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Signal and System Approximation from General Measurements

Dedicated to Professor Paul Butzer on his 85th birthday

Holger Boche* and Ullrich J. Mönich†

Abstract In this paper we analyze the behavior of system approximation processes for stable linear time-invariant (LTI) systems and signals in the Paley–Wiener space \mathcal{PW}_π^1 . We consider approximation processes, where the input signal is not directly used to generate the system output, but instead a sequence of numbers is used that is generated from the input signal by measurement functionals. We consider classical sampling which corresponds to a pointwise evaluation of the signal, as well as several more general measurement functionals. We show that a stable system approximation is not possible for pointwise sampling, because there exist signals and systems such that the approximation process diverges. This remains true even with oversampling. However, if more general measurement functionals are considered, a stable approximation is possible if oversampling is used. Further, we show that without oversampling we have divergence for a large class of practically relevant measurement procedures.

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1 Introduction

Sampling theory plays a fundamental role in modern signal and information processing, because it is the basis for today's digital world [46]. The reconstruction of continuous-time signals from their samples is also essential for other applications and theoretical concepts [29, 26, 34]. The reconstruction of non-bandlimited signals, which was analyzed for example in [15, 17, 18], will not be considered in this paper, instead we focus on bandlimited signals. For an overview of existing sampling theorems see for example [29, 27], and [16].

The core task of digital signal processing is to process data. This means that, usually, the interest is not in a reconstruction of the sampled signal itself, but in some processed version of it. This might be the derivative, the Hilbert transform or the output of any other stable linear system T . Then the goal is to approximate the desired transform Tf of a signal f by an approximation process, which uses only finitely many, not necessarily equidistant, samples of the signal f . Exactly as in the case of signal reconstruction, the convergence and approximation behavior is important for practical applications [14].

Since sampling theory is so fundamental for applications it is essential to have this theory developed rigorously. From the first beginnings in engineering, see for example [11, 10] for historical comments, one main goal in research was to extend the theory to different practically relevant classes of signals and systems. The first author's interest for the topic was aroused in discussions with Paul Butzer in the early 1990s at RWTH Aachen. Since 2005 both authors have done research in this field and contributed with publications, see for example the second author's thesis [35] for a summary.

In order to continue the "digital revolution", enormous capital expenditures and resources are used to maintain the pace of performance increase, which is described by Moore's law. But also the operation of current communication systems requires huge amounts of resources, e.g. energy. It is reasonable to ask whether this is necessary. In this context, from a signal theoretic perspective, three interesting questions are: Do there exist fundamental limits that determine which signals and systems can be implemented digitally? In what technology— analog, digital, or mixed signal— can the systems be implemented? What are the necessary resources in terms of energy and hardware to implement the systems?

Such an implementation theory is of high practical relevance, and it already influences the system design, although there is no general system theoretic approach available yet to answer the posed questions. For example, the question whether to use a system implementation based on the Shannon series operating at Nyquist rate or to use an approach based on oversampling, which comes with higher technological effort, plays a central role in the design of modern information processing systems. A further important question concerns the measurement procedures. Can we use classical sampling-based measurement procedures, where the signal values are taken at certain time instants, or is it better to use more general measurement procedures? As already mentioned, no general methodical approach is known that

could answer these questions. Regardless of these difficulties, Hilbert's vision applies: "We must know. We will know."

In this paper we analyze the convergence behavior of system approximation processes for different kinds of sampling procedures. The structure of this paper is as follows: First, we introduce some notation in Section 2. Then, we treat pointwise sampling in Section 3. In Section 4 we study general sampling functionals and oversampling. In Section 5 we analyze the convergence of subsequences of the approximation process. Finally, in Section 6 we discuss the structure of more general measurement functionals.

2 Notation

In order to continue the discussion, we need some preliminaries and notation. Let \hat{f} denote the Fourier transform of a function f , where \hat{f} is to be understood in the distributional sense. By $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, we denote the usual L^p -spaces with norm $\|\cdot\|_p$. $C[a, b]$ is the space of all continuous functions on $[a, b]$. Further, l^p , $1 \leq p < \infty$, is the space of all sequences that are summable to the p th power.

For $\sigma > 0$ let \mathcal{B}_σ be the set of all entire functions f with the property that for all $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ with $|f(z)| \leq C(\varepsilon) \exp((\sigma + \varepsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space \mathcal{B}_σ^p consists of all functions in \mathcal{B}_σ , whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. A function in \mathcal{B}_σ^p is called bandlimited to σ . By the Paley–Wiener–Schwartz theorem, the Fourier transform of a function bandlimited to σ is supported in $[-\sigma, \sigma]$. For $1 \leq p \leq 2$ the Fourier transformation is defined in the classical and for $p > 2$ in the distributional sense. It is well known that $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^s$ for $1 \leq p \leq s \leq \infty$. Hence, every function $f \in \mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, is bounded.

For $-\infty < \sigma_1 < \sigma_2 < \infty$ and $1 \leq p \leq \infty$ we denote by $\mathcal{PW}_{[\sigma_1, \sigma_2]}^p$ the Paley–Wiener space of functions f with a representation $f(z) = 1/(2\pi) \int_{-\sigma_1}^{\sigma_2} g(\omega) e^{iz\omega} d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[\sigma_1, \sigma_2]$. The norm for $\mathcal{PW}_{[\sigma_1, \sigma_2]}^p$, $1 \leq p < \infty$, is given by $\|f\|_{\mathcal{PW}_{[\sigma_1, \sigma_2]}^p} = (1/(2\pi) \int_{\sigma_1}^{\sigma_2} |\hat{f}(\omega)|^p d\omega)^{1/p}$. For $\mathcal{PW}_{[-\sigma, \sigma]}^p$, $0 < \sigma < \infty$, we use the abbreviation \mathcal{PW}_σ^p . The nomenclature concerning the Bernstein and Paley–Wiener spaces, we introduced so far, is not consistent in the literature. Sometimes the space that we call Bernstein space is called Paley–Wiener space [45]. We adhere to the notation used in [27].

Since our analyses involve stable linear time-invariant (LTI) systems, we briefly review some definitions and facts. A linear system $T : \mathcal{PW}_\pi^p \rightarrow \mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$, is called stable if the operator T is bounded, i.e., if $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^p} \leq 1} \|Tf\|_{\mathcal{PW}_\pi^p} < \infty$. Furthermore, it is called time-invariant if $(Tf(\cdot - a))(t) = (Tf)(t - a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$.

For every stable LTI system $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) e^{i\omega t} d\omega, \quad t \in \mathbb{R}, \quad (1)$$

for all $f \in \mathcal{PW}_\pi^1$ [4]. Conversely, every function $\hat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$. The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}_T\|_{L^\infty[-\pi, \pi]}$. Furthermore, it can be shown that the representation (1) with $\hat{h}_T \in L^\infty[-\pi, \pi]$ is also valid for all stable LTI systems $T : \mathcal{PW}_\pi^2 \rightarrow \mathcal{PW}_\pi^2$. Therefore, every stable LTI system that maps \mathcal{PW}_π^1 in \mathcal{PW}_π^1 maps \mathcal{PW}_π^2 in \mathcal{PW}_π^2 , and vice versa. Note that $\hat{h}_T \in L^\infty[-\pi, \pi] \subset L^2[-\pi, \pi]$, and consequently $h_T \in \mathcal{PW}_\pi^2$.

An LTI system can have different representations. In textbooks, usually the frequency domain representation (1), and the time domain representation in the form of a convolution integral

$$(Tf)(t) = \int_{-\infty}^{\infty} f(\tau) h_T(t - \tau) d\tau \quad (2)$$

are given [23, 39]. Although both are well-defined for stable LTI systems $T : \mathcal{PW}_\pi^2 \rightarrow \mathcal{PW}_\pi^2$ operating on \mathcal{PW}_π^2 , there are systems and signal spaces where these representations are meaningless, because they are divergent [19, 3]. For example, it has been shown that there exist stable LTI systems $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ that do not have a convolution integral representation in the form of (2), because the integral diverges for certain signals $f \in \mathcal{PW}_\pi^1$ [3]. However, the frequency domain representation (1), which we will use in this paper, holds for all stable LTI systems $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$.

3 Sampling-Based Measurements

3.1 Basics of Non-Equidistant Sampling

In the classical non-equidistant sampling setting the goal is to reconstruct a bandlimited signal f from its non-equidistant samples $\{f(t_k)\}_{k \in \mathbb{Z}}$, where $\{t_k\}_{k \in \mathbb{Z}}$ is the sequence of sampling points. One possibility to do the reconstruction is to use the sampling series

$$\sum_{k=-\infty}^{\infty} f(t_k) \phi_k(t), \quad (3)$$

where the ϕ_k , $k \in \mathbb{Z}$, are certain reconstruction functions.

In this paper we restrict ourselves to sampling point sequences $\{t_k\}_{k \in \mathbb{Z}}$ that are real and a complete interpolating sequence for \mathcal{PW}_π^2 .

Definition 1. We say that $\{t_k\}_{k \in \mathbb{Z}}$ is a complete interpolating sequence for \mathcal{PW}_π^2 if the interpolation problem $f(t_k) = c_k, k \in \mathbb{Z}$, has exactly one solution $f \in \mathcal{PW}_\pi^2$ for every sequence $\{c_k\}_{k \in \mathbb{Z}} \in l^2$.

We further assume that the sequence of sampling points $\{t_k\}_{k \in \mathbb{Z}}$ is ordered strictly increasingly, and, without loss of generality, we assume that $t_0 = 0$. Then, it follows that the product

$$\phi(z) = z \lim_{N \rightarrow \infty} \prod_{\substack{|k| \leq N \\ k \neq 0}} \left(1 - \frac{z}{t_k}\right) \quad (4)$$

converges uniformly on $|z| \leq R$ for all $R < \infty$, and ϕ is an entire function of exponential type π [33]. It can be seen from (4) that ϕ , which is often called generating function, has the zeros $\{t_k\}_{k \in \mathbb{Z}}$. Moreover, it follows that

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t - t_k)} \quad (5)$$

is the unique function in \mathcal{PW}_π^2 that solves the interpolation problem $\phi_k(t_l) = \delta_{kl}$, where $\delta_{kl} = 1$ if $k = l$, and $\delta_{kl} = 0$ otherwise.

Definition 2. A system of vectors $\{\phi_k\}_{k \in \mathbb{Z}}$ in a separable Hilbert space \mathcal{H} is called Riesz basis if $\{\phi_k\}_{k \in \mathbb{Z}}$ is complete in \mathcal{H} , and there exist positive constants A and B such that for all $M, N \in \mathbb{N}$ and arbitrary scalars c_k we have

$$A \sum_{k=-M}^N |c_k|^2 \leq \left\| \sum_{k=-M}^N c_k \phi_k \right\|^2 \leq B \sum_{k=-M}^N |c_k|^2. \quad (6)$$

A well-known fact is the following theorem [53, p. 143].

Theorem 1 (Pavlov). *The system $\{e^{i\omega t_k}\}_{k \in \mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$ if and only if $\{t_k\}_{k \in \mathbb{Z}}$ is a complete interpolating sequence for \mathcal{PW}_π^2 .*

It follows immediately from Theorem 1 that $\{\phi_k\}_{k \in \mathbb{Z}}$, as defined in (5), is a Riesz basis for \mathcal{PW}_π^2 if $\{t_k\}_{k \in \mathbb{Z}}$ is a complete interpolating sequence for \mathcal{PW}_π^2 .

For further results and background information on non-equidistant sampling we would like to refer the reader to [27, 34].

3.2 Basics of Sampling-Based System Approximation

In many signal processing applications the goal is to process a signal f . In this paper we consider signals from the space \mathcal{PW}_π^1 . A common method to do such a processing is to use LTI systems. Given a signal $f \in \mathcal{PW}_\pi^1$ and a stable LTI system $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ we can use (1) to calculate the desired system output Tf . Equation (1) can be seen as an analog implementation of the system T . As

described in Section 2, (1) is well defined for all $f \in \mathcal{PW}_\pi^1$ and all stable LTI systems $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$, and we have no convergence problems.

However, often only the samples $\{f(t_k)\}_{k \in \mathbb{Z}}$ of a signal are available, like it is the case in digital signal processing, and not the whole signal. In this situation we seek an implementation of the stable LTI system T which uses only the samples $\{f(t_k)\}_{k \in \mathbb{Z}}$ of the signal f [48]. We call such an implementation an implementation in the digital domain. For example, the sampling series

$$\sum_{k=-\infty}^{\infty} f(t_k)(T\phi_k)(t) \quad (7)$$

is a digital implementation of the system T . However, in contrast to (1), the convergence of (7) is not guaranteed, as we will see in Section 3.4.

In Figure 1 the different approaches that are taken for an analog and a digital system implementation are visualized. The general motive for the development of the “digital world” is the idea that every stable analog system can be implemented digitally, i.e., that the diagram in Figure 1 is commutative.

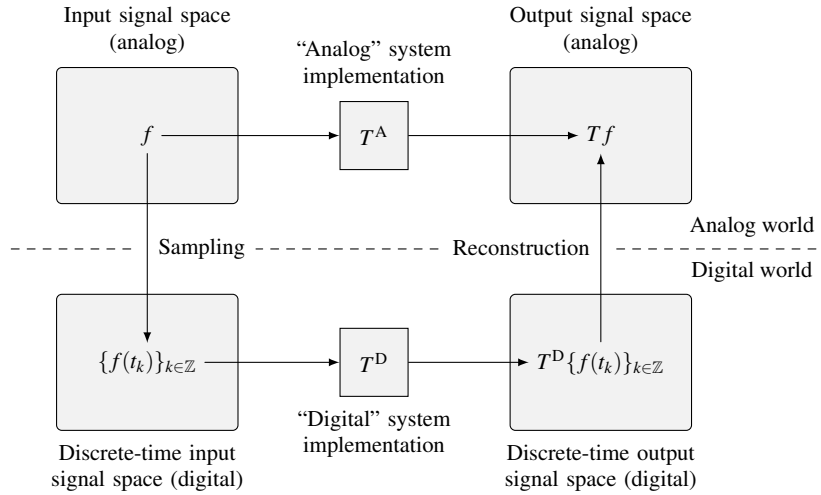


Fig. 1 Analog versus digital system implementation of a stable LTI system T .

Remark 1. In this paper the systems are always linear and well defined. However, there exist practically important systems that do not exist as a linear system [8]. For a discussion about non-linear systems, see [20].

3.3 Two Conjectures

In [5] we posed two conjectures, which we will prove in this paper. The first conjecture is about the divergence of the system approximation process for complete interpolating sequences in the case of classical pointwise sampling.

Conjecture 1. Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_π^2 , ϕ_k as defined in (5), and $0 < \sigma < \pi$. Then, for all $t \in \mathbb{R}$ there exists a stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ and a signal $f_* \in \mathcal{PW}_\sigma^1$ such that

$$\limsup_{N \rightarrow \infty} \left| (T_* f_*)(t) - \sum_{k=-N}^N f_*(t_k) (T_* \phi_k)(t) \right| = \infty.$$

For the special case of equidistant sampling, the system approximation process (7) reduces to

$$\frac{1}{a} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{a}\right) h_T\left(t - \frac{k}{a}\right), \quad (8)$$

where $a \geq 1$ denotes the oversampling factor and h_T is the impulse response of the system T . It has already been shown that the Hilbert transform is a universal system for which there exists, for every amount of oversampling, a signal such that the peak value of (8) diverges [4]. In Conjecture 1 now, the statement is that this divergence even occurs for non-equidistant sampling, which introduces an additional degree of freedom, and even pointwise. However, in this case, the Hilbert transform is no longer the universal divergence creating system.

Conjecture 1 will be proved in Section 3.4.

The second conjecture is about more general measurement procedures and states that with suitable measurement procedures and oversampling we can obtain a convergent approximation process.

Conjecture 2. Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_π^2 , ϕ_k as defined in (5), and $0 < \sigma < \pi$. There exists a sequence of continuous linear functionals $\{c_k\}_{k \in \mathbb{Z}}$ on \mathcal{PW}_π^1 such that for all stable LTI systems $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ and all $f \in \mathcal{PW}_\sigma^1$ we have

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| (Tf)(t) - \sum_{k=-N}^N c_k(f) (T\phi_k)(t) \right| = 0.$$

Conjecture 2 will be proved in Section 4, where we also introduce the general measurement procedures more precisely.

3.4 Approximation for Sampling-Based Measurements

In this section we analyze the system approximation process which is given by the digital implementation (7). The next theorem proves Conjecture 1.

Theorem 2. *Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_π^2 , ϕ_k as defined in (5), and $t \in \mathbb{R}$. Then there exists a stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ such that for every $0 < \sigma < \pi$ there exists a signal $f_* \in \mathcal{PW}_\sigma^1$ such that*

$$\limsup_{N \rightarrow \infty} \left| \sum_{k=-N}^N f_*(t_k) (T_* \phi_k)(t) \right| = \infty. \quad (9)$$

Remark 2. It is interesting to note that the system T_* in Theorem 2 is universal in the sense that it does not depend on σ , i.e., on the amount of oversampling. In other words, we can find a stable LTI system T_* such that regardless of the oversampling factor $1 < \alpha < \infty$ there exists a signal $f_* \in \mathcal{PW}_{\pi/\alpha}^1$ for which the system approximation process diverges as in (9).

Remark 3. Since $\{\phi_k\}_{k \in \mathbb{Z}}$ is a Riesz basis for \mathcal{PW}_π^2 , it follows that the projections of $\{\phi_k\}_{k \in \mathbb{Z}}$ onto \mathcal{PW}_σ^2 form a frame for \mathcal{PW}_σ^2 , $0 < \sigma < \pi$ [25, p. 231]. Theorem 2 shows that the usually nice behavior of frames is destroyed in the presence of a system T . Even though the projections of $\{\phi_k\}_{k \in \mathbb{Z}}$ onto \mathcal{PW}_σ^2 form a frame for \mathcal{PW}_σ^2 , $0 < \sigma < \pi$, we have divergence when we add the system T . This behavior was known before for pointwise sampling: The reconstruction functions in the Shannon sampling series form a Riesz basis for \mathcal{PW}_π^2 , and the convergence of the series is globally uniform for signals in \mathcal{PW}_σ^1 , $0 < \sigma < \pi$, i.e., if oversampling is applied. However, with a system T we can have even pointwise divergence [4]. Theorem 2 illustrates that this is true not only for pointwise sampling but also if more general measurement functionals are used.

Remark 4. The system T_* from Theorem 2 can, as a stable LTI system, of course be implemented, using the analog system implementation (1). However, Theorem 2 shows that a digital, i.e., sampling based, implementation is not possible. This also illustrates the limits of a general sampling-based technology. We will see later, in Section 4.2, that the system can be implemented by using more general measurement functionals and oversampling.

The result of Theorem 2 is also true for bandpass signals. However, in this case the stable LTI system T_* is no longer universal but depends on the actual frequency support of the signal space.

Theorem 3. *Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_π^2 , ϕ_k as defined in (5), $t \in \mathbb{R}$, and $0 < \sigma_1 < \sigma_2 < \pi$. Then there exist a stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ and a signal $f_* \in \mathcal{PW}_{[\sigma_1, \sigma_2]}^1$ such that*

$$\limsup_{N \rightarrow \infty} \left| \sum_{k=-N}^N f_*(t_k) (T_* \phi_k)(t) \right| = \infty.$$

For the proof of Theorems 2 and 3, we need two lemmas, Lemma 1 and Lemma 4. The proof of Lemma 1 heavily relies on a result of Szarek, which was published in [52].

Lemma 1. *Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_π^2 and ϕ_k as defined in (5). Then there exists a positive constant C_1 such that for all $\omega \in [-\pi, \pi]$ and all $N \in \mathbb{N}$ we have*

$$\max_{1 \leq M \leq N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=-M}^M e^{i\omega t_k} \hat{\phi}_k(\omega_1) \right| d\omega_1 \geq C_1 \log(N). \quad (10)$$

Remark 5. Later, in Section 5, we will see what potential implications the presence of the max-operator in (10) can have on the convergence behavior of the approximation process. Currently, our proof technique is not able to show more, however, we conjecture that (10) is also true without $\max_{1 \leq M \leq N}$.

For the proof of Lemma 1 we need Lemmas 2 and 3 from Szarek's paper [52]. For completeness and convenience, we state them next in a slightly simplified version, which is sufficient for our purposes.

Lemma 2 (Szarek). *Let f be a nonnegative measurable function, C_2 a positive constant, and n a natural number such that*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t))^2 dt \leq C_2 n \quad (11)$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t))^{5/4} dt \geq \frac{n^{1/4}}{C_2}. \quad (12)$$

Then there exists a number $\alpha = \alpha(C_2)$, $0 < \alpha < 2^{-3}$ and a natural number s such that

$$\frac{1}{2\pi} \int_{\{t \in [-\pi, \pi] : f(t) > \frac{n}{\alpha^2}\}} f(t) dt \leq \frac{\alpha}{2^4}$$

and

$$\frac{1}{2\pi} \int_{\{t \in [-\pi, \pi] : \frac{\alpha^2 n}{2} < f(t) \leq \frac{\alpha^2 n}{3}\}} f(t) dt \geq s\alpha.$$

Lemma 3 (Szarek). *Let $0 < \alpha < 2^{-3}$ and $\{F_k\}_{k=1}^N$ be a sequence of measurable functions. Further, define $F_{k,n} := F_{k+n} - F_k$. Assume that for all k, n satisfying $1 \leq k, n$ and $1 \leq k+n \leq N$ there exists a natural number $s = s(k, n)$ such that*

$$\frac{1}{2\pi} \int_{\{t \in [-\pi, \pi] : |F_{k,n}(t)| > \frac{n}{\alpha^2}\}} |F_{k,n}(t)| dt \leq \frac{\alpha}{2^4}$$

and

$$\frac{1}{2\pi} \int_{\{t \in [-\pi, \pi] : \frac{\alpha^2 n}{2} < |F_{k,n}(t)| \leq \frac{\alpha^2 n}{3}\}} |F_{k,n}(t)| dt \geq s\alpha.$$

Then there exists a positive constant $C_3 = C_3(\alpha)$ such that

$$\max_{1 \leq k \leq N} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F_k(t)| dt \geq C_3(\alpha) \log(N).$$

Now we are in the position to prove Lemma 1.

Proof (Lemma 1). Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an arbitrary but fixed ordered complete interpolating sequence for \mathcal{PW}_{π}^2 and ϕ_k as defined in (5). Further, let $\omega \in [-\pi, \pi]$ be arbitrary but fixed. For $\omega_1 \in [-\pi, \pi]$ consider the functions

$$G_k(\omega_1, \omega) := \sum_{l=-k}^k e^{i\omega t_l} \hat{\phi}_l(\omega_1),$$

and

$$\begin{aligned} G_{k,n}(\omega_1, \omega) &:= G_{k+n}(\omega_1, \omega) - G_k(\omega_1, \omega) \\ &= \sum_{k < |l| \leq k+n} e^{i\omega t_l} \hat{\phi}_l(\omega_1). \end{aligned}$$

We will show that $|G_{k,n}(\omega_1, \omega)|$ satisfies the conditions (11) and (12) of Lemma 2.

We have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |G_{k,n}(\omega_1, \omega)|^2 d\omega_1 &= \int_{-\infty}^{\infty} \left| \sum_{k < |l| \leq k+n} e^{i\omega t_l} \phi_l(t) \right|^2 dt \\ &\leq B \sum_{k < |l| \leq k+n} 1 \\ &= B2n, \end{aligned} \tag{13}$$

where we used the fact that $\{\phi_l\}_{l \in \mathbb{Z}}$ is a Riesz basis for \mathcal{PW}_{π}^2 .

Next, we analyze the expression

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G_k(\omega_1, \omega)|^p d\omega_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{l=-k}^k e^{i\omega t_l} \hat{\phi}_l(\omega_1) \right|^p d\omega_1$$

for $1 < p < 2$. We set

$$G_{k,n}(\omega_1, \omega) = 0 \quad \text{for } |\omega_1| > \pi \tag{14}$$

and consider the Fourier transform

$$(\mathcal{F}G_{k,n}(\cdot, \omega))(t) = \int_{-\infty}^{\infty} G_{k,n}(\omega_1, \omega) e^{-i\omega_1 t} d\omega_1. \tag{15}$$

Due to (14), the integral in (15) is absolutely convergent. We have

$$(\mathcal{F}G_{k,n}(\cdot, \omega))(t) = 2\pi g_{k,n}(-t, \omega),$$

where

$$g_{k,n}(t, \boldsymbol{\omega}) := \sum_{k < |l| \leq k+n} e^{i\omega_l t} \phi_l(t).$$

Let q be the conjugate of p , i.e., $1/p + 1/q = 1$, then the Hausdorff–Young inequality [9],[27, p. 19] shows that there exists a constant $C_4 = C_4(p)$ such that

$$\left(\int_{-\infty}^{\infty} |(\mathcal{F} G_{k,n}(\cdot, \boldsymbol{\omega}))(t)|^q dt \right)^{\frac{1}{q}} \leq C_4(p) \left(\int_{-\pi}^{\pi} |G_{k,n}(\boldsymbol{\omega}_1, \boldsymbol{\omega})|^p d\boldsymbol{\omega}_1 \right)^{\frac{1}{p}},$$

which implies that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G_{k,n}(\boldsymbol{\omega}_1, \boldsymbol{\omega})|^p d\boldsymbol{\omega}_1 \geq \frac{(2\pi)^{p-1}}{(C_4(p))^p} \left(\int_{-\infty}^{\infty} |g_{k,n}(t, \boldsymbol{\omega})|^q dt \right)^{\frac{p}{q}}. \quad (16)$$

Note that the constant $C_4(p)$ is independent of $\boldsymbol{\omega}$. We analyze the integral on the right-hand side of (16). We have $g_{k,n}(\cdot, \boldsymbol{\omega}) \in \mathcal{B}_{\pi}^q$. Since $\{t_k\}_{k \in \mathbb{Z}}$ is a complete interpolating sequence for \mathcal{PW}_{π}^2 , we have [41]

$$\inf_{k \in \mathbb{Z}} (t_{k+1} - t_k) > 0,$$

and it is known [2, p. 101] that there exists a positive constant $C_5(q)$ that is independent of k, n , and $\boldsymbol{\omega}$ such that

$$\left(\int_{-\infty}^{\infty} |g_{k,n}(t, \boldsymbol{\omega})|^q dt \right)^{\frac{1}{q}} \geq C_5(q) \left(\sum_{l=-\infty}^{\infty} |g_{k,n}(t_l, \boldsymbol{\omega})|^q \right)^{\frac{1}{q}}.$$

Since

$$\sum_{l=-\infty}^{\infty} |g_{k,n}(t_l, \boldsymbol{\omega})|^q = \sum_{k < |l| \leq n+k} 1 = 2n,$$

we obtain

$$\left(\int_{-\infty}^{\infty} |g_{k,n}(t, \boldsymbol{\omega})|^q dt \right)^{\frac{1}{q}} \geq C_5(q) (2n)^{\frac{1}{q}}. \quad (17)$$

Combining (16) and (17) gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G_{k,n}(\boldsymbol{\omega}_1, \boldsymbol{\omega})|^p d\boldsymbol{\omega}_1 \geq \frac{(2\pi)^{p-1} (C_5(q))^p}{(C_4(p))^p} (2n)^{\frac{p}{q}},$$

and for $p = 5/4$ we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G_{k,n}(\boldsymbol{\omega}_1, \boldsymbol{\omega})|^{\frac{5}{4}} d\boldsymbol{\omega}_1 \geq \frac{(4\pi)^{\frac{1}{4}} (C_5(5))^{\frac{5}{4}}}{(C_4(\frac{5}{4}))^{\frac{5}{4}}} n^{\frac{1}{4}}. \quad (18)$$

Choosing

$$C_2 = \max \left\{ 2B^2, \frac{(C_4(\frac{5}{4}))^{\frac{5}{4}}}{(4\pi)^{\frac{1}{4}}(C_5(5))^{\frac{5}{4}}} \right\},$$

we see from (13) and (18) that the function $|G_{k,n}(\omega_1, \omega)|$ satisfies conditions (11) and (12), that is the assumptions of Lemma 2. Hence, as a result of Lemma 2, $|G_k(\omega_1, \omega)|$ also satisfies the assumptions of Lemma 3, and application of Lemma 3 completes the proof. \square

Next, we state the second lemma which we need for the proofs of Theorems 2 and 3. We will use it to analyze the influence of the transfer function \hat{h}_T on the approximation process.

Lemma 4. *Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_{π}^2 and ϕ_k as defined in (5). For all $\omega \in [-\pi, \pi]$, all $t \in \mathbb{R}$, and all $N \in \mathbb{N}$ we have*

$$\begin{aligned} \sup_{\substack{\|\hat{g}\|_{L^\infty[-\pi, \pi]} \leq 1 \\ \hat{g} \in C[-\pi, \pi]}} \left| \sum_{k=-N}^N e^{i\omega t_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| \\ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^N e^{i\omega t_k} \hat{\phi}_k(\omega_1) \right| d\omega_1. \end{aligned}$$

Proof (Lemma 4). Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_{π}^2 , $\omega \in [-\pi, \pi]$, $t \in \mathbb{R}$, and $N \in \mathbb{N}$, all be arbitrary but fixed. Further, let ϕ_k be defined as in (5). For

$$\hat{g}(\omega_1) = \exp \left(-i \arg \left(e^{i\omega_1 t} \sum_{k=-N}^N e^{i\omega t_k} \hat{\phi}_k(\omega_1) \right) \right)$$

we have

$$\left| \sum_{k=-N}^N e^{i\omega t_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^N e^{i\omega t_k} \hat{\phi}_k(\omega_1) \right| d\omega_1. \quad (19)$$

Further, as a consequence of Lusin's theorem [43, p. 56], there exists a sequence of functions $\{\hat{g}_n\}_{n \in \mathbb{N}}$ with $\hat{g}_n \in C[-\pi, \pi]$ and $\|\hat{g}_n\|_{L^\infty[-\pi, \pi]} \leq 1$, such that $\lim_{n \rightarrow \infty} \hat{g}_n(\omega_1) = \hat{g}(\omega_1)$ almost everywhere. It follows from Lebesgue's dominated convergence theorem and (19) that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left| \sum_{k=-N}^N e^{i\omega_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}_n(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| \\
&= \left| \sum_{k=-N}^N e^{i\omega_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^N e^{i\omega_k} \hat{\phi}_k(\omega_1) \right| d\omega_1.
\end{aligned}$$

Hence, taking the limit $n \rightarrow \infty$ on both sides of

$$\begin{aligned}
& \sup_{\substack{\|\hat{g}\|_{L^\infty[-\pi, \pi]} \leq 1 \\ \hat{g} \in C[-\pi, \pi]}} \left| \sum_{k=-N}^N e^{i\omega_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| \\
&\geq \left| \sum_{k=-N}^N e^{i\omega_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}_n(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right|
\end{aligned}$$

completes the proof. \square

Proof (Theorem 2). Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an arbitrary but fixed ordered complete interpolating sequence for \mathcal{PW}_π^2 and ϕ_k as defined in (5). Further, let $t \in \mathbb{R}$ be arbitrary but fixed.

From Lemma 1 we see that

$$\sup_{N \in \mathbb{N}} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^N e^{i\omega_k} \hat{\phi}_k(\omega_1) \right| d\omega_1 = \infty$$

for all $\omega \in [-\pi, \pi]$. Due to Lemma 4 this implies that

$$\sup_{N \in \mathbb{N}} \left(\sup_{\substack{\|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \leq 1 \\ \hat{h}_T \in C[-\pi, \pi]}} \left| \sum_{k=-N}^N e^{i\omega_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| \right) = \infty$$

for all $\omega \in [-\pi, \pi]$. Thus, according to the Banach–Steinhaus theorem [43, p. 98], for all $\omega \in [-\pi, \pi]$ there exists a function $\hat{h}_{T_\omega} \in C[-\pi, \pi]$ such that

$$\limsup_{N \rightarrow \infty} \left(\left| \sum_{k=-N}^N e^{i\omega_k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_{T_\omega}(\omega_1) \hat{\phi}_k(\omega_1) e^{i\omega_1 t} d\omega_1 \right| \right) = \infty.$$

Further, since $\hat{h}_{T_\omega} \in C[-\pi, \pi] \subset L^\infty[-\pi, \pi]$, and since there is the bijection (1) between $L^\infty[-\pi, \pi]$ and the set of stable LTI systems $T_\omega: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$, it follows that for all $\omega \in [-\pi, \pi]$ there exists a stable LTI system T_ω such that

$$\limsup_{N \rightarrow \infty} \left| \sum_{k=-N}^N e^{i\omega t_k} (T_\omega \phi_k)(t) \right| = \infty.$$

In particular, for $\omega = 0$ there exists a stable LTI system $T_* = T_0$ such that

$$\limsup_{N \rightarrow \infty} \left| \sum_{k=-N}^N (T_* \phi_k)(t) \right| = \infty. \quad (20)$$

T_0 is the desired stable LTI system T_* .

Next, let $0 < \sigma < \pi$ be arbitrary but fixed. For $f \in \mathcal{PW}_\sigma^1$ and $N \in \mathbb{N}$ we have

$$\sum_{k=-N}^N f(t_k) (T_* \phi_k)(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \hat{f}(\omega_1) \sum_{k=-N}^N e^{i\omega_1 t_k} (T_* \phi_k)(t) d\omega_1.$$

Hence, it follows that

$$\begin{aligned} \sup_{\|f\|_{\mathcal{PW}_\sigma^1} \leq 1} \left| \sum_{k=-N}^N f(t_k) (T_* \phi_k)(t) \right| &= \max_{\omega_1 \in [-\sigma, \sigma]} \left| \sum_{k=-N}^N e^{i\omega_1 t_k} (T_* \phi_k)(t) \right| \\ &\geq \left| \sum_{k=-N}^N (T_* \phi_k)(t) \right|. \end{aligned}$$

Consequently, from (20) we obtain that

$$\limsup_{N \rightarrow \infty} \left(\sup_{\|f\|_{\mathcal{PW}_\sigma^1} \leq 1} \left| \sum_{k=-N}^N f(t_k) (T_* \phi_k)(t) \right| \right) = \infty.$$

Thus, the Banach–Steinhaus theorem [43, p. 98] implies that there exists a signal $f_* \in \mathcal{PW}_\sigma^1$ such that

$$\limsup_{N \rightarrow \infty} \left| \sum_{k=-N}^N f_*(t_k) (T_* \phi_k)(t) \right| = \infty.$$

This completes the proof. \square

Proof (Theorem 3). The proof of Theorem 3 is identical to the proof of Theorem 2, except that we choose $\omega \in [\sigma_1, \sigma_2]$ instead of $\omega = 0$. Since the divergence creating stable LTI system T_* depends on the actual choice of ω , we see that T_* is no longer universal in the sense that it is independent of σ_1 and σ_2 . \square

4 General Measurement Functionals and Oversampling

4.1 Basic Properties of General Measurement Functionals

A key concept in signal processing is to process analog, i.e., continuous-time signals in the digital domain. The first step in this procedure is to convert the continuous-time signal into a discrete-time signal, i.e., into a sequence of numbers. In Section 3 we analyzed a sampling-based system approximation, where the point evaluation functionals $f \mapsto f(t_k)$ are used to do this conversion. Next, we will proceed to more general measurement functionals [40, 12, 13].

The approximation of Tf by the system approximation process

$$\sum_{k=-N}^N f(t_k)(T\phi_k)(t) \quad (21)$$

can be seen as an approximation that uses the biorthogonal system $\{e^{-i \cdot t_k}, \hat{\phi}_k\}_{k \in \mathbb{Z}}$. In this setting, the sampling functionals, which define a certain measurement procedure, are given by

$$c_k(f) = f(t_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\omega t_k} d\omega, \quad (22)$$

and the functions

$$\phi_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\phi}_k(\omega) e^{i\omega t} d\omega$$

serve as reconstruction functions in the approximation process (21).

In Theorem 2 we have seen that for $f \in \mathcal{PW}_{\pi}^1$ even with oversampling an approximation of Tf using the process (21) is not possible in general, because there are signals $f \in \mathcal{PW}_{\pi}^1$ and stable LTI systems T such that (21) diverges.

Next, we will study more general measurement procedures than (22) in hopes of circumventing the divergence that was observed in Theorem 2. To this end, we consider a complete orthonormal system $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ in $L^2[-\pi, \pi]$.

For $f \in \mathcal{PW}_{\pi}^2$ the situation is simple. The measurement functionals $c_n: \mathcal{PW}_{\pi}^2 \rightarrow \mathbb{C}$ are given by

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \overline{\hat{\theta}_n(\omega)} d\omega = \int_{-\infty}^{\infty} f(t) \overline{\theta_n(t)} dt.$$

Further, we have

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{f}(\omega) - \sum_{n=1}^N c_n(f) \hat{\theta}_n(\omega) \right|^2 d\omega = 0$$

as well as

$$\lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \left| f(t) - \sum_{n=1}^N c_n(f) \theta_n(t) \right|^2 dt = 0$$

for all $f \in \mathcal{PW}_{\pi}^2$.

In order that

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \overline{\hat{\theta}_n(\omega)} d\omega \quad (23)$$

is also a reasonable measurement procedure for $f \in \mathcal{PW}_{\pi}^1$, we need the functionals $c_n: \mathcal{PW}_{\pi}^1 \rightarrow \mathbb{C}$, defined by (23), to be continuous and uniformly bounded in n . Since

$$\sup_{\|f\|_{\mathcal{PW}_{\pi}^1} \leq 1} |c_n(f)| = \|\hat{\theta}_n\|_{L^{\infty}[-\pi, \pi]},$$

this means we additionally have to require that the functions of the complete orthonormal system $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ satisfy

$$\sup_{n \in \mathbb{N}} \|\hat{\theta}_n\|_{L^{\infty}[-\pi, \pi]} < \infty. \quad (24)$$

Using these more general measurement functionals (23), the system approximation process takes the form

$$\sum_{n=1}^{\infty} c_n(f) (T\theta_n)(t). \quad (25)$$

In the next section we study the approximation process (25) and analyze its convergence behavior for signals $f \in \mathcal{PW}_{\sigma}^1$, $0 < \sigma < \pi$. We will see that with these more general linear measurement functionals a stable implementation of LTI systems is possible.

A special case of measurement functionals are local averages. Reconstruction of functions from local averages was, for example, studied in [49, 50, 51, 47].

4.2 Approximation for General Measurement Functionals and Oversampling

The next theorem describes the convergence behavior of the approximation process (25) in the case of oversampling.

Theorem 4. *Let $0 < \sigma < \pi$. There exists a complete orthonormal system $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ in $L^2[-\pi, \pi]$ satisfying (24), an associated sequence of measurement functionals $\{c_n\}_{n \in \mathbb{N}}$ as defined by (23), and a constant C_6 such that for all stable LTI systems $T: \mathcal{PW}_{\pi}^1 \rightarrow \mathcal{PW}_{\pi}^1$ and all $f \in \mathcal{PW}_{\sigma}^1$ we have*

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N c_n(f) (T\theta_n)(t) \right| \leq C_6 \|f\|_{\mathcal{PW}_{\sigma}^1} \|T\|$$

for all $N \in \mathbb{N}$, and further

$$\lim_{N \rightarrow \infty} \left(\sup_{t \in \mathbb{R}} \left| (Tf)(t) - \sum_{n=1}^N c_n(f)(T\theta_n)(t) \right| \right) = 0. \quad (26)$$

Remark 6. Theorem 4 shows that, using oversampling and more general measurement functionals, it is possible to have a stable system approximation with the process (25). This is in contrast to pointwise sampling, which was analyzed in Section 3.4, where even oversampling is not able to prevent the divergence. It is interesting to note that Theorem 4 is not only an abstract existence result. The complete orthonormal system $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ which is used in Theorem 4 can be explicitly constructed by a procedure given in [36, 37].

Remark 7. In Section 4.3 we will see that oversampling is necessary in order to obtain Theorem 4, i.e., a stable system implementation is only possible with oversampling and suitable measurement functionals.

Remark 8. Theorem 4 also shows that, for the space \mathcal{PW}_π^1 , it is sufficient to use a linear process for the system approximation if oversampling is used, which introduces a kind of redundance. However, for other Banach spaces this is not necessarily true. There exist Banach spaces where non-linear processes have to be used, even in the signal reconstruction problem [40].

For the proof of Theorem 4 we need the following theorem from [36, 37].

Theorem 5 (Olevskii). *Let $0 < \delta < 1$. There exists an orthonormal system $\{\psi_n\}_{n \in \mathbb{N}}$ of real-valued functions that is closed in $C[0, 1]$ such that*

$$\sup_{n \in \mathbb{N}} \|\psi_n\|_{L^\infty[0,1]} < \infty$$

and such that there exists a constant C_7 such that for all $x \in [\delta, 1]$ and all $N \in \mathbb{N}$ we have

$$\int_0^1 \left| \sum_{n=1}^N \psi_n(x) \psi_n(\tau) \right| d\tau \leq C_7.$$

Remark 9. In the above theorem, we adopted the notion of “closed” from [38]. In [38] a system $\{\psi_n\}_{n \in \mathbb{N}}$ is called closed in $C[0, 1]$ if every function in $C[0, 1]$ can be uniformly approximated by finite linear combinations of the system $\{\psi_n\}_{n \in \mathbb{N}}$, that is if for every $\varepsilon > 0$ and every $f \in C[0, 1]$ there exists an $N \in \mathbb{N}$ and a sequence $\{\alpha_n\}_{n=1}^N \subset \mathbb{C}$ such that $\|f - \sum_{n=1}^N \alpha_n \psi_n\|_{L^\infty[0,1]} < \varepsilon$.

Proof (Theorem 4). Let $0 < \sigma < \pi$ be arbitrary but fixed and set $\delta = (\pi - \sigma)/(2\pi)$. Using the functions ψ_n from Theorem 5, we define

$$\hat{\theta}_n(\omega) := \psi_n \left(\frac{\omega + \pi}{2\pi} \right), \quad \omega \in [-\pi, \pi].$$

Due to the properties of the functions ψ_n , we see that $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system for $L^2[-\pi, \pi]$, and that

$$\sup_{n \in \mathbb{N}} \|\hat{\theta}_n\|_{L^\infty[-\pi, \pi]} < \infty.$$

Furthermore, for $\omega \in [-\sigma, \sigma]$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=1}^N \hat{\theta}_n(\omega) \hat{\theta}_n(\omega_1) \right| d\omega_1 &= \int_0^1 \left| \sum_{n=1}^N \hat{\psi}_n \left(\frac{\omega + \pi}{2\pi} \right) \psi_n(\tau) \right| d\tau \\ &\leq C_7, \end{aligned} \quad (27)$$

according to Theorem 5, because for $\omega \in [-\sigma, \sigma]$ we have $(\omega + \pi)/(2\pi) \in [\delta, 1]$. Next, we study for $f \in \mathcal{PW}_\sigma^1$ the expression

$$\begin{aligned} (U_N \hat{f})(\omega) &:= \sum_{n=1}^N c_n(f) \hat{\theta}_n(\omega) \\ &= \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \hat{f}(\omega_1) \sum_{n=1}^N \hat{\theta}_n(\omega) \hat{\theta}_n(\omega_1) d\omega_1. \end{aligned}$$

We have

$$|(U_N \hat{f})(\omega)| \leq \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega_1)| \left| \sum_{n=1}^N \hat{\theta}_n(\omega) \hat{\theta}_n(\omega_1) \right| d\omega_1,$$

which implies, using Fubini's theorem and (27), that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |(U_N \hat{f})(\omega)| d\omega &\leq \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega_1)| \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=1}^N \hat{\theta}_n(\omega) \hat{\theta}_n(\omega_1) \right| d\omega \right) d\omega_1 \\ &\leq C_7 \|f\|_{\mathcal{PW}_\sigma^1}. \end{aligned} \quad (28)$$

Now, let $f \in \mathcal{PW}_\sigma^1$ and $\varepsilon > 0$ be arbitrary but fixed. Then there exists an $f_\varepsilon \in \mathcal{PW}_\sigma^2$ such that

$$\|f - f_\varepsilon\|_{\mathcal{PW}_\sigma^1} < \varepsilon. \quad (29)$$

We have

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega) - (U_N \hat{f})(\omega)| d\omega \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega) - \hat{f}_\varepsilon(\omega)| d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}_\varepsilon(\omega) - (U_N \hat{f}_\varepsilon)(\omega)| d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} |(U_N(\hat{f} - \hat{f}_\varepsilon))(\omega)| d\omega \\ &\leq \varepsilon + C_7 \varepsilon + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}_\varepsilon(\omega) - (U_N \hat{f}_\varepsilon)(\omega)|^2 d\omega \right)^{\frac{1}{2}}, \end{aligned}$$

where we used (28) and (29). Since $\mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\pi^2$ and $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L^2[-\pi, \pi]$, there exists a natural number $N_0 = N_0(\varepsilon)$ such that

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}_\varepsilon(\omega) - (U_N \hat{f}_\varepsilon)(\omega)|^2 d\omega \right)^{\frac{1}{2}} < \varepsilon$$

for all $N \geq N_0$. Hence, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega) - (U_N \hat{f})(\omega)| d\omega \leq \varepsilon(2 + C_7)$$

for all $N \geq N_0$. This shows that

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega) - (U_N \hat{f})(\omega)| d\omega = 0. \quad (30)$$

Next, let $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ be an arbitrary but fixed stable LTI system. We have

$$\begin{aligned} (Tf)(t) - \sum_{n=1}^N c_n(f)(T\theta_n)(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{f}(\omega) \hat{h}_T(\omega) e^{i\omega t} - \sum_{n=1}^N c_n(f) \hat{h}_T(\omega) \hat{\theta}_n(\omega) e^{i\omega t} \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\hat{f}(\omega) - (U_N \hat{f})(\omega)) \hat{h}_T(\omega) e^{i\omega t} d\omega \end{aligned}$$

and consequently

$$\left| (Tf)(t) - \sum_{n=1}^N c_n(f)(T\theta_n)(t) \right| \leq \|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega) - (U_N \hat{f})(\omega)| d\omega \quad (31)$$

for all $t \in \mathbb{R}$. From (30) and (31) we see that

$$\lim_{N \rightarrow \infty} \left(\sup_{t \in \mathbb{R}} \left| (Tf)(t) - \sum_{n=1}^N c_n(f)(T\theta_n)(t) \right| \right) = 0.$$

Further, we have

$$\begin{aligned} \left| \sum_{n=1}^N c_n(f)(T\theta_n)(t) \right| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |(U_N \hat{f})(\omega) \hat{h}_T(\omega)| d\omega \\ &\leq C_7 \|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \|f\|_{\mathcal{PW}_\sigma^1}, \end{aligned}$$

where we used (28) in the last inequality. \square

Remark 10. Since $\{\hat{\theta}_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L^2[-\pi, \pi]$, it follows that the projections of the functions $\{\theta_n\}_{n \in \mathbb{N}}$ onto \mathcal{PW}_σ^2 form a Parseval frame

for \mathcal{PW}_σ^2 , $0 < \sigma < \pi$ [25, p. 231]. Although we have seen in Remark 3 that a frame does not necessary lead to a convergent approximation process, Theorem 4 shows that there are even Parseval frames for which we have convergence.

4.3 The Necessity of Oversampling

In Section 4.2 we have seen that if oversampling and generalized measurement functionals are used, we can approximate Tf by (25). The question whether this remains true if no oversampling is used, is the subject of this section. We want to answer this question for a large class of practically relevant measurement functionals.

We start with a biorthogonal system $\{\hat{\gamma}_n, \hat{\phi}_n\}_{n \in \mathbb{N}}$, i.e., a system that satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\gamma}_n(\omega) \overline{\hat{\phi}_m(\omega)} d\omega = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Further, we assume that $\{\hat{\gamma}_n\}_{n \in \mathbb{N}} \subset L^\infty[-\pi, \pi]$ and $\{\hat{\phi}_n\}_{n \in \mathbb{N}} \subset \mathcal{PW}_\pi^2$, and define the measurement functionals by

$$c_n(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \overline{\hat{\gamma}_n(\omega)} d\omega, \quad n \in \mathbb{N}. \quad (32)$$

As discussed in Section 4, we additionally require that

$$\sup_{n \in \mathbb{N}} \|\hat{\gamma}_n\|_{L^\infty[-\pi, \pi]} < \infty, \quad (33)$$

in order that (32) defines reasonable measurement functionals for $f \in \mathcal{PW}_\pi^1$. We further assume that there exists a constant C_8 such that for any finite sequence $\{a_n\}$ we have

$$\int_{-\infty}^{\infty} \left| \sum_n a_n \phi_n(t) \right|^2 dt \leq C_8 \sum_n |a_n|^2. \quad (34)$$

Condition (34) relates the l^2 -norm of the coefficients to the $L^2(\mathbb{R})$ -norm of the continuous-time signal. If (34) is fulfilled, the $L^2(\mathbb{R})$ -norm of the continuous-time signal is always bounded above by the l^2 -norm of the coefficients, i.e., the measurement values. This property is practically interesting, because in digital signal processing we operate on the sequence of coefficients $\{a_n\}_{n \in \mathbb{N}}$ by using stable $l^2 \rightarrow l^2$ mappings, and we always want to be able to control the $L^2(\mathbb{R})$ -norm of the corresponding continuous-time signal. Note that in the special case of equidistant pointwise sampling at Nyquist rate, the norms are equal according to Parseval's equality.

Remark 11. Instead of requiring (34) to hold we could also require that there exists a constant C_9 such that for any finite sequence $\{a_n\}$ we have

$$\int_{-\infty}^{\infty} \left| \sum_n a_n \gamma_n(t) \right|^2 dt \geq C_9 \sum_n |a_n|^2. \quad (35)$$

Indeed (35) is a weaker assumption than (34), because condition (34) implies condition (35) but the reverse direction is not true in general.

Remark 12. Note that the setting which we consider here is a generalization of the setting that arises when starting with complete interpolating sequences.

Theorem 6. Let $\{\hat{\gamma}_n, \hat{\phi}_n\}_{n \in \mathbb{N}}$ be a biorthogonal system that satisfies (33) and (34), and let $\{c_n\}_{n \in \mathbb{N}}$ be the associated sequence of measurement functionals as defined by (32). For every $t \in \mathbb{R}$ there exist a stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ and a signal $f_* \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N \rightarrow \infty} \left| \sum_{n=1}^N c_n(f_*)(T_*\phi_n)(t) \right| = \infty. \quad (36)$$

Remark 13. The orthonormal sequence from Section 4 of course satisfies the conditions of Theorem 6. This shows how important the assumption of oversampling, i.e., $f \in \mathcal{PW}_\sigma^1$, $\sigma < \pi$, is in order to obtain Theorem 4.

For the proof we use the following result from [52], which is included here for convenience, with a slightly modified notation.

Proposition 1 (Szarek). Let (S, \mathcal{B}, m) a probability space and $\{f_n, g_n\}_{n \in \mathbb{N}}$ a biorthogonal sequence of measurable functions on S (i.e., $\int_S f_k \bar{g}_n dm = \delta_{kn}$) such that

1. $\|g_n\|_\infty \leq 1$ for $n = 1, 2, \dots, N$.
2. $\int_S |\sum_{n=1}^N s_n f_n|^2 dm \leq C \sum_{n=1}^N |s_n|^2$ for some $C > 0$ and for all sequences of scalars s_1, \dots, s_N (and, as a consequence, $\int_S |\sum_{n=1}^N t_n g_n|^2 dm \geq C^{-1} \sum_{n=1}^N |t_n|^2$ for all scalars t_1, \dots, t_N).

Then there exists $C' > 0$, depending only on C , such that

$$\max_{1 \leq M \leq N} \int_S \int_S \left| \sum_{n=1}^M \overline{g_n(t)} f_n(s) \right| dm(t) dm(s) \geq C' \log(N).$$

Proof (Theorem 6). Let $\{\hat{\gamma}_n, \hat{\phi}_n\}_{n \in \mathbb{N}}$ be an arbitrary but fixed biorthogonal system that satisfies (33) and (34). According to Proposition 1 we have

$$\max_{1 \leq M \leq N} \left(\frac{1}{2\pi} \right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \sum_{n=1}^M \overline{\hat{\gamma}_n(\omega)} \hat{\phi}_n(\omega_1) \right| d\omega d\omega_1 \geq C_{10} \log(N)$$

with a universal constant C_{10} . This implies that

$$\max_{1 \leq M \leq N} \operatorname{ess\,sup}_{\omega \in [-\pi, \pi]} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=1}^M \overline{\hat{\gamma}_n(\omega)} \hat{\phi}_n(\omega_1) \right| d\omega_1 \geq C_{10} \log(N).$$

As in the proof of Theorem 2 it is shown that there exists a stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ such that

$$\limsup_{N \rightarrow \infty} \operatorname{ess\,sup}_{\omega \in [-\pi, \pi]} \left| \sum_{n=1}^N \overline{\hat{\gamma}_n(\omega)} (T_* \phi_n)(t) \right| = \infty.$$

And again by the same reasoning as in the proof of Theorem 2, there exists a signal $f_* \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N \rightarrow \infty} \left| \sum_{n=1}^N \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_*(\omega) \overline{\hat{\gamma}_n(\omega)} d\omega \right) (T_* \phi_n)(t) \right| = \infty.$$

This completes the proof. \square

5 Convergence of Subsequences of Certain Measurement Procedures

So far, we have seen that a system approximation is possible if we use suitable measurement functionals and oversampling. Further, the previous section has shown that oversampling is necessary, because without oversampling we can always find a stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ and a signal $f_* \in \mathcal{PW}_\pi^1$ such that (36) is true. Since in (36) we have a lim sup, it is legitimate to ask whether there exists an increasing subsequence $\{M_N\}_{N \in \mathbb{N}}$ of the natural numbers such that

$$\lim_{N \rightarrow \infty} \left| (Tf)(t) - \sum_{n=1}^{M_N} c_n(f)(T\phi_n)(t) \right| = 0. \quad (37)$$

If (37) was true it would show that a careful choice of the number of measurements that are used in each step of the approximation could generate a convergent approximation process, even without oversampling. Theorem 7 will answer this question in the affirmative for a special pair of measurement functionals and reconstruction functions.

For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we consider the functions

$$\hat{\theta}_k(\omega) = w_k \left(\frac{\omega + \pi}{2\pi} \right), \quad -\pi \leq \omega < \pi, \quad (38)$$

where w_k are the Walsh functions. Then $\{\hat{\theta}_k\}_{k \in \mathbb{N}_0}$ is a complete orthonormal system in $L^2[-\pi, \pi]$. Further, let $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ be a stable LTI system. For $t \in \mathbb{R}$ we define

$$c_k(f, t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{\theta}_k(\omega) e^{i\omega t} d\omega, \quad (39)$$

and analyze the convergence behavior of

$$\sum_{k=0}^{2^N} c_k(f, 0)(T\theta_k)(t) \quad (40)$$

and

$$\sum_{k=0}^{2^N} c_k(f, t)(T\theta_k)(0) \quad (41)$$

as N tends to infinity. In (40) we have the ordinary system approximation process, except for the difference that the number of measurements, and consequently the number of summands used for the approximation, is doubled in each approximation step. In (41) we have an alternative implementation of the system, where the variable t is included in the measurement functionals. As in (40), the number of measurements is doubled in each step.

We have the following result.

Theorem 7. Let $\{\theta_k\}_{k \in \mathbb{N}_0}$ be defined through its Fourier transform (38) and c_k as in (39). For all $f \in \mathcal{PW}_\pi^1$ and all stable LTI systems $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ we have

$$\lim_{N \rightarrow \infty} \left(\sup_{t \in \mathbb{R}} \left| (Tf)(t) - \sum_{k=0}^{2^N} c_k(f, 0)(T\theta_k)(t) \right| \right) = 0 \quad (42)$$

and

$$\lim_{N \rightarrow \infty} \left(\sup_{t \in \mathbb{R}} \left| (Tf)(t) - \sum_{k=0}^{2^N} c_k(f, t)(T\theta_k)(0) \right| \right) = 0. \quad (43)$$

Theorem 7 shows that there exists a complete orthonormal system that leads to a stable system approximation process for all $f \in \mathcal{PW}_\pi^1$ and all stable LTI systems $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ if we restrict to a suitable subsequence. It is important to note that the subsequence is universal because it neither depends on the signal f nor on the system T . It is also interesting that with this kind of approximation we do not need oversampling in order to have convergence.

Remark 14. For sampling-based signal processing with equidistant sampling points at Nyquist rate such a result cannot exist, because for every subsequence $\{M_N\}_{N \in \mathbb{N}}$ of the natural numbers there exists a signal $f_* \in \mathcal{PW}_\pi^1$ and stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ such that

$$\limsup_{N \rightarrow \infty} \left(\sup_{t \in \mathbb{R}} \left| (T_* f_*)(t) - \sum_{k=-M_N}^{M_N} f_*(k)(T_* \operatorname{sinc}(\cdot - k))(t) \right| \right) = \infty.$$

This follows directly from the fact that there exists a positive constant C_{11} such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^N e^{ik(\omega - \omega_1)} \right| d\omega_1 \geq C_{11} \log(N) \quad (44)$$

for all $\omega \in [-\pi, \pi]$ and all $N \in \mathbb{N}$ [54, p. 67].

Proof (Theorem 7). Let $\{\theta_k\}_{k \in \mathbb{N}_0}$ be defined through its Fourier transform (38) and c_k as in (39). Further, let $f \in \mathcal{PW}_\pi^1$ and $T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ be a stable LTI system, both arbitrary but fixed.

We first prove (42). In [22] it was shown that

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{f}(\omega) - \sum_{k=0}^{2^N} c_k(f, 0) \hat{\theta}_k(\omega) \right| d\omega = 0.$$

Further, since

$$\begin{aligned} & \left| (Tf)(t) - \sum_{k=0}^{2^N} c_k(f, 0) (T\theta_k)(t) \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) e^{i\omega t} d\omega - \sum_{k=0}^{2^N} c_k(f, 0) \hat{\theta}_k(\omega) \hat{h}_T(\omega) e^{i\omega t} d\omega \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{f}(\omega) - \sum_{k=0}^{2^N} c_k(f, 0) \hat{\theta}_k(\omega) \right) \hat{h}_T(\omega) e^{i\omega t} d\omega \right| \\ &\leq \|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{f}(\omega) - \sum_{k=0}^{2^N} c_k(f, 0) \hat{\theta}_k(\omega) \right| d\omega, \end{aligned}$$

the first assertion of the theorem is proved.

Next, we prove (43). Let $\varepsilon > 0$ be arbitrary but fixed. There exists a measurable set $F_\varepsilon \subset [-\pi, \pi]$ such that

$$\frac{1}{2\pi} \int_{F_\varepsilon} |\hat{f}(\omega)| d\omega < \frac{\varepsilon}{2}$$

and

$$\operatorname{ess\,sup}_{\omega \in [-\pi, \pi] \setminus F_\varepsilon} |\hat{f}(\omega)| = C(\hat{f}, F_\varepsilon) < \infty.$$

Further, we have

$$\begin{aligned}
& \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right) \hat{f}(\omega) e^{i\omega t} d\omega \right| \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| d\omega \\
& = \frac{1}{2\pi} \int_{F_\varepsilon} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| d\omega \\
& \quad + \frac{1}{2\pi} \int_{[-\pi, \pi] \setminus F_\varepsilon} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| d\omega. \quad (45)
\end{aligned}$$

Next, we analyze the two summands on the right hand side of (45). For the first summand we have

$$\begin{aligned}
& \frac{1}{2\pi} \int_{F_\varepsilon} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| d\omega \\
& \leq \frac{1}{2\pi} \int_{F_\varepsilon} |\hat{h}_T(\omega)| |\hat{f}(\omega)| d\omega + \frac{1}{2\pi} \int_{F_\varepsilon} \left| \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| d\omega \\
& \leq 2 \|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \frac{1}{2\pi} \int_{F_\varepsilon} |\hat{f}(\omega)| d\omega \\
& < \varepsilon \|\hat{h}_T\|_{L^\infty[-\pi, \pi]}, \quad (46)
\end{aligned}$$

because

$$\begin{aligned}
\left| \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega_1) \sum_{k=0}^{2^N} \hat{\theta}_k(\omega) \hat{\theta}_k(\omega_1) d\omega_1 \right| \\
&\leq \|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{2^N} \hat{\theta}_k(\omega) \hat{\theta}_k(\omega_1) \right| d\omega_1
\end{aligned}$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{2^N} \hat{\theta}_k(\omega) \hat{\theta}_k(\omega_1) \right| d\omega_1 = 1 \quad (47)$$

for all $\omega \in [-\pi, \pi]$ [22, 44]. For the second summand we have

$$\begin{aligned}
& \frac{1}{2\pi} \int_{[-\pi, \pi] \setminus F_\varepsilon} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| \, d\omega \\
& \leq C(\hat{f}, F_\varepsilon) \frac{1}{2\pi} \int_{[-\pi, \pi] \setminus F_\varepsilon} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| \, d\omega \\
& \leq C(\hat{f}, F_\varepsilon) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right|^2 \, d\omega \right)^{\frac{1}{2}},
\end{aligned}$$

and because

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right|^2 \, d\omega = 0,$$

according to [44], there exists a natural number $N_0 = N_0(\varepsilon)$ such that

$$\frac{1}{2\pi} \int_{[-\pi, \pi] \setminus F_\varepsilon} \left| \hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right| |\hat{f}(\omega)| \, d\omega < \varepsilon \quad (48)$$

for all $N \geq N_0$. Combining (45), (46), and (48), we see that

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right) \hat{f}(\omega) e^{i\omega t} \, d\omega \right| \leq (\|\hat{h}_T\|_{L^\infty[-\pi, \pi]} + 1) \varepsilon$$

for all $N \geq N_0$, and since

$$\begin{aligned}
& \left| (Tf)(t) - \sum_{k=0}^{2^N} c_k(f, t)(T\theta_k)(0) \right| \\
& = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{h}_T(\omega) - \sum_{k=0}^{2^N} \hat{\theta}_k(\omega)(T\theta_k)(0) \right) \hat{f}(\omega) e^{i\omega t} \, d\omega \right|,
\end{aligned}$$

the proof is complete. \square

Theorem 7 shows that if the summation is restricted to a suitable subsequence of the natural numbers, we can have a convergent system approximation process if we use measurement functionals. Now the question arises if this is also true for pointwise sampling as analyzed in Section 3. Since in Theorem 2 we only have a limsup this could be the case. However, we have the following conjecture.

Conjecture 3. Let $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ be an ordered complete interpolating sequence for \mathcal{PW}_π^2 . Then there exists a positive constant C_{12} such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^N e^{i\omega t_k} \hat{\phi}_k(\omega_1) \right| \, d\omega_1 \geq C_{12} \log(N) \quad (49)$$

for all $\omega \in [-\pi, \pi]$ and all $N \in \mathbb{N}$.

If this conjecture is true then the derivations in this work imply that a theorem such as Theorem 7 cannot hold for the sampling-based system approximation that was treated in Section 3. Because then, for every subsequence $\{M_N\}_{N \in \mathbb{N}}$ of the natural numbers and all ordered complete interpolating sequences $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ we have

$$\limsup_{N \rightarrow \infty} \left(\sup_{t \in \mathbb{R}} \left| (T_* f_*)(t) - \sum_{k=-M_N}^{M_N} f_*(t_k) (T_* \phi_k)(t) \right| \right) = \infty \quad (50)$$

for some $f_* \in \mathcal{PW}_\pi^1$ and some stable LTI system $T_*: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$. In fact, in order to obtain this negative result for sampling-based system approximation it would suffice to have an arbitrary sequence $\{L_N\}_{N \in \mathbb{N}}$ with $\lim_{N \rightarrow \infty} L_N = \infty$ on the right-hand side of (49). Note that we already know from (44) and Remark 14 that Conjecture 3 and (50) are true for the special case of equidistant sampling.

6 More General Measurement Functionals

In this section we consider even more general measurement functionals than those in Section 4. For this, we restrict ourselves to stable LTI systems T with continuous \hat{h}_T .

Now let $\{\hat{g}_n\}_{n \in \mathbb{N}} \subset C[-\pi, \pi]$ be a sequence of functions with the following properties:

1. $\sup_{n \in \mathbb{N}} \|\hat{g}_n\|_{L^\infty[-\pi, \pi]} < \infty$ and $\inf_{n \in \mathbb{N}} \|\hat{g}_n\|_{L^\infty[-\pi, \pi]} > 0$.
2. $\{\hat{g}_n\}_{n \in \mathbb{N}}$ is closed in $C[-\pi, \pi]$ and minimal, in the sense that for all $m \in \mathbb{N}$ the function \hat{g}_m is not in the closed span of $\{\hat{g}_n\}_{n \neq m}$.
3. There exists a constant $C_{13} > 0$ such that for any finite sequences $\{a_n\}$ we have

$$\left\| \sum_n a_n \hat{g}_n \right\|_{L^\infty[-\pi, \pi]} \geq \frac{1}{C_{13}} \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}}. \quad (51)$$

Property 2 guarantees that there exists a unique sequence of functionals $\{u_n\}_{n \in \mathbb{N}}$ which is biorthogonal to $\{\hat{g}_n\}_{n \in \mathbb{N}}$ [25, p. 155].

We shortly discuss the structure of measurement functionals and approximation processes which are based on sequences $\{\hat{g}_n\}_{n \in \mathbb{N}} \subset C[-\pi, \pi]$ that satisfy the properties 1–3. Let $\{u_n\}_{n \in \mathbb{N}}$ be the unique sequence of functionals which is biorthogonal to $\{\hat{g}_n\}_{n \in \mathbb{N}}$. Since we assume that $\hat{h}_T \in C[-\pi, \pi]$, it follows that there exist finite regular Borel measures μ_n such that

$$u_n(\hat{h}_T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega) d\mu_n(\omega).$$

In [52] it was shown that, due to property 3, there exists a regular Borel measure ν such that

$$\sum_{n=1}^{\infty} |c_n(\hat{h}_T)|^2 \leq C_{14} \int_{-\pi}^{\pi} |\hat{h}_T(\omega)|^2 d\nu(\omega).$$

Further, all Borel measures μ_n are absolutely continuous with respect to ν , and the Radon–Nikodym derivatives of μ_n with respect to ν , which we call F_n , are in $L^2(\nu)$, i.e., we have

$$\int_{-\pi}^{\pi} |F_n(\omega)|^2 d\nu(\omega) < \infty.$$

It follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}_n(\omega) F_l(\omega) d\nu(\omega) = \begin{cases} 1, & n = l, \\ 0, & n \neq l, \end{cases}$$

i.e., the system $\{\hat{g}_n, \overline{F_n}\}_{n \in \mathbb{N}}$ is a biorthogonal system with respect to the measure ν .

Note that this time we have a system that is biorthogonal with respect to the regular Borel measure ν and not with respect to the Lebesgue measure, as before. Thus, if we only require property 3, we cannot find a corresponding biorthogonal system for the Lebesgue measure in general, but only for more general measures. Nevertheless, we can obtain the divergence result that is stated in Theorem 8.

In [52] it was analyzed whether a basis for $C[-\pi, \pi]$ that satisfies the above properties 1–3 could exist, and the nonexistence of such a basis was proved. We employ this result to prove the following theorem, in which we use the abbreviations

$$c_n(f, t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{g}_n(\omega) e^{i\omega t} d\omega.$$

and

$$w_n(\hat{h}_T, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_T(\omega) e^{i\omega t} F_n(\omega) d\nu(\omega). \quad (52)$$

Theorem 8. *Let $\{\hat{g}_n\}_{n \in \mathbb{N}} \subset C[-\pi, \pi]$ be an arbitrary sequence of functions that satisfies the above properties 1–3, and let $t \in \mathbb{R}$. Then we have:*

1. *There exists a stable LTI system $T_{*1}: \mathcal{PW}_{\pi}^1 \rightarrow \mathcal{PW}_{\pi}^1$ with $\hat{h}_{T_{*1}} \in C[-\pi, \pi]$ and a signal $f_{*1} \in \mathcal{PW}_{\pi}^1$ such that*

$$\limsup_{N \rightarrow \infty} \left| \sum_{n=1}^N c_n(f_{*1}, t) w_n(\hat{h}_{T_{*1}}, 0) \right| = \infty. \quad (53)$$

2. *There exists a stable LTI system $T_{*2}: \mathcal{PW}_{\pi}^1 \rightarrow \mathcal{PW}_{\pi}^1$ with $\hat{h}_{T_{*2}} \in C[-\pi, \pi]$ and a signal $f_{*2} \in \mathcal{PW}_{\pi}^1$ such that*

$$\limsup_{N \rightarrow \infty} \left| \sum_{n=1}^N c_n(f_{*2}, 0) w_n(\hat{h}_{T_{*2}}, t) \right| = \infty. \quad (54)$$

Proof. We start with the proof of assertion 1. In [52] it was proved that there exists no basis for $C[-\pi, \pi]$ with the above properties 1–3. That is, if we set

$$(S_N \hat{h}_T)(\omega) = \sum_{n=1}^N w_n(\hat{h}_T, 0) \hat{g}_n(\omega), \quad \omega \in [-\pi, \pi],$$

then, for

$$\|S_N\| = \sup_{\substack{\hat{h}_T \in C[-\pi, \pi], \\ \|\hat{h}_T\|_{L^\infty[-\pi, \pi]} \leq 1}} \|S_N \hat{h}_T\|_{L^\infty[-\pi, \pi]}$$

we have according to [52] that

$$\limsup_{N \rightarrow \infty} \|S_N\| = \infty.$$

Due to the Banach–Steinhaus theorem [43, p. 98] there exists a $\hat{h}_{T_{*1}} \in C[-\pi, \pi]$ such that

$$\limsup_{N \rightarrow \infty} \left(\max_{\omega \in [-\pi, \pi]} \left| \sum_{n=1}^N w_n(\hat{h}_{T_{*1}}, 0) \hat{g}_n(\omega) \right| \right) = \infty. \quad (55)$$

Since

$$\begin{aligned} & \sum_{n=1}^N \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{g}_n(\omega) e^{i\omega t} d\omega \right) w_n(\hat{h}_{T_{*1}}, 0) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\omega t} \left(\sum_{n=1}^N w_n(\hat{h}_{T_{*1}}, 0) \hat{g}_n(\omega) \right) d\omega, \end{aligned}$$

and

$$\begin{aligned} & \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \sum_{n=1}^N \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{g}_n(\omega) e^{i\omega t} d\omega \right) w_n(\hat{h}_{T_{*1}}, 0) \\ &= \max_{\omega \in [-\pi, \pi]} \left| \sum_{n=1}^N w_n(\hat{h}_{T_{*1}}, 0) \hat{g}_n(\omega) \right|, \end{aligned}$$

it follows from (55) and the Banach–Steinhaus theorem [43, p. 98] that there exists an $f_{*1} \in \mathcal{PW}_\pi^1$ such that (53) is true.

Now we prove assertion 2. For $\hat{h}_T \in C[-\pi, \pi]$, it follows for fixed $t \in \mathbb{R}$ that $\hat{h}_T(\omega) e^{i\omega t}$ is a continuous function on $[-\pi, \pi]$, and hence the integral (52) exists. Let $t \in \mathbb{R}$ be arbitrary but fixed, and let $\hat{h}_{T_{*1}} \in C[-\pi, \pi]$ be the function from (55). We define

$$\hat{h}_{T_{*2}}(\omega) = e^{-i\omega t} \hat{h}_{T_{*1}}(\omega), \quad \omega \in [-\pi, \pi],$$

and clearly we have $\hat{h}_{T_{*2}} \in C[-\pi, \pi]$. It follows that

$$\sum_{n=1}^N w_n(\hat{h}_{T_{*2}}, t) \hat{g}_n(\omega) = \sum_{n=1}^N w_n(\hat{h}_{T_{*1}}, 0) \hat{g}_n(\omega)$$

for all $\omega \in [-\pi, \pi]$ and all $N \in \mathbb{N}$. Hence, we see from (55) that

$$\limsup_{N \rightarrow \infty} \left(\max_{\omega \in [-\pi, \pi]} \left| \sum_{n=1}^N w_n(\hat{h}_{T_{*2}}, t) \hat{g}_n(\omega) \right| \right) = \infty,$$

and, by the same reasoning that was used in the proof of assertion 1, there exists an $f_{*2} \in \mathcal{PW}_{\pi}^1$ such that (54) is true. \square

Remark 15. Clearly, the development of an implementation theory, as outlined in the introduction, is a challenging task. Some results are already known. For example, in [8] it was shown that for bounded bandlimited signals a low-pass filter cannot be implemented as a linear system, but only as a non-linear system. Further, problems that arise due to causality constraints were discussed in [42].

At this point, it is worth noting that Arnol'd's [1] and Kolmogorov's [30] solution of Hilbert's thirteenth problem [28] give another implementation for the analog computation of functions. For a discussion of the solution in the context of communication networks, we would like to refer the reader to [24].

Finally, it would also be interesting to connect the ideas of this work with Feynman's "Physics of Computation" [21] and Landauer's principle [31, 32]. Right now we are at the beginning of this development.

"Wir, so gut es gelang, haben das Unsre [(vorerst)] getan."
Friedrich Hölderlin "Der Gang aufs Land - An Landauer"

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