his or her competitive, uninformed counterparts, and hence allows him or her to more actively reduce the price in response to informed endowment shocks.

2.6 Empirical Implications

In this section, we will discuss some of the testable implications of the model. Consider the results about R_E, R_V, and R_τ. These are testable in a straight forward manner. There have been several recent attempts at classifying securities by the amount of informational trade which they experience (see Brennan and Subrahmanyam (1996), Easley, Hvidkjaer, and O'Hara (1999), and Chae (2000)). Furthermore institutional ownership data is easy to obtain (at least about U.S. securities). I propose classifying securities into percentiles based on some measure of informational asymmetry. Then within each percentile, securities can be further classified into groups based on uninformed, institutional ownership (of course it may be difficult to determine which institutional owners are poorly informed). Then the quantities R_E, R_V, and R_τ can be constructed within each information percentile, using the relevant quantities from those securities with the most and least institutional ownership. We can then see whether the predicted patterns for these three quantities are present in the data.

Furthermore, once the subdivision into information percentiles has been done, it is easy to also check whether the implications for the effects of informational asymmetry on expected returns, return volatility, and unexpected turnover actually hold. These quantities can simply be calculated for all securities in a given informational percentile.

An interesting implication discussed in Section 2.4.6 has to do with the dynamics of turnover. In particular we have seen that trade amounts in the strategic economy depended on past trade amounts, whereas in the competitive economy they did not. Hence by regressing daily turnover on lagged daily turnover for individual securities, we may be able to determine which securities are characterized by strategic trading, and which are not. Those securities with significant coefficients in the above regression fall into the strategic group, and those with insignificant coefficients fall into the latter group. It would be interesting to see whether the percentage of institutional ownership in a given security would be able to predict the significance of the coefficient in the above regression. Furthermore, the presence of asymmetric information should manifest intself in a particular pattern of residual autocorrelations in the above regression.

Certainly a verification of some of the above predictions would serve as indirect evidence that certain large institutions are indeed poorly informed (and choose to remain so), and yet trade in a strategic fashion. A more direct approach would yield more interesting results. In particular it would be interesting to assess whether institutions choose to remain poorly informed so as to obtain better terms of trade (i.e. trade surplus) from the rest of the economy. While this is certainly difficult to test, enough data on investment fund expenditures on fundamental research, coupled with data on the terms of trade these institutions face, would make a direct test of this prediction of the model possible. We would expect to see that more research expenditure by funds (which is known to the rest of the market) affects in a non-monotonic way their ability to trade effectively (e.g. to buy at prices below their fundamental or long-term levels).

2.7 Conclusion

In this paper, I have developed a dynamic general equilibrium model which studies the interaction between uninformed and informed agents in a competitive and a strategic setting. I find that in the strategic economy, the strategic uninformed agent has more precise information about asset holdings, and hence about expected returns, and less precise information about cash flow prospects than the uninformed agents in a competitive economy.

I find that the level of informational asymmetry in the economy affects expected returns, return volatility, and unexpected trade volume. The difference between these quantities across the competitive and the strategic economies depends on the level of informational asymmetry. A driving force behind some of these results is that return volatility and unexpected trading volume are less sensitive to the information structure in the strategic economy.

I find that a strategic uninformed agent is less susceptible to informational arbitrage by the informed agents than is the uninformed competitive sector. Interestingly, the presence of asymmetric information allows poorly informed competitive and strategic agents to extract surplus from allocational trade with the informed sector. Such surplus decreases as the information sets of the agents in the economy become equalized. Strategic uninformed agents are affected by changes in information via another channel: their ability to extract surplus from own endowment shocks may increase or decrease.

We find, therefore, that the ex-ante incentives of the uninformed agents to improve their information set depend on the relative demand between their own hedging needs and those of the rest of the market. The ability to credibly commit not to become better informed can sometimes be beneficial to the uninformed agents because it allows them to extract greater surplus from allocational trade initiated by the informed sector. The benefits from such a commitment are higher to the competitive uninformed sector as a whole. To the extent that credibly committing to remain uninformed may be difficult, it would be interesting to study whether current institutional forms facilitate this process. Are broad asset class allocation decisions by pension and endowment funds one form of such commitment?

A very interesting implication of the above results is that if information acquisition were fully endogenized in the current setting (i.e. if agents dynamically chose when to exert effort in order to improve own information sets), then the differences between the competitive and the strategic economies would be substantially increased. It is likely that in the competitive economy the poorly informed competitive sector (as a whole) would choose to have worse information than its strategic counterpart. This is a topic for future research.

2.8 Appendix

2.8.1 The Filtering Problem

Proof of Equivalence of Signals

Here I will show that in the strategic economy, the two signals in (2.20) and in (2.21) contain identical information. The signals are

$$S^{(1)} = (\pi P_F - A_{I,\Omega})F_t + A_{I,I}I_t$$

and

$$S^{(2)} = (P_F - P_\Omega)F_t + P_I I_t$$

Recall also that

$$P_t = \frac{1}{\pi}\boldsymbol{A}_I\boldsymbol{\Psi}_{I,t} + P_F F_t + P_D D_t + \frac{1}{\pi}\delta_{U,t}$$

We note that for $\zeta = \frac{\pi P_F - A_{I,\Omega}}{A_{I,I}}$, we can write that $\hat{I}_t = I_t - \zeta\Omega_t$. Hence $F_{U,I}\hat{I}_t = F_{U,I}I_t - \zeta F_{U,I}\Omega_t$. From the price function above, we will have that

$$\begin{aligned} P_\Omega &= 1/\pi\ A_{I,\Omega} - 1/\pi\ \zeta F_{U,I} \\ P_I &= 1/\pi\ A_{I,I} + 1/\pi\ F_{U,I} \end{aligned}$$

Hence we can write the second signal as

$$S^{(2)} = (P_F - 1/\pi\ A_{I,\Omega} + 1/\pi\ \zeta F_{U,I})F_t + (1/\pi\ A_{I,I} + 1/\pi\ F_{U,I})I_t$$

Using the definition of ζ and multiplying both sides by π we have

$$\pi\ S^{(2)} = \frac{A_{I,I} + F_{U,I}}{A_{I,I}}\left[(\pi P_F - A_{I,\Omega})F_t + A_{I,I}I_t\right]$$

Hence we see that $S^{(2)} = k \times S^{(1)}$ for some known constant k. Therefore, the two signals must contain the same information.

Proof of Theorem 2.2 and the Filtering Problem

The relevant state variables are $\boldsymbol{z}_t = [F_t, I_{\epsilon,t}]'$. Recall that I_ϵ is the part of the informed sector's endowment which is not forecastable from knowing the history of trades of the uninformed agent. The law of motion is given by

$$\boldsymbol{z}_{t+1} = \boldsymbol{A}_Z \boldsymbol{z}_t + \boldsymbol{B}_Z \boldsymbol{\epsilon}_{t+1} \tag{2.A.1}$$

where the noise terms are $\boldsymbol{\epsilon}_t \equiv [\epsilon_U, \epsilon_I, \epsilon_F, \epsilon_\theta, \epsilon_D]'$ and

$$\boldsymbol{A}_Z = \begin{bmatrix} A_F & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \boldsymbol{B}_Z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The signals observed by the uninformed are given by

$$\boldsymbol{S}_t = \boldsymbol{A}_S \boldsymbol{z}_t + \boldsymbol{B}_S \boldsymbol{\epsilon}_t \tag{2.A.2}$$

where

$$\boldsymbol{A}_S = \begin{bmatrix} P_F - \frac{1}{\pi} A_{I,\Omega} & \frac{1}{\pi} A_{I,I} \\ h & 0 \\ h & 0 \end{bmatrix} \quad , \quad \boldsymbol{B}_S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The information set of the uninformed investors is $\mathcal{F}_t = \{\boldsymbol{S}_s : s \leq t\}$. Let us define $\Sigma_Z \equiv \boldsymbol{B}_Z \Sigma \boldsymbol{B}_Z'$ and $\Sigma_S \equiv \boldsymbol{B}_S \Sigma \boldsymbol{B}_S'$. The problem faced by the uninformed agent is to derive a conditional distribution of the state variables, given the history of signals. This is a Kalman filtering problem (for a derivation see Hamilton). Letting $\hat{\boldsymbol{z}}_t \equiv E[\boldsymbol{z}|\mathcal{F}_t]$ and $\boldsymbol{O}_t \equiv E[(\hat{\boldsymbol{z}}_t - \boldsymbol{z}_t)(\hat{\boldsymbol{z}}_t - \boldsymbol{z}_t)'|\mathcal{F}_t]$, the Kalman filter is given by

$$\begin{aligned} \hat{\boldsymbol{z}}_{t+1} &= \boldsymbol{A}_Z \hat{\boldsymbol{z}}_t + \boldsymbol{K}_t(\boldsymbol{S}_{t+1} - E[\boldsymbol{S}_{t+1}|\mathcal{F}_t]) \\ \boldsymbol{O}_{t+1} &= (\boldsymbol{A}_Z \boldsymbol{O}_t \boldsymbol{A}_Z' + \Sigma_Z) - \boldsymbol{K}_{t+1} \boldsymbol{A}_S (\boldsymbol{A}_Z \boldsymbol{O}_t \boldsymbol{A}_Z' + \Sigma_Z) \\ \boldsymbol{K}_{t+1} &= (\boldsymbol{A}_Z \boldsymbol{O}_t \boldsymbol{A}_Z' + \Sigma_Z) \boldsymbol{A}_S' [\boldsymbol{A}_S (\boldsymbol{A}_Z \boldsymbol{O}_t \boldsymbol{A}_Z' + \Sigma_Z) \boldsymbol{A}_S' + \Sigma_S]^{-1} \end{aligned} \tag{2.A.3}$$

where we see that $E[\boldsymbol{S}_{t+1}|\mathcal{F}_t] = \boldsymbol{A}_S \boldsymbol{A}_Z \hat{\boldsymbol{z}}_t$. Given some restrictions on $\boldsymbol{A}_Z$ and a Gaussian prior belief distribution, this filter converges to a steady state. The steady-state Kalman filter is given by

$$\begin{aligned} \hat{\boldsymbol{z}}_{t+1} &= \boldsymbol{A}_Z \hat{\boldsymbol{z}}_t + \boldsymbol{K}(\boldsymbol{S}_{t+1} - \boldsymbol{A}_S \boldsymbol{A}_Z \hat{\boldsymbol{z}}_t) \\ \boldsymbol{O} &= (\boldsymbol{A}_Z \boldsymbol{O} \boldsymbol{A}_Z' + \Sigma_Z) - \boldsymbol{K} \boldsymbol{A}_S (\boldsymbol{A}_Z \boldsymbol{O} \boldsymbol{A}_Z' + \Sigma_Z) \\ \boldsymbol{K} &= (\boldsymbol{A}_Z \boldsymbol{O} \boldsymbol{A}_Z' + \Sigma_Z) \boldsymbol{A}_S' [\boldsymbol{A}_S (\boldsymbol{A}_Z \boldsymbol{O} \boldsymbol{A}_Z' + \Sigma_Z) \boldsymbol{A}_S' + \Sigma_S]^{-1} \end{aligned} \tag{2.A.4}$$

where $\boldsymbol{O}$ and $\boldsymbol{K}$ no longer have a time dependence.[14]

Define the estimation error $\boldsymbol{\omega}_t = \hat{\boldsymbol{z}}_t - \boldsymbol{z}_t$. It follows from (2.A.4) that

$$\begin{aligned}\boldsymbol{\omega}_{t+1} &= (\boldsymbol{A}_Z - \boldsymbol{K}\boldsymbol{A}_S\boldsymbol{A}_Z)\boldsymbol{\omega}_t + [\boldsymbol{K}(\boldsymbol{A}_S\boldsymbol{B}_Z + \boldsymbol{B}_S) - \boldsymbol{B}_Z]\boldsymbol{\epsilon}_{t+1} \\ &= \boldsymbol{A}_\omega\boldsymbol{\omega}_t + \boldsymbol{B}_\omega\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

Because a linear combination of F_t and $I_{\epsilon,t}$ is observed in the signal, the two estimation errors are perfectly correlated. The estimation error of the dividend level $\Omega_t = \hat{F}_t - F_t$ can therefore be written in the form of equation (2.28)

$$\Omega_{t+1} = A_\Omega\,\Omega_t + \boldsymbol{B}_\Omega\,\boldsymbol{\epsilon}_{t+1} \tag{2.A.5}$$

From equation (2.31), we get that

$$\hat{I}_{\epsilon,t} - I_{\epsilon,t} = -\frac{\pi P_F - A_{I,\Omega}}{A_{I,I}}\,\Omega_t$$

Hence the auto-regressive coefficient in (2.A.5) is

$$A_\Omega = \boldsymbol{A}_\omega^{(1,1)} - \frac{\pi P_F - A_{I,\Omega}}{A_{I,I}}\,\boldsymbol{A}_\omega^{(1,2)} \tag{2.A.6}$$

where the notation $\boldsymbol{A}^{(i,j)}$ means the (i,j)th element in the matrix $\boldsymbol{A}$. And $\boldsymbol{B}_\Omega$ is given by the top row in $\boldsymbol{B}_\omega$. If we expand $\boldsymbol{S}_{t+1}$ in (2.A.4), we can write the evolution of the uninformed agent's conditional beliefs as

$$\hat{\boldsymbol{z}}_{t+1} = \boldsymbol{A}_Z\hat{\boldsymbol{z}}_t - \boldsymbol{K}\boldsymbol{A}_S\boldsymbol{A}_Z\boldsymbol{\omega}_t + \boldsymbol{K}(\boldsymbol{A}_S\boldsymbol{B}_Z + \boldsymbol{B}_S)\boldsymbol{\epsilon}_{t+1} \tag{2.A.7}$$

$$= \boldsymbol{A}_Z\hat{\boldsymbol{z}}_t - \boldsymbol{A}_{\hat{z}}\boldsymbol{\omega}_t + \boldsymbol{B}_{\hat{z}}\boldsymbol{\epsilon}_{t+1} \tag{2.A.8}$$

Hence for the uninformed investor we can write $\hat{F}_t$ and $\hat{I}_{\epsilon,t}$ as in (2.32) and (2.33) respectively

$$\begin{aligned}\hat{F}_{t+1} &= A_F\hat{F}_t - C_F\Omega_t + \boldsymbol{B}_F\boldsymbol{\epsilon}_{t+1} \\ \hat{I}_{\epsilon,t+1} &= \hat{I}_{\epsilon,t} - C_I\Omega_t + \boldsymbol{B}_I\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

[14]The convergence of the filter to its steady-state is guaranteed under all conditions on the state and observation equations as long as the variance of the initial belief about the state variables is given by $\boldsymbol{O}$. To prove this simply start with this prior in period 0, and compute the period 1 variance using the updating equation (2.A.3).

where

$$
\begin{aligned}
C_F &= \boldsymbol{A}_{\hat{z}}^{(1,1)} - \frac{\pi P_F - A_{I,\Omega}}{A_{I,I}} \boldsymbol{A}_{\hat{z}}^{(1,2)} \\
C_I &= \boldsymbol{A}_{\hat{z}}^{(2,1)} - \frac{\pi P_F - A_{I,\Omega}}{A_{I,I}} \boldsymbol{A}_{\hat{z}}^{(2,2)}
\end{aligned}
$$

and where $\boldsymbol{B}_F$ is the first row in $\boldsymbol{B}_{\hat{z}}$ and $\boldsymbol{B}_I$ is the second row in $\boldsymbol{B}_{\hat{z}}$.

From the point of view of the uninformed investor, the vector $\boldsymbol{\omega}_t$ is normally distributed with mean 0 with a variance $\boldsymbol{O}$ when conditioning on $\mathcal{F}_t$. Because the two -estimate terms are perfectly correlated, the entire distribution of $\boldsymbol{\omega}_t$ is captured by the fact that Ω_t is normally distributed with mean 0 and variance $\boldsymbol{O}^{(1,1)}$.

2.8.2 The Control Problem of the Uninformed

In this section we will analyze the control problem of the strategic and the competitive uninformed agents. The first section will deal with the first order conditions in the Bellman equation. The second section deals with the coefficient requirements in order to satisfy the Bellman equation.

First Order Conditions

The problem faced by the uninformed agent is

$$
\sup_{c,\delta} E_t \left[\sum_{s=t}^{\infty} -e^{-\rho s h - \gamma c} h \right]
$$

where the expectation is taken with respect to $\mathcal{F}_t^U = \{D_s, P_s, \delta_{U,s} : s \leq t\}$. The dynamics of the state variables are given by

$$
\begin{aligned}
M_{U,t+1} &= e^{rh}(M_{U,t} - c_{U,t}h - (\delta_{U,t} + \epsilon_{U,t+1})P_t) + \\
&\quad (U_t + \delta_{U,t} + \epsilon_{U,t+1})D_{t+1}h \\
U_{t+1} &= U_t + \delta_{U,t} + \epsilon_{U,t+1} \\
U_{t+1}^C &= U_t^C + \delta_{U,t}^C + \epsilon_{U,t+1} \\
\hat{I}_{t+1} &= \hat{I}_t - \delta_{U,t} - C_I \Omega_t + \boldsymbol{B}_I \boldsymbol{\epsilon}_{t+1}
\end{aligned}
$$

Here U_t^C represents the asset holdings of the atomic uninformed agent, $\delta_{U,t}^C$ is the trade amount of an atomic uninformed agent, and $\delta_{U,t}$ is the trade amount of the entire uninformed sector (or of the strategic uninformed agent). Here we define the strategic state variable vector $\boldsymbol{\Psi}_{U,t} = [\ U_t \ \ I_t\ ]$, and the competitive one as $\boldsymbol{\Psi}^C_{U,t} = [\ U_t \ \ U^C_t \ \ I_t\ ]$. In the competitive case, we write $\delta_{U,t}$ in terms of the conjectured trade amount as $\boldsymbol{F}^C_U \boldsymbol{\Psi}^C_{U,t}$. We can write the dynamics of the state variables as

$$\boldsymbol{\Psi}_{U,t+1} = \boldsymbol{A}_{\Psi,U} \boldsymbol{\Psi}_{U,t} + \boldsymbol{i}\delta_{U,t} + \boldsymbol{B}_{\Psi,U} \boldsymbol{\epsilon}_{U,t+1}$$

where

$$\begin{aligned}
\boldsymbol{\epsilon}_{U,t} &= \left[\ \epsilon_{U,t}\ \ \epsilon_{I,t}\ \ \epsilon_{F,t}\ \ \epsilon_{\theta,t}\ \ \epsilon_{D,t}\ \ \Omega_{t-1}\ \right]' \\
\boldsymbol{i} &= [\,1 \ \ -1\,]' \quad \boldsymbol{i}^C = [\,0\ \ 1\ \ 0\,] \\
\boldsymbol{A}_{\Psi,U} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{B}_{\Psi,U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_I \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_I \end{bmatrix} \\
\boldsymbol{A}^C_{\Psi,U} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}^C_U \\ \boldsymbol{0} \\ -\boldsymbol{F}^C_U \end{bmatrix} \quad \boldsymbol{B}_{\Psi,U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_I \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{B}_I \end{bmatrix}
\end{aligned}$$

In the strategic case, the price is given by

$$\hat{P}_t = P_F \hat{F}_t + P_D D_t + \frac{1}{\pi}\left(\tilde{\boldsymbol{A}}_I \boldsymbol{\Psi}_{U,t} + \delta_{U,t}\right)$$

where $\tilde{\boldsymbol{A}}_I = [A_{I,U}\ \ A_{I,I}]$. In the competitive case, the price is given by

$$\hat{P}_t = P_F \hat{F}_t + P_D D_t + \tilde{\boldsymbol{A}}_P\ \boldsymbol{\Psi}_{U,t}$$

where $\tilde{\boldsymbol{A}}_P = [A_{P,U}\ \ A_{P,I}]$, and where $\boldsymbol{A}_P$ is the conjectured price coefficients vector. In what follows, we will use $\tilde{U}_t$ in both cases to represent the uninformed agent's ownership of the risky asset (in the strategic case $\tilde{U}_t = U_t$, and in the competitive case $\tilde{U}_t = U^C_t$). We conjecture that in both the strategic and the competitive cases

$$J(M_{U,t}, D_t, \hat{F}_t, \boldsymbol{\Psi}_{U,t}, t) = -\kappa e^{-\rho t h - \lambda M - \alpha \tilde{U}\hat{F} - \beta \tilde{U} D - \frac{1}{2}\boldsymbol{\Psi}'\boldsymbol{V}\boldsymbol{\Psi}} \tag{2.A.9}$$

where α and β are constants (the same in both cases), and $\boldsymbol{V}$ is a 2×2 symmetric matrix in the strategic case, and a 3×3 symmetric matrix in the competitive case. Let us define

$$\boldsymbol{B}_D = [\, 0 \;\; 0 \;\; h \;\; 0 \;\; 1 \;\; -hA_F \,] \tag{2.A.10}$$

$$\tilde{\boldsymbol{B}}_F = [\, \boldsymbol{B}_F \;\; -C_F \,] \tag{2.A.11}$$

$$\boldsymbol{B}_U = [\, 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,] \tag{2.A.12}$$

$$\begin{aligned}\boldsymbol{\xi} &= \left(\lambda e^{rh} P_t - (\lambda h + \beta)(A_D D_t + hA_F \hat{F}_t) - \alpha A_F \hat{F}_t\right) \boldsymbol{B}'_U - \\ &\quad (\lambda h + \beta)(\tilde{U}_t + \delta)\boldsymbol{B}'_D - \alpha(\tilde{U}_t + \delta)\tilde{\boldsymbol{B}}_F{}' - \boldsymbol{B}'_\Psi \boldsymbol{V}(\boldsymbol{A}_\Psi \boldsymbol{\Psi} + \boldsymbol{i}\delta)\end{aligned} \tag{2.A.13}$$

Notice that in the strategic case, P_t depends on the choice of trade amount by the strategic agent, but in the competitive case it is taken as given. Given the conjectured value function in (2.A.9) and the Normality assumption about $\boldsymbol{\epsilon}_t$, it is possible to show that

$$\begin{aligned}E_t[J_{t+1}] &= -\kappa\nu e^{-\rho(t+1)h} \\ &\quad \exp[-\lambda e^{rh}(M_t - c_t h - \delta P_t) - (\lambda h + \beta)(\tilde{U}_t + \delta)(A_D D_t + hA_F \hat{F}_t) \\ &\quad -\alpha(\tilde{U}_t + \delta)A_F \hat{F}_t - \tfrac{1}{2}(\boldsymbol{A}_\Psi \boldsymbol{\Psi} + \boldsymbol{i}\delta)'\boldsymbol{V}(\boldsymbol{A}_\Psi \boldsymbol{\Psi} + \boldsymbol{i}\delta) + \tfrac{1}{2}\,\boldsymbol{\xi}'\,\boldsymbol{\Xi}\,\boldsymbol{\xi}\,]\end{aligned} \tag{2.A.14}$$

where $\boldsymbol{\Sigma} = E_t[\boldsymbol{\epsilon}_{U,t+1}\boldsymbol{\epsilon}'_{U,t+1}]$, $\boldsymbol{\Xi} = (\boldsymbol{\Sigma}^{-1} + \boldsymbol{B}'_\Psi \boldsymbol{V} \boldsymbol{B}_\Psi)^{-1}$ and $\nu = (|\boldsymbol{\Xi}|/|\boldsymbol{\Sigma}|)^{\frac{1}{2}}$. The Bellman equation becomes

$$0 = \sup_{c,\delta} E_t[-e^{-\rho th - \gamma c}h + J_{t+1} - J_t] \tag{2.A.15}$$

Taking the first-order condition with respect to δ (assuming for the moment that $\boldsymbol{V}$ is symmetric) and collecting the $\hat{F}_t$ and D_t terms we have the following conditions for P_F and P_D. We will return to these later in the second half of the proof.

$$P_D = \frac{e^{-rh}A_D}{\lambda}(\lambda h + \beta) \tag{2.A.16}$$

$$P_F = \frac{e^{-rh}A_F}{\lambda}(h(\lambda h + \beta) + \alpha) \tag{2.A.17}$$

Once we have done this, all terms involving $\hat{F}_t$ and D_t will have canceled out of the Bellman equation. In particular, notice that

$$\boldsymbol{\xi} = \lambda e^{rh} P_t \boldsymbol{B}'_U - (\lambda h + \beta)(\tilde{U}_t + \delta)\boldsymbol{B}'_D - \alpha(\tilde{U}_t + \delta)\boldsymbol{B}'_F - \boldsymbol{B}'_\Psi \boldsymbol{V}(\boldsymbol{A}_\Psi \boldsymbol{\Psi} + \boldsymbol{i}\delta)$$

Noticing that in the strategic case we can write $\tilde{U}_t = \mathbf{\Lambda}_U \mathbf{\Psi}$ for $\mathbf{\Lambda}_U = [\ 1 \ \ 0\ ]$, and that in the competitive case we can write $\tilde{U}_t = \mathbf{\Lambda}_U \mathbf{\Psi}^C$ for $\mathbf{\Lambda}_U = [\ 0 \ \ 1 \ \ 0\ ]$, we find that the first order conditions gives us that $\delta_{U,t} = \boldsymbol{F}_U \mathbf{\Psi}_{U,t}$ where

$$\begin{aligned}\boldsymbol{F}_U &= (-2\lambda e^{rh}\frac{\partial P_t}{\partial \delta} + \boldsymbol{i}'\boldsymbol{V}\boldsymbol{i} - \frac{\partial \boldsymbol{\xi}'}{\partial \delta}\Xi\frac{\partial \boldsymbol{\xi}}{\partial \delta})^{-1}\left(\lambda e^{rh}\boldsymbol{C} - \boldsymbol{i}'V\boldsymbol{A}_\Psi + \frac{\partial \boldsymbol{\xi}'}{\partial \delta}\Xi\boldsymbol{\zeta}_U\right) \\ \boldsymbol{\zeta}_U &= \lambda e^{rh}\boldsymbol{B}_U'\boldsymbol{C} - (\lambda h + \beta)\boldsymbol{B}_D'\Lambda_U - \alpha\boldsymbol{B}_F'\Lambda_U - \boldsymbol{B}_\Psi'\boldsymbol{V}\boldsymbol{A}_\Psi\end{aligned}$$

where in the strategic case $\boldsymbol{C} = \frac{1}{\pi}\tilde{\boldsymbol{A}}_I$, and $\boldsymbol{C} = \tilde{\boldsymbol{A}}_P$ in the competitive case. Notice that $\frac{\partial P_t}{\partial \delta} = 1/\pi$ in the strategic case, and $\frac{\partial P_t}{\partial \delta} = 0$ in the competitive case. We use δ_U and (2.A.16–2.A.17) in (2.A.14) to get

$$\begin{aligned}E_t[J_{t+1}] &= -\kappa\nu\exp[-\rho(t+1)h - \lambda e^{rh}(M - c_t h) \\ &\quad -(\lambda h + \beta)\tilde{U}_t(A_D D_t + hA_F\hat{F}_t) - \alpha\tilde{U}_t A_F \hat{F}_t - \tfrac{1}{2}\mathbf{\Psi}'\boldsymbol{G}\mathbf{\Psi}] \qquad (2.A.18)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{G} &= -\lambda e^{rh}(\boldsymbol{F}_U'\boldsymbol{C} + \boldsymbol{C}'\boldsymbol{F}_U + 2\frac{\partial P_t}{\partial \delta}\boldsymbol{F}_U'\boldsymbol{F}_U) + \\ &\quad (\boldsymbol{A}_\Psi + \boldsymbol{i}\boldsymbol{F}_U)'\boldsymbol{V}(\boldsymbol{A}_\Psi + \boldsymbol{i}\boldsymbol{F}_U) - (\boldsymbol{\zeta}_U + \frac{\partial \boldsymbol{\xi}}{\partial \delta}\boldsymbol{F}_U)'\Xi(\boldsymbol{\zeta}_U + \frac{\partial \boldsymbol{\xi}}{\partial \delta}\boldsymbol{F}_U) \qquad (2.A.19)\end{aligned}$$

Taking the first-order condition in (2.A.15),we find that

$$\begin{aligned}c_t &= -\frac{\rho t h}{\gamma} - \frac{1}{\gamma}\ln\left[\frac{1}{\gamma}\frac{\partial}{\partial M}E_t[J_{t+1}]\right] \qquad (2.A.20) \\ &= \frac{1}{\gamma + \lambda e^{rh}h}(\ln(\frac{\gamma}{\kappa\nu\lambda}) + (\rho - r)h + \lambda e^{rh}M_t + \tfrac{1}{2}\mathbf{\Psi}'\boldsymbol{G}\mathbf{\Psi} \\ &\quad +(\lambda h + \beta)\tilde{U}_t(A_D D_t + hA_F\hat{F}_t) + \alpha\tilde{U}_t A_F\hat{F}_t)\end{aligned}$$

The Bellman Equation

In this part of the proof we now need to solve for the coefficients of the value function so that the Bellman equation (2.A.15) is satisfied and the optimal δ and c. Plugging the expressions for the optimal trade amount and the optimal consumption level back into (2.A.15), we find that in order to satisfy the Bellman equation we will have the following conditions on λ, α, β, κ, and on $\boldsymbol{V}$

$$\lambda = \gamma\frac{1-e^{-rh}}{h} \tag{2.A.21}$$

$$0 = -\beta + e^{-rh}[(\lambda h + \beta)A_D] \tag{2.A.22}$$

$$0 = -\alpha + e^{-rh}[(\lambda h + \beta)hA_F + \alpha A_F] \tag{2.A.23}$$

$$0 = \ln(\gamma/\lambda) - \ln(\kappa) + e^{-rh}\log(\nu\kappa) - e^{-rh}\ln(\gamma/\lambda) - e^{-rh}(\rho - r)h \tag{2.A.24}$$

$$0 = e^{-rh}\boldsymbol{G} - \boldsymbol{V} \tag{2.A.25}$$

Using these together with (2.A.16–2.A.17), we can solve for P_D, P_F, α, β, and κ

$$\begin{aligned}
\beta &= \frac{\lambda h A_D}{e^{rh} - A_D} = \lambda P_D \\
\alpha &= \frac{(\lambda h + \beta)hA_F}{e^{rh} - A_F} = \lambda P_F \\
\kappa &= \frac{\gamma}{\lambda}\exp\left[\frac{e^{-rh}}{1-e^{-rh}}(\ln\nu - h(\rho - r))\right] \\
P_D &= \frac{A_D h}{e^{rh} - A_D} \\
P_F &= \frac{A_F e^{rh} h^2}{(e^{rh} - A_F)(e^{rh} - A_D)}
\end{aligned}$$

Finally we see that from (2.A.19) that $\boldsymbol{G}$ is a symmetric matrix, and then (2.A.25) gives us an implicit equation for the unknowns in $\boldsymbol{V}$. Also we conclude that $\boldsymbol{V}$ is indeed a symmetric matrix (which justifies the earlier assumption), and hence (2.A.25) gives us three equations in the strategic case, and 6 equations in the competitive case.

To check the transversality condition that $\lim_{T\to\infty} E_t[J_T] = 0$, let us first note that

$$\frac{\partial}{\partial M}E_t[J_{t+1}] = -\lambda e^{rh}E_t[J_{t+1}]$$

Then using c_t in (2.A.20) and λ from (2.A.21), we can substitute into (2.A.15) to find that

$$E_t[J_{t+1}] = e^{-rh}J_t$$

Noting that r and h are both positive, applying the above equation recursively, and using the law of iterated expectations, we can verify the transversality condition.

2.8.3 The Control Problem of the Informed

The analysis here proceeds in much the same way as for the uninformed agent above.

First Order Conditions

The informed investor solves the following problem ($\mathcal{I}$):

$$\sup_{c,\delta} E_t \left[\sum_{s=t}^{\infty} -e^{-\rho s h - \gamma c} h \right]$$

where the expectation is taken with respect to the informed agent's information set in period t. The dynamics of the state variables are given by

$$\begin{aligned}
M_{I,t+1} &= e^{rh}(M_{I,t} - c_{I,t}h - \delta_{I,t}P_t - \epsilon_{I,t+1}(P_F F_t + P_D D_t)) + \\
&\quad (I_t + \delta_{I,t} + \epsilon_{I,t+1})D_{t+1}h \\
U_{t+1} &= U_t + \delta_{U,t} + \epsilon_{U,t+1} \\
I_{t+1} &= I_t - \delta_{U,t} + \epsilon_{I,t+1} \\
\tilde{I}_{t+1} &= \tilde{I}_t + \delta_{I,t} + \epsilon_{I,t+1} \\
\Omega_{t+1} &= A_\Omega \Omega_t + \boldsymbol{B}_\Omega \epsilon_{t+1}
\end{aligned}$$

Here U_t and I_t are the asset holdings of the uninformed and the informed sectors respectively. The trade amount $\delta_{U,t}$ is known and given by the vector $\tilde{\boldsymbol{F}}_U$ as some linear function of U_t, I_t, and Ω_t. The atomic informed agent's holdings are given by $\tilde{I}_t$, and his or her trade amount is given by $d_{I,t}$. Let us define $\tilde{\boldsymbol{\Psi}}_t = [\ U_t \quad I_t \quad \Omega_t\ ]$. Clearly we can write

$$\tilde{\boldsymbol{\Psi}}_{t+1} = \boldsymbol{A}_{\tilde{\Psi}} \tilde{\boldsymbol{\Psi}}_t + \boldsymbol{B}_{\tilde{\Psi}} \boldsymbol{\epsilon}_{t+1} \tag{2.A.26}$$

for some matrixes $\boldsymbol{A}_{\tilde{\Psi}}$ and $\boldsymbol{B}_{\tilde{\Psi}}$. We know that the equilibrium risk discount in the price is some linear function of the state variables $\tilde{\boldsymbol{\Psi}}$

$$\tilde{P}_t = \frac{1}{\pi} \left(\boldsymbol{A}_I + \tilde{\boldsymbol{F}}_U \right) \tilde{\boldsymbol{\Psi}}_t = \boldsymbol{A}_P \tilde{\boldsymbol{\Psi}}_t$$

Let us define the state vector $\boldsymbol{\Psi}_t = [\ \tilde{\boldsymbol{\Psi}}_t \quad \tilde{P}_t\ ]$. We can then write the risk discount in the price as

$$\begin{aligned}
\tilde{P}_{t+1} &= \boldsymbol{A}_P \tilde{\boldsymbol{\Psi}}_{t+1} = \boldsymbol{A}_P \boldsymbol{A}_{\tilde{\Psi}} \tilde{\boldsymbol{\Psi}}_t + \boldsymbol{A}_P \boldsymbol{B}_{\tilde{\Psi}} \boldsymbol{\epsilon}_{t+1} \\
&= \boldsymbol{A}_{EP} \boldsymbol{\Psi}_t + \boldsymbol{B}_P \boldsymbol{\epsilon}_{t+1} \qquad (2.A.27)
\end{aligned}$$

for the appropriately defined vectors $\boldsymbol{A}_{EP}$ and $\boldsymbol{B}_P$. Hence we can write $\tilde{P}_{t+1}$ in terms of only the period t values of U_t, I_t, and Ω_t (i.e. $\tilde{\boldsymbol{\Psi}}$) and the period $t+1$ error terms $\boldsymbol{\epsilon}_{t+1}$ as (2.38) claims. Given this equation for the state dynamics the derivation now proceeds in much the same way as in the uninformed case.

We conjecture that

$$J(M_{I,t}, D_t, \hat{F}_t, \tilde{I}_t, \boldsymbol{\Psi}_{I,t}, t) = -\kappa e^{-\rho t h - \lambda M - \alpha \tilde{I}_t F_t - \beta \tilde{I}_t D_t - \zeta \tilde{I}_t \tilde{P}_t - \frac{1}{2}\boldsymbol{\Psi}' \boldsymbol{V} \boldsymbol{\Psi}}$$

where α and β are constants, and $\boldsymbol{V}$ is a 4×4 symmetric matrix.

Let us define

$$\begin{aligned} \boldsymbol{\xi} \;=\; & \left(\lambda e^{rh}(P_F F_t + P_D D_t) - (\lambda h + \beta)(A_D D_t + h A_F F_t) - \alpha A_F F_t\right) \boldsymbol{B}_I' - \\ & \zeta E_t[\tilde{P}_{t+1}]\boldsymbol{B}_I' - (\lambda h + \beta)(\tilde{I}_t + \delta)\boldsymbol{B}_D' - \alpha(\tilde{I}_t + \delta)\boldsymbol{B}_F' - \zeta(\tilde{I}_t + \delta)\boldsymbol{B}_P' - \\ & \boldsymbol{B}_\Psi' \boldsymbol{V} \boldsymbol{A}_\Psi \boldsymbol{\Psi} \end{aligned} \tag{2.A.28}$$

Then it is possible to show that

$$\begin{aligned} E_t[J_{t+1}] \;=\; & -\kappa \nu e^{-\rho(t+1)h} \\ & \exp[-\lambda e^{rh}(M_t - c_t h - \delta P_t) - \tfrac{1}{2}(\boldsymbol{A}_\Psi \boldsymbol{\Psi})' \boldsymbol{V} (\boldsymbol{A}_\Psi \boldsymbol{\Psi}) \\ & -(\lambda h + \beta)(\tilde{I}_t + \delta)(A_D D_t + h A_F F_t) - \alpha(\tilde{I}_t + \delta) A_F F_t \\ & -\zeta(\tilde{I}_t + \delta) E_t[\tilde{P}_{t+1}] + \tfrac{1}{2}\boldsymbol{\xi}' \Xi \, \boldsymbol{\xi}] \end{aligned} \tag{2.A.29}$$

where $\Xi = (\Sigma^{-1} + \boldsymbol{B}_\Psi' \boldsymbol{V} \boldsymbol{B}_\Psi)^{-1}$ and $\nu = (|\Xi|/|\Sigma|)^{\frac{1}{2}}$. The Bellman equation becomes

$$0 = \sup_{c,\delta} E_t[-e^{-\rho t h - \gamma c} h + J_{t+1} - J_t] \tag{2.A.30}$$

We will assume for not that $\boldsymbol{V}$ is symmetric, and will verify this later. Taking the first-order condition with respect to δ, we find that the same values for P_D and P_F obtain as in the case of the strategic agent (in (2.A.16)–(2.A.17)). As before this cancels out all of the D_t and F_t terms in the first-order condition, as well as in $\boldsymbol{\xi}$. It then follows from the first-order condition that $\delta_{I,t} = \boldsymbol{F}_I \boldsymbol{\Psi}_t - \tilde{I}_t$, where

$$\boldsymbol{F}_I = \left(\frac{\partial \boldsymbol{\xi}'}{\partial \delta} \Xi \frac{\partial \boldsymbol{\xi}}{\partial \delta}\right)^{-1} \left(\zeta \boldsymbol{A}_{EP} - \lambda e^{rh} \boldsymbol{A}_P + \frac{\partial \boldsymbol{\xi}'}{\partial \delta} \Xi [\zeta \boldsymbol{B}_I' \boldsymbol{A}_{EP} + \boldsymbol{B}_\Psi' \boldsymbol{V} \boldsymbol{A}_\Psi]\right) \tag{2.A.31}$$

where $\boldsymbol{A}_P = [\,0 \;\; 0 \;\; 0 \;\; 0 \;\; 1\,]$.

Noticing that $\tilde{I}_t + \delta = \boldsymbol{F}_I \boldsymbol{\Psi}_t$, we can now re-write (2.A.29) as

$$E_t[J_{t+1}] = -\kappa\, \nu \exp[-\rho(t+1)h - \lambda e^{rh}(M - c_t h + \tilde{I}_t P_t) - \tfrac{1}{2}\boldsymbol{\Psi}'\boldsymbol{G}\boldsymbol{\Psi}] \tag{2.A.32}$$

where the matrix $\boldsymbol{G}$ will be given by

$$\begin{aligned} \boldsymbol{G} &= -\lambda e^{rh}(\boldsymbol{F}'\boldsymbol{A}_P + \boldsymbol{A}_P'\boldsymbol{F}) + \boldsymbol{A}_\Psi'\boldsymbol{V}\boldsymbol{A}_\Psi + \zeta(\boldsymbol{F}_I'\boldsymbol{A}_{EP} + \boldsymbol{A}_{EP}'\boldsymbol{F}_I) - \boldsymbol{g}'\Xi\boldsymbol{g} \qquad (2.A.33) \\ \boldsymbol{g} &= -\zeta\boldsymbol{B}_I'\boldsymbol{A}_{EP} - (\lambda h + \beta)\boldsymbol{B}_D'\boldsymbol{F}_I - \alpha\boldsymbol{B}_F'\boldsymbol{F}_I - \zeta\boldsymbol{B}_P'\boldsymbol{F}_I - \boldsymbol{B}_\Psi'\boldsymbol{V}\boldsymbol{A}_\Psi \end{aligned}$$

Taking the first-order condition with respect to c in (2.A.30), we find that

$$c_t = \frac{1}{\gamma + \lambda e^{rh} h}(\ln(\frac{\gamma}{\kappa\nu\lambda}) + (\rho - r)h + \lambda e^{rh}(M_t + \tilde{I}_t P_t) + \tfrac{1}{2}\boldsymbol{\Psi}'\boldsymbol{G}\boldsymbol{\Psi}) \tag{2.A.34}$$

The Bellman Equation

Plugging the optimal consumption and trade amount into the Bellman equation (2.A.30), we find the following restrictions on the coefficients of the model.

$$0 = e^{-rh}\boldsymbol{G}_I - \boldsymbol{V}_I \tag{2.A.35}$$

$$\lambda = \gamma\,\frac{1 - e^{-rh}}{h} \tag{2.A.36}$$

$$\kappa = \frac{\gamma}{\lambda}\exp\left[\frac{e^{-rh}}{e^{-rh} - 1}(h(\rho - r) - \ln\nu)\right] \tag{2.A.37}$$

$$\zeta = \lambda \tag{2.A.38}$$

$$\alpha = \lambda P_F \tag{2.A.39}$$

$$\beta = \lambda P_D \tag{2.A.40}$$

Notice that the value function quantities $\tilde{I}_t(\alpha F_t + \beta D_t + \zeta\tilde{P}_t)$ are equal to $\tilde{I}_t\lambda P_t$. The symmetry of $\boldsymbol{G}$ and (2.A.35) imply the symmetry of $\boldsymbol{V}$. The transversality condition is verified in the same way as in the case of the strategic agent. This completes the proof.

2.8.4 Determination of Equilibrium

In the strategic economy, the equilibrium of the economy is the solution $\boldsymbol{V}_U(2 \times 2), \boldsymbol{V}_I(4 \times 4), \boldsymbol{A}_I, \pi$ of the following set of 17 non-linear algebraic equations

$$e^{-rh}\boldsymbol{G}_U = \boldsymbol{V}_U \tag{2.A.41}$$

$$e^{-rh}\boldsymbol{G}_I = \boldsymbol{V}_I \tag{2.A.42}$$

$$[\, F_{I,U} \;\; F_{I,I} - 1 \;\; F_{I,\Omega} \,] = \boldsymbol{A}_I \tag{2.A.43}$$

$$F_{I,P} = -\pi \tag{2.A.44}$$

where the scalars $F_{I,\cdot}$ are the elements of the vector $\boldsymbol{F}_I$ $\boldsymbol{A}_I$ is the conjectured trade amount of the informed sector, and π is the conjectured demand elasticity of the informed sector.

In the competitive economy, the conjectured quantities are the price, $\boldsymbol{A}_P$, and the trade amount, $\boldsymbol{F}$. Equilibrium is given by the quantities (21 unknowns) $\boldsymbol{V}_U(3 \times 3), \boldsymbol{V}_I(4 \times 4), \boldsymbol{A}_P, \boldsymbol{F}$ such that the following set of 21 non-linear algebraic equations are satisfied

$$e^{-rh}\boldsymbol{G}_U = \boldsymbol{V}_U \tag{2.A.45}$$

$$e^{-rh}\boldsymbol{G}_I = \boldsymbol{V}_I \tag{2.A.46}$$

$$[\, F_{U,U} - 1 \;\; F_{U,I} \,] = \boldsymbol{F} \tag{2.A.47}$$

$$\tilde{\boldsymbol{F}}_U = \tilde{\boldsymbol{F}}_I \tag{2.A.48}$$

where $\tilde{\boldsymbol{F}}_U$ is the demand of the atomic uninformed agent expressed in terms of the true state variables in the economy (U_t, I_t, and Ω_t, or $\tilde{\boldsymbol{\Psi}}_t$), and the demand of the atomic informed agent is given by $\delta_{I,t} = \tilde{\boldsymbol{F}}_I \tilde{\boldsymbol{\Psi}}_t$.

This verifies the claim made in Theorem 2.1.

2.8.5 Computation of Impulse Responses

We know from our analysis of the competitive agent's problem that $\boldsymbol{\Psi}_{I,t} = \{U_t, I_t, \Omega_t, \tilde{P}_t\}$, and from (2.38) the dynamics of this state vector are given by

$$\boldsymbol{\Psi}_{I,t+1} = \boldsymbol{A}_{\Psi,I}\boldsymbol{\Psi}_{I,t} + \boldsymbol{B}_{\Psi,I}\boldsymbol{\epsilon}_{t+1}$$

where $\boldsymbol{\epsilon} = [\ \epsilon_U \ \ \epsilon_I \ \ \epsilon_F \ \ \epsilon_\theta \ \ \epsilon_D\ ]'$. We are interested in the equilibrium behavior of the economy. Thus $P_t = P_F F_t + P_D D_t + \frac{1}{\pi}(\boldsymbol{A}_I + \tilde{\boldsymbol{F}}_U)\boldsymbol{x}_t$, where the state variables of the economy are $\boldsymbol{x}_t = [\ U_t \ \ I_t \ \ \Omega_t\ ]'$. Their dynamics are given by (2.A.26). The timing is as follows

- in period 0, the state variables are all zero, or $\boldsymbol{x}_0 = \mathbf{0}$
- in period 1 a vector of shocks $\boldsymbol{\epsilon}_1$ is realized
- hence $\boldsymbol{x}_1 = \boldsymbol{B}_x \boldsymbol{\epsilon}_1$
- agents observe their state variables and make trade decisions
- the state variables (including the controlled ones) then evolve according to the state equations, assuming that all future shocks $\boldsymbol{\epsilon}_t$ $\forall t > 0$ are zero.

The state of the economy t periods after the initial shock will therefore be

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}_x^t \boldsymbol{B}_x \boldsymbol{\epsilon}_1 \tag{2.A.49}$$

where $\boldsymbol{A}_x^0$ is understood to be the identity matrix of the proper order.

2.8.6 Excess Returns

Recall that we have defines excess share returns as

$$Q_t \equiv \sum_{s=1}^{t} \left(D_s h + P_s - P_{s-1} - (e^{rh} - 1) P_{s-1} \right) \tag{2.A.50}$$

Therefore we have that $dQ_t \equiv Q_t - Q_{t-1} = D_t h + P_t - e^{rh} P_{t-1}$. From the dynamics of the state variables, it is straightforward to show that the price is given by

$$\begin{aligned} P_t &= (P_F A_F + P_D h A_F) F_{t-1} + P_D A_D D_{t-1} + \\ &\quad P_U U_{t-1} + P_I I_{t-1} + (P_U - P_I)\delta_{U,t-1} + P_\Omega A_\Omega \Omega_{t-1} + \boldsymbol{B}_P \boldsymbol{\epsilon}_t \end{aligned}$$

where $\boldsymbol{B}_P = [\ P_U \ \ P_I \ \ P_F + P_D h \ \ 0 \ \ P_D\ ] + P_\Omega \boldsymbol{B}_\Omega$. Since $D_t = A_D D_{t-1} + h(A_F F_{t-1} + \epsilon_{F,t}) + \epsilon_{D,t}$, we can write the excess return as

$$\begin{aligned} dQ_t &= (P_F A_F + P_D h A_F + h^2 A_F - e^{rh} P_F) F_{t-1} + (P_D A_D + h A_D - e^{rh} P_D) D_{t-1} + \\ &\quad (P_U - e^{rh} P_U) U_{t-1} + (P_I - e^{rh} P_I) I_{t-1} + (P_U - P_I)\delta_{U,t-1} + \\ &\quad (P_\Omega A_\Omega - e^{rh} P_\Omega)\Omega_{t-1} + \boldsymbol{B}_Q \boldsymbol{\epsilon}_t \end{aligned}$$

where $\boldsymbol{B}_Q = \boldsymbol{B}_P + [\,0 \;\; 0 \;\; h^2 \;\; 0 \;\; h\,]$. It is easy to show that the loadings on F_{t-1} and on D_{t-1} are both zero. Keeping in mind that $\delta_{U,t-1} = \boldsymbol{F}_U \Psi_{t-1}$, we get that

$$dQ_t = \boldsymbol{A}_Q \Psi_{t-1} + \boldsymbol{B}_Q \epsilon_t$$

for the appropriate vectors $\boldsymbol{A}_Q$ and $\boldsymbol{B}_Q$.

References

Amihud, Y. and H. Mendelson (1980), "Dealership market: market making with inventory," *Journal of Financial Economics*, 8, 31–53.

Back, K. (1992), "Insider trading in continuous time," *Review of Financial Studies*, 5, 387–410.

Back, K., H. Cao and G. Willard (1999), "Imperfect competition among informed traders," *Journal of Finance*, forthcoming.

Basak, S. (1997), "Consumption choice and asset pricing with a non-price taking agent," *Economic Theory*, 10, 437–462.

Bertsimas, D. and A. Lo (1998), "Optimal control of execution costs," *Journal of Financial Markets*, 1, 1–50.

Brennan, M.J. and A. Subrahmanyam (1996), "Market microstructure and asset pricing: On the compensation for illiquidity in stock returns," *Journal of Financial Economics*, 41, 441–464..

Brennan, M.J. and A. Subrahmanyam (1998), "The determinants of average trade size," *Journal of Business*, 71 (1), 1–25.

Carhart, M.M. (1997), "On persistence in mutual fund performance," *Journal of Finance*, 52, 57–82.

Chan, L.K.C. and J. Lakonishok (1995), "The behavior of stock prices around institutional trades," *Journal of Finance*, 50, 1147–1174.

Chae, J. (2000), "Asymmetric information...," mimeo, MIT.

Chau, M. (1999), "Dynamic trading and market-making with inventory costs and private information," *mimeo*, ESSEC.

Detemple, J. (1986), "Asset pricing in a production economy with incomplete information," *Journal of Finance*, 41 (2), 383–391.

Dothan, M.U. and D. Feldman (1986), "Equilibrium interest rates and multiperiod bonds in a partially observable economy," *Journal of Finance*, 41 (2), 369–382.

Easley, D. and M. O'Hara (1987), "Price, trade size and information in securities markets," *Journal of Financial Economics*, 19, 69–90.

Easley, D., S. Hvidkjaer, and M. O'Hara (1999), "Is information risk a determinant of asset returns," *mimeo*, Cornell.

Fama, E. (1991), "Efficient Capital Markets: II," *Journal of Finance*, 46, 1575–1617.

Gennotte, G. (1986), "Optimal portfolio choice under incomplete information," *Journal of Finance*, 41 (3), 733–746.

Glosten, L.R. (1989), "Insider trading, liquidity, and the role of the monopolist specialist," *Journal of Business*, 62 (2), 211–235.

Grossman, S.J. and J.E. Stiglitz (1980), "On the impossibility of informationally efficient markets," *American Economic Review*, 70, 393–408.

Hausman, J., A. Lo, and A.C. MacKinlay (1992), "An ordered probit analysis of transaction stock prices," *Journal of Financial Economics*, 31, 319–379.

He, H. and J. Wang (1995), "Differential information and dynamic behavior of stock trading volume," *Review of Financial Studies*, 8 (4), 919–972.

Hellwig, M.F. (1980), "On the aggregation of information in competitive markets," *Journal of Economic Theory*, 22, 477–498.

Ho, T. and H.R. Stoll (1981), "Optimal dealer pricing under transactions and return uncertainty," *Journal of Financial Economics*, 9, 47–73.

Holthausen, R., R. Leftwich, and D. Mayers (1990), "Large block transactions, the speed of response of temporary and permanent stock-price effects," *Journal of Financial Economics*, 26, 71–95.

Hong, H. (1999), "A model of returns and trading in futures markets," *Journal of Finance*, forthcoming.

Keim, D. and A. Madhavan (1996), "The upstairs market for large-block transactions: Analysis and measurement of price effects," *Review of Financial Studies*, 9, 1–36.

Kyle, A.S. (1985), "Continuous auctions and insider trading," *Econometrica*, 53 (6), 1315–1335.

Kyle, A.S. (1989), "Informed speculation with imperfect information," *Review of Economic Studies*, 56, 317–356.

Madhavan, A. and S. Smidt (1993), "An analysis of changes in specialist inventories and quotations," *Journal of Finance*, 48 (5), 1595–1628.

O'Hara, M. (1997), *Market Microstructure Theory*, Blackwell Publishers.

Scholes, M. (1972), "The market for securities: Substitution versus price pressure and the effects of information on share prices," *Journal of Business*, 45, 179–211.

Schwartz, R. and J. Shapiro (1992), "The challenge of institutionalization for the equity markets," in Saunders, A. (ed.), *Recent Developments in Finance*, New York University Salomon Center, New York, NY, and Business One Irwin, Homewood, IL.

Spiegel, M. and A. Subrahmanyam (1992), "Informed speculation and hedging in a noncompetitive securities market," *The Review of Financial Studies*, 5 (2), 307–329.

Vayanos, D. (1999a), "Strategic trading and welfare in a dynamic market," *Review of Economic Studies*, 66, 219–254.

Vayanos, D. (1999b), "Optimal decentralization of information processing in the presence of synergies," *mimeo*, MIT.

Vayanos, D. (2000), "Strategic Trading in a Dynamic Noisy Market," forthcoming in *Journal of Finance.*

Wang, J. (1993), "A model of intertemporal asset prices under asymmetric information," *Review of Economic Studies*, 60, 249–282.

Wang, J. (1994), "A model of competitive stock trading volume," *Journal of Political Economy*, 102 (1), 127–168.

Zhou, C. (1998), "Dynamic portfolio choice and asset pricing with differential information," *Journal of Economic Dynamics and Control*, 22, 1027–1051.

Chapter 3

Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation

3.1 Introduction

One of the greatest gulfs between academic finance and industry practice is the separation that exists between technical analysts and their academic critics. In contrast to fundamental analysis, which was quick to be adopted by the scholars of modern quantitative finance, technical analysis has been an orphan from the very start. It has been argued that the difference between fundamental analysis and technical analysis is not unlike the difference between astronomy and astrology. Among some circles, technical analysis is known as "voodoo finance." And in his influential book *A Random Walk Down Wall Street*, Burton Malkiel (1996) concludes that "[u]nder scientific scrutiny, chart-reading must share a pedestal with alchemy."

However, several academic studies suggest that despite its jargon and methods, technical analysis may well be an effective means for extracting useful information from market prices. For example, in rejecting the Random Walk Hypothesis for weekly US stock indexes, Lo and MacKinlay (1988, 1999) have shown that past prices may be used to forecast future returns to some degree, a fact that all technical analysts take for granted. Studies by Tabell and Tabell (1964), Treynor and Ferguson (1985), Brown and Jennings (1989), Jegadeesh and Titman (1993), Blume, Easley, and O'Hara (1994), Chan, Jegadeesh, and Lakonishok (1996), Lo and

MacKinlay (1997), Grundy and Martin (1998), and Rouwenhorst (1998) have also provided indirect support for technical analysis, and more direct support has been given by Pruitt and White (1988), Neftci (1991), Brock, Lakonishok, and LeBaron (1992), Neely, Weber, and Dittmar (1997), Neely and Weller (1998), Chang and Osler (1994), Osler and Chang (1995), and Allen and Karjalainen (1999).

One explanation for this state of controversy and confusion is the unique and sometimes impenetrable jargon used by technical analysts, some of which has developed into a standard lexicon that can be translated. But there are many "homegrown" variations, each with its own patois, which can often frustrate the uninitiated. Campbell, Lo, and MacKinlay (1997, pp. 43–44) provide a striking example of the linguistic barriers between technical analysts and academic finance by contrasting this statement:

> The presence of clearly identified support and resistance levels, coupled with a one-third retracement parameter when prices lie between them, suggests the presence of strong buying and selling opportunities in the near-term.

with this one:

> The magnitudes and decay pattern of the first twelve autocorrelations and the statistical significance of the Box-Pierce Q-statistic suggest the presence of a high-frequency predictable component in stock returns.

Despite the fact that both statements have the same meaning—that past prices contain information for predicting future returns—most readers find one statement plausible and the other puzzling, or worse, offensive.

These linguistic barriers underscore an important difference between technical analysis and quantitative finance: technical analysis is primarily *visual*, while quantitative finance is primarily algebraic and numerical. Therefore, technical analysis employs the tools of geometry and pattern recognition, while quantitative finance employs the tools of mathematical analysis and probability and statistics. In the wake of recent breakthroughs in financial engineering, computer technology, and numerical algorithms, it is no wonder that quantitative finance has overtaken technical analysis in popularity—the principles of portfolio optimization are far easier to program into a computer than the basic tenets of technical analysis. Nevertheless, technical analysis has survived through the years, perhaps because its visual mode of analysis is more conducive to human cognition, and because pattern recognition is one of the few repetitive activities for which computers do not have an absolute advantage (yet).

Indeed, it is difficult to dispute the potential value of price/volume charts when confronted with the visual evidence. For example, compare the two hypothetical price charts given in Figure 3.1. Despite the fact that the two price series are identical over the first half of the sample, the volume patterns differ, and this seems to be informative. In particular, the lower chart, which shows high volume accompanying a positive price trend, suggests that there may be more information content in the trend, e.g., broader participation among investors. The fact that the joint distribution of prices and volume contains important information is hardly controversial among academics. Why, then, is the value of a visual depiction of that joint distribution so hotly contested?

In this paper, we hope to bridge this gulf between technical analysis and quantitative finance by developing a systematic and scientific approach to the practice of technical analysis, and by employing the now-standard methods of empirical analysis to gauge the efficacy of technical indicators over time and across securities. In doing so, our goal is not only to develop a lingua franca with which disciples of both disciplines can engage in productive dialogue, but also to extend the reach of technical analysis by augmenting its tool kit with some modern techniques in pattern recognition.

The general goal of technical analysis is to identify regularities in the time series of prices by extracting nonlinear patterns from noisy data. Implicit in this goal is the recognition that some price movements are significant—they contribute to the formation of a specific pattern—and others are merely random fluctuations to be ignored. In many cases, the human eye can perform this "signal extraction" quickly and accurately, and until recently, computer algorithms could not. However, a class of statistical estimators, called *smoothing estimators*, is ideally suited to this task because they extract nonlinear relations $\hat{m}(\cdot)$ by "averaging out" the noise. Therefore, we propose using these estimators to mimic, and in some cases, sharpen the skills of a trained technical analyst in identifying certain patterns in historical price series.

In Section 3.2, we provide a brief review of smoothing estimators and describe in detail the specific smoothing estimator we use in our analysis: kernel regression. Our algorithm for automating technical analysis is described in Section 3.3. We apply this algorithm to the daily returns of several hundred U.S. stocks from 1962 to 1996 and report the results in Section 3.4. To check the accuracy of our statistical inferences, we perform several Monte Carlo simulation experiments and the results are given in Section 3.5. We conclude in Section 3.6.

3.2 Smoothing Estimators and Kernel Regression

The starting point for any study of technical analysis is the recognition that prices evolve in a nonlinear fashion over time and that the nonlinearities contain certain regularities or patterns. To capture such regularities quantitatively, we begin by asserting that prices $\{P_t\}$ satisfy the following expression:

$$P_t = m(X_t) + \epsilon_t \ , \quad t = 1, \ldots, T \tag{3.1}$$

where $m(X_t)$ is an arbitrary fixed but unknown nonlinear function of a state variable X_t, and $\{\epsilon_t\}$ is white noise.

For the purposes of pattern recognition in which our goal is to construct a smooth function $\hat{m}(\cdot)$ to approximate the time series of prices $\{p_t\}$, we set the state variable equal to time, $X_t = t$. However, to keep our notation consistent with that of the kernel regression literature, we will continue to use X_t in our exposition.

When prices are expressed as (3.1), it is apparent that geometric patterns can emerge from a visual inspection of historical prices series—prices are the sum of the nonlinear pattern $m(X_t)$ and white noise—and that such patterns may provide useful information about the unknown function $m(\cdot)$ to be estimated. But just how useful is this information?

To answer this question empirically and systematically, we must first develop a method for automating the identification of technical indicators, i.e., we require a pattern recognition algorithm. Once such an algorithm is developed, it can be applied to a large number of securities over many time periods to determine the efficacy of various technical indicators. Moreover, quantitative comparisons of the performance of several indicators can be conducted, and the statistical significance of such performance can be assessed through Monte Carlo simulation and bootstrap techniques.[1]

[1] A similar approach has been proposed by Chang and Osler (1994) and Osler and Chang (1995) for the case of foreign-currency trading rules based on a head-and-shoulders pattern. They develop an algorithm for

In Section 3.2.1, we provide a brief review of a general class of pattern-recognition techniques known as *smoothing*, and in Section 3.2.2 we describe in some detail a particular method called *nonparametric kernel regression* on which our algorithm is based. Kernel regression estimators are calibrated by a *bandwidth* parameter and we discuss how the bandwidth is selected in 3.2.3.

3.2.1 Smoothing Estimators

One of the most common methods for estimating nonlinear relations such as (3.1) is *smoothing*, in which observational errors are reduced by averaging the data in sophisticated ways. Kernel regression, orthogonal series expansion, projection pursuit, nearest-neighbor estimators, average derivative estimators, splines, and neural networks are all examples of smoothing estimators. In addition to possessing certain statistical optimality properties, smoothing estimators are motivated by their close correspondence to the way human cognition extracts regularities from noisy data.[2] Therefore, they are ideal for our purposes.

To provide some intuition for how averaging can recover nonlinear relations such as the function $m(\cdot)$ in (3.1), suppose we wish to estimate $m(\cdot)$ at a particular date t_0 when $X_{t_0}=x_0$. Now suppose that for this one observation, X_{t_0}, we can obtain *repeated* independent observations of the price P_{t_0}, say $P_{t_0}^1 = p_1, \ldots, P_{t_0}^n = p_n$ (note that these are n independent realizations of the price at the *same* date t_0, clearly an impossibility in practice, but let us continue with this thought experiment for a few more steps). Then a natural estimator of the function $m(\cdot)$ at the point x_0 is

$$\hat{m}(x_0) = \frac{1}{n}\sum_{i=1}^{n} p_i = \frac{1}{n}\sum_{i=1}^{n}[\,m(x_0) + \epsilon_t^i\,] \tag{3.2}$$

$$= m(x_0) + \frac{1}{n}\sum_{i=1}^{n}\epsilon_t^i, \tag{3.3}$$

and by the Law of Large Numbers, the second term in (3.3) becomes negligible for large n.

Of course, if $\{P_t\}$ is a time series, we do not have the luxury of repeated observations for a given X_t. However, if we assume that the function $m(\cdot)$ is sufficiently smooth, then

automatically detecting geometric patterns in price or exchange data by looking at properly defined local extrema.

[2] See, for example, Beymer and Poggio (1996), Poggio and Beymer (1996), and Riesenhuber and Poggio (1997).

for time-series observations X_t near the value x_0, the corresponding values of P_t should be close to $m(x_0)$. In other words, if $m(\cdot)$ is sufficiently smooth, then in a small neighborhood around x_0, $m(x_0)$ will be nearly constant and may be estimated by taking an average of the P_t's that correspond to those X_t's near x_0. The closer the X_t's are to the value x_0, the closer an average of corresponding P_t's will be to $m(x_0)$. This argues for a *weighted* average of the P_t's, where the weights decline as the X_t's get farther away from x_0. This weighted-average or "local averaging" procedure of estimating $m(x)$ is the essence of smoothing.

More formally, for any arbitrary x, a smoothing estimator of $m(x)$ may be expressed as

$$\hat{m}(x) \equiv \frac{1}{T}\sum_{t=1}^{T}\omega_t(x)P_t \tag{3.4}$$

where the weights $\{\omega_t(x)\}$ are large for those P_t's paired with X_t's near x, and small for those P_t's with X_t's far from x. To implement such a procedure, we must define what we mean by "near" and "far." If we choose too large a neighborhood around x to compute the average, the weighted average will be too smooth and will not exhibit the genuine nonlinearities of $m(\cdot)$. If we choose too small a neighborhood around x, the weighted average will be too variable, reflecting noise as well as the variations in $m(\cdot)$. Therefore, the weights $\{\omega_t(x)\}$ must be chosen carefully to balance these two considerations.

3.2.2 Kernel Regression

For the *kernel regression* estimator, the weight function $\omega_t(x)$ is constructed from a probability density function $\mathrm{K}(x)$, also called a *kernel*:[3]

$$\mathrm{K}(x) \geq 0 \quad , \quad \int \mathrm{K}(u)du = 1. \tag{3.5}$$

By rescaling the kernel with respect to a parameter $h>0$, we can change its spread, i.e., let:

$$\mathrm{K}_h(u) \equiv \frac{1}{h}\mathrm{K}(u/h) \quad , \quad \int \mathrm{K}_h(u)du = 1 \tag{3.6}$$

and define the weight function to be used in the weighted average (3.4) as

$$\omega_{t,h}(x) \equiv \mathrm{K}_h(x-X_t)/g_h(x) \tag{3.7}$$

$$g_h(x) \equiv \frac{1}{T}\sum_{t=1}^{T}\mathrm{K}_h(x-X_t)\ . \tag{3.8}$$

[3]Despite the fact that $\mathrm{K}(x)$ is a probability density function, it plays no probabilistic role in the subsequent analysis—it is merely a convenient method for computing a weighted average, and does *not* imply, for example, that X is distributed according to $\mathrm{K}(x)$ (which would be a parametric assumption).

If h is very small, the averaging will be done with respect to a rather small neighborhood around each of the X_t's. If h is very large, the averaging will be over larger neighborhoods of the X_t's. Therefore, controlling the degree of averaging amounts to adjusting the smoothing parameter h, also known as the *bandwidth*. Choosing the appropriate bandwidth is an important aspect of any local-averaging technique and is discussed more fully in Section 3.2.3.

Substituting (3.8) into (3.4) yields the *Nadaraya-Watson* kernel estimator $\hat{m}_h(x)$ of $m(x)$:

$$\hat{m}_h(x) \quad = \quad \frac{1}{T}\sum_{t=1}^{T} \omega_{t,h}(x) Y_t \quad = \quad \frac{\sum_{t=1}^{T} \mathrm{K}_h(x - X_t)Y_t}{\sum_{t=1}^{T} \mathrm{K}_h(x - X_t)} \ . \tag{3.9}$$

Under certain regularity conditions on the shape of the kernel K and the magnitudes and behavior of the weights as the sample size grows, it may be shown that $\hat{m}_h(x)$ converges to $m(x)$ asymptotically in several ways (see Härdle (1990) for further details). This convergence property holds for a wide class of kernels, but for the remainder of this paper we shall use the most popular choice of kernel, the Gaussian kernel:

$$\mathrm{K}_h(x) \quad = \quad \frac{1}{h\sqrt{2\pi}} e^{-\frac{x^2}{2h^2}} \tag{3.10}$$

3.2.3 Selecting the Bandwidth

Selecting the appropriate bandwidth h in (3.9) is clearly central to the success of $\hat{m}_h(\cdot)$ in approximating $m(\cdot)$—too little averaging yields a function that is too choppy, and too much averaging yields a function that is too smooth. To illustrate these two extremes, Figure 3.2 displays the Nadaraya-Watson kernel estimator applied to 500 data points generated from the relation:

$$Y_t \quad = \quad \mathrm{Sin}(X_t) \ + \ 0.5\,\epsilon_t \quad , \quad \epsilon_t \sim \mathcal{N}(0,1) \tag{3.11}$$

where X_t is evenly spaced in the interval $[0, 2\pi]$. Panel 3.2(a) plots the raw data and the function to be approximated.

Kernel estimators for three different bandwidths are plotted as solid lines in Panels 3.2(b)–(c). The bandwidth in 3.2(b) is clearly too small; the function is too variable, fitting the "noise" $0.5\,\epsilon_t$ as well as the "signal" Sin(·). Increasing the bandwidth slightly yields a much more accurate approximation to Sin(·) as Panel 3.2(c) illustrates. However, Panel 3.2(d) shows that if the bandwidth is increased beyond some point, there is too much averaging and information is lost.

There are several methods for automating the choice of bandwidth h in (3.9), but the most popular is the *cross-validation* method in which h is chosen to minimize the cross-validation function:

$$\mathrm{CV}(h) = \frac{1}{T}\sum_{t=1}^{T}(P_t - \hat{m}_{h,t})^2 \tag{3.12}$$

where

$$\hat{m}_{h,t} \equiv \frac{1}{T}\sum_{\tau \neq t}^{T}\omega_{\tau,h}Y_\tau \; . \tag{3.13}$$

The estimator $\hat{m}_{h,t}$ is the kernel regression estimator applied to the price history $\{P_\tau\}$ with the t-th observation omitted, and the summands in (3.12) are the squared errors of the $\hat{m}_{h,t}$'s, each evaluated at the omitted observation. For a given bandwidth parameter h, the cross-validation function is a measure of the ability of the kernel regression estimator to fit each observation P_t when that observation is not used to construct the kernel estimator. By selecting the bandwidth that minimizes this function, we obtain a kernel estimator that satisfies certain optimality properties, e.g., minimum asymptotic mean-squared error.[4]

Interestingly, the bandwidths obtained from minimizing the cross-validation function are generally too large for our application to technical analysis—when we presented several professional technical analysts with plots of cross-validation-fitted functions $\hat{m}_h(\cdot)$, they all concluded that the fitted functions were too smooth. In other words, the cross-validation-determined bandwidth places too much weight on prices far away from any given time t, inducing too much averaging and discarding valuable information in local price movements. Through trial and error, and by polling professional technical analysts, we have found that an acceptable solution to this problem is to use a bandwidth of $0.3 \times h^*$, where h^* minimizes $\mathrm{CV}(h)$.[5] Admittedly, this is an ad hoc approach, and it remains an important challenge for future research to develop a more rigorous procedure.

[4]However, there are other bandwidth-selection methods that yield the same asymptotic optimality properties but which have different implications for the finite-sample properties of kernel estimators. See Härdle (1990) for further discussion.

[5]Specifically, we produced fitted curves for various bandwidths and compared their extrema to the original price series visually to see if we were fitting more "noise" than "signal," and asked several professional technical analysts to do the same. Through this informal process, we settled on the bandwidth of $0.3 \times h^*$ and used it for the remainder of our analysis. This procedure was followed before we performed the statistical analysis of Section 3.4, and we made no revision to the choice of bandwidth afterwards.

Another promising direction for future research is to consider alternatives to kernel regression. Although useful for its simplicity and intuitive appeal, kernel estimators suffer from a number of well-known deficiencies, e.g., boundary bias, lack of local variability in the degree of smoothing, etc. A popular alternative that overcomes these particular deficiencies is *local polynomial regression* in which local averaging of polynomials is performed to obtain an estimator of $m(x)$.[6] Such alternatives may yield important improvements the pattern-recognition algorithm described in Section 3.3.

3.3 Automating Technical Analysis

Armed with a mathematical representation $\hat{m}(\cdot)$ of $\{P_t\}$ with which geometric properties can be characterized in an objective manner, we can now construct an algorithm for automating the detection of technical patterns. Specifically, our algorithm contains three steps:

1. Define each technical pattern in terms of its geometric properties, e.g., local extrema (maxima and minima).
2. Construct a kernel estimator $\hat{m}(\cdot)$ of a given time series of prices so that its extrema can be determined numerically.
3. Analyze $\hat{m}(\cdot)$ for occurrences of each technical pattern.

The last two steps are rather straightforward applications of kernel regression. The first step is likely to be the most controversial because it is here that the skills and judgment of a professional technical analyst come into play. Although we will argue in Section 3.3.1 that most technical indicators can be characterized by specific *sequences* of local extrema, technical analysts may argue that these are poor approximations to the kinds of patterns that trained human analysts can identify.

While pattern-recognition techniques have been successful in automating a number of tasks previously considered to be uniquely human endeavors—fingerprint identification, handwriting analysis, face recognition, and so on—nevertheless it is possible that no algorithm can completely capture the skills of an experienced technical analyst. We acknowledge

[6]See Simonoff (1996) for a discussion of the problems with kernel estimators and alternatives such as local polynomial regression.

that any automated procedure for pattern recognition may miss some of the more subtle nuances that human cognition is capable of discerning, but whether an algorithm is a poor approximation to human judgment can only be determined by investigating the approximation errors empirically. As long as an algorithm can provide a reasonable approximation to *some* of the cognitive abilities of a human analyst, we can use such an algorithm to investigate the empirical performance of those aspects of technical analysis for which the algorithm is a good approximation. Moreover, if technical analysis is an art form that can be taught, then surely its basic precepts can be quantified and automated to some degree. And as increasingly sophisticated pattern-recognition techniques are developed, a larger fraction of the art will become a science.

More importantly, from a practical perspective, there may be significant benefits to developing an algorithmic approach to technical analysis because of the leverage that technology can provide. As with many other successful technologies, the automation of technical pattern recognition may not replace the skills of a technical analyst, but can amplify them considerably.

In Section 3.3.1, we propose definitions of ten technical patterns based on their extrema. In Section 3.3.2, we describe a specific algorithm to identify technical patterns based on the local extrema of kernel regression estimators, and provide specific examples of the algorithm at work in Section 3.3.3.

3.3.1 Definitions of Technical Patterns

We focus on five pairs of technical patterns that are among the most popular patterns of traditional technical analysis (see, for example, Edwards and Magee (1966, Chapters VII–X)): head-and-shoulders (HS) and inverse head-and-shoulders (IHS), broadening tops (BT) and bottoms (BB), triangle tops (TT) and bottoms (TB), rectangle tops (RT) and bottoms (RB), and double tops (DT) and bottoms (DB). There are many other technical indicators that may be easier to detect algorithmically—moving averages, support and resistance levels, and oscillators, for example—but because we wish to illustrate the power of smoothing techniques in automating technical analysis, we focus on precisely those patterns that are most difficult to quantify analytically.

Consider the systematic component $m(\cdot)$ of a price history $\{P_t\}$ and suppose we have identified n local extrema, i.e., the local maxima and minima, of $m(\cdot)$. Denote by $E_1, E_2, \ldots, E_n$ the n extrema and $t_1^*, t_2^*, \ldots, t_n^*$ the dates on which these extrema occur. Then we have the following definitions:

Definition 1 (Head-and-Shoulders) *Head-and-shoulders (HS) and inverted head-and-shoulders (IHS) patterns are characterized by a sequence of five consecutive local extrema $E_1, \ldots, E_5$ such that:*

$$HS \equiv \begin{cases} E_1 \text{ a maximum} \\ E_3 > E_1 \ , \ E_3 > E_5 \\ E_1 \text{ and } E_5 \text{ within } 1.5\% \text{ of their average} \\ E_2 \text{ and } E_4 \text{ within } 1.5\% \text{ of their average} \end{cases}$$

$$IHS \equiv \begin{cases} E_1 \text{ a minimum} \\ E_3 < E_1 \ , \ E_3 < E_5 \\ E_1 \text{ and } E_5 \text{ within } 1.5\% \text{ of their average} \\ E_2 \text{ and } E_4 \text{ within } 1.5\% \text{ of their average} \end{cases}$$

Observe that only five consecutive extrema are required to identify a head-and-shoulders pattern. This follows from the formalization of the geometry of a head-and-shoulders pattern: three peaks, with the middle peak higher than the other two. Because consecutive extrema must alternate between maxima and minima for smooth functions,[7] the three-peaks pattern corresponds to a sequence of five local extrema: maximum, minimum, highest maximum, minimum, and maximum. The inverse head-and-shoulders is simply the mirror image of the head-and-shoulders, with the initial local extrema a minimum.

Because broadening, rectangle, and triangle patterns can begin on either a local maximum or minimum, we allow for both of these possibilities in our definitions by distinguishing between broadening tops and bottoms:

Definition 2 (Broadening) *Broadening tops (BTOP) and bottoms (BBOT) are characterized by a sequence of five consecutive local extrema $E_1, \ldots, E_5$ such that:*

$$BTOP \equiv \begin{cases} E_1 \text{ a maximum} \\ E_1 < E_3 < E_5 \\ E_2 > E_4 \end{cases} , \qquad BBOT \equiv \begin{cases} E_1 \text{ a minimum} \\ E_1 > E_3 > E_5 \\ E_2 < E_4 \end{cases}$$

Definitions for triangle and rectangle patterns follow naturally:

[7] After all, for two consecutive maxima to be local maxima, there must be a local minimum in between, and vice versa for two consecutive minima.

Definition 3 (Triangle) *Triangle tops (TTOP) and bottoms (TBOT) are characterized by a sequence of five consecutive local extrema $E_1, \ldots, E_5$ such that:*

$$TTOP \equiv \begin{cases} E_1 \text{ a maximum} \\ E_1 > E_3 > E_5 \\ E_2 < E_4 \end{cases} \quad , \quad TBOT \equiv \begin{cases} E_1 \text{ a minimum} \\ E_1 < E_3 < E_5 \\ E_2 > E_4 \end{cases}$$

Definition 4 (Rectangle) *Rectangle tops (RTOP) and bottoms (RBOT) are characterized by a sequence of five consecutive local extrema $E_1, \ldots, E_5$ such that:*

$$RTOP \equiv \begin{cases} E_1 \text{ a maximum} \\ \text{tops within 0.75 \% of their average} \\ \text{bottoms within 0.75 \% of their average} \\ \text{lowest top} > \text{highest bottom} \end{cases}$$

$$RBOT \equiv \begin{cases} E_1 \text{ a minimum} \\ \text{tops within 0.75 \% of their average} \\ \text{bottoms within 0.75 \% of their average} \\ \text{lowest top} > \text{highest bottom} \end{cases}$$

The definition for double tops and bottoms is slightly more involved. Consider first the double top. Starting at a local maximum E_1, we locate the highest local maximum E_a occurring after E_1 in the set of all local extrema in the sample. We require that the two tops, E_1 and E_a, be within 1.5 percent of their average. Finally, following Edwards and Magee (1966), we require that the two tops occur at least a month, or 22 trading days, apart. Therefore, we have:

Definition 5 (Double Top and Bottom) *Double tops (DTOP) and bottoms (DBOT) are characterized by an initial local extremum E_1 and a subsequent local extrema E_a and E_b such that:*

$$E_a \equiv \sup \{ P_{t_k^*} : t_k^* > t_1^* , \ k = 2, \ldots, n \}$$
$$E_b \equiv \inf \{ P_{t_k^*} : t_k^* > t_1^* , \ k = 2, \ldots, n \}$$

and

$$DTOP \equiv \begin{cases} E_1 \text{ a maximum} \\ E_1 \text{ and } E_a \text{ within 1.5 \% of their average} \\ t_a^* - t_1^* > 22 \end{cases}$$

$$DBOT \equiv \begin{cases} E_1 \text{ a minimum} \\ E_1 \text{ and } E_b \text{ within 1.5 \% of their average} \\ t_a^* - t_1^* > 22 \end{cases}$$

3.3.2 The Identification Algorithm

Our algorithm begins with a sample of prices $\{P_1, \ldots, P_T\}$ for which we fit kernel regressions, one for each subsample or *window* from t to $t+l+d-1$, where t varies from 1 to $T-l-d+1$, and l and d are fixed parameters whose purpose is explained below. In the empirical analysis of Section 3.4, we set $l=35$ and $d=3$, hence each window consists of 38 trading days.

The motivation for fitting kernel regressions to rolling windows of data is to narrow our focus to patterns that are completed within the span of the window—$l+d$ trading days in our case. If we fit a single kernel regression to the entire dataset, many patterns of various durations may emerge, and without imposing some additional structure on the nature of the patterns, it is virtually impossible to distinguish signal from noise in this case. Therefore, our algorithm fixes the length of the window at $l+d$, but kernel regressions are estimated on a rolling basis and we search for patterns in each window.

Of course, for any fixed window, we can only find patterns that are completed within $l+d$ trading days. Without further structure on the systematic component of prices $m(\cdot)$, this is a restriction that any empirical analysis must contend with.[8] We choose a shorter window length of $l=35$ trading days to focus on short-horizon patterns that may be more relevant for active equity traders, and leave the analysis of longer-horizon patterns to future research.

The parameter d controls for the fact that in practice we do not observe a realization of a given pattern as soon as it has completed. Instead, we assume that there may be a lag between the pattern completion and the time of pattern detection. To account for this lag, we require that the final extremum that completes a pattern occurs on day $t+l-1$; hence d is the number of days following the completion of a pattern that must pass before the pattern is detected. This will become more important in Section 3.4 when we compute conditional returns, conditioned on the realization of each pattern. In particular, we compute post-pattern returns starting from the end of trading day $t+l+d$, i.e., one day after the pattern has completed. For example, if we determine that a head-and-shoulder pattern has completed on day $t+l-1$ (having used prices from time t through time $t+l+d-1$), we compute the conditional one-day gross return as $Z_1 \equiv Y_{t+l+d+1}/Y_{t+l+d}$. Hence we do *not* use any forward information in computing returns conditional on pattern completion. In other words, the

[8]If we are willing to place additional restrictions on $m(\cdot)$, e.g., linearity, we can obtain considerably more accurate inferences even for partially completed patterns in any fixed window.

lag d ensures that we are computing our conditional returns completely out-of-sample and without any "look-ahead" bias.

Within each window, we estimate a kernel regression using the prices in that window, hence:

$$\hat{m}_h(\tau) \quad = \quad \frac{\sum_{s=t}^{t+l+d-1} \mathrm{K}_h(\tau - s) P_s}{\sum_{s=t}^{t+l+d-1} \mathrm{K}_h(\tau - s)} \quad , \quad t = 1, \ldots , T-l-d+1 \tag{3.14}$$

where $\mathrm{K}_h(z)$ is given in (3.10) and h is the bandwidth parameter (see Section 3.2.3). It is clear that $\hat{m}_h(\tau)$ is a differentiable function of τ.

Once the function $\hat{m}_h(\tau)$ has been computed, its local extrema can be readily identified by finding times τ such that $\mathrm{Sgn}\big(\hat{m}_h'(\tau)\big) = -\mathrm{Sgn}\big(\hat{m}_h'(\tau+1)\big)$, where $\hat{m}_h'$ denotes the derivative of $\hat{m}_h$ with respect to τ and $\mathrm{Sgn}(\cdot)$ is the signum function. If the signs of $\hat{m}_h'(\tau)$ $\hat{m}_h'(\tau+1)$ are $+1$ and -1, respectively, then we have found a local maximum, and if they are -1 and $+1$, respectively, then we have found a local minimum. Once such a time τ has been identified, we proceed to identify a maximum or minimum in the original price series $\{P_t\}$ in the range $[t-1, t+1]$, and the extrema in the original price series are used to determine whether or not a pattern has occurred according to the definitions of Section 3.3.1.

If $\hat{m}_h'(\tau) = 0$ for a given τ, which occurs if closing prices stay the same for several consecutive days, we need to check whether the price we have found is a local minimum or maximum. We look for the date s such that $s = \inf \{ \, s > \tau \; : \; \hat{m}_h'(s) \neq 0 \, \}$. We then apply the same method as discussed above, except here we compare $\mathrm{Sgn}\big(\hat{m}_h'(\tau-1)\big)$ and $\mathrm{Sgn}\big(\hat{m}_h'(s)\big)$.

One useful consequence of this algorithm is that the series of extrema which it identifies contains alternating minima and maxima. That is, if the k^{th} extremum is a maximum, then it is always the case that the $(k+1)^{\text{th}}$ extremum is a minimum, and vice versa.

An important advantage of using this kernel regression approach to identify patterns is the fact that it ignores extrema that are "too local." For example, a simpler alternative is to identify local extrema from the raw price data directly, i.e., identify a price P_t as a local maximum if $P_{t-1} < P_t$ and $P_t > P_{t+1}$, and vice versa for a local minimum. The problem with this approach is that it identifies too many extrema, and also yields patterns that are not visually consistent with the kind of patterns that technical analysts find compelling.

Once we have identified all of the local extrema in the window $[t, t{+}l{+}d{-}1]$, we can proceed to check for the presence of the various technical patterns using the definitions of Section 3.3.1. This procedure is then repeated for the next window $[t+1, t+l+d]$, and continues until the end of the sample is reached at the window $[T-l-d+1, T]$.

3.3.3 Empirical Examples

To see how our algorithm performs in practice, we apply it to the daily returns of a single security, CTX, during the five-year period from 1992 to 1996. Figures 3.3–3.7 plot occurrences of the five pairs of patterns defined in Section 3.3.1 that were identified by our algorithm. Note that there were no rectangle bottoms detected for CTX during this period, so for completeness we substituted a rectangle bottom for CDO stock which occurred during the same period.

In each of these graphs, the solid lines are the raw prices, the dashed lines are the kernel estimators $\hat{m}_h(\cdot)$, the circles indicate the local extrema, and the vertical line marks date $t+l-1$, the day that the final extremum occurs to complete the pattern.

Casual inspection by several professional technical analysts seems to confirm the ability of our automated procedure to match human judgment in identifying the five pairs of patterns in Section 3.3.1. Of course, this is merely anecdotal evidence and not meant to be conclusive—we provide these figures simply to illustrate the output of a technical pattern recognition algorithm based on kernel regression.

3.4 Is Technical Analysis Informative?

Although there have been many tests of technical analysis over the years, most of these tests have focused on the profitability of technical trading rules.[9] While some of these studies do find that technical indicators can generate statistically significant trading profits, they beg

[9] For example, Chang and Osler (1994) and Osler and Chang (1995) propose an algorithm for automatically detecting head-and-shoulders patterns in foreign exchange data by looking at properly defined local extrema. To assess the efficacy of a head-and-shoulders trading rule, they take a stand on a class of trading strategies and compute the profitability of these across a sample of exchange rates against the U.S. dollar. The null return distribution is computed by a bootstrap that samples returns randomly from the original data so as to induce temporal independence in the bootstrapped time series. By comparing the actual returns from trading strategies to the bootstrapped distribution, the authors find that for two of the six currencies in their sample (the yen and the Deutsche mark), trading strategies based on a head and shoulders pattern can lead to statistically significant profits. See, also, Neftci and Policano (1984), Pruitt and White (1988), and Brock, Lakonishok, and LeBaron (1992).

the question of whether or not such profits are merely the equilibrium rents that accrue to investors willing to bear the risks associated with such strategies. Without specifying a fully articulated dynamic general equilibrium asset-pricing model, it is impossible to determine the economic source of trading profits.

Instead, we propose a more fundamental test in this section, one that attempts to gauge the information content in the technical patterns of Section 3.3.1 by comparing the unconditional empirical distribution of returns with the corresponding conditional empirical distribution, conditioned on the occurrence of a technical pattern. If technical patterns are informative, conditioning on them should alter the empirical distribution of returns; if the information contained in such patterns has already been incorporated into returns, the conditional and unconditional distribution of returns should be close. Although this is a weaker test of the effectiveness of technical analysis—informativeness does not guarantee a profitable trading strategy—it is, nevertheless, a natural first step in a quantitative assessment of technical analysis.

To measure the distance between the two distributions, we propose two goodness-of-fit measures in Section 3.4.1. We apply these diagnostics to the daily returns of individual stocks from 1962 to 1996 using a procedure described in Sections 3.4.2 to 3.4.4, and the results are reported in Sections 3.4.5 and 3.4.6.

3.4.1 Goodness-of-Fit Tests

A simple diagnostic to test the informativeness of the ten technical patterns is to compare the quantiles of the conditional returns with their unconditional counterparts. If conditioning on these technical patterns provides no incremental information, the quantiles of the conditional returns should be similar to those of unconditional returns. In particular, we compute the deciles of unconditional returns and tabulate the relative frequency $\hat{\delta}_j$ of *conditional* returns falling into decile j of the unconditional returns, $j = 1, \ldots, 10$:

$$\hat{\delta}_j \equiv \frac{\text{number of conditional returns in decile } j}{\text{total number of conditional returns}} . \tag{3.15}$$

Under the null hypothesis that the returns are independently and identically distributed and the conditional and unconditional distributions are identical, the asymptotic distributions of

$\hat{\delta}_j$ and the corresponding goodness-of-fit test statistic Q are given by:

$$\sqrt{n}(\hat{\delta}_j - 0.10) \overset{a}{\sim} \mathcal{N}(0, 0.10(1-0.10)) \tag{3.16}$$

$$Q \equiv \sum_{j=1}^{10} \frac{(n_j - 0.10n)^2}{0.10n} \overset{a}{\sim} \chi_9^2 \tag{3.17}$$

where n_j is the number of observations that fall in decile j and n is the total number of observations (see, for example, DeGroot (1986)).

Another comparison of the conditional and unconditional distributions of returns is provided by the Kolmogorov-Smirnov test. Denote by $\{Z_{1t}\}_{t=1}^{n_1}$ and $\{Z_{2t}\}_{t=1}^{n_2}$ two samples that are each independently and identically distributed with cumulative distribution functions $F_1(z)$ and $F_2(z)$, respectively. The Kolmogorov-Smirnov statistic is designed to test the null hypothesis that $F_1 = F_2$, and is based on the empirical cumulative distribution functions $\hat{F}_i$ of both samples:

$$\hat{F}_i(z) \equiv \frac{1}{n_i} \sum_{k=1}^{n_i} \mathbf{1}(Z_{ik} \leq z) \quad , \quad i = 1, 2 \tag{3.18}$$

where $\mathbf{1}(\cdot)$ is the indicator function. The statistic is given by the expression:

$$\gamma_{n_1,n_2} = \left(\frac{n_1 \, n_2}{n_1 + n_2}\right)^{1/2} \sup_{-\infty<z<\infty} |\hat{F}_1(z) - \hat{F}_2(z)| \ . \tag{3.19}$$

Under the null hypothesis $F_1 = F_2$, the statistic γ_{n_1,n_2} should be small. Moreover, Smirnov (1939a, 1939b) derives the limiting distribution of the statistic to be:

$$\lim_{\min(n_1,n_2)\to\infty} \text{Prob}\,(\,\gamma_{n_1,n_2} \leq x\,) = \sum_{k=-\infty}^{\infty} (-1)^k \exp(-2k^2x^2) \quad , \quad x > 0 \tag{3.20}$$

An approximate α-level test of the null hypothesis can be performed by computing the statistic and rejecting the null if it exceeds the upper 100α-th percentile for the null distribution given by (3.20) (see Hollander and Wolfe (1973, Table A.23), Csáki (1984), and Press et al. (1986, Chapter 13.5)).

Note that the sampling distributions of both the goodness-of-fit and Kolmogorov-Smirnov statistics are derived under the assumption that returns are independently and identically distributed, which is not plausible for financial data. We attempt to address this problem by normalizing the returns of each security, i.e., by subtracting its mean and dividing by its standard deviation (see Section 3.4.3), but this does not eliminate the dependence or heterogeneity. We hope to extend our analysis to the more general non-IID case in future research.

3.4.2 The Data and Sampling Procedure

We apply the goodness-of-fit and Kolmogorov-Smirnov tests to the daily returns of individual NYSE/AMEX and Nasdaq stocks from 1962 to 1996 using data from the Center for Research in Securities Prices (CRSP). To ameliorate the effects of nonstationarities induced by changing market structure and institutions, we split the data into NYSE/AMEX stocks and Nasdaq stocks and into seven five-year periods: 1962 to 1966, 1967 to 1971, and so on. To obtain a broad cross-section of securities, in each five-year subperiod, we randomly select ten stocks from each of five market-capitalization quintiles (using mean market-capitalization over the subperiod), with the further restriction that at least 75 percent of the price observations must be non-missing during the subperiod.[10] This procedure yields a sample of 50 stocks for each subperiod across seven subperiods (note that we sample with replacement, hence there may be names in common across subperiods).

As a check on the robustness of our inferences, we perform this sampling procedure twice to construct two samples, and apply our empirical analysis to both. Although we report results only from the first sample to conserve space, the results of the second sample are qualitatively consistent with the first and are available upon request.

3.4.3 Computing Conditional Returns

For each stock in each subperiod, we apply the procedure outlined in Section 3.3 to identify all occurrences of the ten patterns defined in Section 3.3.1. For each pattern detected, we compute the one-day continuously compounded return d days after the pattern has completed. Specifically, consider a window of prices $\{P_t\}$ from t to $t+l+d-1$, and suppose that the identified pattern p is completed at $t+l-1$. Then we take the conditional return R^p as $\log(1 + R_{t+l+d+1})$. Therefore, for each stock, we have ten sets of such conditional returns, each conditioned on one of the ten patterns of Section 3.3.1.

[10] If the first price observation of a stock is missing, we set it equal to the first non-missing price in the series. If the t-th price observation is missing, we set it equal to the first non-missing price prior to t.

For each stock, we construct a sample of *unconditional* continuously compounded returns using non-overlapping intervals of length τ, and we compare the empirical distribution function of these returns with those of the conditional returns. To facilitate such comparisons, we standardize all returns—both conditional and unconditional—by subtracting means and dividing by standard deviations, hence:

$$X_{it} \quad = \quad \frac{R_{it} - \text{Mean}[R_{it}]}{\text{SD}[R_{it}]} \qquad (3.21)$$

where the means and standard deviations are computed for each individual stock within each subperiod. Therefore, by construction, each normalized return series has zero mean and unit variance.

Finally, to increase the power of our goodness-of-fit tests, we combine the normalized returns of all 50 stocks within each subperiod; hence for each subperiod we have two samples—unconditional and conditional returns—and from these we compute two empirical distribution functions that we compare using our diagnostic test statistics.

3.4.4 Conditioning on Volume

Given the prominent role that volume plays in technical analysis, we also construct returns conditioned on increasing or decreasing volume. Specifically, for each stock in each subperiod, we compute its average share-turnover during the first and second halves of each subperiod, τ_1 and τ_2, respectively.[11] If $\tau_1 > 1.2 \times \tau_2$, we categorize this as a "decreasing volume" event; if $\tau_2 > 1.2 \times \tau_1$, we categorize this as an "increasing volume" event. If neither of these conditions holds, then neither event is considered to have occurred.

Using these events, we can construct conditional returns conditioned on two pieces of information: the occurrence of a technical pattern and the occurrence of increasing or decreasing volume. Therefore, we shall compare the empirical distribution of unconditional returns with three conditional-return distributions: the distribution of returns conditioned on technical patterns, the distribution conditioned on technical patterns and increasing volume, and the distribution conditioned on technical patterns and decreasing volume.

Of course, other conditioning variables can easily be incorporated into this procedure,

[11]For the Nasdaq stocks, τ_1 is the average turnover over the first third of the sample, and τ_2 is the average turnover over the final third of the sample.

though the "curse of dimensionality" imposes certain practical limits on the ability to estimate multivariate conditional distributions nonparametrically.

3.4.5 Summary Statistics

In Tables I and II, we report frequency counts for the number of patterns detected over the entire 1962 to 1996 sample, and within each subperiod and each market-capitalization quintile, for the ten patterns defined in Section 3.3.1. Table I contains results for the NYSE/AMEX stocks, and Table II contains corresponding results for Nasdaq stocks.

Table I shows that the most common patterns across all stocks and over the entire sample period are double tops and bottoms (see the row labeled "Entire"), with over 2,000 occurrences of each. The second most common patterns are the head-and-shoulders and inverted head-and-shoulders, with over 1,600 occurrences of each. These total counts correspond roughly to four to six occurrences of each of these patterns for each stock during each five-year subperiod (divide the total number of occurrences by 7×50), not an unreasonable frequency from the point of view of professional technical analysts. Table I shows that most of the ten patterns are more frequent for larger stocks than for smaller ones, and that they are relatively evenly distributed over the five-year subperiods. When volume trend is considered jointly with the occurrences of the ten patterns, Table I shows that the frequency of patterns is not evenly distributed between increasing (the row labeled "$\tau(\nearrow)$") and decreasing (the row labeled "$\tau(\searrow)$") volume-trend cases. For example, for the entire sample of stocks over the 1962 to 1996 sample period, there are 143 occurrences of a broadening top with decreasing volume trend, but 409 occurrences of a broadening top with increasing volume trend.

For purposes of comparison, Table I also reports frequency counts for the number of patterns detected in a sample of simulated geometric Brownian motion, calibrated to match the mean and standard deviation of each stock in each five-year subperiod.[12] The entries in the

[12]In particular, let the price process satisfy

$$dP(t) \;=\; \mu P(t)\,dt \;+\; \sigma P(t)\,dW(t) \tag{3.22}$$

where $W(t)$ is a standard Brownian motion. To generate simulated prices for a single security in a given period, we estimate the security's drift and diffusion coefficients by maximum likelihood and then simulate prices using the estimated parameter values. An independent price series is simulated for each of the 350 securities in both the NYSE/AMEX and the Nasdaq samples. Finally, we use our pattern recognition algorithm to detect the occurrence of each of the ten patterns in the simulated price series.

row labeled "Sim. GBM" show that the random walk model yields very different implications for the frequency counts of several technical patterns. For example, the simulated sample has only 577 head-and-shoulders and 578 inverted-head-and-shoulders patterns, whereas the actual data have considerably more, 1,611 and 1,654, respectively. On the other hand, for broadening tops and bottoms, the simulated sample contains many more occurrences than the actual data, 1,227 and 1,028 as compared to 725 and 748, respectively. The number of triangles is roughly comparable across the two samples, but for rectangles and double tops and bottoms, the differences are dramatic. Of course, the simulated sample is only one realization of geometric Brownian motion, so it is difficult to draw general conclusions about the relative frequencies. Nevertheless, these simulations point to important differences between the data and independently and identically distributed lognormal returns.

To develop further intuition for these patterns, Figures 3.8 and 3.9 display the cross-sectional and time-series distribution of each of the ten patterns for the NYSE/AMEX and Nasdaq samples, respectively. Each symbol represents a pattern detected by our algorithm, the vertical axis is divided into five quintiles, the horizontal axis is calendar time, and alternating symbols (diamonds and asterisks) represent distinct subperiods. These graphs show that the distribution of patterns is not clustered in time or among a subset of securities.

Table II provides the same frequency counts for Nasdaq stocks, and despite the fact that we have the same number of stocks in this sample (50 per subperiod over seven subperiods), there are considerably fewer patterns detected than in the NYSE/AMEX case. For example, the Nasdaq sample yields only 919 head-and-shoulders patterns, whereas the NYSE/AMEX sample contains 1,611. Not surprisingly, the frequency counts for the sample of simulated geometric Brownian motion are similar to those in Table I.

Tables III and IV report summary statistics—means, standard deviations, skewness, and excess kurtosis—of unconditional and conditional normalized returns of NYSE/AMEX and Nasdaq stocks, respectively. These statistics show considerable variation in the different return populations. For example, the first four moments of normalized raw returns are 0.000, 1.000, 0.345, and 8.122, respectively. The same four moments of post-BTOP returns are −0.005, 1.035, −1.151, and 16.701, respectively, and those of post-DTOP returns are 0.017, 0.910, 0.206, and 3.386, respectively. The differences in these statistics among the ten conditional return populations, and the differences between the conditional and unconditional return populations, suggest that conditioning on the ten technical indicators does have some effect on the distribution of returns.

3.4.6 Empirical Results

Tables V and VI reports the results of the goodness-of-fit test (3.16)–(3.17) for our sample of NYSE and AMEX (Table V) and Nasdaq (Table VI) stocks, respectively, from 1962 to 1996 for each of the ten technical patterns. Table V shows that in the NYSE/AMEX sample, the relative frequencies of the conditional returns are significantly different from those of the unconditional returns for seven of the ten patterns considered. The three exceptions are the conditional returns from the BBOT, TTOP, and DBOT patterns, for which the p-values of the test statistics Q are 5.1 percent, 21.2 percent, and 16.6 percent respectively. These results yield mixed support for the overall efficacy of technical indicators. However, the results of Table VI tell a different story: there is overwhelming significance for all ten indicators in the Nasdaq sample, with p-values that are zero to three significant digits, and test statistics Q that range from 34.12 to 92.09. In contrast, the test statistics in Table V range from 12.03 to 50.97.

One possible explanation for the difference between the NYSE/AMEX and Nasdaq samples is a difference in the power of the test because of different sample sizes. If the NYSE/AMEX sample contained fewer conditional returns, i.e., fewer patterns, the corresponding test statistics might be subject to greater sampling variation and lower power. However, this explanation can be ruled out from the frequency counts of Tables I and II—the number of patterns in the NYSE/AMEX sample is considerably larger than those of the Nasdaq sample for all ten patterns. Tables V and VI seem to suggest important differences in the informativeness of technical indicators for NYSE/AMEX and Nasdaq stocks.

Table VII and VIII report the results of the Kolmogorov-Smirnov test (3.19) of the equality of the conditional and unconditional return distributions for NYSE/AMEX (Table VII) and Nasdaq (Table VIII) stocks, respectively, from 1962 to 1996, in five-year subperiods, and in market-capitalization quintiles. Recall that conditional returns are defined as the one-day return starting three days following the conclusion of an occurrence of a pattern. The p-values are with respect to the asymptotic distribution of the Kolmogorov-Smirnov test statistic given in (3.20).

Table VII shows that for NYSE/AMEX stocks, five of the ten patterns—HS, BBOT, RTOP, RBOT, and DTOP—yield statistically significant test statistics, with p-values ranging from 0.000 for RBOT to 0.021 for DTOP patterns. However, for the other five patterns, the p-values range from 0.104 for IHS to 0.393 for DBOT, which implies an inability to distinguish between the conditional and unconditional distributions of normalized returns.

When we condition on declining volume trend as well, the statistical significance declines for most patterns, but increases the statistical significance of TBOT patterns. In contrast, conditioning on increasing volume trend yields an increase in the statistical significance of BTOP patterns. This difference may suggest an important role for volume trend in TBOT and BTOP patterns. The difference between the increasing and decreasing volume-trend conditional distributions is statistically insignificant for almost all the patterns (the sole exception is the TBOT pattern). This drop in statistical significance may be due to a lack of power of the K-S test given the relatively small sample sizes of these conditional returns (see Table I for frequency counts).

Table VIII reports corresponding results for the Nasdaq sample and as in Table VI, in contrast to the NYSE/AMEX results, here all the patterns are statistically significant at the 5 percent level. This is especially significant because the the Nasdaq sample exhibits far fewer patterns than the NYSE/AMEX sample (see Tables I and II), hence the K-S test is likely to have lower power in this case.

As with the NYSE/AMEX sample, volume trend seems to provide little incremental information for the Nasdaq sample except in one case: increasing volume and BTOP. And except for the TTOP pattern, the K-S test still cannot distinguish between the decreasing and increasing volume-trend conditional distributions, as the last pair of rows of Table VIII's first panel indicates.

3.5 Monte Carlo Analysis

Tables IX and X contain bootstrap percentiles for the Kolmogorov-Smirnov test of the equality of conditional and unconditional one-day return distributions for NYSE/AMEX and Nasdaq stocks, respectively, from 1962 to 1996, and for market-capitalization quintiles, under the null hypothesis of equality. For each of the two sets of market data, two sample sizes,

m_1 and m_2, have been chosen to span the range of frequency counts of patterns reported in Tables I and II. For each sample size m_i, we resample one-day normalized returns (with replacement) to obtain a bootstrap sample of m_i observations, compute the Kolmogorov-Smirnov test statistic (against the entire sample of one-day normalized returns), and repeat this procedure 1,000 times. The percentiles of the asymptotic distribution are also reported for comparison under the column "γ".

Tables IX and X show that for a broad range of sample sizes and across size quintiles, subperiod, and exchanges, the bootstrap distribution of the Kolmogorov-Smirnov statistic is well approximated by its asymptotic distribution (3.20).

3.6 Conclusion

In this paper, we have proposed a new approach to evaluating the efficacy of technical analysis. Based on smoothing techniques such as nonparametric kernel regression, our approach incorporates the essence of technical analysis: to identify regularities in the time series of prices by extracting nonlinear patterns from noisy data. While human judgment is still superior to most computational algorithms in the area of visual pattern recognition, recent advances in statistical learning theory have had successful applications in fingerprint identification, handwriting analysis, and face recognition. Technical analysis may well be the next frontier for such methods.

When applied to many stocks over many time periods, we find that certain technical patterns do provide incremental information, especially for Nasdaq stocks. While this does not necessarily imply that technical analysis can be used to generate "excess" trading profits, it does raise the possibility that technical analysis can add value to the investment process.

Moreover, our methods suggest that technical analysis can be improved by using automated algorithms such as ours, and that traditional patterns such as head-and-shoulders and rectangles, while sometimes effective, need not be optimal. In particular, it may be possible to determine "optimal patterns" for detecting certain types of phenomena in financial time series, e.g., an optimal shape for detecting stochastic volatility or changes in regime. Moreover, patterns that are optimal for detecting statistical anomalies need not be optimal for trading profits, and vice versa. Such considerations may lead to an entirely new branch of technical analysis, one based on selecting pattern recognition algorithms to optimize specific objective functions. We hope to explore these issues more fully in future research.

References

Allen, Franklin and Risto Karjalainen (1999), Using genetic algorithms to find technical trading rules, *Journal of Financial Economics* 51, 245–271.

Beymer, David and Tomaso Poggio (1996), "Image representation for visual learning", *Science* 272, 1905–1909.

Blume, Lawrence, Easley, David and Maureen O'Hara (1994), "Market statistics and technical analysis: The role of volume", *Journal of Finance* 49, 153–181.

Brock, William, Lakonishok, Joseph and Blake LeBaron (1992), "Simple technical trading rules and the stochastic properties of stock returns", *Journal of Finance* 47, 1731–1764.

Brown, David and Robert Jennings (1989), "On technical analysis", *Review of Financial Studies* 2, 527–551.

Campbell, John, Lo, Andrew W. and A. Craig MacKinlay (1997), *The Econometrics of Financial Markets.* Princeton, NJ: Princeton University Press.

Chan, Louis, Jegadeesh, Narasimhan and Joseph Lakonishok (1996), "Momentum strategies", *Journal of Finance* 51, 1681–1713.

Chang, Kevin and Carol Osler (1994), "Evaluating chart-based technical analysis: The head-and-shoulders pattern in foreign exchange markets", working paper, Federal Reserve Bank of New York.

Csáki, E. (1984), "Empirical distribution function", in P. Krishnaiah and P. Sen, eds., *Handbook of Statistics*, Volume 4 (Elsevier Science Publishers, Amsterdam, The Netherlands).

DeGroot, Morris (1986), *Probability and Statistics* (Addison Wesley Publishing Company, Reading, MA).

Edwards, Robert and John Magee (1966), *Technical Analysis of Stock Trends*, 5th Edition (John Magee Inc., Boston, MA).

Grundy, Bruce and S. Martin (1998), "Understanding the nature of the risks and the source of the rewards to momentum investing", unpublished working paper, Wharton School, University of Pennsylvania.

Härdle, Wolfgang (1990), *Applied Nonparametric Regression* (Cambridge University Press, Cambridge, UK).

Hollander, Myles and Douglas Wolfe (1973), *Nonparametric Statistical Methods* (John Wiley & Sons, New York, NY).

Jegadeesh, Narasimhan and Sheridan Titman (1993), "Returns to buying winners and selling losers: Implications for stock market efficiency", *Journal of Finance* 48, 65–91.

Lo, Andrew W. and A. Craig MacKinlay (1988), "Stock market prices do not follow random walks: Evidence from a simple specification test", *Review of Financial Studies* 1, 41–66.

Lo, Andrew W. and A. Craig MacKinlay (1997), "Maximizing predictability in the stock and bond markets", *Macroeconomic Dynamics* 1(1997), 102–134.

Lo, Andrew W. and A. Craig MacKinlay (1999), *A Non-Random Walk Down Wall Street* (Princeton University Press, Princeton, NJ).

Malkiel, Burton (1996), *A Random Walk Down Wall Street: Including a Life-Cycle Guide to Personal Investing* (W.W. Norton, New York, NY).

Neely, Christopher, Weller, Peter and Robert Dittmar (1997), "Is technical analysis in the foreign exchange market profitable? A genetic programming approach", *Journal of Financial and Quantitative Analysis* 32, 405–426.

Neely, Christopher and Peter Weller (1998), "Technical trading rules in the European monetary system", working paper, Federal Bank of St. Louis.

Neftci, Salih (1991), "Naive trading rules in financial markets and Wiener-Kolmogorov prediction theory: A study of technical analysis", *Journal of Business* 64, 549–571.

Neftci, Salih and Andrew Policano (1984), "Can chartists outperform the market? Market efficiency tests for 'technical analyst' ", *Journal of Future Markets* 4, 465–478.

Osler, Carol and Kevin Chang (1995), "Head and shoulders: Not just a flaky pattern", Staff Report No. 4, Federal Reserve Bank of New York.

Poggio, Tomaso and David Beymer (1996), "Regularization networks for visual learning", in Shree Nayar and Tomaso Poggio, eds., *Early Visual Learning* (Oxford University Press, Oxford, UK).

Press, William, Flannery, Brian, Teukolsky, Saul and William Vetterling, 1986, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, Cambridge, UK).

Pruitt, Stephen and Robert White (1988), "The CRISMA trading system: Who says technical analysis can't beat the market?", *Journal of Portfolio Management* 14, 55–58.

Riesenhuber, Maximilian and Tomaso Poggio (1997), "Common computational strategies in machine and biological vision", in *Proceedings of International Symposium on System Life*. Tokyo, Japan, 67–75.

Rouwenhorst, Geert (1998), "International momentum strategies", *Journal of Finance* 53, 267–284.

Simonoff, Jeffrey (1996), *Smoothing Methods in Statistics* (Springer-Verlag, New York, NY).

Smirnov, N. (1939a), "Sur les écarts de la courbe de distribution empirique", *Rec. Math. (Mat. Sborn.)* 6, 3–26.

Smirnov, N. (1939b), "On the estimation of the discrepancy between empirical curves of distribution for two independent samples", *Bulletin. Math. Univ. Moscow* 2, 3–14.

Tabell, Anthony and Edward Tabell (1964), "The case for technical analysis", *Financial Analyst Journal* 20, 67–76.

Treynor, Jack and Robert Ferguson (1985), "In defense of technical analysis", *Journal of Finance* 40, 757–773.

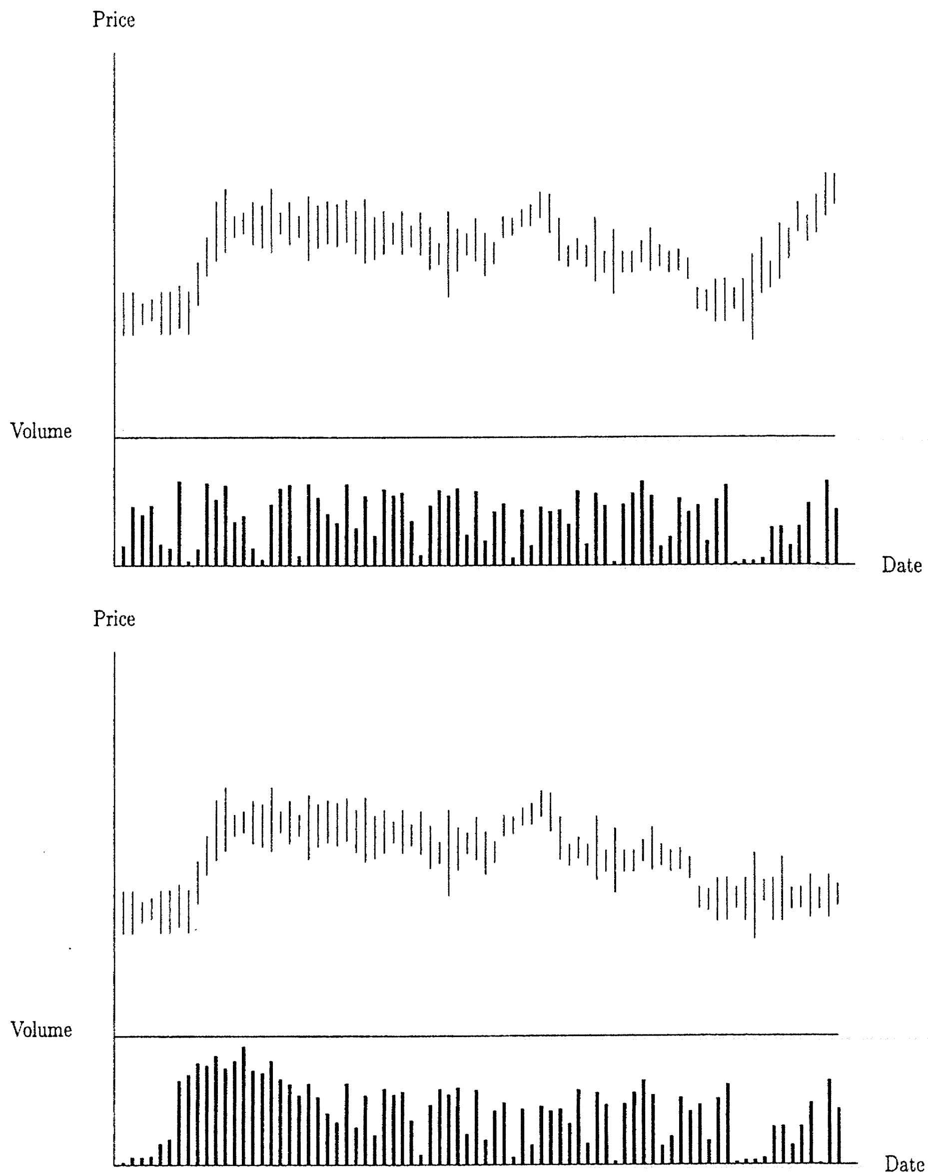
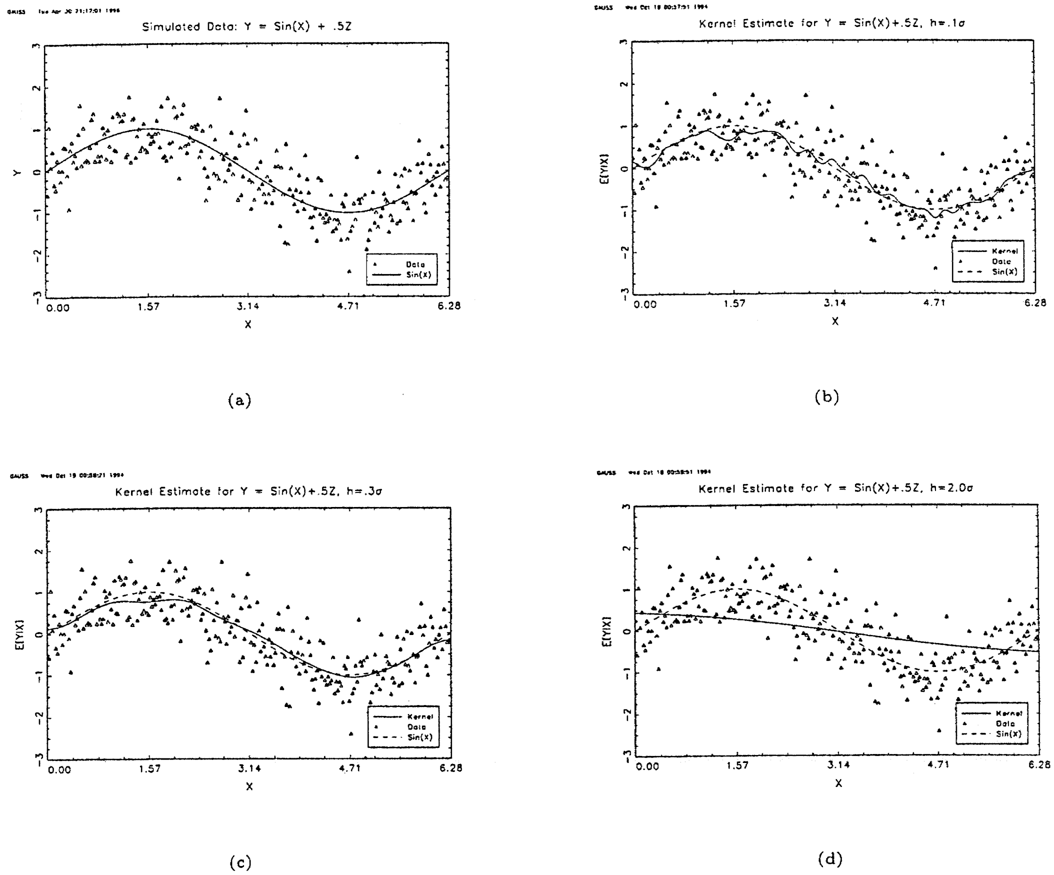


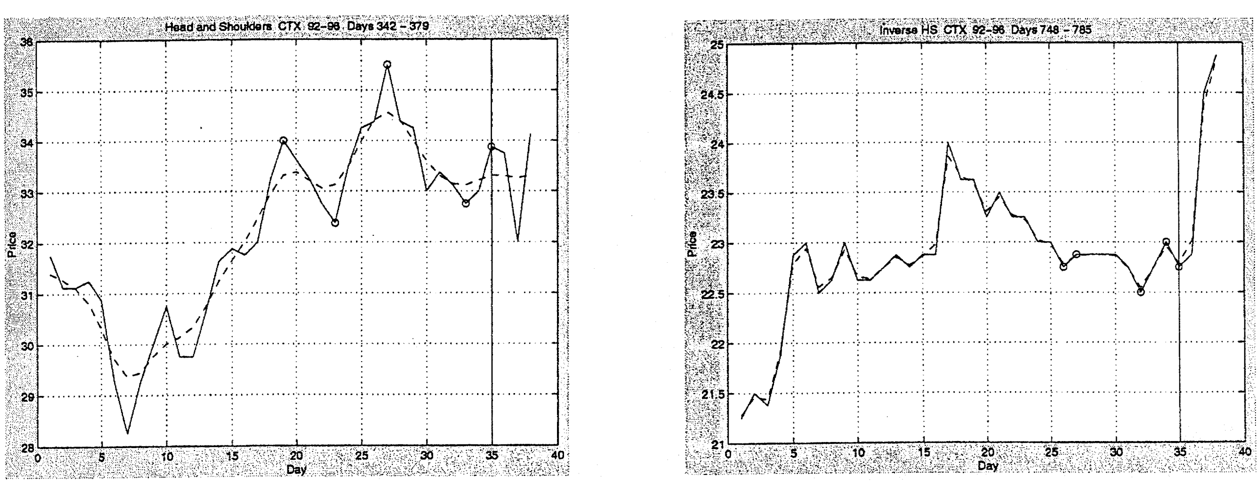
Figure 3.1: Two hypothetical price/volume charts.



(a) (b)

(c) (d)

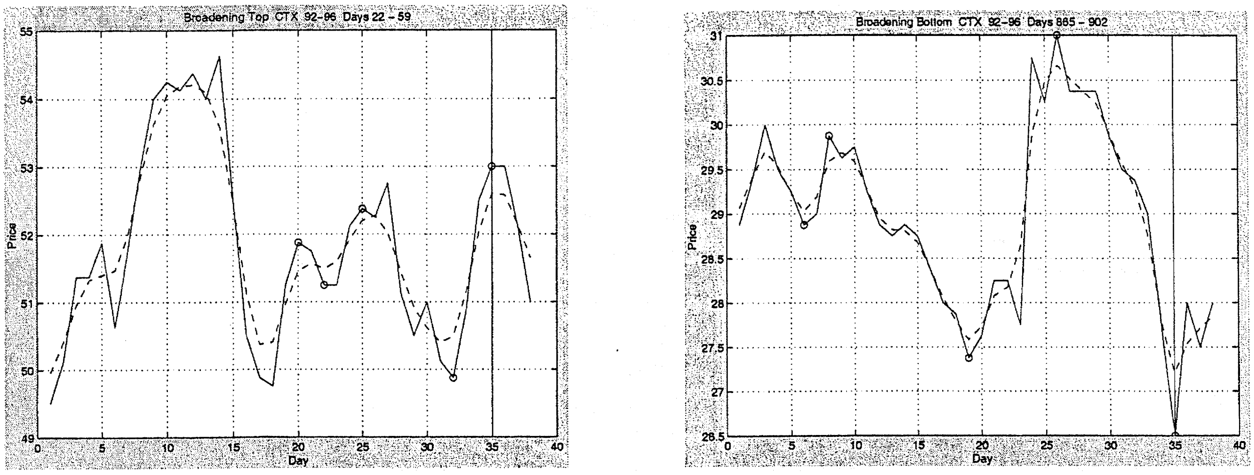
Figure 3.2: Illustration of bandwidth selection for kernel regression.



(a) Head-and-Shoulders

(b) Inverse Head-and-Shoulders

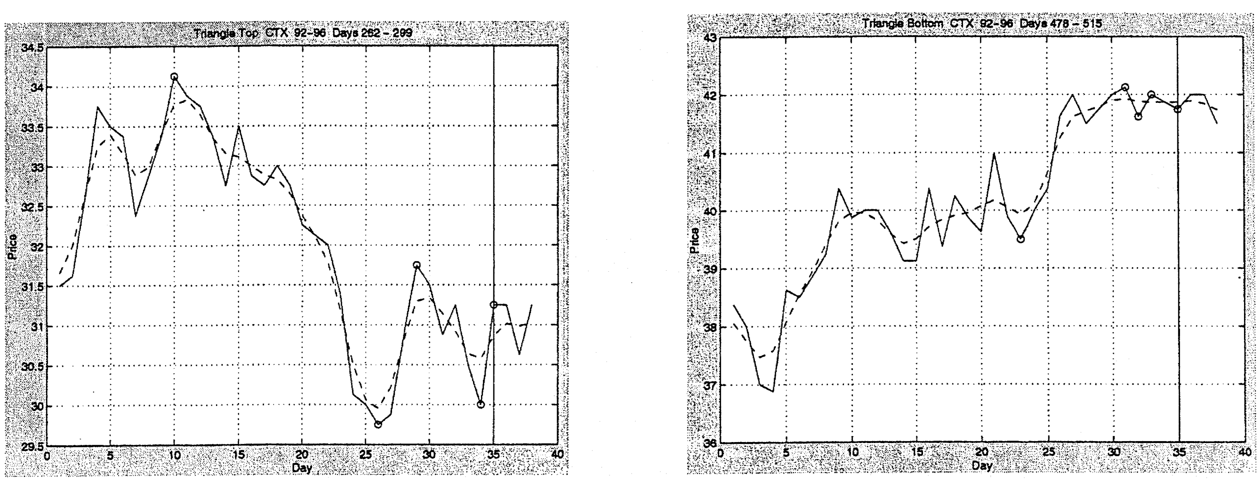
Figure 3.3: Head-and-shoulders and inverse head-and-shoulders.



(a) Broadening Top

(b) Broadening Bottom

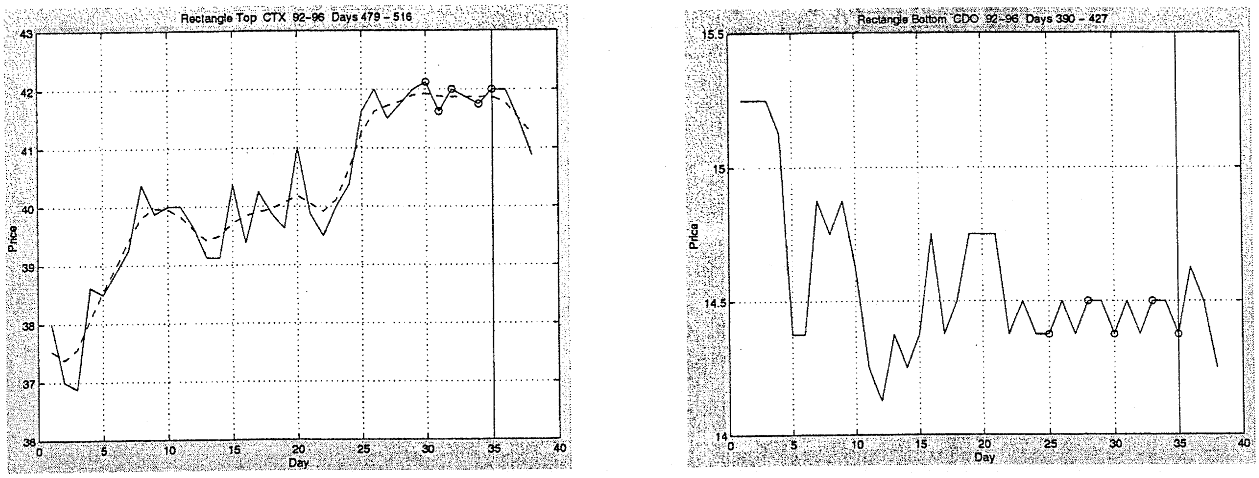
Figure 3.4: Broadening tops and bottoms.



(a) Triangle Top

(b) Triangle Bottom

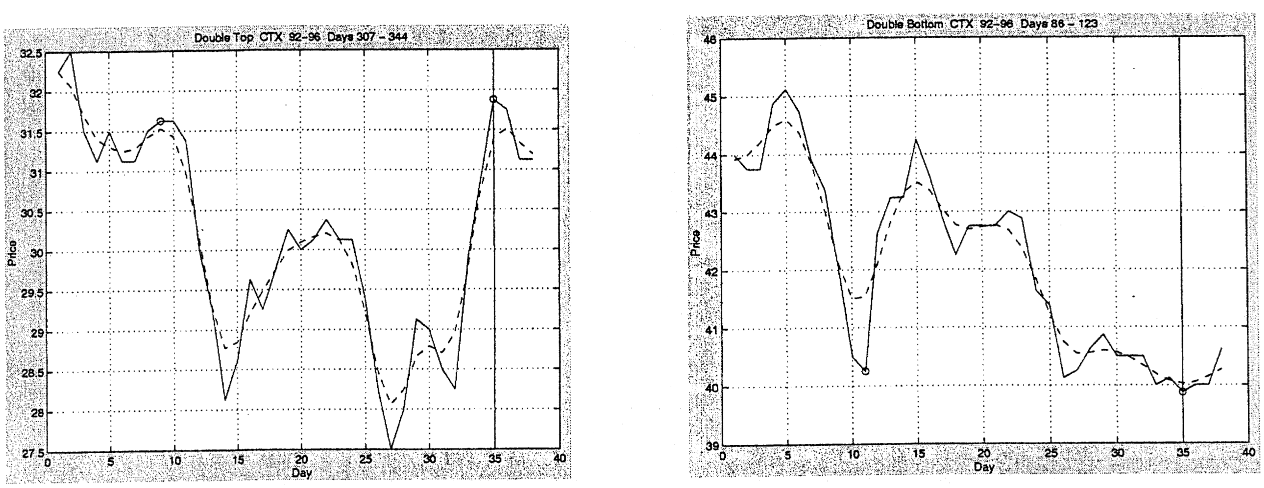
Figure 3.5: Triangle tops and bottoms.



(a) Rectangle Top

(b) Rectangle Bottom

Figure 3.6: Rectangle tops and bottoms.



(a) Double Top

(b) Double Bottom

Figure 3.7: Double tops and bottoms.

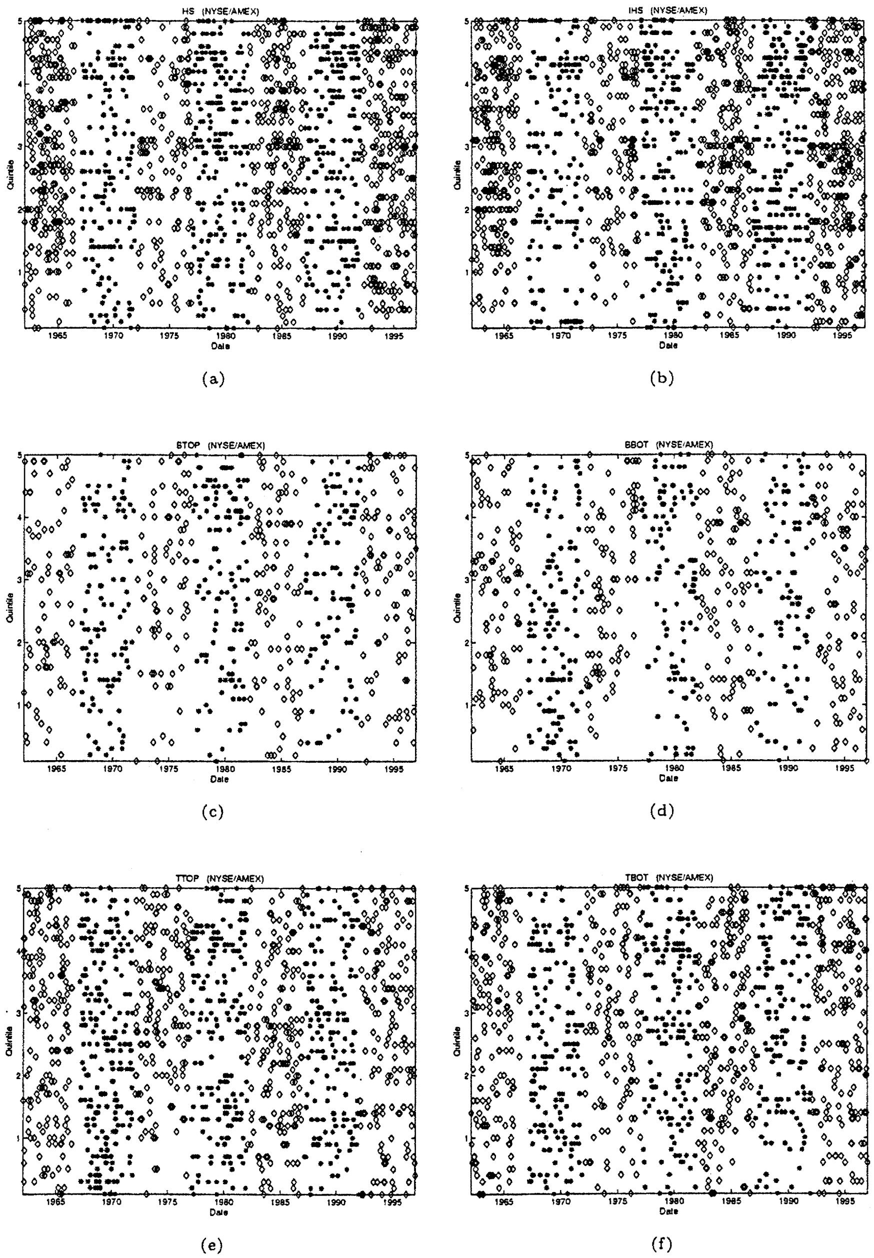


Figure 3.8: Distribution of patterns in NYSE/AMEX sample.

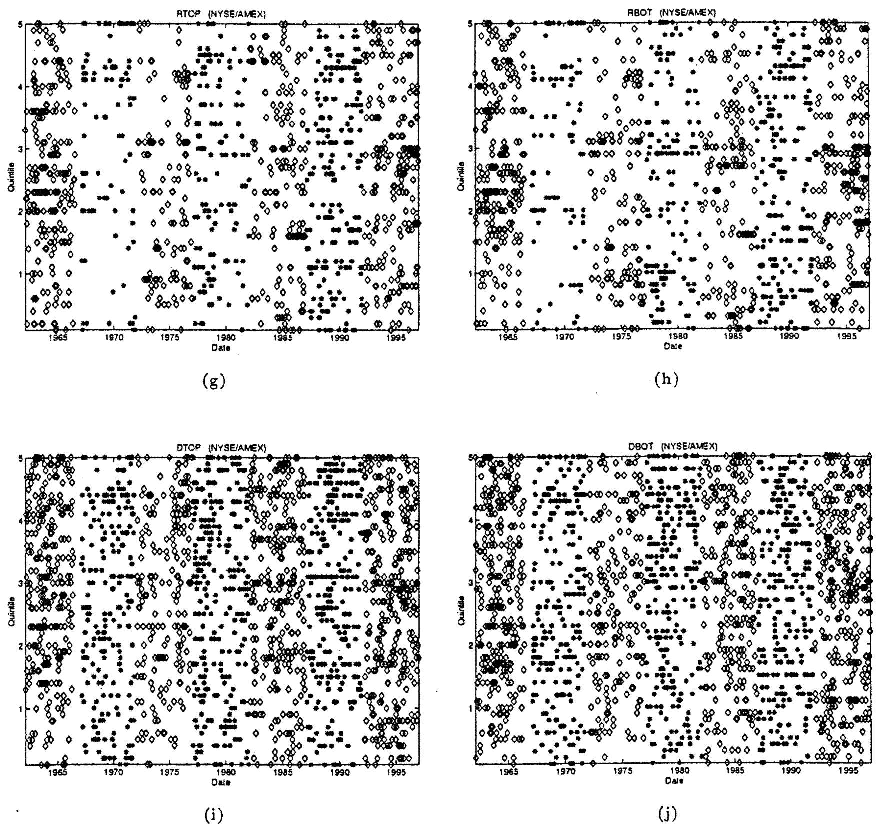
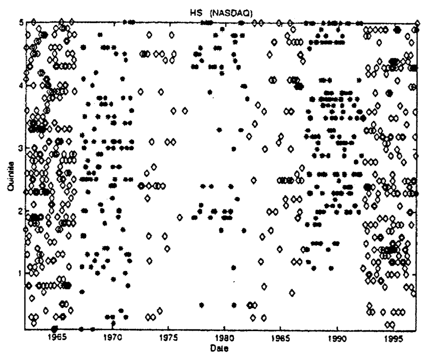
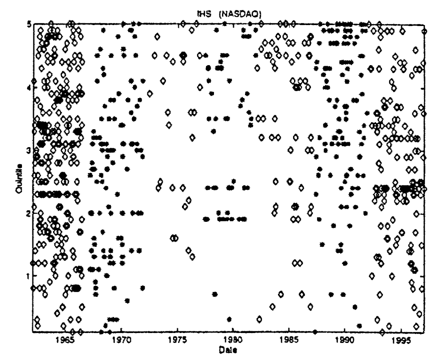


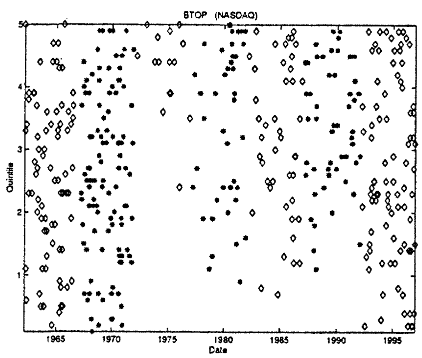
Figure 3.8: (continued).



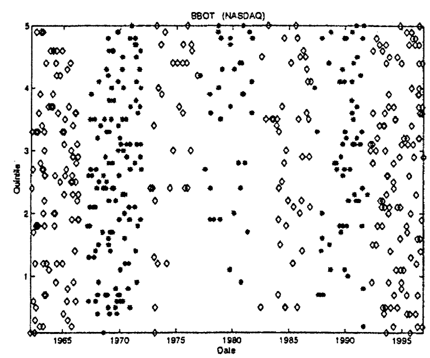
(a)



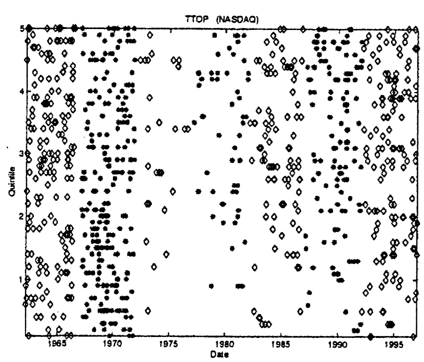
(b)



(c)



(d)



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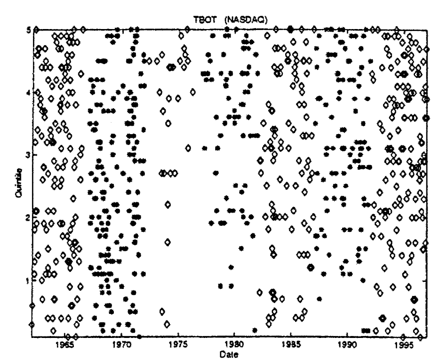

(f)

Figure 3.9: Distribution of patterns in NASDAQ sample.

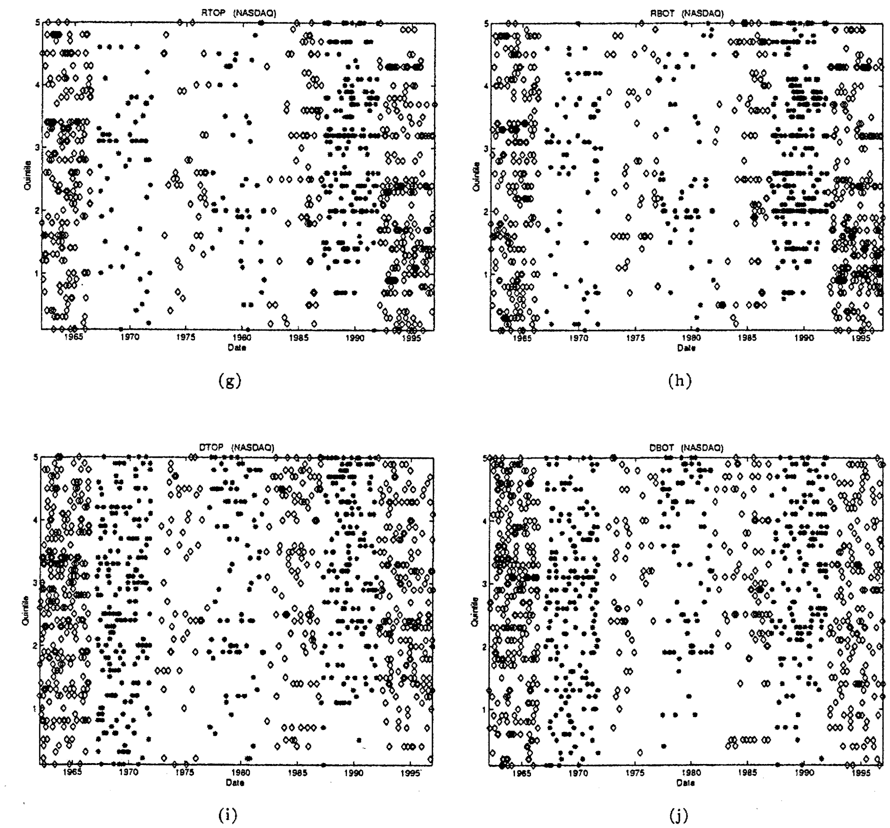

Figure 3.9: (continued).

Table 1a

Frequency counts for 10 technical indicators detected among NYSE/AMEX stocks from 1962 to 1996, in 5-year subperiods, in size quintiles, and in a sample of simulated geometric Brownian motion. In each 5-year subperiod, 10 stocks per quintile are selected at random among stocks with at least 80% non-missing prices, and each stock's price history is scanned for any occurrence of the following 10 technical indicators within the subperiod: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT). The 'Sample' column indicates whether the frequency counts are conditioned on decreasing volume trend ('$\tau(\searrow)$'), increasing volume trend ('$\tau(\nearrow)$'), unconditional ('Entire'), or for a sample of simulated geometric Brownian motion with parameters calibrated to match the data ('Sim. GBM').

Sample	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1996						
Entire	423,556	1611	1654	725	748	1294	1193	1482	1616	2076	2075
Sim. GBM	423,556	577	578	1227	1028	1049	1176	122	113	535	574
$\tau(\searrow)$	—	655	593	143	220	666	710	582	637	691	974
$\tau(\nearrow)$	—	553	614	409	337	300	222	523	552	776	533
					Smallest Quintile, 1962 to 1996						
Entire	84,363	182	181	78	97	203	159	265	320	261	271
Sim. GBM	84,363	82	99	279	256	269	295	18	16	129	127
$\tau(\searrow)$	—	90	81	13	42	122	119	113	131	78	161
$\tau(\nearrow)$	—	58	76	51	37	41	22	99	120	124	64
					2nd Quintile, 1962 to 1996						
Entire	83,986	309	321	146	150	255	228	299	322	372	420
Sim. GBM	83,986	108	105	291	251	261	278	20	17	106	126
$\tau(\searrow)$	—	133	126	25	48	135	147	130	149	113	211
$\tau(\nearrow)$	—	112	126	90	63	55	39	104	110	153	107
					3rd Quintile, 1962 to 1996						
Entire	84,420	361	388	145	161	291	247	334	399	458	443
Sim. GBM	84,420	122	120	268	222	212	249	24	31	115	125
$\tau(\searrow)$	—	152	131	20	49	151	149	130	160	154	215
$\tau(\nearrow)$	—	125	146	83	66	67	44	121	142	179	106
					4th Quintile, 1962 to 1996						
Entire	84,780	332	317	176	173	262	255	259	264	424	420
Sim. GBM	84,780	143	127	249	210	183	210	35	24	116	122
$\tau(\searrow)$	—	131	115	36	42	138	145	85	97	144	184
$\tau(\nearrow)$	—	110	126	103	89	56	55	102	96	147	118
					Largest Quintile, 1962 to 1996						
Entire	86,007	427	447	180	167	283	304	325	311	561	521
Sim. GBM	86,007	122	127	140	89	124	144	25	25	69	74
$\tau(\searrow)$	—	149	140	49	39	120	150	124	100	202	203
$\tau(\nearrow)$	—	148	140	82	82	81	62	97	84	173	138

Table 1a (continued)

Sample	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
All Stocks, 1962 to 1966											
Entire	55,254	276	278	85	103	179	165	316	354	356	352
Sim. GBM	55,254	56	58	144	126	129	139	9	16	60	68
$\tau(\searrow)$	—	104	88	26	29	93	109	130	141	113	188
$\tau(\nearrow)$	—	96	112	44	39	37	25	130	122	137	88
All Stocks, 1967 to 1971											
Entire	60,299	179	175	112	134	227	172	115	117	239	258
Sim. GBM	60,299	92	70	167	148	150	180	19	16	84	77
$\tau(\searrow)$	—	68	64	16	45	126	111	42	39	80	143
$\tau(\nearrow)$	—	71	69	68	57	47	29	41	41	87	53
All Stocks, 1972 to 1976											
Entire	59,915	152	162	82	93	165	136	171	182	218	223
Sim. GBM	59,915	75	85	183	154	156	178	16	10	70	71
$\tau(\searrow)$	—	64	55	16	23	88	78	60	64	53	97
$\tau(\nearrow)$	—	54	62	42	50	32	21	61	67	80	59
All Stocks, 1977 to 1981											
Entire	62,133	223	206	134	110	188	167	146	182	274	290
Sim. GBM	62,133	83	88	245	200	188	210	18	12	90	115
$\tau(\searrow)$	—	114	61	24	39	100	97	54	60	82	140
$\tau(\nearrow)$	—	56	93	78	44	35	36	53	71	113	76
All Stocks, 1982 to 1986											
Entire	61,984	242	256	106	108	182	190	182	207	313	299
Sim. GBM	61,984	115	120	188	144	152	169	31	23	99	87
$\tau(\searrow)$	—	101	104	28	30	93	104	70	95	109	124
$\tau(\nearrow)$	—	89	94	51	62	46	40	73	68	116	85
All Stocks, 1987 to 1991											
Entire	61,780	240	241	104	98	180	169	260	259	287	285
Sim. GBM	61,780	68	79	168	132	131	150	11	10	76	68
$\tau(\searrow)$	—	95	89	16	30	86	101	103	102	105	137
$\tau(\nearrow)$	—	81	79	68	43	53	36	73	87	100	68
All Stocks, 1992 to 1996											
Entire	62,191	299	336	102	102	173	194	292	315	389	368
Sim. GBM	62,191	88	78	132	124	143	150	18	26	56	88
$\tau(\searrow)$	—	109	132	17	24	80	110	123	136	149	145
$\tau(\nearrow)$	—	106	105	58	42	50	35	92	96	143	104

Table 1b

Frequency counts for 10 technical indicators detected among NASDAQ stocks from 1962 to 1996, in 5-year subperiods, in size quintiles, and in a sample of simulated geometric Brownian motion. In each 5-year subperiod, 10 stocks per quintile are selected at random among stocks with at least 80% non-missing prices, and each stock's price history is scanned for any occurrence of the following 10 technical indicators within the subperiod: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT). The 'Sample' column indicates whether the frequency counts are conditioned on decreasing volume trend ('$\tau(\searrow)$'), increasing volume trend ('$\tau(\nearrow)$'), unconditional ('Entire'), or for a sample of simulated geometric Brownian motion with parameters calibrated to match the data ('Sim. GBM').

Sample	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1996						
Entire	411,010	919	817	414	508	850	789	1134	1320	1208	1147
Sim. GBM	411,010	434	447	1297	1139	1169	1309	96	91	567	579
$\tau(\searrow)$	—	408	268	69	133	429	460	488	550	339	580
$\tau(\nearrow)$	—	284	325	234	209	185	125	391	461	474	229
					Smallest Quintile, 1962 to 1996						
Entire	81,754	84	64	41	73	111	93	165	218	113	125
Sim. GBM	81,754	85	84	341	289	334	367	11	12	140	125
$\tau(\searrow)$	—	36	25	6	20	56	59	77	102	31	81
$\tau(\nearrow)$	—	31	23	31	30	24	15	59	85	46	17
					2nd Quintile, 1962 to 1996						
Entire	81,336	191	138	68	88	161	148	242	305	219	176
Sim. GBM	81,336	67	84	243	225	219	229	24	12	99	124
$\tau(\searrow)$	—	94	51	11	28	86	109	111	131	69	101
$\tau(\nearrow)$	—	66	57	46	38	45	22	85	120	90	42
					3rd Quintile, 1962 to 1996						
Entire	81,772	224	186	105	121	183	155	235	244	279	267
Sim. GBM	81,772	69	86	227	210	214	239	15	14	105	100
$\tau(\searrow)$	—	108	66	23	35	87	91	90	84	78	145
$\tau(\nearrow)$	—	71	79	56	49	39	29	84	86	122	58
					4th Quintile, 1962 to 1996						
Entire	82,727	212	214	92	116	187	179	296	303	289	297
Sim. GBM	82,727	104	92	242	219	209	255	23	26	115	97
$\tau(\searrow)$	—	88	68	12	26	101	101	127	141	77	143
$\tau(\nearrow)$	—	62	83	57	56	34	22	104	93	118	66
					Largest Quintile, 1962 to 1996						
Entire	83,421	208	215	108	110	208	214	196	250	308	282
Sim. GBM	83,421	109	101	244	196	193	219	23	27	108	133
$\tau(\searrow)$	—	82	58	17	24	99	100	83	92	84	110
$\tau(\nearrow)$	—	54	83	44	36	43	37	59	77	98	46

Table 1b (continued)

Sample	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
All Stocks, 1962 to 1966											
Entire	55,969	274	268	72	99	182	144	288	329	326	342
Sim. GBM	55,969	69	63	163	123	137	149	24	22	77	90
$\tau(\searrow)$	—	129	99	10	23	104	98	115	136	96	210
$\tau(\nearrow)$	—	83	103	48	51	37	23	101	116	144	64
All Stocks, 1967 to 1971											
Entire	60,563	115	120	104	123	227	171	65	83	196	200
Sim. GBM	60,563	58	61	194	184	181	188	9	8	90	83
$\tau(\searrow)$	—	61	29	15	40	127	123	26	39	49	137
$\tau(\nearrow)$	—	24	57	71	51	45	19	25	16	86	17
All Stocks, 1972 to 1976											
Entire	51,446	34	30	14	30	29	28	51	55	55	58
Sim. GBM	51,446	32	37	115	113	107	110	5	6	46	46
$\tau(\searrow)$	—	5	4	0	4	5	7	12	8	3	8
$\tau(\nearrow)$	—	8	7	1	2	2	0	5	12	8	3
All Stocks, 1977 to 1981											
Entire	61,972	56	53	41	36	52	73	57	65	89	96
Sim. GBM	61,972	90	84	236	165	176	212	19	19	110	98
$\tau(\searrow)$	—	7	7	1	2	4	8	12	12	7	9
$\tau(\nearrow)$	—	6	6	5	1	4	0	5	8	7	6
All Stocks, 1982 to 1986											
Entire	61,110	71	64	46	44	97	107	109	115	120	97
Sim. GBM	61,110	86	90	162	168	147	174	23	21	97	98
$\tau(\searrow)$	—	37	19	8	14	46	58	45	52	40	48
$\tau(\nearrow)$	—	21	25	24	18	26	22	42	42	38	24
All Stocks, 1987 to 1991											
Entire	60,862	158	120	50	61	120	109	265	312	177	155
Sim. GBM	60,862	59	57	229	187	205	244	7	7	79	88
$\tau(\searrow)$	—	79	46	11	19	73	69	130	140	50	69
$\tau(\nearrow)$	—	58	56	33	30	26	28	100	122	89	55
All Stocks, 1992 to 1996											
Entire	59,088	211	162	87	115	143	157	299	361	245	199
Sim. GBM	59,088	40	55	198	199	216	232	9	8	68	76
$\tau(\searrow)$	—	90	64	24	31	70	97	148	163	94	99
$\tau(\nearrow)$	—	84	71	52	56	45	33	113	145	102	60

Table 2a

Summary statistics (mean, standard deviation, skewness, and excess kurtosis) of raw and conditional 1-day normalized returns of NYSE/AMEX stocks from 1962 to 1996, in 5-year subperiods, and in size quintiles. Conditional returns are defined as the daily return three days following the conclusion of an occurrence of one of 10 technical indicators: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT). All returns have been normalized by subtraction of their means and division by their standard deviations.

Moment	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1996						
Mean	−0.000	−0.038	0.040	−0.005	−0.062	0.021	−0.009	0.009	0.014	0.017	−0.001
S.D.	1.000	0.867	0.937	1.035	0.979	0.955	0.959	0.865	0.883	0.910	0.999
Skew	0.345	0.135	0.660	−1.151	0.090	0.137	0.643	−0.420	0.110	0.206	0.460
Kurt	8.122	2.428	4.527	16.701	3.169	3.293	7.061	7.360	4.194	3.386	7.374
					Smallest Quintile, 1962 to 1996						
Mean	−0.000	−0.014	0.036	−0.093	−0.188	0.036	−0.020	0.037	−0.093	0.043	−0.055
S.D.	1.000	0.854	1.002	0.940	0.850	0.937	1.157	0.833	0.986	0.950	0.962
Skew	0.697	0.802	1.337	−1.771	−0.367	0.861	2.592	−0.187	0.445	0.511	0.002
Kurt	10.873	3.870	7.143	6.701	0.575	4.185	12.532	1.793	4.384	2.581	3.989
					2nd Quintile, 1962 to 1996						
Mean	−0.000	−0.069	0.144	0.061	−0.113	0.003	0.035	0.018	0.019	0.067	−0.011
S.D.	1.000	0.772	1.031	1.278	1.004	0.913	0.965	0.979	0.868	0.776	1.069
Skew	0.392	0.223	1.128	−3.296	0.485	−0.529	0.166	−1.375	0.452	0.392	1.728
Kurt	7.836	0.657	6.734	32.750	3.779	3.024	4.987	17.040	3.914	2.151	15.544
					3rd Quintile, 1962 to 1996						
Mean	−0.000	−0.048	−0.043	−0.076	−0.056	0.036	0.012	0.075	0.028	−0.039	−0.034
S.D.	1.000	0.888	0.856	0.894	0.925	0.973	0.796	0.798	0.892	0.956	1.026
Skew	0.246	−0.465	0.107	−0.023	0.233	0.538	0.166	0.678	−0.618	0.013	−0.242
Kurt	7.466	3.239	1.612	1.024	0.611	2.995	0.586	3.010	4.769	4.517	3.663
					4th Quintile, 1962 to 1996						
Mean	−0.000	−0.012	0.022	0.115	0.028	0.022	−0.014	−0.113	0.065	0.015	−0.006
S.D.	1.000	0.964	0.903	0.990	1.093	0.986	0.959	0.854	0.821	0.858	0.992
Skew	0.222	0.055	0.592	0.458	0.537	−0.217	−0.456	−0.415	0.820	0.550	−0.062
Kurt	6.452	1.444	1.745	1.251	2.168	4.237	8.324	4.311	3.632	1.719	4.691
					Largest Quintile, 1962 to 1996						
Mean	−0.000	−0.038	0.054	−0.081	−0.042	0.010	−0.049	0.009	0.060	0.018	0.067
S.D.	1.000	0.843	0.927	0.997	0.951	0.964	0.965	0.850	0.820	0.971	0.941
Skew	0.174	0.438	0.182	0.470	−1.099	0.089	0.357	−0.167	−0.140	0.011	0.511
Kurt	7.992	2.621	3.465	3.275	6.603	2.107	2.509	0.816	3.179	3.498	5.035

Table 2a (continued)

Moment	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1966						
Mean	−0.000	0.070	0.090	0.159	0.079	−0.033	−0.039	−0.041	0.019	−0.071	−0.100
S.D.	1.000	0.797	0.925	0.825	1.085	1.068	1.011	0.961	0.814	0.859	0.962
Skew	0.563	0.159	0.462	0.363	1.151	−0.158	1.264	−1.337	−0.341	−0.427	−0.876
Kurt	9.161	0.612	1.728	0.657	5.063	2.674	4.826	17.161	1.400	3.416	5.622
					All Stocks, 1967 to 1971						
Mean	−0.000	−0.044	0.079	−0.035	−0.056	0.025	0.057	−0.101	0.110	0.093	0.079
S.D.	1.000	0.809	0.944	0.793	0.850	0.885	0.886	0.831	0.863	1.083	0.835
Skew	0.342	0.754	0.666	0.304	0.085	0.650	0.697	−1.393	0.395	1.360	0.701
Kurt	5.810	3.684	2.725	0.706	0.141	3.099	1.659	8.596	3.254	4.487	1.853
					All Stocks, 1972 to 1976						
Mean	−0.000	−0.035	0.043	0.101	−0.138	−0.045	−0.010	−0.025	−0.003	−0.051	−0.108
S.D.	1.000	1.015	0.810	0.985	0.918	0.945	0.922	0.870	0.754	0.914	0.903
Skew	0.316	−0.334	0.717	−0.699	0.272	−1.014	0.676	0.234	0.199	0.056	−0.366
Kurt	6.520	2.286	1.565	6.562	1.453	5.261	4.912	3.627	2.337	3.520	5.047
					All Stocks, 1977 to 1981						
Mean	−0.000	−0.138	−0.040	0.076	−0.114	0.135	−0.050	−0.004	0.026	0.042	0.178
S.D.	1.000	0.786	0.863	1.015	0.989	1.041	1.011	0.755	0.956	0.827	1.095
Skew	0.466	−0.304	0.052	1.599	−0.033	0.776	0.110	−0.084	0.534	0.761	2.214
Kurt	6.419	1.132	1.048	4.961	−0.125	2.964	0.989	1.870	2.184	2.369	15.290
					All Stocks, 1982 to 1986						
Mean	−0.000	−0.099	−0.007	0.011	0.095	−0.114	−0.067	0.050	0.005	0.011	−0.013
S.D.	1.000	0.883	1.002	1.109	0.956	0.924	0.801	0.826	0.934	0.850	1.026
Skew	0.460	0.464	0.441	0.372	−0.165	0.473	−1.249	0.231	0.467	0.528	0.867
Kurt	6.799	2.280	6.128	2.566	2.735	3.208	5.278	1.108	4.234	1.515	7.400
					All Stocks, 1987 to 1991						
Mean	−0.000	−0.037	0.033	−0.091	−0.040	0.053	0.003	0.040	−0.020	−0.022	−0.017
S.D.	1.000	0.848	0.895	0.955	0.818	0.857	0.981	0.894	0.833	0.873	1.052
Skew	−0.018	−0.526	0.272	0.108	0.231	0.165	−1.216	0.293	0.124	−1.184	−0.368
Kurt	13.478	3.835	4.395	2.247	1.469	4.422	9.586	1.646	3.973	4.808	4.297
					All Stocks, 1992 to 1996						
Mean	−0.000	−0.014	0.069	−0.231	−0.272	0.122	0.041	0.082	0.011	0.102	−0.016
S.D.	1.000	0.935	1.021	1.406	1.187	0.953	1.078	0.814	0.996	0.960	1.035
Skew	0.308	0.545	1.305	−3.988	−0.502	−0.190	2.460	−0.167	−0.129	−0.091	0.379
Kurt	8.683	2.249	6.684	27.022	3.947	1.235	12.883	0.506	6.399	1.507	3.358

Table 2b

Summary statistics (mean, standard deviation, skewness, and excess kurtosis) of raw and conditional 1-day normalized returns of NASDAQ stocks from 1962 to 1996, in 5-year subperiods, and in size quintiles. Conditional returns are defined as the daily return three days following the conclusion of an occurrence of one of 10 technical indicators: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT). All returns have been normalized by subtraction of their means and division by their standard deviations.

Moment	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1996						
Mean	0.000	−0.016	0.042	−0.009	0.009	−0.020	0.017	0.052	0.043	0.003	−0.035
S.D.	1.000	0.907	0.994	0.960	0.995	0.984	0.932	0.948	0.929	0.933	0.880
Skew	0.608	−0.017	1.290	0.397	0.586	0.895	0.716	0.710	0.755	0.405	−0.104
Kurt	12.728	3.039	8.774	3.246	2.783	6.692	3.844	5.173	4.368	4.150	2.052
					Smallest Quintile, 1962 to 1996						
Mean	−0.000	0.018	−0.032	0.087	−0.153	0.059	0.108	0.136	0.013	0.040	0.043
S.D.	1.000	0.845	1.319	0.874	0.894	1.113	1.044	1.187	0.982	0.773	0.906
Skew	0.754	0.325	1.756	−0.239	−0.109	2.727	2.300	1.741	0.199	0.126	−0.368
Kurt	15.859	1.096	4.221	1.490	0.571	14.270	10.594	8.670	1.918	0.127	0.730
					2nd Quintile, 1962 to 1996						
Mean	−0.000	−0.064	0.076	−0.109	−0.093	−0.085	−0.038	−0.066	−0.015	0.039	−0.034
S.D.	1.000	0.848	0.991	1.106	1.026	0.805	0.997	0.898	0.897	1.119	0.821
Skew	0.844	0.406	1.892	−0.122	0.635	0.036	0.455	−0.579	0.416	1.196	0.190
Kurt	16.738	2.127	11.561	2.496	3.458	0.689	1.332	2.699	3.871	3.910	0.777
					3rd Quintile, 1962 to 1996						
Mean	−0.000	0.033	0.028	0.078	0.210	−0.030	0.068	0.117	0.210	−0.109	−0.075
S.D.	1.000	0.933	0.906	0.931	0.971	0.825	1.002	0.992	0.970	0.997	0.973
Skew	0.698	0.223	0.529	0.656	0.326	0.539	0.442	0.885	0.820	−0.163	0.123
Kurt	12.161	1.520	1.526	1.003	0.430	1.673	1.038	2.908	4.915	5.266	2.573
					4th Quintile, 1962 to 1996						
Mean	0.000	−0.079	0.037	−0.006	−0.044	−0.080	0.007	0.084	0.044	0.038	−0.048
S.D.	1.000	0.911	0.957	0.992	0.975	1.076	0.824	0.890	0.851	0.857	0.819
Skew	0.655	−0.456	2.671	−0.174	0.385	0.554	0.717	0.290	1.034	0.154	−0.149
Kurt	11.043	2.525	19.593	2.163	1.601	7.723	3.930	1.555	2.982	2.807	2.139
					Largest Quintile, 1962 to 1996						
Mean	0.000	0.026	0.058	−0.070	0.031	0.052	−0.013	0.001	−0.024	0.032	−0.018
S.D.	1.000	0.952	1.002	0.895	1.060	1.076	0.871	0.794	0.958	0.844	0.877
Skew	0.100	−0.266	−0.144	1.699	1.225	0.409	0.025	0.105	1.300	0.315	−0.363
Kurt	7.976	5.807	4.367	8.371	5.778	1.970	2.696	1.336	7.503	2.091	2.241

Table 2b (continued)

Moment	Raw	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1966						
Mean	−0.000	0.116	0.041	0.099	0.090	0.028	−0.066	0.100	0.010	0.096	0.027
S.D.	1.000	0.912	0.949	0.989	1.039	1.015	0.839	0.925	0.873	1.039	0.840
Skew	0.575	0.711	1.794	0.252	1.258	1.601	0.247	2.016	1.021	0.533	−0.351
Kurt	6.555	1.538	9.115	2.560	6.445	7.974	1.324	13.653	5.603	6.277	2.243
					All Stocks, 1967 to 1971						
Mean	−0.000	−0.127	0.114	0.121	0.016	0.045	0.077	0.154	0.136	−0.000	0.006
S.D.	1.000	0.864	0.805	0.995	1.013	0.976	0.955	1.016	1.118	0.882	0.930
Skew	0.734	−0.097	1.080	0.574	0.843	1.607	0.545	0.810	1.925	0.465	0.431
Kurt	5.194	1.060	2.509	0.380	2.928	10.129	1.908	1.712	5.815	1.585	2.476
					All Stocks, 1972 to 1976						
Mean	0.000	0.014	0.089	−0.403	−0.034	−0.132	−0.422	−0.076	0.108	−0.004	−0.163
S.D.	1.000	0.575	0.908	0.569	0.803	0.618	0.830	0.886	0.910	0.924	0.564
Skew	0.466	−0.281	0.973	−1.176	0.046	−0.064	−1.503	−2.728	2.047	−0.551	−0.791
Kurt	17.228	2.194	1.828	0.077	0.587	−0.444	2.137	13.320	9.510	1.434	2.010
					All Stocks, 1977 to 1981						
Mean	−0.000	0.025	−0.212	−0.112	−0.056	−0.110	0.086	0.055	0.177	0.081	0.040
S.D.	1.000	0.769	1.025	1.091	0.838	0.683	0.834	1.036	1.047	0.986	0.880
Skew	1.092	0.230	−1.516	−0.731	0.368	0.430	0.249	2.391	2.571	1.520	−0.291
Kurt	20.043	1.618	4.397	3.766	0.460	0.962	4.722	9.137	10.961	7.127	3.682
					All Stocks, 1982 to 1986						
Mean	0.000	−0.147	0.204	−0.137	−0.001	−0.053	−0.022	−0.028	0.116	−0.224	−0.052
S.D.	1.000	1.073	1.442	0.804	1.040	0.982	1.158	0.910	0.830	0.868	1.082
Skew	1.267	−1.400	2.192	0.001	0.048	1.370	1.690	−0.120	0.048	0.001	−0.091
Kurt	21.789	4.899	10.530	0.863	0.732	8.460	7.086	0.780	0.444	1.174	0.818
					All Stocks, 1987 to 1991						
Mean	0.000	0.012	0.120	−0.080	−0.031	−0.052	0.038	0.098	0.049	−0.048	−0.122
S.D.	1.000	0.907	1.136	0.925	0.826	1.007	0.878	0.936	1.000	0.772	0.860
Skew	0.104	−0.326	0.976	−0.342	0.234	−0.248	1.002	0.233	0.023	−0.105	−0.375
Kurt	12.688	3.922	5.183	1.839	0.734	2.796	2.768	1.038	2.350	0.313	2.598
					All Stocks, 1992 to 1996						
Mean	0.000	−0.119	−0.058	−0.033	−0.013	−0.078	0.086	−0.006	−0.011	0.003	−0.105
S.D.	1.000	0.926	0.854	0.964	1.106	1.093	0.901	0.973	0.879	0.932	0.875
Skew	−0.036	0.079	−0.015	1.399	0.158	−0.127	0.150	0.283	0.236	0.039	−0.097
Kurt	5.377	2.818	−0.059	7.584	0.626	2.019	1.040	1.266	1.445	1.583	0.205

Table 3a

Goodness-of-fit diagnostics for the conditional 1-day normalized returns, conditional on 10 technical indicators, for a sample of 350 NYSE/AMEX stocks from 1962 to 1996 (10 stocks per size-quintile with at least 80% non-missing prices are randomly chosen in each five-year subperiod, yielding 50 stocks per subperiod over 7 subperiods). For each pattern, the percentage of conditional returns that fall within each of the 10 unconditional-return deciles is tabulated. If conditioning on the pattern provides no information, the expected percentage falling in each decile is 10%. Asymptotic t-statistics for this null hypothesis are reported in parentheses, and the χ^2 goodness-of-fitness test statistic Q is reported in the last column with the p-value in parentheses below the statistic. The 10 technical indicators are: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT).

Pattern	Decile:										Q
	1	2	3	4	5	6	7	8	9	10	(p-Value)
HS	8.9	10.4	11.2	11.7	12.2	7.9	9.2	10.4	10.8	7.1	39.31
	(−1.49)	(0.56)	(1.49)	(2.16)	(2.73)	(−3.05)	(−1.04)	(0.48)	(1.04)	(−4.46)	(0.000)
IHS	8.6	9.7	9.4	11.2	13.7	7.7	9.1	11.1	9.6	10.0	40.95
	(−2.05)	(−0.36)	(−0.88)	(1.60)	(4.34)	(−3.44)	(−1.32)	(1.38)	(−0.62)	(−0.03)	(0.000)
BTOP	9.4	10.6	10.6	11.9	8.7	6.6	9.2	13.7	9.2	10.1	23.40
	(−0.57)	(0.54)	(0.54)	(1.55)	(−1.25)	(−3.66)	(−0.71)	(2.87)	(−0.71)	(0.06)	(0.005)
BBOT	11.5	9.9	13.0	11.1	7.8	9.2	8.3	9.0	10.7	9.6	16.87
	(1.28)	(−0.10)	(2.42)	(0.95)	(−2.30)	(−0.73)	(−1.70)	(−1.00)	(0.62)	(−0.35)	(0.051)
TTOP	7.8	10.4	10.9	11.3	9.0	9.9	10.0	10.7	10.5	9.7	12.03
	(−2.94)	(0.42)	(1.03)	(1.46)	(−1.30)	(−0.13)	(−0.04)	(0.77)	(0.60)	(−0.41)	(0.212)
TBOT	8.9	10.6	10.9	12.2	9.2	8.7	9.3	11.6	8.7	9.8	17.12
	(−1.35)	(0.72)	(0.99)	(2.36)	(−0.93)	(−1.57)	(−0.83)	(1.69)	(−1.57)	(−0.22)	(0.047)
RTOP	8.4	9.9	9.2	10.5	12.5	10.1	10.0	10.0	11.4	8.1	22.72
	(−2.27)	(−0.10)	(−1.10)	(0.58)	(2.89)	(0.16)	(−0.02)	(−0.02)	(1.70)	(−2.69)	(0.007)
RBOT	8.6	9.6	7.8	10.5	12.9	10.8	11.6	9.3	10.3	8.7	33.94
	(−2.01)	(−0.56)	(−3.30)	(0.60)	(3.45)	(1.07)	(1.98)	(−0.99)	(0.44)	(−1.91)	(0.000)
DTOP	8.2	10.9	9.6	12.4	11.8	7.5	8.2	11.3	10.3	9.7	50.97
	(−2.92)	(1.36)	(−0.64)	(3.29)	(2.61)	(−4.39)	(−2.92)	(1.83)	(0.46)	(−0.41)	(0.000)
DBOT	9.7	9.9	10.0	10.9	11.4	8.5	9.2	10.0	10.7	9.8	12.92
	(−0.48)	(−0.18)	(−0.04)	(1.37)	(1.97)	(−2.40)	(−1.33)	(0.04)	(0.96)	(−0.33)	(0.166)

Table 3b

Goodness-of-fit diagnostics for the conditional 1-day normalized returns, conditional on 10 technical indicators, for a sample of 350 NASDAQ stocks from 1962 to 1996 (10 stocks per size-quintile with at least 80% non-missing prices are randomly chosen in each five-year subperiod, yielding 50 stocks per subperiod over 7 subperiods). For each pattern, the percentage of conditional returns that fall within each of the 10 unconditional-return deciles is tabulated. If conditioning on the pattern provides no information, the expected percentage falling in each decile is 10%. Asymptotic t-statistics for this null hypothesis are reported in parentheses, and the χ^2 goodness-of-fitness test statistic Q is reported in the last column with the p-value in parentheses below the statistic. The 10 technical indicators are: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT).

Pattern	Decile: 1	2	3	4	5	6	7	8	9	10	Q (p-Value)
HS	10.8	10.8	13.7	8.6	8.5	6.0	6.0	12.5	13.5	9.7	64.41
	(0.76)	(0.76)	(3.27)	(−1.52)	(−1.65)	(−5.13)	(−5.13)	(2.30)	(3.10)	(−0.32)	(0.000)
IHS	9.4	14.1	12.5	8.0	7.7	4.8	6.4	13.5	12.5	11.3	75.84
	(−0.56)	(3.35)	(2.15)	(−2.16)	(−2.45)	(−7.01)	(−4.26)	(2.90)	(2.15)	(1.14)	(0.000)
BTOP	11.6	12.3	12.8	7.7	8.2	6.8	4.3	13.3	12.1	10.9	34.12
	(1.01)	(1.44)	(1.71)	(−1.73)	(−1.32)	(−2.62)	(−5.64)	(1.97)	(1.30)	(0.57)	(0.000)
BBOT	11.4	11.4	14.8	5.9	6.7	9.6	5.7	11.4	9.8	13.2	43.26
	(1.00)	(1.00)	(3.03)	(−3.91)	(−2.98)	(−0.27)	(−4.17)	(1.00)	(−0.12)	(2.12)	(0.000)
TTOP	10.7	12.1	16.2	6.2	7.9	8.7	4.0	12.5	11.4	10.2	92.09
	(0.67)	(1.89)	(4.93)	(−4.54)	(−2.29)	(−1.34)	(−8.93)	(2.18)	(1.29)	(0.23)	(0.000)
TBOT	9.9	11.3	15.6	7.9	7.7	5.7	5.3	14.6	12.0	10.0	85.26
	(−0.11)	(1.14)	(4.33)	(−2.24)	(−2.39)	(−5.20)	(−5.85)	(3.64)	(1.76)	(0.01)	(0.000)
RTOP	11.2	10.8	8.8	8.3	10.2	7.1	7.7	9.3	15.3	11.3	57.08
	(1.28)	(0.92)	(−1.40)	(−2.09)	(0.25)	(−3.87)	(−2.95)	(−0.75)	(4.92)	(1.37)	(0.000)
RBOT	8.9	12.3	8.9	8.9	11.6	8.9	7.0	9.5	13.6	10.3	45.79
	(−1.35)	(2.52)	(−1.35)	(−1.45)	(1.81)	(−1.35)	(−4.19)	(−0.66)	(3.85)	(0.36)	(0.000)
DTOP	11.0	12.6	11.7	9.0	9.2	5.5	5.8	11.6	12.3	11.3	71.29
	(1.12)	(2.71)	(1.81)	(−1.18)	(−0.98)	(−6.76)	(−6.26)	(1.73)	(2.39)	(1.47)	(0.000)
DBOT	10.9	11.5	13.1	8.0	8.1	7.1	7.6	11.5	12.8	9.3	51.23
	(0.98)	(1.60)	(3.09)	(−2.47)	(−2.35)	(−3.75)	(−3.09)	(1.60)	(2.85)	(−0.78)	(0.000)

Table 4a

Kolmogorov-Smirnov test of the equality of conditional and unconditional 1-day return distributions for NYSE/AMEX stocks from 1962 to 1996, in 5-year subperiods, and in size quintiles. Conditional returns are defined as the daily return three days following the conclusion of an occurrence of one of 10 technical indicators: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT). All returns have been normalized by subtraction of their means and division by their standard deviations. p-values are with respect to the asymptotic distribution of the Kolmogorov-Smirnov test statistic. The symbols '$\tau(\searrow)$' and '$\tau(\nearrow)$' indicate that the conditional distribution is also conditioned on decreasing and increasing volume trend, respectively.

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1996					
γ	1.89	1.22	1.15	1.76	0.90	1.09	1.84	2.45	1.51	1.06
p-value	0.002	0.104	0.139	0.004	0.393	0.185	0.002	0.000	0.021	0.215
$\gamma\ \tau(\searrow)$	1.49	0.95	0.44	0.62	0.73	1.33	1.37	1.77	0.96	0.78
p-value	0.024	0.327	0.989	0.839	0.657	0.059	0.047	0.004	0.319	0.579
$\gamma\ \tau(\nearrow)$	0.72	1.05	1.33	1.59	0.92	1.29	1.13	1.24	0.74	0.84
p-value	0.671	0.220	0.059	0.013	0.368	0.073	0.156	0.090	0.638	0.481
γ Diff.	0.88	0.54	0.59	0.94	0.75	1.37	0.79	1.20	0.82	0.71
p-value	0.418	0.935	0.879	0.342	0.628	0.046	0.557	0.111	0.512	0.698

Table 4a (continued)

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					Smallest Quintile, 1962 to 1996					
γ	0.59	1.19	0.72	1.20	0.98	1.43	1.09	1.19	0.84	0.78
p-value	0.872	0.116	0.679	0.114	0.290	0.033	0.188	0.120	0.485	0.583
$\gamma\ \tau(\searrow)$	0.67	0.80	1.16	0.69	1.00	1.46	1.31	0.94	1.12	0.73
p-value	0.765	0.540	0.136	0.723	0.271	0.029	0.065	0.339	0.165	0.663
$\gamma\ \tau(\nearrow)$	0.43	0.95	0.67	1.03	0.47	0.88	0.51	0.93	0.94	0.58
p-value	0.994	0.325	0.756	0.236	0.981	0.423	0.959	0.356	0.342	0.892
γ Diff.	0.52	0.48	1.14	0.68	0.48	0.98	0.98	0.79	1.16	0.62
p-value	0.951	0.974	0.151	0.741	0.976	0.291	0.294	0.552	0.133	0.840
					2nd Quintile, 1962 to 1996					
γ	1.82	1.63	0.93	0.92	0.82	0.84	0.88	1.29	1.46	0.84
p-value	0.003	0.010	0.353	0.365	0.505	0.485	0.417	0.073	0.029	0.478
$\gamma\ \tau(\searrow)$	1.62	1.03	0.88	0.42	0.91	0.90	0.71	0.86	1.50	0.97
p-value	0.010	0.242	0.427	0.994	0.378	0.394	0.703	0.443	0.022	0.298
$\gamma\ \tau(\nearrow)$	1.06	1.63	0.96	0.83	0.89	0.98	1.19	1.15	0.96	0.99
p-value	0.213	0.010	0.317	0.497	0.407	0.289	0.119	0.141	0.317	0.286
γ Diff.	0.78	0.94	1.04	0.71	1.22	0.92	0.99	0.79	1.18	0.68
p-value	0.576	0.334	0.228	0.687	0.102	0.361	0.276	0.564	0.126	0.745
					3rd Quintile, 1962 to 1996					
γ	0.83	1.56	1.00	1.28	0.57	1.03	1.96	1.50	1.55	1.14
p-value	0.502	0.016	0.266	0.074	0.903	0.243	0.001	0.023	0.016	0.150
$\gamma\ \tau(\searrow)$	0.95	0.94	0.66	0.76	0.61	0.82	1.45	1.61	1.17	1.01
p-value	0.326	0.346	0.775	0.613	0.854	0.520	0.031	0.012	0.131	0.258
$\gamma\ \tau(\nearrow)$	1.05	1.43	0.93	1.14	0.63	0.80	0.93	0.78	0.59	0.86
p-value	0.223	0.033	0.350	0.147	0.826	0.544	0.354	0.578	0.878	0.450
γ Diff.	1.02	1.14	0.45	0.48	0.50	0.89	0.66	0.91	0.72	1.15
p-value	0.246	0.148	0.986	0.974	0.964	0.413	0.774	0.383	0.670	0.143
					4th Quintile, 1962 to 1996					
γ	0.72	0.61	1.29	0.84	0.61	0.84	1.37	1.37	0.72	0.53
p-value	0.683	0.852	0.071	0.479	0.855	0.480	0.048	0.047	0.682	0.943
$\gamma\ \tau(\searrow)$	1.01	0.95	0.83	0.96	0.78	0.84	1.34	0.72	0.62	1.01
p-value	0.255	0.330	0.504	0.311	0.585	0.487	0.056	0.680	0.841	0.258
$\gamma\ \tau(\nearrow)$	0.93	0.66	1.29	0.96	1.16	0.69	0.64	1.16	0.69	0.85
p-value	0.349	0.772	0.072	0.316	0.137	0.731	0.810	0.136	0.720	0.468
γ Diff.	1.10	0.97	0.64	1.16	1.31	0.78	0.64	0.92	0.66	1.10
p-value	0.175	0.301	0.804	0.138	0.065	0.571	0.806	0.363	0.780	0.176
					Largest Quintile, 1962 to 1996					
γ	1.25	1.16	0.98	0.48	0.50	0.80	0.94	1.76	0.90	1.28
p-value	0.088	0.136	0.287	0.977	0.964	0.544	0.346	0.004	0.395	0.077
$\gamma\ \tau(\searrow)$	1.12	0.90	0.57	0.78	0.64	1.17	0.91	0.87	0.64	1.20
p-value	0.164	0.386	0.906	0.580	0.806	0.127	0.379	0.442	0.802	0.114
$\gamma\ \tau(\nearrow)$	0.81	0.93	0.83	0.61	0.69	0.81	0.73	0.87	0.46	0.88
p-value	0.522	0.350	0.495	0.854	0.729	0.532	0.661	0.432	0.982	0.418
γ Diff.	0.71	0.54	0.59	0.64	0.76	1.21	0.85	1.11	0.54	0.79
p-value	0.699	0.934	0.874	0.800	0.607	0.110	0.467	0.170	0.929	0.552

Table 4a (continued)

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1966					
γ	1.29	1.67	1.07	0.72	0.75	1.32	1.20	1.53	2.04	1.73
p-value	0.072	0.007	0.202	0.671	0.634	0.062	0.112	0.018	0.001	0.005
$\gamma\ \tau(\searrow)$	0.83	1.01	1.04	0.80	0.63	1.80	0.66	1.84	1.03	1.54
p-value	0.499	0.260	0.232	0.539	0.826	0.003	0.771	0.002	0.244	0.017
$\gamma\ \tau(\nearrow)$	1.13	1.13	0.84	0.84	0.58	1.40	1.12	0.83	1.09	1.16
p-value	0.156	0.153	0.480	0.475	0.894	0.040	0.163	0.492	0.183	0.135
γ Diff.	0.65	0.71	0.75	0.76	0.60	1.90	0.68	1.35	0.73	0.83
p-value	0.799	0.691	0.629	0.615	0.863	0.001	0.741	0.052	0.657	0.503
					All Stocks, 1967 to 1971					
γ	1.10	0.96	0.60	0.65	0.98	0.76	1.29	1.65	0.87	1.22
p-value	0.177	0.317	0.867	0.797	0.292	0.606	0.071	0.009	0.436	0.101
$\gamma\ \tau(\searrow)$	1.02	0.80	0.53	0.85	0.97	0.77	0.71	1.42	0.97	1.06
p-value	0.248	0.551	0.943	0.464	0.303	0.590	0.700	0.035	0.300	0.214
$\gamma\ \tau(\nearrow)$	1.08	0.86	0.68	0.91	1.11	0.82	0.79	0.73	0.71	0.96
p-value	0.190	0.454	0.750	0.373	0.169	0.508	0.554	0.660	0.699	0.315
γ Diff.	1.36	0.51	0.53	0.76	0.68	0.71	0.71	0.98	1.06	1.12
p-value	0.049	0.956	0.942	0.616	0.751	0.699	0.701	0.290	0.210	0.163
					All Stocks, 1972 to 1976					
γ	0.47	0.75	0.87	1.56	1.21	0.75	0.87	0.94	1.64	1.20
p-value	0.980	0.620	0.441	0.015	0.106	0.627	0.441	0.341	0.009	0.113
$\gamma\ \tau(\searrow)$	0.80	0.40	0.50	1.24	1.21	0.65	1.26	0.63	0.70	1.39
p-value	0.539	0.998	0.966	0.093	0.106	0.794	0.084	0.821	0.718	0.041
$\gamma\ \tau(\nearrow)$	0.49	0.78	0.94	1.21	1.12	1.03	0.81	0.95	0.84	0.70
p-value	0.970	0.577	0.340	0.108	0.159	0.244	0.521	0.331	0.485	0.719
γ Diff.	0.55	0.56	0.51	0.95	0.81	1.11	1.15	0.62	0.67	1.31
p-value	0.925	0.915	0.960	0.333	0.525	0.170	0.141	0.836	0.767	0.065
					All Stocks, 1977 to 1981					
γ	1.16	0.73	0.76	1.16	0.82	1.14	1.01	0.87	0.86	1.79
p-value	0.138	0.665	0.617	0.136	0.506	0.147	0.263	0.428	0.449	0.003
$\gamma\ \tau(\searrow)$	1.04	0.73	1.00	1.31	1.10	1.32	0.83	0.80	1.20	1.81
p-value	0.228	0.654	0.274	0.065	0.176	0.062	0.494	0.550	0.113	0.003
$\gamma\ \tau(\nearrow)$	0.75	0.84	0.88	0.65	0.67	0.76	1.51	1.41	0.86	0.99
p-value	0.623	0.476	0.426	0.799	0.754	0.602	0.020	0.037	0.450	0.280
γ Diff.	0.67	0.94	0.88	0.70	0.65	0.70	1.11	1.29	1.16	0.70
p-value	0.767	0.335	0.423	0.708	0.785	0.716	0.172	0.073	0.137	0.713

Table 4a (continued)

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1982 to 1986					
γ	1.57	0.99	0.59	1.46	1.47	1.04	0.87	0.68	0.76	0.90
p-value	0.015	0.276	0.883	0.029	0.027	0.232	0.431	0.742	0.617	0.387
$\gamma\ \tau(\searrow)$	1.17	0.68	0.44	1.30	1.53	1.21	1.08	0.93	0.84	0.88
p-value	0.129	0.741	0.991	0.070	0.018	0.106	0.190	0.356	0.478	0.421
$\gamma\ \tau(\nearrow)$	0.81	1.03	0.74	0.62	0.83	1.23	0.77	0.79	0.63	0.81
p-value	0.533	0.243	0.640	0.831	0.499	0.097	0.597	0.564	0.821	0.528
γ Diff.	0.51	0.79	0.70	0.81	0.74	1.21	0.73	0.75	0.93	0.74
p-value	0.961	0.567	0.717	0.532	0.643	0.107	0.657	0.623	0.352	0.642
					All Stocks, 1987 to 1991					
γ	1.36	1.53	1.05	0.67	0.75	0.86	0.60	1.09	1.20	0.67
p-value	0.048	0.019	0.219	0.756	0.627	0.456	0.862	0.185	0.111	0.764
$\gamma\ \tau(\searrow)$	0.52	1.16	1.25	0.72	1.03	0.81	0.81	0.61	1.07	0.68
p-value	0.953	0.135	0.087	0.673	0.235	0.522	0.527	0.848	0.201	0.751
$\gamma\ \tau(\nearrow)$	1.72	1.03	0.64	1.37	0.74	1.10	1.04	1.20	1.02	1.32
p-value	0.006	0.241	0.813	0.046	0.639	0.181	0.232	0.111	0.250	0.062
γ Diff.	1.11	1.29	1.07	1.06	0.67	0.93	0.89	0.74	0.84	1.17
p-value	0.168	0.072	0.201	0.215	0.753	0.357	0.403	0.638	0.483	0.129
					All Stocks, 1992 to 1996					
γ	1.50	1.31	1.05	1.89	1.27	0.94	1.23	0.66	1.72	1.54
p-value	0.022	0.066	0.222	0.002	0.078	0.343	0.095	0.782	0.005	0.018
$\gamma\ \tau(\searrow)$	0.87	1.05	0.60	0.89	1.11	1.03	0.90	0.65	0.99	1.12
p-value	0.443	0.218	0.858	0.404	0.174	0.242	0.390	0.787	0.283	0.165
$\gamma\ \tau(\nearrow)$	0.72	0.66	0.75	1.42	1.02	0.58	0.61	0.64	1.36	0.93
p-value	0.670	0.778	0.624	0.036	0.246	0.895	0.854	0.813	0.048	0.357
γ Diff.	0.58	0.88	0.50	0.49	0.43	0.81	0.60	0.46	0.96	0.99
p-value	0.887	0.422	0.966	0.971	0.993	0.528	0.858	0.984	0.314	0.282

Table 4b

Kolmogorov-Smirnov test of the equality of conditional and unconditional 1-day return distributions for NASDAQ stocks from 1962 to 1996, in 5-year subperiods, and in size quintiles. Conditional returns are defined as the daily return three days following the conclusion of an occurrence of one of 10 technical indicators: head-and-shoulders (HS), inverted head-and-shoulders (IHS), broadening top (BTOP), broadening bottom (BBOT), triangle top (TTOP), triangle bottom (TBOT), rectangle top (RTOP), rectangle bottom (RBOT), double top (DTOP), and double bottom (DBOT). All returns have been normalized by subtraction of their means and division by their standard deviations. p-values are with respect to the asymptotic distribution of the Kolmogorov-Smirnov test statistic. The symbols '$\tau(\searrow)$' and '$\tau(\nearrow)$' indicate that the conditional distribution is also conditioned on decreasing and increasing volume trend, respectively.

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1996					
γ	2.31	2.68	1.60	1.84	2.81	2.34	2.69	1.90	2.29	2.06
p-value	0.000	0.000	0.012	0.002	0.000	0.000	0.000	0.001	0.000	0.000
$\gamma\ \tau(\searrow)$	1.86	1.53	1.35	0.99	1.97	1.95	2.16	1.73	1.38	1.94
p-value	0.002	0.019	0.052	0.281	0.001	0.001	0.000	0.005	0.045	0.001
$\gamma\ \tau(\nearrow)$	1.59	2.10	1.82	1.59	1.89	1.18	1.57	1.22	2.15	1.46
p-value	0.013	0.000	0.003	0.013	0.002	0.126	0.014	0.102	0.000	0.028
γ Diff.	1.08	0.86	1.10	0.80	1.73	0.74	0.91	0.75	0.76	1.52
p-value	0.195	0.450	0.175	0.542	0.005	0.637	0.379	0.621	0.619	0.020

Table 4b (continued)

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					Smallest Quintile, 1962 to 1996					
γ	1.51	2.16	1.72	1.68	1.22	1.55	2.13	1.70	1.74	1.98
p-value	0.021	0.000	0.006	0.007	0.101	0.016	0.000	0.006	0.005	0.001
$\gamma\ \tau(\searrow)$	1.16	1.30	0.85	1.14	1.25	1.62	1.43	1.05	1.08	1.95
p-value	0.139	0.070	0.463	0.150	0.089	0.010	0.033	0.216	0.191	0.001
$\gamma\ \tau(\nearrow)$	0.85	1.73	1.61	2.00	1.34	0.79	1.58	1.52	1.47	1.20
p-value	0.462	0.005	0.012	0.001	0.055	0.553	0.014	0.019	0.026	0.115
γ Diff.	1.04	0.95	0.83	1.44	1.39	0.78	0.95	0.73	0.94	1.09
p-value	0.227	0.334	0.493	0.031	0.042	0.574	0.326	0.654	0.338	0.184
					2nd Quintile, 1962 to 1996					
γ	1.55	1.46	0.94	1.44	1.24	1.08	1.20	1.10	1.90	1.27
p-value	0.016	0.029	0.341	0.031	0.095	0.192	0.113	0.175	0.001	0.078
$\gamma\ \tau(\searrow)$	1.11	1.13	1.08	0.92	1.23	0.79	1.34	1.19	1.09	1.61
p-value	0.173	0.157	0.192	0.371	0.097	0.557	0.055	0.117	0.185	0.011
$\gamma\ \tau(\nearrow)$	1.37	0.87	0.73	0.97	1.38	1.29	1.12	0.91	1.12	0.94
p-value	0.048	0.439	0.665	0.309	0.044	0.073	0.162	0.381	0.165	0.343
γ Diff.	1.23	0.62	0.97	0.69	1.02	1.05	1.09	0.78	0.58	0.51
p-value	0.095	0.835	0.309	0.733	0.248	0.224	0.183	0.579	0.894	0.955
					3rd Quintile, 1962 to 1996					
γ	1.25	1.72	0.82	1.71	1.41	1.52	1.25	1.84	1.86	1.82
p-value	0.087	0.005	0.510	0.006	0.038	0.020	0.089	0.002	0.002	0.003
$\gamma\ \tau(\searrow)$	0.93	1.08	0.54	1.23	1.06	1.02	0.79	1.47	1.38	0.88
p-value	0.348	0.194	0.930	0.097	0.213	0.245	0.560	0.026	0.044	0.423
$\gamma\ \tau(\nearrow)$	0.59	1.14	0.97	1.37	0.75	1.01	1.13	1.34	1.37	1.78
p-value	0.873	0.146	0.309	0.047	0.633	0.262	0.159	0.054	0.047	0.003
γ Diff.	0.61	0.89	0.58	0.46	0.61	0.89	0.52	0.38	0.60	1.09
p-value	0.852	0.405	0.890	0.984	0.844	0.404	0.947	0.999	0.864	0.188
					4th Quintile, 1962 to 1996					
γ	1.04	0.82	1.20	0.98	1.30	1.25	1.88	0.79	0.94	0.66
p-value	0.233	0.510	0.111	0.298	0.067	0.087	0.002	0.553	0.341	0.779
$\gamma\ \tau(\searrow)$	0.81	0.54	0.57	1.05	0.92	1.06	1.23	0.72	1.53	0.87
p-value	0.528	0.935	0.897	0.217	0.367	0.215	0.097	0.672	0.019	0.431
$\gamma\ \tau(\nearrow)$	0.97	1.04	1.29	0.53	2.25	0.71	1.05	0.77	1.20	0.97
p-value	0.306	0.229	0.071	0.938	0.000	0.696	0.219	0.589	0.114	0.309
γ Diff.	1.17	0.89	0.98	0.97	1.86	0.62	0.93	0.73	1.31	0.92
p-value	0.128	0.400	0.292	0.301	0.002	0.843	0.352	0.653	0.065	0.371
					Largest Quintile, 1962 to 1996					
γ	1.08	1.01	1.03	0.66	0.92	0.68	0.85	1.16	1.14	0.67
p-value	0.190	0.255	0.242	0.778	0.360	0.742	0.462	0.137	0.150	0.756
$\gamma\ \tau(\searrow)$	1.03	0.54	0.93	0.47	0.77	0.76	0.85	0.62	0.85	1.14
p-value	0.237	0.931	0.356	0.981	0.587	0.612	0.468	0.840	0.465	0.149
$\gamma\ \tau(\nearrow)$	1.18	1.39	0.50	0.93	0.88	1.25	0.77	1.13	0.98	1.12
p-value	0.123	0.041	0.967	0.358	0.415	0.089	0.597	0.156	0.292	0.160
γ Diff.	0.94	1.25	0.73	0.84	0.76	1.11	0.73	0.86	0.86	0.77
p-value	0.342	0.090	0.668	0.476	0.617	0.169	0.662	0.457	0.454	0.598

Table 4b (continued)

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1962 to 1966					
γ	1.01	0.84	1.08	0.82	0.71	0.70	1.59	0.89	1.12	1.10
p-value	0.261	0.481	0.193	0.508	0.697	0.718	0.013	0.411	0.166	0.175
$\gamma\ \tau(\searrow)$	0.95	0.65	0.41	1.05	0.51	1.13	0.79	0.93	0.93	1.21
p-value	0.322	0.798	0.997	0.224	0.956	0.155	0.556	0.350	0.350	0.108
$\gamma\ \tau(\nearrow)$	0.77	0.96	0.83	0.73	1.35	0.49	1.17	0.62	1.18	1.15
p-value	0.586	0.314	0.489	0.663	0.052	0.972	0.130	0.843	0.121	0.140
γ Diff.	1.10	0.67	0.32	0.69	1.29	0.58	0.80	0.75	0.98	1.06
p-value	0.174	0.761	1.000	0.735	0.071	0.892	0.551	0.620	0.298	0.208
					All Stocks, 1967 to 1971					
γ	0.75	1.10	1.00	0.74	1.27	1.35	1.16	0.74	0.74	1.21
p-value	0.636	0.175	0.273	0.637	0.079	0.052	0.136	0.642	0.638	0.107
$\gamma\ \tau(\searrow)$	1.03	0.52	0.70	0.87	1.24	1.33	1.29	0.83	0.72	1.45
p-value	0.241	0.947	0.714	0.438	0.092	0.058	0.072	0.490	0.684	0.031
$\gamma\ \tau(\nearrow)$	1.05	1.08	1.12	0.64	0.79	0.65	0.55	0.53	0.75	0.69
p-value	0.217	0.192	0.165	0.810	0.566	0.797	0.923	0.941	0.631	0.723
γ Diff.	1.24	0.89	0.66	0.78	1.07	0.88	0.88	0.40	0.91	0.76
p-value	0.093	0.413	0.770	0.585	0.203	0.418	0.423	0.997	0.385	0.602
					All Stocks, 1972 to 1976					
γ	0.82	1.28	1.84	1.13	1.45	1.53	1.31	0.96	0.85	1.76
p-value	0.509	0.077	0.002	0.156	0.029	0.019	0.064	0.314	0.464	0.004
$\gamma\ \tau(\searrow)$	0.59	0.73	−99.00	0.91	1.39	0.73	1.37	0.98	1.22	0.94
p-value	0.875	0.669	0.000	0.376	0.042	0.654	0.046	0.292	0.100	0.344
$\gamma\ \tau(\nearrow)$	0.65	0.73	−99.00	−99.00	−99.00	−99.00	0.59	0.76	0.78	0.65
p-value	0.800	0.653	0.000	0.000	0.000	0.000	0.878	0.611	0.573	0.798
γ Diff.	0.48	0.57	−99.00	−99.00	−99.00	−99.00	0.63	0.55	0.92	0.37
p-value	0.974	0.902	0.000	0.000	0.000	0.000	0.828	0.925	0.362	0.999
					All Stocks, 1977 to 1981					
γ	1.35	1.40	1.03	1.02	1.55	2.07	0.74	0.62	0.92	1.28
p-value	0.053	0.039	0.236	0.249	0.016	0.000	0.636	0.842	0.369	0.077
$\gamma\ \tau(\searrow)$	1.19	1.47	−99.00	−99.00	0.96	0.98	0.86	0.79	0.81	0.68
p-value	0.117	0.027	0.000	0.000	0.317	0.290	0.453	0.554	0.522	0.748
$\gamma\ \tau(\nearrow)$	0.69	0.94	0.80	−99.00	1.46	−99.00	0.56	0.82	1.06	0.94
p-value	0.728	0.341	0.542	0.000	0.028	0.000	0.918	0.514	0.207	0.336
γ Diff.	0.73	0.90	−99.00	−99.00	0.35	−99.00	0.44	0.37	0.80	0.53
p-value	0.665	0.395	0.000	0.000	1.000	0.000	0.991	0.999	0.541	0.944

Table 4b (continued)

Statistic	HS	IHS	BTOP	BBOT	TTOP	TBOT	RTOP	RBOT	DTOP	DBOT
					All Stocks, 1982 to 1986					
γ	1.66	1.59	1.17	0.73	1.46	1.69	1.04	1.24	2.44	1.27
p-value	0.008	0.013	0.129	0.654	0.028	0.006	0.232	0.093	0.000	0.078
$\gamma\ \tau(\searrow)$	1.65	1.10	0.46	0.74	0.95	1.47	0.83	1.18	1.20	0.59
p-value	0.009	0.176	0.984	0.641	0.330	0.027	0.503	0.121	0.112	0.873
$\gamma\ \tau(\nearrow)$	1.13	1.31	0.86	0.42	1.17	1.04	0.97	1.13	1.68	0.89
p-value	0.153	0.065	0.445	0.995	0.129	0.231	0.302	0.155	0.007	0.405
γ Diff.	0.67	0.39	0.51	0.42	0.85	0.43	0.41	0.67	0.66	0.75
p-value	0.755	0.998	0.957	0.994	0.462	0.993	0.996	0.766	0.782	0.627
					All Stocks, 1987 to 1991					
γ	1.24	1.29	0.91	0.88	1.28	1.41	2.01	1.49	1.55	1.53
p-value	0.091	0.070	0.384	0.421	0.074	0.039	0.001	0.024	0.017	0.019
$\gamma\ \tau(\searrow)$	1.05	1.00	1.00	0.78	1.68	0.92	1.67	1.25	0.61	0.86
p-value	0.221	0.266	0.274	0.580	0.007	0.369	0.008	0.087	0.849	0.448
$\gamma\ \tau(\nearrow)$	1.23	1.26	1.06	1.32	0.65	1.27	1.10	1.26	1.67	1.81
p-value	0.099	0.084	0.208	0.060	0.787	0.078	0.176	0.085	0.007	0.003
γ Diff.	0.80	0.91	1.22	1.28	1.22	0.92	0.87	0.81	1.07	1.05
p-value	0.552	0.375	0.103	0.075	0.102	0.360	0.431	0.520	0.202	0.217
					All Stocks, 1992 to 1996					
γ	1.21	1.61	0.84	0.90	0.97	0.91	1.60	1.51	1.13	1.00
p-value	0.108	0.011	0.476	0.394	0.299	0.379	0.012	0.021	0.156	0.265
$\gamma\ \tau(\searrow)$	0.68	1.02	0.81	0.78	0.81	0.93	0.79	1.07	0.94	0.64
p-value	0.752	0.246	0.530	0.578	0.532	0.357	0.558	0.201	0.340	0.814
$\gamma\ \tau(\nearrow)$	1.56	0.85	0.71	1.00	1.10	1.04	1.43	0.93	0.90	1.44
p-value	0.015	0.470	0.688	0.275	0.180	0.231	0.034	0.352	0.392	0.031
γ Diff.	1.45	0.59	0.94	0.62	1.15	1.14	0.64	0.52	0.59	1.35
p-value	0.030	0.879	0.346	0.840	0.139	0.148	0.814	0.953	0.874	0.052

Table 5a

Bootstrap percentiles for the Kolmogorov-Smirnov test of the equality of conditional and unconditional 1-day return distributions for NYSE/AMEX and NASDAQ stocks from 1962 to 1996, and for size quintiles, under the null hypothesis of equality. For each of the two sets of market data, two sample sizes, m_1 and m_2, have been chosen to span the range of frequency counts of patterns reported in Table 1. For each sample size m_i, we resample 1-day normalized returns (with replacement) to obtain a bootstrap sample of m_i observations, compute the Kolmogorov-Smirnov test statistic (against the entire sample of 1-day normalized returns), and repeat this procedure 1,000 times. The percentiles of the asymptotic distribution are also reported for comparison.

Percentile	NYSE/AMEX Sample					NASDAQ Sample				
	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ
				All Stocks, 1962 to 1996						
0.01	2076	0.433	725	0.435	0.441	1320	0.430	414	0.438	0.441
0.05	2076	0.515	725	0.535	0.520	1320	0.514	414	0.522	0.520
0.10	2076	0.568	725	0.590	0.571	1320	0.573	414	0.566	0.571
0.50	2076	0.827	725	0.836	0.828	1320	0.840	414	0.826	0.828
0.90	2076	1.219	725	1.237	1.224	1320	1.244	414	1.229	1.224
0.95	2076	1.385	725	1.395	1.358	1320	1.373	414	1.340	1.358
0.99	2076	1.608	725	1.611	1.628	1320	1.645	414	1.600	1.628
				Smallest Quintile, 1962 to 1996						
0.01	320	0.456	78	0.406	0.441	218	0.459	41	0.436	0.441
0.05	320	0.535	78	0.502	0.520	218	0.533	41	0.498	0.520
0.10	320	0.586	78	0.559	0.571	218	0.590	41	0.543	0.571
0.50	320	0.848	78	0.814	0.828	218	0.847	41	0.801	0.828
0.90	320	1.231	78	1.204	1.224	218	1.229	41	1.216	1.224
0.95	320	1.357	78	1.330	1.358	218	1.381	41	1.332	1.358
0.99	320	1.661	78	1.590	1.628	218	1.708	41	1.571	1.628
				2nd Quintile, 1962 to 1996						
0.01	420	0.445	146	0.428	0.441	305	0.458	68	0.426	0.441
0.05	420	0.530	146	0.505	0.520	305	0.557	68	0.501	0.520
0.10	420	0.580	146	0.553	0.571	305	0.610	68	0.559	0.571
0.50	420	0.831	146	0.823	0.828	305	0.862	68	0.804	0.828
0.90	420	1.197	146	1.210	1.224	305	1.265	68	1.210	1.224
0.95	420	1.349	146	1.343	1.358	305	1.407	68	1.409	1.358
0.99	420	1.634	146	1.626	1.628	305	1.686	68	1.614	1.628
				3rd Quintile, 1962 to 1996						
0.01	458	0.442	145	0.458	0.441	279	0.464	105	0.425	0.441
0.05	458	0.516	145	0.508	0.520	279	0.539	105	0.525	0.520
0.10	458	0.559	145	0.557	0.571	279	0.586	105	0.570	0.571
0.50	458	0.838	145	0.835	0.828	279	0.832	105	0.818	0.828
0.90	458	1.216	145	1.251	1.224	279	1.220	105	1.233	1.224
0.95	458	1.406	145	1.397	1.358	279	1.357	105	1.355	1.358
0.99	458	1.660	145	1.661	1.628	279	1.606	105	1.638	1.628

Table 5a (continued)

Percentile	NYSE/AMEX Sample					NASDAQ Sample				
	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ
					4th Quintile, 1962 to 1996					
0.01	424	0.429	173	0.418	0.441	303	0.454	92	0.446	0.441
0.05	424	0.506	173	0.516	0.520	303	0.526	92	0.506	0.520
0.10	424	0.552	173	0.559	0.571	303	0.563	92	0.554	0.571
0.50	424	0.823	173	0.815	0.828	303	0.840	92	0.818	0.828
0.90	424	1.197	173	1.183	1.224	303	1.217	92	1.178	1.224
0.95	424	1.336	173	1.313	1.358	303	1.350	92	1.327	1.358
0.99	424	1.664	173	1.592	1.628	303	1.659	92	1.606	1.628
					Largest Quintile, 1962 to 1996					
0.01	561	0.421	167	0.425	0.441	308	0.441	108	0.429	0.441
0.05	561	0.509	167	0.500	0.520	308	0.520	108	0.508	0.520
0.10	561	0.557	167	0.554	0.571	308	0.573	108	0.558	0.571
0.50	561	0.830	167	0.817	0.828	308	0.842	108	0.816	0.828
0.90	561	1.218	167	1.202	1.224	308	1.231	108	1.226	1.224
0.95	561	1.369	167	1.308	1.358	308	1.408	108	1.357	1.358
0.99	561	1.565	167	1.615	1.628	308	1.724	108	1.630	1.628

Table 5b

Bootstrap percentiles for the Kolmogorov-Smirnov test of the equality of conditional and unconditional 1-day return distributions for NYSE/AMEX and NASDAQ stocks from 1962 to 1996, for 5-year subperiods, under the null hypothesis of equality. For each of the two sets of market data, two sample sizes, m_1 and m_2, have been chosen to span the range of frequency counts of patterns reported in Table 1. For each sample size m_i, we resample 1-day normalized returns (with replacement) to obtain a bootstrap sample of m_i observations, compute the Kolmogorov-Smirnov test statistic (against the entire sample of 1-day normalized returns), and repeat this procedure 1,000 times. The percentiles of the asymptotic distribution are also reported for comparison.

Percentile	NYSE/AMEX Sample					NASDAQ Sample				
	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ
					All Stocks, 1962 to 1966					
0.01	356	0.431	85	0.427	0.441	342	0.460	72	0.417	0.441
0.05	356	0.516	85	0.509	0.520	342	0.539	72	0.501	0.520
0.10	356	0.576	85	0.559	0.571	342	0.589	72	0.565	0.571
0.50	356	0.827	85	0.813	0.828	342	0.849	72	0.802	0.828
0.90	356	1.233	85	1.221	1.224	342	1.242	72	1.192	1.224
0.95	356	1.359	85	1.363	1.358	342	1.384	72	1.339	1.358
0.99	356	1.635	85	1.711	1.628	342	1.582	72	1.684	1.628
					All Stocks, 1967 to 1971					
0.01	258	0.432	112	0.423	0.441	227	0.435	65	0.424	0.441
0.05	258	0.522	112	0.508	0.520	227	0.512	65	0.498	0.520
0.10	258	0.588	112	0.562	0.571	227	0.571	65	0.546	0.571
0.50	258	0.841	112	0.819	0.828	227	0.811	65	0.812	0.828
0.90	258	1.194	112	1.253	1.224	227	1.179	65	1.219	1.224
0.95	258	1.315	112	1.385	1.358	227	1.346	65	1.357	1.358
0.99	258	1.703	112	1.563	1.628	227	1.625	65	1.669	1.628
					All Stocks, 1972 to 1976					
0.01	223	0.439	82	0.440	0.441	58	0.433	25	0.405	0.441
0.05	223	0.518	82	0.503	0.520	58	0.495	25	0.479	0.520
0.10	223	0.588	82	0.554	0.571	58	0.542	25	0.526	0.571
0.50	223	0.854	82	0.798	0.828	58	0.793	25	0.783	0.828
0.90	223	1.249	82	1.208	1.224	58	1.168	25	1.203	1.224
0.95	223	1.406	82	1.364	1.358	58	1.272	25	1.345	1.358
0.99	223	1.685	82	1.635	1.628	58	1.618	25	1.616	1.628
					All Stocks, 1977 to 1981					
0.01	290	0.426	110	0.435	0.441	96	0.430	36	0.417	0.441
0.05	290	0.519	110	0.504	0.520	96	0.504	36	0.485	0.520
0.10	290	0.573	110	0.555	0.571	96	0.570	36	0.542	0.571
0.50	290	0.841	110	0.793	0.828	96	0.821	36	0.810	0.828
0.90	290	1.262	110	1.184	1.224	96	1.197	36	1.201	1.224
0.95	290	1.383	110	1.342	1.358	96	1.352	36	1.371	1.358
0.99	290	1.598	110	1.645	1.628	96	1.540	36	1.545	1.628

Table 5b (continued)

Percentile	NYSE/AMEX Sample					NASDAQ Sample				
	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ	m_1	$\Delta_{m_1,n}$	m_2	$\Delta_{m_2,n}$	Δ
					All Stocks, 1982 to 1986					
0.01	313	0.462	106	0.437	0.441	120	0.448	44	0.417	0.441
0.05	313	0.542	106	0.506	0.520	120	0.514	44	0.499	0.520
0.10	313	0.585	106	0.559	0.571	120	0.579	44	0.555	0.571
0.50	313	0.844	106	0.819	0.828	120	0.825	44	0.802	0.828
0.90	313	1.266	106	1.220	1.224	120	1.253	44	1.197	1.224
0.95	313	1.397	106	1.369	1.358	120	1.366	44	1.337	1.358
0.99	313	1.727	106	1.615	1.628	120	1.692	44	1.631	1.628
					All Stocks, 1987 to 1991					
0.01	287	0.443	98	0.449	0.441	312	0.455	50	0.432	0.441
0.05	287	0.513	98	0.522	0.520	312	0.542	50	0.517	0.520
0.10	287	0.565	98	0.566	0.571	312	0.610	50	0.563	0.571
0.50	287	0.837	98	0.813	0.828	312	0.878	50	0.814	0.828
0.90	287	1.200	98	1.217	1.224	312	1.319	50	1.216	1.224
0.95	287	1.336	98	1.348	1.358	312	1.457	50	1.323	1.358
0.99	287	1.626	98	1.563	1.628	312	1.701	50	1.648	1.628
					All Stocks, 1992 to 1996					
0.01	389	0.438	102	0.432	0.441	361	0.447	87	0.428	0.441
0.05	389	0.522	102	0.506	0.520	361	0.518	87	0.492	0.520
0.10	389	0.567	102	0.558	0.571	361	0.559	87	0.550	0.571
0.50	389	0.824	102	0.818	0.828	361	0.817	87	0.799	0.828
0.90	389	1.220	102	1.213	1.224	361	1.226	87	1.216	1.224
0.95	389	1.321	102	1.310	1.358	361	1.353	87	1.341	1.358
0.99	389	1.580	102	1.616	1.628	361	1.617	87	1.572	1.628