Macro-Scale Investigation of High Speed Gas Bearings for MEMS Devices

by

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Abstract

A macro-scale experimental test facility for investigating high-speed gas bearings for MEMS devices such as the MIT Micro-Engine is presented along with results from subsequent experiments. It is shown that the bearings required by such MEMS devices fall outside the usual range of design parameters for conventional gas lubrication systems. Due to the unorthodox design of the bearings, a new “hybrid” mode of operation is introduced along with the traditional hydrodynamic regime. The new hybrid mode is exploited to implement a novel in-situ rotor balancing scheme which enables hydrodynamic operation. Analysis for both modes of operation is presented along with experimental results. A high-order, efficient scheme for computing both the steady and unsteady hydrodynamic properties of the fully coupled, rotor/gas film dynamical system is presented along with comprehensive calculations for this class of plain, cylindrical, gas journal bearing. The scheme is then used to perform a generalized eigenvalue analysis on the compressible, unsteady system which reveals a new type a hydrodynamic instability. From a fundamental understanding of the bearing physics, strategies for operating MEMS devices with this class of bearing are deduced and minimum requirements for the accompanying measurement systems are established. Ancillary issues such as axial equilibrium of the rotor are discussed in detail.

Thesis Supervisor: Prof. Kenneth S. Breuer
Title: Visiting Associate Professor
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in industry and experience in rotor-dynamics provided a valuable perspective for me while operating the macro-bearing test rig. I am very appreciative of his assistance.

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"I have, alas, studied philosophy,
Jurisprudence and medicine, too,
And, worst of all, theology
With keen endeavor, through and through -
And here I am, for all my lore,
The wretched fool I was before.
Called Master of Arts, and Doctor to boot,
For ten years almost I confute
And up and down, wherever it goes,
I drag my students by the nose -
And see that for all our science and art
We can know nothing. It burns my heart.
Of course, I am smarter than all the shysters,
The doctors, and teachers, and scribes, and Christers;
No scruple nor doubt could make me ill,
I am not afraid of the Devil or hell -
But therefore I also lack all delight,
Do not fancy that I know anything right,
Do not fancy that I could teach or assert
What would better mankind or what might convert.
I also have neither money nor treasures,
Nor worldly honors or earthly pleasures;
No dog would want to live this way!
Hence I have yielded to magic to see
Whether the spirit's mouth and might
Would bring some mysteries to light,
That I need not with work and woe
Go on to say what I don't know;
That I might see what secret force
Hides in the world and rules it's course.
Envisage the creative blazes
Instead of rummaging in phrases.' "

- Goethe's Faust
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Chapter 1

Introduction

1.1 Motivation

The MIT Micro-Engine Project is an aggressive, multi-disciplinary effort in micro-electro-mechanical systems (MEMS) with the goal of demonstrating a working gas turbine engine on a micro-scale [23]. Applications for such devices include micro-power generators, compressors for micro-coolers, and propulsion systems for micro-air vehicles (MAVs). Because the micro-engine turbomachinery requires a transonic Mach number at the outer edge of the spool [23], smaller device dimensions dictate higher rotational speeds. Specifically, the micro-engine and related devices have design rotational speeds ranging from 1.2 million rpm to 2.4 million rpm [24]. Consequently, any feasible micro-engine design will require high-speed, low friction bearings. In addition, due to limitations of the deep reactive ion etching (DRIE) techniques used to fabricate the micro-engine, component geometries are limited to very low aspect ratio, nominally planar features [24].

In order to select an appropriate class of bearing technology to develop for the micro-engine, one must consider the range of alternatives available. Such options include rolling element bearings, oil film bearings, gas bearings, and electro-static bearings.

Rolling element bearings (REBs) have several attractive features. First of all, they provide stable, low friction operation at low speeds and are capable of sus-
porting thrust loads. However, from the standpoint of the micro-engine, REBs have serious disadvantages. For example, even high performance REBs are limited to rotational speeds on the order of $10^5$ rpm [22]. Furthermore, such state-of-the-art REBs often require auxiliary systems for oil mist lubrication and cooling which add to the space requirements of an already sizeable system. This creates additional complexity and introduces the possibility of contamination of other micro-engine components by lubricant oil. Lastly, REBs are incompatible with the planar fabrication technologies used for the micro-engine and installing off-the-shelf miniaturized REBs would create a complicated manual assembly step which would seriously impede mass production of micro-engines.

Oil film bearings (OFBs) remedy some of the deficiencies of REBs. OFBs are more compact than REBs, less complex in design and have substantially lower friction than REBs. OFBs are capable of achieving rotational speeds on the order of $10^6$ rpm. Moreover, the inability of the oil film to support negative pressure regions leads to cavitation which enhances the stability properties of the system by increasing direct stiffness [28]. Despite this improvement in performance over REBs, OFBs still require an externally supplied source of oil lubricant, create oil contamination problems and introduce the problematic issue of fractional speed whirl due to the hydrodynamic instability of the oil film/rotor system [51].

Gas bearings have many properties compatible with the micro-engine concept. First of all, since the viscosity of gases is three orders of magnitude less than that typical of oil lubricants, the power dissipated by a gas bearing is correspondingly less than that of a comparable OFB [28]. Secondly, a gas bearing can utilize the working fluid of the micro-turbine as a lubricant so that contamination is not an issue as was the case with the REB. Gas bearings are also very compact and simple in construction so that they pose minimal demands on the overall micro-engine design. Although gas bearings provide low load capacity relative to OFBs, the load-bearing characteristics of the micro-engine lubrication system are of secondary importance when compared to the high, rotational speeds needed by the micro-engine turbomachinery [24]. Thus, in the context of the Micro-Engine Project, the primary drawback of gas bearings is
that they are highly susceptible to instabilities similar to those observed in the OFB [51].

Considered in isolation, electro-static bearings are very attractive because they can provide little or no friction, are capable of extremely high speeds and are dependent on the inertial properties of the rotor. Due to the planar nature of the micro-engine components, virtually any rotor design under consideration will be approximated by an oblate body of revolution which, in the absence of external torques, will be stable when rotated about its major axis of inertia [69].

This all bodes well for electro-static bearings except when one realizes that it is not feasible to evacuate the journal bearing gap of the micro-engine of all gas. Thus, any force generated by a electro-static bearing on the rotor will have to compete with the hydrodynamic forces developed in the journal bearing gap by interaction between the spinning rotor and the working fluid of the micro-engine. The greatest electro-static forces occur during close approach of the journal with the bearing housing. This also happens to be when the hydrodynamic bearing forces are strongest. Since the hydrodynamic forces are substantially greater than the electro-static forces for any feasible design of electro-static bearing [38], the electro-static forces will play a secondary role to the hydrodynamic forces ergo the bearing is primarily a gas bearing with electro-static augmentation. Therefore, the option of electro-static bearings should be considered, at best, a way to enhance the performance of the gas bearing. Whether or not any beneficial effect that could be achieved is enough to offset the additional complexity of the circuitry associated with the electro-static bearing is another issue.

In light of the above discussion, it is clear that gas bearings provide the most attractive alternative for meeting the lubrication needs of the Micro-Engine. However, stable, high-speed operation of the bearing represents a formidable challenge.
1.2 Previous Work

The roots of lubrication theory began with the work of Petrov [43, 44, 45, 46], Tower [65, 66] and Reynolds [56]. Petrov deduced the power consumption of a journal bearing lubrication film contemporaneous with the work of Reynolds who derived the celebrated equation that now bears his name in an attempt to describe the experimental observations of Tower. Although the work of Reynolds was limited to an incompressible fluid, it was later extended to a compressible fluid by Harrison [29].

1.2.1 Theory

Sommerfeld developed an ingenious change of variables that allowed an analytical solution for the incompressible lubrication film of a “long” journal bearing for both cavitated and uncavitated cases [59]. That is, journal bearings for which the axial Poiseuille terms in the Reynolds equation may be neglected in favor of the circumferential Poiseuille terms. However, despite the elegance of Sommerfeld’s transformation, the tedious nature and limited scope of this approach are apparent when one considers the recursive structure of the “journal bearing integral” [5] in conjunction with the Cauchy residue theorem. As a result, one almost never uses the Sommerfeld substitution in more general lubrication settings, but instead, a combined approach of recursive tables and contour integration in the complex plane. Dubois and Ocvirk [15] specialized the approach of Sommerfeld with their “short journal bearing” approximation which involves neglecting the circumferential Poiseuille terms in favor of the axial Poiseuille and circumferential Couette terms.

In the early days of lubrication theory, the compressible lubrication equations eluded solution due to the nonlinearity and unsteady properties of the governing equations. With the advent of computers in the 1950’s, numerical solutions of the steady gas lubrication equations began to appear in the literature [62]. Perhaps the most comprehensive work was that of Raimondi and Boyd who published design charts for the static characteristics of gas journal bearings of finite length [52, 53, 54]. One should note that not all approximate solutions to the gas lubrication equations
were exclusively numerical. Ausman developed the “linearized PH theory” for approximating gas journal bearing films using perturbation techniques to generate and successively solve coupled, linear differential equations [2, 3]. However, the utility of Ausman’s technique was severely restricted due to the requirement of low journal eccentricity ratios.

After the problem of determining the steady behavior of gas lubricated journal bearings was solved in the 1950's, research in the 1960's addressed the unsteady properties of the journal bearing. In particular, the hydrodynamic stability of the film/rotor system for both oil film bearings as well as gas lubricated bearings was investigated. Experiments had long documented the problem of “fractional speed whirl,” the signature of hydrodynamic instability in self-acting journal bearings [27]. The work of Poritsky [27] explained the physical mechanisms behind the hydrodynamic instability while Lund [35] quantitatively solved the hydrodynamic stability problem for cavitated short oil film journal bearings by introducing the now-familiar concepts of “equivalent hydrodynamic stiffness” and “critical mass.” In addition, Lund used the averaging techniques of Krylov, Bogoliubov and Mitropolsky [35] to address the nonlinear problem of Liapunov stability.

In the area of hydrodynamic stability of gas journal bearings, Marsh [37] made an early attempt to solve the problem, producing mixed results. Ng [41] weighed in with a finite length version of Ausman’s earlier infinite length stability theory [4]. Of course, this approach inherited the limitations of Ausman’s technique. Cheng and Pan laid the problem of hydrodynamic stability of plain cylindrical gas journal bearings to rest with their landmark 1965 paper [7]. By extending the earlier work of Cheng and Trumpler [8] on infinite length bearings, Cheng and Pan used Galerkin’s method for not only the solution of the fixed points of the nonlinear dynamical system of the coupled, unsteady, gas film and rotor but also for the linear stability of these fixed points.

Cheng and Pan’s technique remedied the two fundamental deficiencies plaguing all previous attempts at solving the gas journal bearing stability problem. First of all, the primary weakness of earlier methods [37, 41, 42] was the linearization of the
fluid equations. Cheng and Pan acknowledged that in order to solve for the linear stability of the system, one must start with a proper calculation of the fixed points for the nonlinear system. In other words, the previous approaches did not adequately approximate the “basic state.” Secondly, prior investigators tried to decouple the rotor from the film. This was probably motivated by the complexity of the coupled system as well as the tidy de-coupling exhibited in Lund’s work on short oil film bearings. Whatever the reason, Cheng and Pan realized that any scheme which decouples the rotor from the compressible lubrication film neglects one of the defining characteristics of a gas journal bearing - unsteadiness. Close examination of Cheng and Pan’s results reveals the explicit role of unsteadiness in the coupling of the gas film and rotor. This is another statement of the well-known fact that the equation of motion of the rotor and fluid film equations must be solved simultaneously for proper treatment of gas journal bearing stability problems. Thus, “coupling” becomes the shibboleth of linearized analyses.

The shortcoming of Cheng and Pan’s calculations was that the number of expansion terms they retained in their approximate solution was low. Jacobson [30] recently revisited Cheng and Pan’s scheme using a commercial symbolic manipulation package to perform the algebra associated with the Galerkin system. With current computing technology, this approach led to a surprisingly modest increase in the number of terms retained in the Galerkin truncation compared to the order of Cheng and Pan’s [7] manually-derived system. However, it will be shown in this thesis that using a suitable advanced symbolic programming language [71] and a thorough understanding of journal bearing integrals [5], one can carry out Cheng and Pan’s calculations to virtually arbitrary accuracy.

Richardson [57] developed a lumped-parameter analysis for externally pressurized or “hydrostatic” gas bearings. In this work, both inherent as well as orifice compensation was studied. Steady and unsteady properties were characterized and a low-order eigenvalue analysis was presented. It is important to note that this study neglected the effect of journal rotation so that no hydrodynamic forces were included in the analysis. Consequently, the characteristics of the lubrication film were lumped into a
single hydraulic resistance that acted to throttle the restrictor exhaust.

Lemon [32] performed a lumped-parameter analysis of both inherent and orifice compensated externally pressurized gas bearings similar to that of Richardson [57]. However, Lemon also included a comparison to experimental data, demonstrating good agreement for static load properties, compensation pressure losses and mass flows. A scheme for optimizing the stiffness and load capacity of such bearings was also described. Once again, the effect of journal rotation was ignored.

For hybrid bearings, systems characterized by both hydrostatic and hydrodynamic forces, Larson and Richardson [31] developed a simple but general low-order theory. This theory was based upon the four-degree-of-freedom equations of motion that describe the cylindrical motion of the rotor. In effect, this work augmented the earlier efforts of Richardson [57] and Lemon [32] to include the hydrodynamic effects introduced by journal rotation. Because the film forces were calculated independently of the rotor motion, this approach represented a de-coupling of the rotor and lubrication film. Furthermore, since the hydrostatic and hydrodynamic forces were simply superimposed upon each other, a second “de-coupling” assumption was made by not including interaction between the hydrostatic and hydrodynamic mechanisms. By avoiding assumptions pertaining to the exact nature of the external pressurization, this model effectively combined a generic hydrostatic stiffness and the hydrodynamic forces for a given rotor speed and static position. With the fluid dynamics described in this manner, Larson and Richardson then conducted an eigenvalue analysis of the 4 x 4 dynamical system in order to quantify the effect of hydrostatic augmentation on the stability properties of the hybrid scheme. The primary conclusions of this analysis were as follows:

1. Radial hydrostatic stiffness can be used to increase the stable operating range of the bearing.

2. Hydrostatic augmentation with inherent compensation will eventually reach a maximum stable ratio of rotor speed to natural frequency as the external supply pressure is increased, establishing the top speed of the hybrid system.
3. The hybrid system loses stability due to the hydrodynamic tangential force.

Larson and Richardson's [31] calculations have several limitations. For example, the hydrodynamic forces were obtained using incompressible theory. This is, at best, a first approximation to the properties of a compressible gas film. Secondly, the scheme assumed that the hydrostatic and hydrodynamic fluid flows are independent of each other. The validity of this assumption is not clear for the radial injection hydrostatic system considered by the authors. In spite of these assumptions, it will be shown in this thesis that for the micro-engine class of bearings, these conjectures are justifiable.

Lund [34] combined hydrostatic and hydrodynamic effects in a hybrid analysis using the compressible Reynolds equation for the gas film, unlike the analysis of Larson and Richardson [31], which implemented incompressible solutions in the journal gap. However, Lund's theory included hydrodynamic effects via a first-order perturbation of the nonlinear gas lubrication equation, limiting the regime of applicability to low eccentricity ratios. The differential equation produced by the low-order term of the perturbation analysis was solved numerically. Design charts were presented for static properties and a linearized, vibrations analysis was used to evaluate squeeze-film effects.

Mori and Mori [39] developed a stabilization scheme for hybrid bearings that exploited the well-known principle [27] that the additional damping produced by supporting a journal bearing on a rubber o-ring can delay rotor-dynamic instability. Citing uncertainty in the properties of rubber due to wear and aging, Mori and Mori replaced the o-ring by an externally pressurized gas film around the bearing itself. This secondary "sub-bearing" was included in low-order, eigenvalue analysis of the primary bearing. The accompanying calculations showed that the sub-bearing could increase the threshold speed of the primary bearing.

1.2.2 Experiment

One of the most extensive early studies in experimental gas bearing research was the work of Sternlicht and Elwell in 1958 [63]. This series of experiments was for hydro-
dynamic, plain, cylindrical gas journal bearings with $L/D$ ratios of 1.0, 1.5, and 2.0. Presented results included static circumferential pressure distributions taken at mid-span, static eccentricity ratio/attitude angle charts, and threshold whirl boundaries.

Reynolds and Gross [55] conducted similar experiments on this type of bearing at $L/D$ ratios of 0.25, 0.5, 0.75, and 1.0. The effects of various operational and design parameters on the threshold whirl speed were explored and hysteresis near the whirl boundary was reported. Typical journal orbits observed were also presented along with explanations that involved both the cylindrical and conical modes of the rotor. It was observed that unbalance had the effect of suppressing fractional speed hydrodynamic whirl.

Larson and Richardson [31] conducted experiments on a hybrid gas bearing test rig ($L/D = 0.8$) that implemented external pressurization through radial ports located at the bearing mid-span. Both inherent and external hydrostatic compensation techniques were used. Since the test rig rotor was essentially a prolate body of revolution, cylindrical as well as conical modes were studied. The locations of resonances and stability boundaries were thoroughly mapped. The results were compared with the accompanying low-order theory. Mori and Mori [39] performed experiments on this type of bearing, showing that onset of instability could be a delayed by floating the main bearing on an externally pressurized sub-bearing.

The work of Wilson [70] marked a departure from earlier work on plain, cylindrical, journal bearings by addressing more advanced topics such as the effect on hydrodynamic whirl instability of rotor non-circularity, rotor taper, bearing surface finish, and moment loading. With these experiments, Wilson considered non-ideal bearings that took into account the limitations of that era's fabrication techniques. Wilson's test rig utilized bearings with an $L/D$ ratio of 1.0.

In 1983, Yoshimoto and Nakano performed experiments on unsymmetrical rotors supported by two identical, plain, cylindrical, gas journal bearings. Threshold whirl boundaries were compared to computations of other authors as well as those specially developed for the study. Rather than examining non-ideal conditions related to the fabrication techniques used to construct the bearing, the stated goal of these authors...
Table 1.1: Values of bearing $L/D$ ratios for past hydrodynamic gas bearing experiments.

<table>
<thead>
<tr>
<th>researcher(s)</th>
<th>$L/D$</th>
<th>year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sternlicht, Elwell</td>
<td>1.0,1.5,2.0</td>
<td>1958</td>
</tr>
<tr>
<td>Reynolds, Gross</td>
<td>0.25,0.5,0.75,1.0</td>
<td>1962</td>
</tr>
<tr>
<td>Larson, Richardson</td>
<td>0.8</td>
<td>1962</td>
</tr>
<tr>
<td>Wilson</td>
<td>1.0</td>
<td>1966</td>
</tr>
<tr>
<td>Yoshimoto, Nakano</td>
<td>1.0,1.5</td>
<td>1983</td>
</tr>
<tr>
<td>Dimofte</td>
<td>1.0</td>
<td>1995</td>
</tr>
<tr>
<td>Orr</td>
<td>0.075</td>
<td>1999</td>
</tr>
</tbody>
</table>

was to study complications arising in gas bearing systems for machines requiring an unsymmetrical rotor. That is, the asymmetries involved were larger than those that would occur due to tolerances in the manufacture of the devices. The $L/D$ ratios considered were 1.0 and 1.5.

Finally, Dimofte [13] conducted theoretical and experimental work on bearing non-circularity. However, instead of studying the effects of non-circularity from manufacturing tolerances, Dimofte intentionally introduced large-scale geometric features in order to manipulate the bearing characteristics. The resulting “wave bearings” were analyzed and tested for an $L/D$ ratio of 1.0.

Table 1.1 compares the $L/D$ ratios studied in previous experiments on self-acting gas bearings to this thesis. Note that the lowest aspect ratio previously tested was in the work of Reynolds and Gross [55] for $L/D = 0.25$. Even in the classical oil film bearing experiments of Dubois and Ocvirk [15] that were performed to verify their “short bearing” theory, the lowest bearing aspect ratio was $L/D = 0.25$. Clearly, the baseline micro-bearing design for the MIT Micro-Engine of $L/D = 0.075$ considered in the current study is a significant departure from the existing literature.

1.3 Uniqueness of the Micro-Engine Regime

Because the MIT Micro-Engine represents a radical departure from existing machines, one would expect the gas lubrication requirements to fall outside the realm of previ-
ously studied systems. Table 1.1 affirms this assertion by quantifying the impact of the DRIE fabrication techniques on a crucial aspect ratio of the system - the bearing $L/D$ ratio. A common theme throughout this thesis is how this fabrication-imposed, low $L/D$ requirement impacts the operation as well as analysis of this class of bearing.

It will be shown in later chapters that not only does this class of bearing require operation far outside the "safe" range of system parameters as prescribed by the predominant air bearing orthodoxy, but that achievement of high bearing speeds with such operation is possible. In addition, the low $L/D$ ratio permits the first experimental observation of a novel inertial effect, confirming the results of computations conducted three decades ago. This inertial mechanism is subsequently demonstrated to be quite useful for enabling effective high-speed operation of the bearing and providing robust operation at low speeds where hydrodynamic instabilities are especially problematic. Finally, in the area of analysis, the extreme operating conditions required by the low $L/D$, Micro-Engine class of gas bearings are the impetus for high-order computations that reveal a new, second hydrodynamic instability that joins the well-known, hydrodynamic whirl mode which has occupied the literature for the past 40 years.

1.4 Problem Statement

The primary objective of this thesis is to demonstrate the applicability of low aspect ratio, high-speed, gas bearings to the lubrication needs of the MEMS technologies embodied by the MIT Micro-Engine Project. In addition, operating strategies for such bearings are required to lay a foundation of practical guidelines for the implementation of micro-bearings in MEMS devices.

1.5 Methodology

Conventional gas turbine engines are the embodiment of a sophisticated and harmonious confluence of many diverse technologies and disciplines. Similarly, the devel-
opment of micro-engines is an inherently multi-disciplinary endeavor. Unique constraints that arise due to limitations of micro-fabrication techniques make "solved" problems on the macro-scale the subject of original research on the micro-scale. Consequently, the development of the MIT Micro-Engine has proceeded as a broad range of detailed sub-system studies in the required disciplines under the unification of higher level systems integration with emphasis on overall device performance, cost-effectiveness, and ease of manufacture.

In this thesis, the detailed analysis of the lubrication sub-systems of the MIT Micro-Engine proceeds on several fronts. First of all, due to the difficulty in the instrumentation of test rigs on the micro-scale, a macro-scale bearing test facility has been developed to study the unique, low aspect ratio gas journal bearing required by the MIT Micro-Engine. By matching the pertinent non-dimensional parameters of the gas lubrication/rotor dynamics problem at hand, one may achieve dynamic similitude between the macro and micro scale bearings so that successful operational modes and strategies implemented in macro-experiments can be extended to experiments for similar test rigs at the micro-scale. Finally, to facilitate experimental efforts, computational modeling is used to guide not only the design of the bearings but also the development of strategies for operation of bearing test rigs.
Chapter 2

Experimental Facility

2.1 Introduction

The 26:1 macro-scale experimental facility used to investigate the operation of the short, high-speed gas journal bearing required by the micro-engine is centered around a gas journal bearing test rig. The test rig, designed by Fred Ehrich in tandem with the Bearing Research Group of the MIT Micro-Engine Project, includes supporting systems for tasks such as the delivery/exhaust of compressed air, measurement of operational parameters and acquisition/storage of data from instrumentation. To give a sense of the scaling, a comparison of the macro and micro devices is shown in Figure 2-1. From this perspective, the difficulties of installing instrumentation in the micro-device are as obvious as the advantages of conducting macro-scale studies.

2.2 Test Rig

The macro-bearing test rig is a radial inflow turbine driven by a compressed air source. Figure 2-2 shows a cut-away view of the test rig while Figure 2-3 is a detailed design drawing of the device. The upper side of the rig where the main turbine air supply is located is termed the “forward” side while the opposite end is referred to as the “aft” side. The direction along the spin axis of the rotor is known as the “axial” direction. The rotor is supported about its circumference by a gas journal bearing.
while thrust (axial) loads are balanced by hydrostatic gas bearings (Figure 2-2). The main turbine drive air enters a supply plenum axially through two ports on the forward side of the device. The plenum allows for mixing and uniform dispersion of the two supply flows. The drive air exits the plenum through a circular array of exhaust holes and then makes a right angle turn, radially inward toward the rotor center. This radial inflow passes through the stator section or nozzles of the turbomachinery so that the incident flow angle is properly adjusted for impingement on the turbine blades. It is important to note the the supply air travels across the forward end of the journal bearing before reaching the turbine blades on the rotor. Thus, the forward pressure boundary conditions on the journal will, in general, be dependent on the rotational speed of the rotor. This region between the stator and turbine is called the “inter-row” station. Once the supply air has passed through the turbine blades, exerting a torque on the rotor, it makes another right angle turn and exits in the axial direction, anti-parallel whence it entered. The main turbine air exhausts from
Figure 2-2: Cut-away view of macro-bearing test rig. In-plane location of radial displacement probes shown at stations 1 and 2. Axial displacement probes overlook rotor from aft cover plate at stations 3 and 4. RPM sensors overlook rotor blades at station 5 and 6.

the device through two main exhaust ports on the forward side of the rig.

Figure 2-4 shows an exploded view of the micro-bearing test rig presented by Lin [33] and shown in comparison to the macro-rig in Figure 2-1. At 1/26 the size of the macro-rig, this device is essentially identical to its macro-scale counterpart depicted in Figures 2-2 and 2-3. As will be evident in later macro-rig figures, there are some minor differences related to “packaging” issues, but the main gas path is essentially the same for both devices. Consequently, the perspective provided by Figure 2-4 is helpful in understanding the geometry and functionality of the macro-rig components.

An actual photograph of the installed test rig of Figures 2-2 and 2-3 is given in Figure 2-5. In addition to the figures of this chapter, a comprehensive set of macro-rig design drawings is presented in the appendices for reference.
1. Use shims (item 16) as required at assembly to achieve build specification of total axial clearances between item 1 and item 3 or item 4 in range .002 to .010.

2. SAE/MS male connector (item 19) to be CAT (SWAGELOK) NO. SS-500-1-5ST (SWAGELOK).

3. SAE/MS male connector (item 20) to be CAT (SWAGELOK) NO. SS-161D-1-12ST (SWAGELOK).

4. Shoulder screw (item 27) to be 5/16" X 3-1/2" long (HOLO--KROME or equiv).

5. Shoulder screws (item 27) to be 5/16" DIAM.

6. Install probe and plug assemblies (items 14, 15 & 16) per build specification, with probe immersion shim ground to provide immersion depth - .002 to .005 (see enlarged view).

7. Bond radial probe mounts (20), item 11, into stator, item 2.
Figure 2-4: Exploded view of *micro-rig* assembly.
Figure 2-5: Installed macro-bearing test rig.
2.2.1 Forward Foundation Plate

The forward foundation plate, shown in Figures 2-6 and 2-7 serves as the mounting platform for the rest of the plates in the device stack. The entire assembly is bolted together through common through-holes and aligned with a pair of alignment pins. The forward foundation plate also serves as the “hard point” for mounting the device to the tilt table used to control the rig attitude as well as hosting numerous air supply and instrumentation fittings. In particular, Figure 2-6 shows the main turbine supply ports, the main turbine exhaust, the forward thrust bearing supply, turbine blade overlook stations for RPM probes, and pressure measurement stations.

2.2.2 Forward Cover Plate

The forward cover plate (Figures 2-8, 2-9 and 2-10) has two primary functions. First of all, the forward side of the forward cover plate forms the main turbine air supply plenum with the forward foundation plate. The exhaust holes of this plenum can
be seen in Figure 2-9. Secondly, the forward cover plate forms the forward thrust bearing. The supply plenum for the thrust bearing can be seen in Figure 2-8 which shows the forward side of the forward cover plate. The thrust pad and restrictors are shown in Figure 2-9 and in close-up view in Figure 2-19. The blade-overlook stations that house RPM probes and inter-row stations for pressure probes are also evident in the figure. Note that the forward thrust bearing exhausts into the main turbine exhaust (Figure 2-2). In general, the main turbine mass flow will be much larger than the mass flow for the thrust bearing so that the back-pressure that the forward thrust bearing experiences will be dependent on the rotor rotational speed.

Note that the aft side of the forward cover plate is coated with a hard, plasma-sprayed, chrome oxide coating to give the forward thrust surface improved wear tolerance. The coating was found to be quite robust, surviving numerous high-speed touchdowns.
Figure 2-8: Forward side of forward cover plate.

Figure 2-9: Aft side of forward cover plate.
Figure 2.10: Design drawing for forward cover plate.

Notes:
1. Material: Type 304 stainless steel, as specified by ASTM A240-73 (AISI) in annealed (hardness Rockwell B93). 2. Surface to be ground and lapped flat. 3. Breakout of .065 dia holes to surface should be sharp edge. 4. Coat surface as follows: phosphate treatment, spray chrome, oil. Finish dimensions as indicated on the drawing. 5. Coating should have a uniform hardness, not greater than 150 (C 140 maximum). The surfaces should be lapped to better than fine finish. The coating thickness after lapping should be in the range 0.004 to 0.006.
Figure 2.11: Design drawing for macro-tig stator.

- Material: Type 303 as specified by ASTM A582-93 (A582 M) Annealed (Hardness Rockwell RB92)
- With rotor (DWG No. GTL-301) located in place, surfaces and are to be ground and lapped flat and parallel to one another to the same thickness of rotor.

**NOTES:**

1. Material Type 303 as specified by ASTM A582-93 (A582 M) Annealed (Hardness Rockwell RB92).
2. With rotor (DWG No. GTL-301) located in place, surfaces and are to be ground and lapped flat and parallel to one another to the same thickness of rotor.
2.2.3 Stator/Liner

The stator plate contains the nozzles that properly direct the air toward the turbine blades on the rotor (Figures 2-12 and 2-11). The nozzles were designed with MISES [72]. The stator plate also holds fittings for radial probes for measurement of quantities such as the rotor position and lubrication film pressure or temperature. Finally, the stator plate contains a removable journal bearing liner (Figure 2-12) which features radial instrumentation ports as well as a chrome oxide coated bearing surface to increase tolerance to wear. The liner was designed with a slip-fit tolerance for installation in the stator and requires shims to adjust the circularity of the bearing geometry. Even with a careful application of this technique, there may be significant residual non-circularity. Figure 2-13 shows a sample liner contour measured by rolling the rotor on the liner at low speed. Clearly, there is a substantial deviation from perfect circularity.
Figure 2-13: Macro-rig liner contour. Radial coordinate in mils, angular coordinate in degrees.
2.2.4 Rotor

The macro-rig rotor is machined from 17 – 4PH stainless steel and has all potential rub surfaces protected with a chrome oxide ceramic coating. Like the nozzles of the stator plate, the turbine blades of the rotor were designed with MISES [72]. The forward surface of the turbine blades (Figures 2-14 and 2-16) is cut back below the level of the forward thrust plate so that the thrust plate will sustain any rubbing contact in the event of loss of axial equilibrium. Similarly, the outer circumference of the aft rotor surface (Figure 2-15) is also reduced below the level of the aft thrust surface in order to avoid touchdown of the rotor surfaces with the highest tangential speeds.

2.2.5 Aft Cover Plate

Like the forward cover plate, the aft cover plate (Figures 2-17, 2-18, and 2-20) contains a hydrostatic thrust bearing and various instrumentation ports to monitor the dynamics of the rotor. A close-up view of the thrust bearing pad is shown in Fig-
Figure 2-15: Aft side of rotor.

Note that some wear of the chrome oxide protective coating is visible. In addition, the aft cover plate also contains two externally regulated pressure chambers (Figure 2-17). One pressure chamber, termed the “high side-pressure chamber,” covers two-thirds of the aft journal bearing boundary circumference. The other chamber, the “low side-pressure chamber,” covers the other one-third of the aft journal bearing boundary circumference. By maintaining a pressure difference between the two chambers, one can produce a net side-force on the rotor. In an attempt to reduce the rotor moments generated by this side-loading scheme, the high pressure chamber is extended around the entire circumference of the rotor just inside the aft end of the journal bearing. This allows the journal bearing to be exposed to differential pressures while maintaining a relatively uniform pressure over the aft side of the rotor. However, it should be noted that this scheme will produce a net axial or thrust load when the average pressure on the aft side of the rotor is significantly different that that on the forward side of the rotor.

As with the other potential rub surfaces, the forward side of the aft cover plate is also protected by a ceramic coating. The coating proved to be durable and only
Figure 2-16: Design drawing for macro-rig rotor.

NOTES:
1. MATERIAL: UNS S71750 (17-4 PH) SOLUTION TREATED AS SPECIFIED BY AEM 132 (HARDNESS: ROCKWELL R30)
2. SURFACE - TO BE DRAFED AND LAPPED FLAT TO THE SAME THICKNESS OF STATOR (DRAWING NO. 5/32 X 300) WITHIN ± 0.001 - 0.002 OF STATOR
3. SURFACE - TO BE FLAT AND PARALLEL TO SURFACE WITHIN 0.002 BELOW SURFACE
4. SURFACES - TO BE FLAT AND PARALLEL TO SURFACE WITHIN 0.002 BELOW SURFACE
5. SURFACES - TO BE FLAT AND PARALLEL TO SURFACE WITHIN 0.002 BELOW SURFACE
6. TRACKING OF TC RIG PARTS TO BE IN THE RANGE OF 0.004" - 0.006"
2.2.6 Aft Foundation Plate

The aft foundation plate is the final level in the macro-rig stack and serves as a seal for the pressure chambers of the aft cover plate as well as a foundation for various instrumentation and plumbing fittings (Figures 2-21 and 2-22). In particular, fittings for the aft thrust bearing supply and exhaust as well as those for the axial displacement and pressure probes are located in this level of the stack. The aft foundation plate also functions as a mounting point for brackets that clamp four radial capacitive displacement probes firmly in place.

failed during a high speed touchdown after significant wear had already occurred.

Figure 2-17: Forward side of aft cover plate.
Figure 2-18: Aft side of aft cover plate.

Figure 2-19: Close-up view of hydrostatic thrust bearing.
Figure 2-21: Forward side of aft foundation plate.

Figure 2-22: Aft side of aft foundation plate.
2.3 Instrumentation

2.3.1 Summary of Measurement Requirements

The macro-bearing test rig represents a complicated coupling between a mechanical system and various fluid flows. The mechanical system is described by the equations of motion of the rotor while the fluid flows are solutions of conservation equations describing the turbomachinery, gas journal bearing, hydrostatic thrust bearings, and pressure chambers. As a result, the measurement requirements include instantaneous rotor position and attitude along with gas pressures, temperatures, and mass flows at strategic locations throughout the system.

2.3.2 Rotor Dynamics

In order to measure the instantaneous position and attitude of the rotor, eight capacitive displacement probes were used. The probes, manufactured by Capacitec, have a range of $0 - 0.020$ in. described by a linear output from $0 - 10$ volts. The frequency response of the probes is $3 \text{ kHz}$. All specifications quoted by the manufacturer were verified in bench-top experiments.

Four of the capacitive displacement probes were placed at radial locations in the mid-span of the journal bearing liner at 90 degree intervals. The pair of radial displacement probes that are in-plane with the cut-away view of Figure 2-2 are labeled as stations 1 and 2. The tips of the probes were recessed into the journal bearing liner to a depth of $0.003 - 0.005$ in. to prevent contact with the rotor. Measurements from orthogonal probe pairs describe the motion of the rotor in the plane of the stator while opposing pairs of probes detect the elastic deformation of the rotor under centrifugal loads during high speed operation. Three capacitive displacement probes were positioned facing the aft side of the rotor in the axial direction to detect any "wobble" of the rotor. The two in-plane locations for the axial displacement probes in Figure 2-2 are marked as stations 3 and 4. All seven of these probes were connected to the A/D card in the data acquisition computer so that the output could
be digitized and stored on disk for later analysis.

The last of the eight capacitive displacement probes was used as an RPM indicator by placing it in one of the two turbine blade overlook stations shown in Figure 2-9 and labeled as station 5 in Figure 2-2. However, because the rotor has 24 turbine blades, a Capacitec probe used in this manner is limited to rotor rotational speeds of 7,500 RPM. To overcome this limitation, the other turbine blade overlook position (station 6 in Figure 2-2) was occupied by a Kaman inductive displacement probe with a frequency response of 50 kHz so that rotor rotational speeds of up to 125,000 RPM could be measured. Signals from these two probes were processed independently in real-time by both a Hewlett-Packard spectrum analyzer as well as a Tektronix digital oscilloscope. The output of these devices was communicated back to the data acquisition computer via the general purpose interface bus (GPIB) for real-time display using the graphical user interface (GUI) of the data acquisition software. By performing the signal processing with remote devices, the data acquisition CPU is free for other tasks and no A/D channels need be devoted to a direct measurement of the rotor rotational speed.

2.3.3 Mass Flow

Mass flow measurements were made using Fischer-Porter rotameters of appropriate sizes. The flowmeters are adequate for measuring steady mass flows and require corrections for variations in flow conditions [25] to an accuracy of ±0.5%. Measurements of flow temperature and pressure necessary for these corrections were made as described below.

2.3.4 Pressure

Pressure measurements were made throughout the experimental set-up using a Scanivalve pressure transducer in conjunction with a Scanivalve pressure switcher driven by a Scanivalve Digital Interface Unit (SDIU). The SDIU was interfaced with the data acquisition PC over the GPIB. This system allowed quick and efficient scanning
of literally dozens of pressures in the system using a single pressure transducer. The transducer was calibrated at the factory prior to delivery and this calibration was verified using a reference pressure gauge in bench-top experiments. The accuracy of the transducer is ±0.05 psi.

### 2.3.5 Temperature

Temperature measurements were made using a Fluke Helios thermocouple scanner with K-type thermocouples. This instrument interfaced with the data acquisition PC over the RS-232 port and therefore did not require any A/D channels for the relatively low frequency measurements. The uncertainty in the temperature measurements is ±0.1 degrees Celsius.

### 2.3.6 Error Analysis

Uncertainties in measured quantities were estimated by combining sources of error for a given measurement in a root-mean-square sense. To estimate the uncertainty in a calculated quantity based upon experimental measurements, error bounds were computed using worst-case scenarios for the various sources of error.

### 2.3.7 Rig Attitude

The attitude of the entire macro-rig stack was measured with a pair of orthogonally-mounted pendulous capacitive inclinometers made by Reiker. These instruments were used to find the reference of the test rig mounting system with respect to the local direction of gravity to an accuracy of ±0.05 degrees. This “zero” provides an absolute reference so that the system orientation may be adjusted relative to this benchmark.

### 2.4 Supporting Hardware

Because the macro-bearing test rig is a complex fluid/mechanical system, carefully controlled experiments dictate that many gas flows be managed as massive amounts
of data are simultaneously acquired, processed, stored and finally echoed back to the operator. Such a device requires secondary air-handling systems for supply as well as exhaust along with a centralized control panel where the operator has easy access to crucial valves and regulators.

2.4.1 Test Platform

The macro-bearing rig stack is rigidly mounted to a precision two-axis tilt table (Figure 2-5) referenced to the direction of gravity as determined by the inclinometer system described above. The tilt table allows attitude adjustment to an accuracy of 30 arcseconds. During a “zeroing” operation, the tilt table is used to rotate the entire rig about the rotor rotational axis. The inclinometers, mounted to the forward foundation plate, are used to sense the projection of the gravity vector in the plane normal to this spin axis. If the spin axis of the rig is aligned with the direction of gravity, the inclinometers will not sense any variation as the rig is rotated on the table. However, if the inclinometers do sense a variation in the projection of the gravity vector onto the detection plane, the second axis of the tilt table is used to rotate the rig about a judiciously-chosen line within the detection plane to null out the observed variations.

2.4.2 Control Panel

A schematic of the macro-rig control panel is shown in Figure 2-23. The turbine supply air is controlled by a main supply regulator upstream of three throttle valves of “fine,” “medium,” and “coarse” sizes. The forward and aft hydrostatic thrust bearings are independently controlled by separate pressure regulators as are the high and low pressure chambers used for side-loading the rotor. The panel also features a safety valve for nulling the pressure difference across the turbine in the event of an emergency shut-down.
Figure 2-23: Macro-bearing test rig control panel.
2.5 Data Acquisition System

2.5.1 Acquisition Computer

Data acquisition operations are executed using a single PC through various communications protocols and expansion cards (Figure 2-24). The PC initializes instruments, activates acquisition modes, retrieves data, processes and stores data, echoes key data to the operator and adjusts instrument settings based upon data received. Thus, as shown in Figure 2-24, the PC integrates an assortment of measurement devices of varying complexity into a single, coordinated, data acquisition operation.
2.5.2 High-Speed Acquisition

The most resource-intensive aspect of acquiring data during macro-bearing rig experiments is in handling the massive amounts of data produced by the capacitive displacement probes that monitor rotor movement. Because the rig is capable of rotational speeds on the order of $10^5$ RPM, the Nyquist sampling theorem dictates that sampling occurs at twice the highest frequency to be measured. As a practical matter, since it is desirable to have relatively continuous reconstructions of the rotor orbit geometry for comparison with calculated orbits, the system is typically over-sampled relative to the Nyquist criterion. For example, during high-speed runs where rotational speeds may reach 30 – 50 kRPM (500 – 833 Hz), the capacitive displacement probes are typically sampled at 12 kHz.

The A/D card digitizes the probe signals and stores the data in a circular RAM buffer. This buffer is broken into smaller buffers which are split in half and serviced in a double-buffered scheme [12]. This arrangement allows high-speed data to be acquired for fixed periods of time less than the length of the circular RAM buffer or as a “rolling window” in time representing the most recently acquired data for an experiment lasting longer than the length of the circular buffer. Consequently, the scheme acts as a “flight recorder” when the circular buffer begins to over-write itself. Displacement probe signals describing both the axial and radial position of the rotor are sent to analog displays to give the operator real-time feedback from control actuations.

2.5.3 Low-Speed Acquisition

In addition to the high-speed acquisition needed to describe the rotor dynamics of the system, Figure 2-24 shows the other measurements made by remote devices which digitize and process signals from instruments measuring quantities such as flow temperature, pressure, and rotor rotational speed. The results of these measurements are reported back to the data acquisition PC using the appropriate communications protocol. This provides a degree of preliminary data reduction and reduces the com-
putational and storage demands on the PC.

2.5.4 Software

A data acquisition code was written in the C language using the programming environment and libraries of LabWindows CVI from National Instruments. This code served to synchronize, service, and optimize settings for the array of instruments shown in Figure 2-24. Data were echoed through the GUI to guide the operator and all hardware settings were set automatically via software so that a "hands off" approach could be implemented. This greatly streamlined experiments and allowed emphasis to be placed on operation of the test rig itself rather than on the instrumentation in the data acquisition system.
Chapter 3

Thrust Bearings

3.1 Axial Equilibrium

During operation of the macro-bearing test rig, the rotor is exposed to pressures from a variety of sources. For example, the gas journal bearing generates pressures that support loads in the plane normal to the rotor axis of rotation. In addition, the aft and forward sides of the rotor will be exposed to pressures generated by the turbomachinery and the side-loading scheme, creating the potential for a net axial force sufficient to initiate rubbing between the rotor and thrust plates. Concomitantly, asymmetry in axial loading may produce a net moment on the rotor, initiating "conical" modes which might also result in contact at the thrust surfaces. Therefore, although the journal bearing is the focus of this study, maintenance of axial balance is a crucial pre-condition for successful journal bearing operation.

The primary tool for maintaining axial balance is the pair of opposing hydrostatic thrust bearings shown in Figures 2-9 and 2-17. Driven by external pressure sources, the thrust bearings create secondary flows in the device that act to center the rotor between the thrust surfaces. At high rotor rotational speeds when the force on the rotor due to the turbomachinery is much larger than that from the side-pressure scheme, the rotor will tend to "sink" toward the aft cover plate. If the thrust bearings are not adequately stiff to prevent this axial translation from occurring, additional pressure from an external source may be introduced to back-pressure the aft thrust
bearing inner exhaust (Figure 2-17) to provide additional load capacity.

Because the thrust bearings have a small radius compared to that of the rotor, only a modest amount of "tipping stiffness" is provided. A potential mechanism for introducing tipping moments on the rotor is the side-loading scheme contained in the aft cover plate (Figure 2-17). When one side-load chamber is held at a higher pressure than the other chamber, a net side-force is created as well as the possibility of a moment on the rotor. In order to alleviate this problem, an anti-torque channel contiguous with one of the side-pressure chambers is extended around the circumference of the aft rotor face (Figure 2-17). This allows the journal bearing to be exposed to the differential pressures of the side-loading chambers while the aft rotor face is exposed to a pressure distribution that is relatively symmetric, reducing the resulting moment. Finally, as is well-known from rigid body dynamics, the oblate geometry of the rotor produces a high degree of stability for rotation about the major axis of inertia, reducing the need for high tipping stiffness in the thrust bearings [69].

3.2 Principles of Hydrostatic Thrust Bearings

Unlike hydrodynamic bearings, hydrostatic bearings have the advantage that no relative motion of the bearing surfaces is required to produce load capacity. In the case of a hydrostatic bearing, it is the flow field established by the external pressure source that creates load capacity. This eliminates the unlubricated friction between the bearing surfaces that occurs before a hydrodynamic bearing "takes off."

When an above-ambient pressure, $P_s$, is applied to the supply plenum of the thrust bearing restrictors shown in Figure 2-9 while both the center and outer exhausts are held at ambient pressure, $P_a$, a flow is established from the restrictors to the exhaust. One may think of the pressure losses that occur along this flow-path as being divided into the two categories of "compensation" and "film" losses.


3.2.1 Compensation Losses

Compensation losses occur in the restrictors and the “film entrance” region just downstream of the restrictor exhausts where the flow enters the lubrication film that separates the rotor from the cover plate (Figure 2-2). As the flow enters a restrictor tube after exiting the relatively large thrust bearing supply plenum, the flow may separate at the sharp inlet corner of the tube, creating a pressure loss. In addition, there may be pressure losses due to inertial effects as the inviscid core flow accelerates into the restrictor. When the flow becomes “fully developed,” viscous effects dominate and the corresponding pressure drop is that of a Poiseuille flow. In Figure 3-1 the flat pressure plateau between 1.15 and 1.35 inches where \( p = P_s \) suggests that this pressure drop within the restrictor itself is small for the bearing shown.

As the flow exits the sharp-edged exhaust orifice of the restrictor, separation will occur with the accompanying pressure decrease. The flow then accelerates into the lubrication film, causing another inertial pressure loss in the film entrance region. This is exhibited in Figure 3-1 where it is seen that the measured static pressure (solid round markers) just outside the restrictor is substantially below that predicted by the inertialess theory (dashed line). When the experimental data are corrected for inertia effects using an ad-hoc, semi-empirical scheme [67], there is good agreement between experiment and theory (solid line).

Because compensation losses are caused by complicated fluid dynamic mechanisms, they are usually estimated using ad-hoc analyses and/or empirical correlations. For example, losses in the inlet and exhaust of the restrictor are frequently treated using the concept of hydraulic resistance [26]. Experiments and ad-hoc analyses are frequently combined [11, 67] with reasonable results as demonstrated by Figure 3-1. Of course, modern CFD techniques may be used at considerable expense to directly determine pressure losses by computing the complete, three-dimensional Navier-Stokes flow. However, the effective implementation of hydrostatic thrust bearings by engineers for the past 40 years using relatively simple design tools suggests that such computational largesse is unnecessary in the context of conventional design...
Compensation losses are typically summarized in the form of a pressure loss coefficient [26, 67]. That is, the pressure drop associated with delivery of the supply air to the lubrication film is expressed in terms of a fraction of the dynamic head at the film entrance as shown below:

\[ P_s - P_i = \bar{K}P_{dyn} \]  \hspace{1cm} (3.1)

In Equation 3.1, \( P_s \) is the restrictor supply static pressure, \( P_i \) is the film entrance static pressure, \( \bar{K} \) is the loss coefficient and \( P_{dyn} \) is the film entrance dynamic pressure.

### 3.2.2 Film Losses

Once the losses in the vicinity of the restrictor exhaust are incurred, viscous effects again dominate for the remainder of the lubrication film until the flow exits to ambient pressure. Evaluation of these “film losses” requires computation of a compressible, planar lubrication film and is fairly straightforward. Neglecting Couette terms due to rotation of the rotor, the governing equation is:

\[ \nabla \cdot (h^3 \nabla P^2) = 0 \]  \hspace{1cm} (3.2)

where \( P \) is the local film static pressure and \( h \) is the local film thickness.

A low-order analysis is easily accomplished by modeling the circular crown of restrictors (Figure 2-19) as a circular line source with the appropriate inlet boundary condition provided by a compensation model and setting the exhaust boundary condition to ambient pressure. The gas lubrication equation is then easily integrated in polar coordinates to yield the familiar logarithmic pressure distribution with net loads expressed in terms of the Gaussian error function [51]. The accuracy of the analysis is obviously dependent on the number of restrictors for a given bearing radius since the line-source model represents the infinite restrictor limit [11].
Figure 3-1: Vohr's experimentally measured hydrostatic thrust bearing restrictor/film pressure distribution with comparison to theory (adapted from [67]).
above low-order analysis, one may easily compute the film surrounding the discrete restrictors in many ways. In the case of classical analysis, the circular crown of restrictors may be conformally mapped into a linear array of restrictors and slit along its symmetry line to create a simply connected domain so that the film solution may be expressed in terms of Jacobi elliptic functions prior to transformation back into the physical domain [11].

Alternatively, since lubrication approximations reduce the governing PDE's from three-dimensional Navier-Stokes equations to a two-dimensional system, direct numerical solution of Equation 3.2 is fairly inexpensive. With this approach, one transforms the nonlinear PDE into an elliptic system (the form shown in Equation 3.2) with Dirichlet boundary conditions on the multiply-connected domain using the classical "hydrodynamic analogy" [10] prior to numerical solution. In the simplest case, one may exploit the fact that the planar, steady, constant film thickness, gas lubrication equation is harmonic in the square of the pressure.

Standard packages such as the finite element routines in MATLAB [1] can dispatch the numerics in an expedient fashion. Such a solution technique is demonstrated in Figure 3-2 for the thrust bearing design used in the macro-bearing test rig (Figure 2-19) in the case of constant film thickness. The film entrance pressure in the calculation of Figure 3-2 is 5 atm. while the exhaust pressure is 1 atm. The pressure contours are spaced at intervals of 0.2 atm. Clearly, the discrete nature of the restrictor crown has a substantial impact on the pressure field which would not be captured by the low-order analytical solution.

3.2.3 Combined Restrictor/Film System

The operation of a hydrostatic thrust bearing is the result of a coupling between the restrictors and lubrication film described above. Consequently, one may think of the supply pressure, $P_s$, as a "pressure budget" to be divided between the restrictors and film. If a large load is applied to the thrust bearing, the bearing clearance will close so that the restrictor mass flow will decrease. This will in turn reduce the pressure losses in the restrictors so that a larger fraction of the supply pressure is delivered
Figure 3-2: Pressure contours from finite element solution of thrust bearing lubrication film. Pressure at supply inlets (white) is 5 atm. Exhaust pressure at inner/outer circumference of pad (black) is 1 atm. Pressure contours are spaced equally at intervals of 0.2 atm.
to the film, increasing the bearing reaction. Therefore, the losses in the restrictors function as a “pressure reserve” that may be delivered to the film in the event of heavy loading.

To see how the thrust bearing flows scale between macro and micro devices, note that the absence of Couette terms in Equation 3.2 removes any bearing number [28] dependence. Also, the assumption of steadiness precludes any appearance of the squeeze number [28]. Consequently, the lubrication film portion of the thrust bearing flow obeys simple geometric scaling laws. That is, for a given set of exhaust and film entrance pressure boundary conditions, load capacity is proportional to thrust pad area. Since compensation losses include inertial as well as viscous effects, it is not surprising that the film entrance Reynolds number is used to scale this mechanism [67].

3.3 Experimental Results

Due to the crucial role of axial balance in journal bearing stability experiments, it is important to exhaustively document the performance of the hydrostatic thrust bearings in the macro-bearing test rig. This provides heuristics for safe operation of the rig, especially in the area of off-design performance. Additionally, a major contribution to the Micro-Engine Project resulting from the macro bearing experiments is the establishment of an extensive data base which may be used to validate computational models. In the context of thrust bearing calculations, such a data base is particularly important for constructing effective, low-order, semi-empirical models of the compensation losses.

3.3.1 Load-Displacement Characteristic

The static load characteristics of the macro-rig hydrostatic thrust bearings were measured under loads for a wide range of pressure combinations for both the forward and aft thrust bearing supply pressures. The loads were applied to the rotor in-situ by calibration masses supported by push-rods that were installed in the two main tur-
bine exhaust ports (Figure 2-6). The push-rods were designed to create a minimum amount of flow blockage in these exhaust ports which also double as the exhaust for the forward thrust bearing. For each pair of forward and aft thrust bearing supply pressure combinations tested, a range of loads were applied. Pressure measurements were made with the Scanivalve system and rotor displacement was measured using the axial displacement probes under the aft rotor face. Since the rotor had a tendency to wobble during tests, the axial displacement used to describe the rotor position is that computed for the geometric center of the rotor using the displacement probe readings. Results based on this metric were found to be repeatable and physically consistent with the applied loading conditions.

The forward thrust bearing supply pressures, $P_{FTB}$, aft thrust bearing supply pressures, $P_{ATB}$, and static loads, $F$, investigated in the experiments were chosen at the discrete values in Table 3.1. The load capacity of the thrust bearings was measured for all possible combinations of the parameters given in Table 3.1. The total clearance in all tests was $3.4 \times 10^{-3}$ in. (nominal axial clearance of $1.7 \times 10^{-3}$ in. with rotor centered between thrust plates). The uncertainty in measuring the rotor displacement is $\pm 0.05 \times 10^{-3}$ in., or 3% of the nominal clearance. The uncertainty in measuring the applied load is $\pm 0.011$ lbf. The thrust bearing pressures are accurate to $\pm 0.05$ psig.

Table 3.1: Parameter values for macro-thrust bearing tests.

<table>
<thead>
<tr>
<th>parameter</th>
<th>values</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{FTB}$</td>
<td>15.0, 30.0, 46.0, 62.0 ± 0.05</td>
<td>psig</td>
</tr>
<tr>
<td>$P_{ATB}$</td>
<td>5.0, 10.0, 15.0, 20.0, 25.0, 30.0, 35.0, 40.0, 45.0, 50.0 ± 0.05</td>
<td>psig</td>
</tr>
<tr>
<td>$F$</td>
<td>1.153, 2.036, 3.359, 4.241, 5.564, 6.446, 7.770, 8.652 ± 0.011</td>
<td>lbf.</td>
</tr>
</tbody>
</table>

Figure 3-3 shows the normalized rotor displacement between the thrust bearings for a fixed $P_{ATB}$ of 5 psig. The different styles of data markers correspond to the four values of $P_{FTB}$ given in Table 3.1 while the vertical axis denotes the eight different loads also specified in Table 3.1. As can be seen in the figure, the thrust bearings cannot maintain axial equilibrium for any of the applied loads, including the load.
Figure 3-3: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 5$ psig).

Figure 3-4: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 20$ psig).
Figure 3-5: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 35$ psig).

Figure 3-6: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 50$ psig).
of \( F = 1.15 \) lbf, which corresponds to the weight of the rotor. Clearly, the forward thrust bearing is at high pressure relative to the aft thrust bearing and the system cannot levitate the rotor.

Figure 3-4 is a similar plot except now \( P_{ATB} \) has been elevated to 20 psig. In this case, the rotor centers itself between the thrust plates for the lowest value of \( P_{FTB} = 15 \) psig and the case of an applied load consisting only of the rotor weight. As expected, the rotor is slightly closer to the forward thrust plate than the aft thrust plate under these lightly loaded conditions since the aft thrust bearing is at a supply pressure 5 psig higher than the forward thrust bearing. If the forward and aft supply pressures had been equal at only 20 psig, the weight of the rotor would have been negligible relative to the load capacity of the bearings so that the rotor was perfectly centered.

The tests in Figure 3-5 have \( P_{ATB} = 35 \) psig which is approximately midway in the range over which \( P_{FTB} \) varies. As a result, there is a larger number of supply pressure combinations that can levitate the rotor under a correspondingly wider range of loading conditions. It is notable that for an applied load of only the rotor weight, altering \( P_{FTB} \) much below or above \( P_{ATB} = 35 \) psig produces a fairly large “snap-over” response of the rotor position to whichever thrust plate is favored by the thrust bearing supply pressure differential. Consequently, care should be taken when operating the thrust bearings with asymmetric supply pressures.

In Figure 3-6, it is evident that the bearings are quite stiff. For example, at \( P_{FTB} = 62 \) psig, the thrust bearings support loads an order of magnitude greater than the rotor weight while only allowing a normalized displacement of 0.2 from the centered position. Again, for \( P_{FTB} = 46 \) psig \( \approx P_{ATB} = 50 \) psig, the rotor is only slightly closer to the forward thrust plate at low loading conditions, as expected. The complete data set for these static load experiments is given in the appendices.

### 3.3.2 Measurement of the Restrictor Loss Coefficient

The static load tests presented above give information characteristic of the complete restrictor/film system. Since the lubrication film is readily computed as previously
Figure 3-7: Pressure contours from finite element solution of thrust bearing lubrication film with center exhaust hole eliminated. Pressure at supply inlets (white) is 5 atm. while exhaust pressure at outer circumference of pad (black) is 1 atm. Pressure contours are spaced equally at intervals of 0.2 atm.

described, it is desirable to gain information about the restrictor losses which are substantially more expensive to rigorously analyze. That is, one would like to find a way to measure the restrictor losses without having to radically alter the existing experimental set-up.

In the experiments of Vohr [67], the apparatus was specially designed with a movable pressure port on the surface facing the restrictor exhaust. Such a strategy is not practical in the macro-bearing rig because the surface facing the restrictors is the spinning rotor. Instead, an alternative strategy is revealed when the lubrication film calculation of Figure 3-2 is revisited.
Figure 3-7 shows the finite element solution for the lubrication film with the center exhaust eliminated from the domain. The outer exhaust boundary condition remains at 1 atm while the film entrance pressure is again set to 5 atm. Each pressure contour corresponds to a pressure change of 0.2 atm. From the figure it is seen that the pressure in the center of the thrust pad equilibrates to a value fairly close to the film entrance pressure. As a result, if one isolates the center exhaust hole of the macro-rig from the outer exhaust hole and uses the center exhaust to measure pressure rather than set pressure, a reasonable estimate of the film entrance pressure is obtained. This approximation of the restrictor losses can be correlated to the film entrance Reynolds number which is calculated using corresponding restrictor mass flow measurements.

The restrictor loss coefficient, $K$, is defined by Equation 3.1. The experimentally measured values of $K$ for a range of operating conditions are shown in Figure 3-8. Although the uncertainties include contributions from measurements of temperature, pressure and rotor position, the primary source of error is due to the mass flow mea-
surements. The $\tilde{K}$ correlation of Figure 3-8 is in good agreement with the empirical correlation of Vohr [67] where $\tilde{K} \approx 0.6$ for film entrance Reynolds numbers over 2000.

3.4 Summary of Results

Due to the importance of axial equilibrium in journal bearing experiments with the macro-bearing test rig, a wide range of off-design operating conditions has been examined under varied axial loads. The results were found to be repeatable and physically consistent. The thrust bearings demonstrated load capacities an order of magnitude greater than the rotor weight at moderate axial eccentricity. The rotor exhibited a "snap-over" phenomenon when the bearing supply pressures were substantially disparate. The restrictor loss coefficient was measured and found to be in agreement with the literature. Taken as a whole, the results of the thrust bearing experiments documented in this chapter constitute a useful data base for validation of calculations as well as providing strategies for maintaining axial equilibrium of the rotor during journal bearing experiments.
Chapter 4

Hydrodynamic Operation: Theory

4.1 Introduction

The notion that a lubrication film could separate bearing surfaces and act as a pressure generation mechanism capable of transmitting applied loads with very low friction was first established by the pressure measurements of Tower [65, 66] and the friction measurements of Petrov [43, 44, 45, 46]. When Reynolds [56] derived the governing differential equation for such lubrication films, the physical origins of load-bearing pressure were exposed. In short, when no external pressures other than ambient are imposed on the lubrication film, all load capacity is the direct result of relative motion between the bearing surfaces. A bearing operating in this fashion is considered "self-acting" or "hydrodynamic" in the sense that the lubricant pressure distribution is a result of bearing surface kinematics.

The hydrodynamic mode is antithetic to the hydrostatic mechanism previously described in Chapter 3 where super-ambient external pressures were required to produce load capacity. It should be noted that the term "hydrostatic" is a misnomer when contrasted with hydrodynamic effects since it implies that loads are supported by a lubricant which is motionless. As seen in Chapter 3, the load-bearing properties of hydrostatic bearings depend upon pressure losses due to lubricant flows.

A schematic of a plain, cylindrical journal bearing is given in Figure 4-1. The journal bearing is simply a rotor or journal which fits inside a slightly over-sized
outer sleeve. In the figure, the lubrication gap between the sleeve and rotor is greatly exaggerated for clarity. Typically, the sleeve radius differs from that of the rotor by approximately one part per thousand.

Under loaded conditions when the geometric center of the rotor, $O'$, is pushed away from that of the bearing, $O$, the displacement is quantified by an eccentricity vector, $\vec{e}$. The modulus of this vector is $e$ while the direction of the vector with respect to the inertial axes, $(i_1, i_2)$, is $\alpha$. The scalar displacement, $e$, is often normalized by the nominal bearing clearance, $c$, that results when the rotor is centered in the sleeve. This normalized distance is termed the “eccentricity ratio” and is given by the symbol $\epsilon$. The rotor has a mass of $M$, a radius of $R$ and turns in a counterclockwise sense at angular velocity $\omega$. The geometric center of the rotor is tracked by a non-inertial set of axes, $(\hat{s}_1, \hat{s}_2)$, whose origin is fixed at $O$. The angle $\theta$ locates field points in the lubrication film with respect to the “line of centers” that connects the geometric centers of the journal and bearing. The axial length of the bearing, $L$, runs perpendicular to the plane of Figure 4-1 which is taken to be at mid-span of the bearing. Specific values of these parameters for the macro-bearing test rig are shown in Table 4.1.

### Table 4.1: Values of bearing parameters for macro-bearing test rig.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$0.004 \pm 0.0001$</td>
<td>in.</td>
</tr>
<tr>
<td>$D = 2R$</td>
<td>$4.096 \pm 0.0001$</td>
<td>in.</td>
</tr>
<tr>
<td>$L$</td>
<td>$0.308 \pm 0.0001$</td>
<td>in.</td>
</tr>
<tr>
<td>$M$</td>
<td>$0.523 \pm 0.00002$</td>
<td>kg</td>
</tr>
</tbody>
</table>

4.2 The Rigid Body

In the case of a balanced, rigid journal restricted to rotation about its longitudinal axis and translational motion in the plane of Figure 4-1 ("cylindrical" motion), the journal/film combination represents an autonomous dynamical system of four coupled, nonlinear, ordinary differential equations (ODE's).
The equations of motion of the system are given by:

\[
\frac{d\dot{\epsilon}}{d\tau} = \frac{1}{\Gamma^2} \left( \frac{1}{2\pi P_m} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \frac{\Psi}{H} \cos \theta \ d\theta \ d\xi + \cos \alpha \right) + \epsilon \dot{\alpha}^2 \tag{4.1}
\]

\[
\frac{d\epsilon}{d\tau} = \dot{\epsilon} \tag{4.2}
\]

\[
\frac{d\dot{\alpha}}{d\tau} = \frac{1}{\epsilon} \left[ \frac{1}{\Gamma^2} \left( \frac{1}{2\pi P_m} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \frac{\Psi}{H} \sin \theta \ d\theta \ d\xi - \sin \alpha \right) - 2\dot{\alpha} \dot{\epsilon} \right] \tag{4.3}
\]

\[
\frac{d\alpha}{d\tau} = \dot{\alpha} \tag{4.4}
\]

In Equations 4.1, 4.2, 4.3, and 4.4, non-dimensional time is defined by \( \tau = \omega t \) where \( \omega \) is the rotational speed of the rotor and \( t \) is time. Differentiation with respect to \( \tau \) is denoted by an over-dot. The product of non-dimensional pressure, \( P \), and
non-dimensional film thickness, \( H \), is expressed by the symbol \( \Psi = PH \). In light of this, it is evident that the double-integrals of \( \Psi/H \) in Equations 4.1 and 4.3 represent integration of the film pressure distribution over the journal. The symbol \( \Gamma = \frac{\omega}{\sqrt{Mc}} \) contains the effect of the rotor speed, \( \omega \), rotor mass, \( M \), and the dimensional load, \( F \). Because \( \Gamma \) has the form of a non-dimensional rotational speed, it is sometimes referred to as the “speed-parameter.”

Thus, the equations of motion quantify the net acceleration due to a steady applied load of \( P_m \) and the integrated pressure force from the lubrication film. A list of symbols is given in Table 4.2. It should be noted that the reaction of the self-acting lubrication film obtained from the governing fluid equations is determined by the kinematics of the bearing surfaces - solutions of the above equations of motion. Because the equations of motion contain the film reaction, the rotor and film are coupled and Newton’s second law must be considered simultaneously with the governing equations of the film.

### 4.3 The Lubrication Film

The unsteady, compressible lubrication film that separates the rotor and bearing of Figure 4-1 is governed by the well-known Reynolds Equation:

\[
\left( \frac{\pi D}{2L} \right)^2 \frac{\partial}{\partial \xi} \left( H \Psi \frac{\partial \Psi}{\partial \xi} \right) + \frac{\partial}{\partial \theta} \left[ \Psi \left( H \frac{\partial \Psi}{\partial \theta} - \Psi \frac{\partial H}{\partial \theta} \right) \right] = \Lambda \left[ (1 - 2\alpha) \frac{\partial \Psi}{\partial \theta} + 2 \frac{\partial \Psi}{\partial \tau} \right].
\] (4.5)

This represents an unsteady, nonlinear, partial differential equation (PDE) in two spatial dimensions. The derivation of this PDE can be found in any elementary text on lubrication theory [28].

It is worth noting that the variable of interest is the product of local film pressure and film thickness (\( \Psi = PH \)) rather than the pressure itself. This is because the sharp changes in the pressure distribution occur precisely where the film thickness is smallest. By casting the equation in terms of the product of film thickness and pressure, a better-behaved formulation results. Of course, the standard assumptions
Table 4.2: Nomenclature list for journal bearing equations of motion and lubrication film.

<table>
<thead>
<tr>
<th>symbol/definition</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \equiv \frac{p}{p_a} )</td>
<td>non-dimensional pressure</td>
</tr>
<tr>
<td>( \epsilon \equiv \frac{\xi}{c} )</td>
<td>eccentricity ratio</td>
</tr>
<tr>
<td>( c )</td>
<td>dimensional bearing eccentricity</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>nominal bearing clearance</td>
</tr>
<tr>
<td>( \tau \equiv \omega t )</td>
<td>non-dimensional time</td>
</tr>
<tr>
<td>( \omega )</td>
<td>rotor rotational speed</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( H \equiv \frac{h}{c} = 1 + \epsilon \cos \theta )</td>
<td>non-dimensional film thickness</td>
</tr>
<tr>
<td>( h = c + \epsilon \cos \theta )</td>
<td>dimensional film thickness</td>
</tr>
<tr>
<td>( \theta )</td>
<td>angular film coordinate</td>
</tr>
<tr>
<td>( \tilde{\xi} \equiv \frac{\pi_z}{L} )</td>
<td>non-dimensional axial coordinate</td>
</tr>
<tr>
<td>( z )</td>
<td>axial coordinate</td>
</tr>
<tr>
<td>( P_m \equiv \frac{F}{p_a L D} )</td>
<td>non-dimensional applied load</td>
</tr>
<tr>
<td>( F )</td>
<td>applied load</td>
</tr>
<tr>
<td>( p_a )</td>
<td>ambient pressure</td>
</tr>
<tr>
<td>( L )</td>
<td>bearing length</td>
</tr>
<tr>
<td>( R )</td>
<td>bearing radius</td>
</tr>
<tr>
<td>( D = 2R )</td>
<td>bearing diameter</td>
</tr>
<tr>
<td>( \Gamma \equiv \frac{\omega}{\sqrt{F / M c}} )</td>
<td>speed-parameter</td>
</tr>
<tr>
<td>( M )</td>
<td>bearing mass</td>
</tr>
<tr>
<td>( \Lambda \equiv \frac{6 \mu \omega}{p_a} \left( \frac{R}{c} \right)^2 )</td>
<td>bearing number</td>
</tr>
<tr>
<td>( \mu )</td>
<td>absolute viscosity of lubricant</td>
</tr>
</tbody>
</table>

of lubrication theory apply. For example, cross-film pressure gradients are neglected and the fluid is assumed to be inertia-free. In addition, since the radii of the journal and bearing differ by only a fraction of a percent, curvature terms normally appearing in rotating flows may also be neglected. Thus, the annular lubrication film shown in Figure 4-1 may be "unwrapped", described in Cartesian coordinates and treated as a channel of varying width with periodic inflow/outflow boundary conditions.

For a given bearing design of fixed \( L/D \), the pertinent non-dimensional parameter in Equation (4.5) is the "bearing number", denoted by the symbol \( \Lambda = \frac{6 \mu \omega}{p_a} \left( \frac{R}{c} \right)^2 \). Since the inertia terms have been dropped from the more general Navier-Stokes equations in the derivation of this PDE, the remaining terms in the approximation repre-
sent pressure-driven ("Poiseuille") and shear-driven ("Couette") flows. Consequently, the bearing number, $\Lambda$, measures the ratio of shearing to pressure effects. Put another way, when the journal rotates, causing Couette flows, Equation (4.5) dictates that the film must respond with Poiseuille flows on the other side of the "hydrodynamic ledger." The pressure rise associated with these Poiseuille effects provides load capacity. It should be emphasized that this simple explanation is only a caricature of one particular hydrodynamic mechanism. For example, film pressures may also be generated by squeezing action as the journal approaches the bearing wall and lubricant is expelled from the squeeze gap. In addition, rotor whirl causes the point of minimum film thickness to precess and may generate load capacity in a dynamic sense.

Finally, because the bearing number is proportional to the rotor rotational speed, it is often interpreted as an non-dimensional operational parameter acting as a proxy for rotational speed, $\omega$. Because compressibility effects in the lubrication film increase with $\omega$, $\Lambda$ is often referred to as the "compressibility number." In the case of a journal bearing, this is somewhat misleading because, for a fixed $\Lambda$, it is possible to produce flows in either the compressible or incompressible regimes by varying the eccentricity ratio. This result is demonstrated in computational results from LubePack discussed later in this chapter. Furthermore, it can be shown that the incompressible lubrication equation can also be cast in terms of $\Lambda$, making the term "compressibility number" rather inaccurate.

4.3.1 Numerical Solution of the Steady Lubrication Film

The nonlinearity of Equation (4.5) made accurate solution of hydrodynamic, compressible, lubrication films impractical until numerical methods became accessible in the 1950's and 1960's. When considering steady solutions of Equation (4.5), all hydrodynamic effects are the result of the rigid body rotation of the rotor. Consequently, the equations of motion (Equations (4.1), (4.2), (4.3), (4.4)) reduce to integration of the film pressure distribution under static equilibrium. This allows direct computation of the film pressure followed by a posteriori determination of the concomitant static attitude angle and bearing load capacity.
<table>
<thead>
<tr>
<th>symbol/definition</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>axial Galerkin truncation index</td>
</tr>
<tr>
<td>$M$</td>
<td>circumferential Galerkin truncation index</td>
</tr>
<tr>
<td>$n$</td>
<td>axial Galerkin index</td>
</tr>
<tr>
<td>$m$</td>
<td>circumferential Galerkin index</td>
</tr>
<tr>
<td>$C_n$</td>
<td>axial Galerkin coefficient</td>
</tr>
<tr>
<td>$A_{n,m}$, $B_{n,m}$</td>
<td>circumferential Galerkin coefficients</td>
</tr>
</tbody>
</table>

Table 4.3: Nomenclature list for Galerkin solution of lubrication film.

Raimondi and Boyd [52, 53, 54] used a finite difference technique to produce comprehensive design charts for the static solutions of conventional $L/D$, plain, cylindrical, gas journal bearings. This approach provided accurate, steady solutions but did not readily yield information on the stability of these results; the task of deciding which areas of the design charts were physically realizable and which were not remained an open issue.

Seven years later, Cheng and Pan [7] implemented a Galerkin technique to solve Equation (4.5) for plain, cylindrical, gas journal bearings. This work extended that of Cheng and Trumpler [8] to include finite length bearings and unsteady effects. In particular, this technique allowed for an efficient formulation of stability conditions for the steady-state solutions. As a result, this is the technique selected for the current study. The more recent approaches of Jacobson [30], Piekos et al. [48] and Piekos [47] are discussed in the context of LubePack results to follow.

Ostensibly, application of Galerkin's method [7] to determine the static and dynamic characteristics of a plain, cylindrical, gas journal bearing is straightforward. One uses a double Fourier series to represent the $PH$ function in Equation (4.5), as follows:

$$\Psi = PH = H + \sum_{n=1}^{N} \cos (2n - 1) \xi \left\{ C_n + \sum_{m=1}^{M} \left[ A_{n,m} \cos m\theta + B_{n,m} \sin m\theta \right] \right\}.$$  \hspace{1cm} (4.6)

A nomenclature list is provided in Table 4.3. When a truncation of this series (denoted by truncation indices $N, M$) for $\Psi$ is substituted into the governing PDE, a residual,
\( R(\Psi) \), is defined as:

\[
R(\Psi) = \left( \frac{\pi D}{2L} \right)^2 \frac{\partial}{\partial \xi} \left( H \frac{\partial \Psi}{\partial \zeta} \right) + \frac{\partial}{\partial \theta} \left[ \Psi \left( H \frac{\partial \Psi}{\partial \theta} - \frac{\partial H}{\partial \theta} \right) \right] - \Lambda \left( 1 - 2\dot{\alpha} \right) \frac{\partial \Psi}{\partial \theta} + 2 \frac{\partial \Psi}{\partial r} .
\] (4.7)

For static solutions, the coefficients of the \( PH \) series, namely, \( A_{n,m} \), \( B_{n,m} \), and \( C_{n} \), are determined in the usual way by solving the \( N + 2MN \) dimensional nonlinear optimization problem embodied by the coefficient orthogonality relations shown as follows:

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\pi} R(\Psi) \cos (2r - 1) \xi \, d\xi \, d\theta = 0 \quad r = 1, \ldots, N \] (4.8)

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\pi} R(\Psi) \cos (2r - 1) \xi \cos p\theta \, d\xi \, d\theta = 0 \quad r = 1, \ldots, N \quad p = 1, \ldots, M \] (4.9)

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\pi} R(\Psi) \cos (2r - 1) \xi \sin q\theta \, d\xi \, d\theta = 0 \quad r = 1, \ldots, N \quad q = 1, \ldots, M . \] (4.10)

With the static solutions in hand, one may then proceed to dynamic analysis to test the stability of these results.

### 4.4 Dynamical System Formulation for Stability Analysis

While the steady-state solutions obtainable by the Galerkin technique described in the previous section are useful, utility of these calculations is limited by the well-known fact that such bearings are prone to self-excited instabilities. Thus, one must determine which of the static solutions are physically realizable.

Equations (4.1), (4.2), (4.3), (4.4), (4.8), (4.9), and (4.10) represent a \( 4+N+2MN \) order nonlinear, autonomous dynamical system. This system of coupled ODE’s has the form of the time rate-of-change of a state vector, \( \ddot{\vec{x}} \), defined by that state vector, \( \vec{x} \), under the action of a nonlinear vector field, \( \vec{F} \), for a given parameter vector, \( \vec{M} \).
\[ \dot{x} = F(x; \bar{M}) \]  \hspace{1cm} (4.11)

with nomenclature summarized by Table 4.4.

The previously described steady-state solutions represent the fixed points of the dynamical system of Equation (4.11). These special cases represent points in the system phase space that are mapped to themselves for all time by \( \bar{F} \). This condition is expressed as:

\[ \bar{F} (\bar{x}_0; \bar{M}_0) = 0. \]  \hspace{1cm} (4.12)

To examine the stability of the fixed points, a small perturbation, \( \bar{y}(\tau) \), is superimposed on the static solution:

\[ \bar{x}(\tau) = \bar{x}_0 + \bar{y}(\tau). \]  \hspace{1cm} (4.13)

Substituting this expression into Equation (4.11) results in the system:

\[ \dot{\bar{y}} = \bar{F} \left( \bar{x}_0 + \bar{y}; \bar{M}_0 \right). \]  \hspace{1cm} (4.14)

It is evident that the fixed point, \( \bar{x}_0 \), has been transformed to the fixed point \( \bar{y} = 0 \).

Assuming that \( \bar{F} \) is sufficiently well-behaved, the first-order Taylor expansion of

Table 4.4: Nomenclature list for dynamical system formulation.

<table>
<thead>
<tr>
<th>symbol/definition</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{x} = (x_1, \ldots, x_n) )</td>
<td>state vector</td>
</tr>
<tr>
<td>( \bar{x}_0 = ((x_1)_0, \ldots, (x_n)_0) )</td>
<td>state vector corresponding to fixed point</td>
</tr>
<tr>
<td>( \bar{M} )</td>
<td>parameter vector</td>
</tr>
<tr>
<td>( \bar{M}_0 )</td>
<td>parameter vector corresponding to fixed point</td>
</tr>
<tr>
<td>( \bar{F} = (F_1 (x_1, \ldots, x_n), \ldots, F_n (x_1, \ldots, x_n)) )</td>
<td>vector field</td>
</tr>
<tr>
<td>( \bar{A} )</td>
<td>Jacobian of dynamical system</td>
</tr>
</tbody>
</table>
\[ \ddot{y} = F(x; \bar{M}_o) + D_x \ddot{F}(x; \bar{M}_o) \dot{y} + O(|\dot{y}|^2). \]  

(4.15)

Using the definition of a fixed point and truncating the expansion to first order, Equation (4.15) reduces to:

\[ \ddot{y} \approx D_x \ddot{F}(\bar{x}_0; \bar{M}_o) \dot{y} = A \dot{y}. \]  

(4.16)

In this equation, the Jacobian matrix of the dynamical system, \( A \), is introduced. A more explicit form of \( A \) is given by:

\[
A = \begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n}
\end{pmatrix}. 
\]  

(4.17)

The stability of the fixed point is then obtained by applying Routh’s criterion to determine whether or not the real parts of the eigenvalues of \( A \) are negative (stable) or positive (unstable) [7]. At the threshold of instability, the imaginary part of the neutrally stable eigenvalues represents the response frequency of the system. In the classical treatment of the plain, cylindrical, gas journal bearing, this corresponds to the whirl frequency of the rotor [7]. Thus, the fixed points lose stability via Hopf bifurcation [40].

### 4.5 Cheng and Pan’s Scheme: Computational Issues

#### 4.5.1 Application of Galerkin’s Method

In Cheng and Pan’s original work [7], the symbolic integration required by the equations of motion and orthogonality relations had to be performed by hand. The subse-
quent numerics were then executed on a computer. Due to the vast number of integrals produced by substituting the Fourier series into the various nonlinear operators in the Galerkin system, the major limitation was the task of symbolic integration. In fact, Cheng and Pan used truncation indices of \( N = 1, M = 2 \) in order to make the calculation tractable. This produced large truncation errors, although it should be noted that many of the trends exhibited in their results were correct for low values of \( \epsilon \).

With the advent and proliferation of symbolic manipulation packages in the past 20 years and the accompanying wide availability of computing power, one may be led to believe that the symbolic integration workload which stifled Cheng and Pan may now be easily swept aside using standard, commercial symbolic integration routines. To test this hypothesis, benchmark tests representing characteristic aspects of the Galerkin system were developed using the built-in functions of a state-of-the-art symbolic manipulation package, Mathematica by Wolfram Research [71].

### 4.5.2 Benchmarks for Computer-Based Symbolic Integration

The first benchmark uses the Mathematica function `Integrate[]` to perform the symbolic integration necessary to construct the orthogonality relations given by Equations (4.8), (4.9) and (4.10) for the highest order axial, circumferential sine, and circumferential cosine relations. The two metrics used to evaluate performance were CPU time (as measured by the Mathematica function `Timing[]`) and total memory required. The number of axial modes was fixed at \( N = 1 \) while the circumferential truncation index, \( M \), was varied. The calculations were performed on a 266 MHz Intel Pentium II PC with 96 MB of RAM.

Figure 4-2 shows the results for the CPU time required to perform the film benchmarks. As is evident from the figure, the nonlinear operators of the Galerkin system are surprisingly unforgiving, exhibiting a sharp climb in CPU time at only 7 circumferential modes. This is a sobering result since it represents only an improvement of 5 modes over what Cheng and Pan were able to accomplish armed only with pencil and paper [7].
Figure 4-2: Mathematica's `Integrate[ ]`: CPU Time for Unsteady Film Integrals.

Figure 4-3: Mathematica's `Integrate[ ]`: RAM Required for Unsteady Film Integrals.
4.5.3 Causes of Poor Performance

Considerable light is shed on the CPU timing results of Figure 4-2 by considering the memory requirements of the benchmark. Figure 4-3 shows that the memory necessary for evaluation of the film equation integrals increases exponentially with the order of the calculation, nearly doubling for each increase in the circumferential truncation index. In order for the Galerkin system to be tractable, all calculations must be done in RAM since hard disk operations have access times that are orders of magnitude longer. In fact, during the benchmark tests with \( M = 7 \), the computer ran out of RAM and had to begin using a disk cache, explaining the sudden increase in CPU time shown in Figure 4-2. However, even if one had a computer with 10 GB of RAM, according to the trend of Figure 4-3, this would allow the evaluation of only 11 total circumferential modes. Clearly, purchasing additional hardware at exponential rates is not a viable way to expand the order of the calculations. It should be emphasized that the benchmark tests only evaluate the highest order equations in the system; there will be a substantial increase in computational load to evaluate the entire equation set.

One may be tempted to eliminate the need for disk caches by subdividing the calculation and only attempting to evaluate limited numbers of integrals in succession. However, this is not an effective solution because the computations in RAM will still increase exponentially with \( M \), requiring a commensurate amount of CPU time.

A primary cause of the exponential memory requirements shown in Figure 4-3 is the fact that the built-in symbolic integration routines of packages such as Mathematica are quite general in nature to provide the user with robustness. When the Mathematica kernel begins to symbolically evaluate an integral, it launches into a hierarchy of transformations to try and reduce the integrand into a form that is conducive to evaluation [71]. As the complexity of the integrand increases, as is the case with successively higher order Galerkin systems, the number of possible simplifying transformations increases dramatically along with the number of complicated integrals to compute. This produces the enormous demand for memory shown in Fig-
Similar results were obtained for a benchmark thatformulates the equations of motion associated with the Galerkin system described above.

4.5.4 Symbolic Integration Benchmarks: Conclusions

The hypothesis posed at the start of this section is whether or not the built-in routines of commercial symbolic manipulation packages and current computing technology could easily dispatch the massive numbers of integrals produced by the Galerkin system of a plain, cylindrical, gas journal bearing. Based upon the benchmarks, the answer is a resounding "No." What is clearly needed is a highly optimized set of transformation rules specially designed for symbolic evaluation of the classes of integrals appearing in Galerkin systems for lubrication problems. The derivation of such a scheme is described in the next section and given the appellation "LubePack."

4.6 Development of Efficient, Symbolic Integration Routines: LubePack

The primary result of the previous section is that in order to implement a tractable Galerkin formulation of the rotor/film dynamical system, one must avoid the standard symbolic integration routines of commercially available symbolic manipulation packages in favor of custom-made integration rules. In the current study, this means bypassing the standard Integrate[] function of Mathematica, and instead, working directly with the pattern recognition capability of the Mathematica kernel. Thus, as soon as the kernel recognizes certain patterns in the algebraic terms generated by the Galerkin operators, an immediate replacement is executed as prescribed by a hierarchical set of transformation rules known as “LubePack,” specially developed for this study.

This approach is quite general and owes much of its flexibility to the underlying fixed point constructs - repeatedly applying a transformation to an expression until the result no longer changes. In large symbolic computations, the analyst may never
even see the expressions involved since the primary interest is often the properties of
the objects being manipulated rather than the objects themselves. One may define
these properties by establishing a set of transformation rules under which working
expressions must be invariant before being declared as “simplest form.” In this inves-
tigation, the governing set of transformation rules is LubePack while “simplest form”
is some expression of interest such as the complete, nonlinear system of equations
governing the dynamical system or the Jacobian associated with this system.

4.6.1 Symbolic Integration for the Equations of Motion

In order to efficiently dispatch the problematic integrals that constitute the Galerkin
system of Equations (4.1), (4.2), (4.3), (4.4), (4.8), (4.9), and (4.10), it is instructive
to consider the general form of the so-called “journal bearing integrals” studied
by Sommerfeld [59] and Booker [5]. For arbitrary limits of integration, the form of these
integrals is denoted by \( I^{i,j}_{k}^{b} |_{a} \) and given as:

\[
I^{i,j}_{k}^{b} |_{a} = \int_{a}^{b} \frac{\sin^{i} \theta \cos^{j} \theta}{(1 + \epsilon \cos \theta)^{k}} \, d\theta \quad \forall (i, j, k) \in \mathcal{N} . \tag{4.18}
\]

While Sommerfeld developed techniques for evaluating the relatively simple in-
tegrals that arise in incompressible analysis, Booker provided limited integral tables
that presented results in terms of recursive relations derived using partial fraction
identities. When fully discharged, these recursive formulae result in expressions in-
volving two fundamental integrals that may be easily evaluated using residue theory.
Although Booker’s tables are cumbersome to implement manually, they do demon-
strate an effective algorithm for symbolic evaluation of integrals involving inverse
powers of film thickness.

In light of this, the first transformation rules of LubePack, written in the Math-
ematica language, implement the recursive system of partial fraction identities de-
scribed by Booker [5] and are shown in the appendices as Equations (B.28) and (B.29).
These transformation rules are vital for efficient symbolic evaluation of the Galerkin
formulation for the equations of motion shown in Equations (4.1), (4.2), (4.3) and (4.4).
4.6.2 Symbolic Integration for the Fluid Equations

The equations from the lubrication film, Equations 4.8, 4.9, and 4.10, produce a special case of the journal bearing integral, \( I_{0}^{i,j|a} \). Although some mathematics handbooks tabulate results for this class of integrals for special values of \( a \) and \( b \), it is shown in the appendices that the closed form solution for arbitrary limits of integration in the case of \( i(=2m) \) even, \( j(=2n) \) even, is given by:

\[
I_{0}^{2m,2n|a} = -\frac{\Gamma \left( m + \frac{1}{2} \right)}{2 \cdot \Gamma(n + m + 1)} \cdot \sum_{s=1}^{m} \frac{\Gamma \left( n + (m - s) + 1 \right)}{\Gamma \left( m - (s - 1) + \frac{1}{2} \right)} \cdot \sin^{2(m-s)+1} \theta \cos^{j+1} \theta |_{a}^{b} \\
+ \frac{1}{\pi} B \left( m + \frac{1}{2}, n + \frac{1}{2} \right) \cdot (b - a)
\]

(4.19)

where \( B \) denotes the Euler beta function. Generalizing for arbitrary limits of integration is useful because it facilitates application of the rules for axial integration over \((-\frac{\pi}{2}, \frac{\pi}{2})\) and circumferential integration over \((0, 2\pi)\) in the case of plain cylindrical journal bearings. In addition, this flexibility allows extension to schemes for segmented pad bearings defined by the integration limits \((a, b)\). Similar results may be obtained for the combinations of \( i \) even - \( j \) odd, \( i \) odd - \( j \) even, and \( i \) odd - \( j \) odd.

For the limits of integration appearing in Galerkin systems for plain, cylindrical, journal bearings the general integration formulae reduce to the following compact expressions:

\[
I_{0}^{2m,2n|2\pi} = 2B \left( m + \frac{1}{2}, n + \frac{1}{2} \right) \\
I_{0}^{2m,2n+1|2\pi} = 0 \\
I_{0}^{2m+1,2n|2\pi} = 0 \\
I_{0}^{2m+1,2n+1|2\pi} = 0
\]

(4.20)

\[
I_{0}^{2m,2n|\frac{\pi}{2}} = B \left( m + \frac{1}{2}, n + \frac{1}{2} \right) \\
I_{0}^{2m,2n+1|\frac{\pi}{2}} = B \left( m + \frac{1}{2}, n + 1 \right)
\]
These simple equations embody a remarkable result when one considers the profound implications for practical computation of design properties for this class of bearing. Specifically, the entire system of nonlinear, unsteady, compressible, lubrication equations shown in Equations (4.8), (4.9), and (4.10) may be expressed in terms of the Euler beta function. This allows for extremely efficient symbolic evaluation of the plethora of integrals generated by the nonlinear operators in the film equations.

Secondly, it is evident from Equations (4.20) and (4.21) that the system of film equations is “sparse” in the sense that many of the integrals in the Galerkin system are trivial. Because of this, LubePack has a “clobber list” that consists of a set of pattern recognition objects that yield trivial results when integrated. Figure 4-4 shows a schematic representation of this concept. Using the clobber list, one may take advantage of this “sparseness” by immediately dropping any such terms as soon as they are generated during manipulations, in stark contrast to standard symbolic integration packages which may “chew” on the expression for an inordinate amount of time only to spit out a trivial result. In this way, the only retained terms are the “players,” that yield non-trivial results.

It should be noted that a further computational efficiency could be realized by noting that some of the “players” will exactly cancel each other even though they may yield a non-trivial result when considered in isolation. If these “annihilation pairs” could be identified, they could be considered members of the clobber list and more needless computations could be avoided. Figure 4-4 depicts this extension of the clobber list. Obviously, this concept could be extended to higher order “annihilation sets” where multiple terms combine to produce a trivial result. A simple algorithm for realizing this strategy would be to rewrite the Galerkin operator in terms of nested series using Cauchy product identities. When implementing the orthogonality relations, the order of integration and summation could be interchanged to produce sums of integrals. Standard simplification tactics of index shifting in the nested
series could then be used to combine “like” series so that the annihilation of terms could be carried out on entire summations at a time. The author has manually verified that such a strategy would work on the axial equations but the issue of the circumferential equations remains an open question. However, automating such a strategy would require far more sophisticated pattern recognition objects than those found in LubePack which only clobbers terms that are inherently trivial.

The results of Equations (4.20) and (4.21) are incorporated into Mathematica transformation rules shown in the appendices as Equations (B.26) and (B.27). These rules combine with those of Equations (B.28) and (B.29) to form the core of LubePack.

4.7 LubePack: Baseline Results

It is important to evaluate the performance of LubePack on several different levels. Perhaps the most fundamental aspect of performance is how efficiently LubePack executes the algebraic manipulations required for the generation of the governing
equations of the dynamical system. Secondly, typical results derived from the output of LubePack must be examined for physical consistency. Finally, the accuracy of these LubePack calculations must be determined and compared with results published in the literature.

4.7.1 Symbolic Manipulations: Generation of Governing Equations

To examine the effectiveness of LubePack in generating the governing equations of the dynamical system represented by a plain, cylindrical, gas journal bearing, the symbolic manipulation benchmarks previously used to evaluate Mathematica’s \texttt{Integrate[ ]} function are revisited. Figure 4-5 compares the CPU time required for executing the film benchmark with \texttt{Integrate[ ]} and, alternatively, LubePack. It is seen from the figure that LubePack achieves a large reduction in CPU time as the number of circumferential modes in the Galerkin expansion is increased. The boundary of practical calculation is pushed substantially higher, allowing much more accuracy in the resulting solution.

The reason for LubePack’s performance advantage over \texttt{Integrate[ ]} may be seen in Figure 4-6 which shows the memory requirements for the two schemes. The exponential memory demands of \texttt{Integrate[ ]} dwarf the nearly linear memory requirements of LubePack. This is in part due to the previously mentioned sparseness of LubePack’s beta function formulation as well as the inability of \texttt{Integrate[ ]} to cope with the increasing complexity and volume of integrals generated by the Galerkin system as one proceeds to higher order solutions.

It should be noted that given the current state of computing technology, the inefficiency of \texttt{Integrate[ ]} in dealing with journal bearing integrals is no small inconvenience - in the context of the Galerkin system, it is literally the death knell for this routine. For example, given the exponential memory demands shown in Figure 4-6, \texttt{Integrate[ ]} would require approximately 10 \textit{terabytes} (10,000 gigabytes) of RAM to perform the film benchmark for the case of \( n = 1, m = 24 \). It is important to recall
that the benchmark only considers a fraction of the equations in the entire system so that one could count on using approximately 100 terabytes of memory to derive the entire system of film equations. Because similar results hold for the equation of motion benchmark, one may need substantially more memory to formulate the entire dynamical system using Integrate[]. It will be shown below that this $n = 1, m = 24$ truncation can be efficiently handled using LubePack on a modest PC system.

4.7.2 Numerical Analysis: Steady Solutions

Although the Reynolds equation is solved in terms of the variable $PH$, the fundamental quantity of interest is the non-dimensional pressure, $P$. The LubePack solution for the compressible, steady, pressure distribution for a plain, cylindrical, journal bearing with $L/D = 0.075$, $n = 1$, $m = 24$, $\epsilon = 0.85$ and $\Lambda = 1.0$ is shown in Figure 4-7. The distribution is shown for the unwrapped lubrication film so that the tangential velocity of the rotor surface is directed from the periodic boundary condition at the
left end of the film to the corresponding boundary condition at the right extreme of the film.

As one proceeds along the circumferential \((\theta)\) direction from left to right in Figure 4-7 a large pressure rise is observed due to fluid being forced into the squeeze gap resulting from the non-zero eccentricity ratio of the bearing. As the lubricant moves into the divergent portion of the squeeze gap downstream of the point of minimum film thickness, a sub-ambient, "suction" region is encountered before recovery to nominally ambient conditions at the right end of the figure.

In the axial or \(\xi\) direction in Figure 4-7 it is evident that the pressure extremes are located at the bearing mid-span with axial relieving effects enforcing ambient conditions at the ends of the bearing at \(\xi = -\frac{\pi}{2}\) and \(\xi = \frac{\pi}{2}\). Because the bearing under consideration is so short \((L/D = 0.075)\), one would expect that the film might have difficulty developing compressibility effects at conditions other than those of high \(\epsilon\) or high \(\Lambda\). In other words, the lubricant would likely be expelled out the ends of the short bearing rather than remain in the bearing interior and undergo compression.
Figure 4-7: Pressure distribution of compressible lubrication film in plain cylindrical journal bearing for $\epsilon = 0.85$ and $\Lambda = 1.0$.

due to Couette action. In fact, this is an informal expression of the classical "short bearing" assumption of lubrication theory where circumferential Poiseuille terms are negligible in comparison to axial Poiseuille terms.

The incompressible, full-Sommerfeld, short bearing pressure solution is plotted in Figure 4-8 for the same conditions used in Figure 4-7. Indeed, the two pressure distributions look strikingly similar as implied by the above physical argument. For a more quantitative comparison, a circumferential slice of the compressible and incompressible pressure distributions at the bearing mid-plane station of $\zeta = 0$ is shown in Figure 4-9 for $L/D = 0.075$, $\epsilon = 0.5$, and $\Lambda = 1.0$. The figure shows that the compressible LubePack solution is virtually identical to the incompressible analytical solution, as expected.

As $\epsilon$ is increased to 0.75, Figure 4-10 reveals the first signs of departure of the compressible profile from the incompressible solution. These effects become more pronounced when $\epsilon$ is increased to 0.85 as illustrated in Figure 4-11. Now, the compressible distribution lacks the symmetry of the full-Sommerfeld solution, a well-known signature of compressibility effects. In addition, the pressure peaks of the LubePack
Figure 4-8: Pressure distribution of incompressible lubrication film in a plain cylindrical journal bearing for $\epsilon = 0.85$ and $\Lambda = 1.0$.

Figure 4-9: Comparison between incompressible and compressible pressure distributions at the axial mid-plane ($\xi = 0$) of a plain cylindrical journal bearing ($L/D = 0.075$) for $\epsilon = 0.50$ and $\Lambda = 1.0$. 
solution are shifted with respect to the incompressible peaks, indicating a change in attitude angle - another sign of compressibility effects.

The adherence of the LubePack solution to the full-Sommerfeld results in the expected parameter ranges of low/moderate $\epsilon$ and moderate $\Lambda$ explain why only one axial mode ($n = 1$) is sufficient for accurate results. It is well-known that the incompressible solution has a parabolic axial dependence which is represented by the first two terms of the Taylor series for the single cosine axial mode used for the Galerkin solution. This observation is worthy of note since it suggests that adding axial modes will not significantly impact the accuracy of the Galerkin solution due to the unusually low $L/D$ range demanded by the micro-fabrication techniques used for the micro-engine bearings. Additionally, from Figure 4-11 it is seen that at high $\epsilon$, the pressure solutions exhibit a multiple-scale behavior in the circumferential direction; the sharp gradients in the vicinity of the squeeze gap are reminiscent of internal boundary layers studied in perturbation theory. This provides even more evidence that the key to improving the accuracy of the LubePack solutions is to increase the number of circumferential modes ($m$) rather than axial modes ($n$). Of course, as noted by Piekos [48], [47], the accuracy of this approach diminishes with larger $L/D$ as the system moves out of the “short-bearing” regime.

The results of Figures 4-9, 4-10, and 4-11 provide a certain degree of validation for LubePack; LubePack is observed to be consistent with common sense tribology arguments. Further confidence is gained by considering the comprehensive design charts depicted in Figures 4-12 and 4-13 for static journal bearing properties generated with LubePack for $n = 1$ and $m = 24$. These charts are similar to those published by Cheng and Pan [7].

Figure 4-12 illustrates the expected strong dependence of load capacity on eccentricity ratio. Unlike the static load charts of Cheng and Pan, the LubePack results do not show a substantial roll-off of load capacity at higher values of $\Lambda$. This is indicative of the far greater number of circumferential modes retained in the LubePack solution ($n = 1, m = 24$) relative to the severe truncation used by Cheng and Pan ($n = 1, m = 2$).
Figure 4-10: Comparison between incompressible and compressible pressure distributions at the axial mid-plane ($\bar{\xi} = 0$) of a plain cylindrical journal bearing ($L/D = 0.075$) for $\epsilon = 0.75$ and $\Lambda = 1.0$.

Figure 4-11: Comparison between incompressible and compressible pressure distributions at the axial mid-plane ($\bar{\xi} = 0$) of a plain cylindrical journal bearing ($L/D = 0.075$) for $\epsilon = 0.85$ and $\Lambda = 1.0$. 
Figure 4-12: Design chart for load parameter vs. bearing number of a plain cylindrical journal bearing at various $\varepsilon$ for $L/D = 0.075$.

Figure 4-13 documents the corresponding attitude angle for the case of Figure 4-12. Again, intuition is confirmed by the zero-$\Lambda$ asymptote of $\alpha = 90^\circ$ which is in agreement with the full-Sommerfeld attitude angle of $90^\circ$. Also, it is observed that the compressible attitude angles are very near the incompressible value of $90^\circ$ at larger $\Lambda$ as long as $\varepsilon$ is small. Again, this is consistent with the previous observation that due to the shortness of the bearings under consideration, the compressible solution can be expected to mimic that of the incompressible system for low/moderate $\varepsilon$ and moderate $\Lambda$.

4.7.3 Numerical Analysis: Unsteady Solutions

For dynamic analysis, LubePack is used to calculate the Jacobian (Equation (4.17)) of the dynamical system represented by Equations (4.1), (4.2), (4.3), (4.8), (4.9), and (4.10). For a given $L/D$ and operational parameters $\varepsilon$ and $\Lambda$, the speed-parameter, $\Gamma$, is used as a bifurcation parameter for the Jacobian matrix. With this formulation,
Figure 4-13: Design chart for attitude angle vs. bearing number of a plain cylindrical journal bearing at various $\epsilon$ for $\frac{L}{D} = 0.075$.

Hydrodynamic stability calculations can be presented in terms of speed-parameter at bifurcation versus bearing number for a fixed value of eccentricity ratio and a given design class characterized by the $L/D$ ratio [7].

However, as noted by Piekos [47], this speed-parameter formulation is inconvenient because both the $\Gamma$ and $\Lambda$ axes have strong dependence on the rotational speed, $\omega$. In order to remedy this, Piekos non-dimensionalized the equations of motion so that the bifurcation parameter containing the effect of the rotor mass was given by $\bar{M} = \frac{M_{pm} \omega^5}{72L^2 \mu R}$, with symbols defined by Table 4.2. Neglecting deformation of the rotor geometry due to centripetal acceleration and dependence of ambient pressure at the bearing ends on $\omega$, this $\bar{M}$ formulation confines the influence of $\omega$ to the $\Lambda$-axis. To this level of approximation, $\bar{M}$ can be interpreted as a non-dimensional rotor mass.

In an incompressible calculation, the rotor is decoupled from the film so that the film parameters may be computed a priori [35]. This leads to a four-dimensional phase space so that the associated Jacobian matrix has four eigenvalues. Simple
approximations to compressible systems also accomplish a de-coupling of the lubrication film from the rotor, usually by linearizing the governing fluid equations prior to solution [36]. As a consequence, there are typically gross errors in estimates for the fixed points of the dynamical system when \( c \) is large. Any stability results for such a solution will obviously be limited by the accuracy of the computed fixed point itself.

In contrast, LubePack allows one to compute high-order solutions of the full, nonlinear fixed points as well as the complete, nonlinear, unsteady dynamical system and a symbolic representation of the associated Jacobian. Because LubePack derives a highly accurate, closed-form formula for the Jacobian, a substantial savings in computation is achieved in addition to eliminating sources of numerical error. For example, numerically computing the Jacobian from the given nonlinear system would involve perturbing one component of the state vector and numerically estimating the rates of change of all the other state variables for a given set of operational parameters \( \epsilon, A \) and a fixed design of \( L/D, \tilde{M} \). This process would be repeated for all components of the state vector. Then, all these calculations would need to be repeated for all conceivable values of the operational parameters, \( \epsilon, A \). The entirety of these calculations would need to be swept on a sufficiently dense mesh in \( L/D \) and \( \tilde{M} \) if the design space is to be exhausted.

Clearly, such a full, numerical approach would be computationally intensive for high-order systems and one would need to implement an elaborate error control scheme for the numerical differentiation since the system is known to exhibit sharp gradients in the state variables. In other words, it would be necessary to develop a sophisticated definition of the state variable perturbation for a given design and set of operating conditions in order to ensure the accuracy of a numerical scheme for calculating the system Jacobian. In contrast, LubePack eliminates these numerical difficulties by use of exact symbolic manipulation. Furthermore, LubePack achieves a high level of efficiency relative to fully numerical schemes since the Jacobian formula represents "pre-computed" differentiation in the sense that once the formula is derived, no further differentiation is necessary. Finally, the Jacobian formula itself is algebraically reduced to simplest form by LubePack so that overhead incurred as the
computer's floating-point hardware evaluates the expression is minimized.

With this approach, LubePack allows a proper treatment of the fully coupled rotor/film system without the often debilitating, simplifying assumptions that often afflicted earlier work. In this general framework of the fully coupled system, the rotor will again account for four eigenvalues in the Jacobian matrix but the film will have an infinite number of eigenvalues. The film and rotor eigenvalues are coupled so that computation of both must be executed simultaneously. The number of film eigenvalues in a practical calculation will depend on the number of Galerkin modes retained to represent the film.

Figure 4-14 depicts a fixed point for a plain, cylindrical, gas journal bearing in a 53-dimensional phase space. This “spectral footprint” uses 49 film eigenvalues \((n = 1, m = 24)\) and is observed to be at the threshold of bifurcation. With a further increase in \(\Gamma\), the two right-most eigenvalues will cross into the right half of the complex plane and the fixed point will lose stability. The transversality condition [40] is easily verified so that the loss of stability comes in the form of a Hopf bifurcation which is easily detected using Routh’s criterion [40]. In addition, by examining the eigenvectors, it is clear that these eigenvalues which undergo bifurcation are associated with the state variable representing rotor whirl. Note that LubePack facilitates a complete spectral analysis of the dynamical system. That is, LubePack allows one to study the behavior of all the eigenvalues in the high-order approximation. In particular, one may check for other types of bifurcations other than the classical hydrodynamic whirl instability described above.

For a given \(L/D\) value, one may use LubePack solutions to compute the locus of speed-parameter values where the Hopf bifurcations occur for fixed \(\varepsilon\) and varying \(\Lambda\). These sweeps may be repeated for the entire \(\varepsilon\) range of interest as shown in Figure 4-15. In this way, the hydrodynamic stability conditions for an entire \(L/D\) family of bearings may be determined in a very efficient manner using LubePack.
4.7.4 Validation of LubePack with Previous Work

The most accurate calculations to date for the unique class of low $L/D$ bearings required by the MIT Micro-Engine are those of Piekos [47]. In this work, a pseudo-spectral code, SPECTRES, was developed and used to determine the static properties of this bearing class along with the associated stability conditions. This work also introduced the stability parameter, $\bar{M}$, as a more convenient format for presenting stability results in comparison to the speed-parameter charts of Cheng and Pan [7].

Figure 4-16 compares the SPECTRES load capacity calculations of Piekos et al. [48] to those obtained using LubePack with $n = 1$, $m = 24$. As can be seen in the figure, the agreement is excellent. Figure 4-17 demonstrates similar agreement between the two calculations for the associated static attitude angles. Note that the dramatic depression of $\alpha$ at high $\epsilon$ and high $\Lambda$ shows that LubePack is accurately capturing compressibility effects.

For stability calculations, Figure 4-18 shows LubePack again matching the accu-
Figure 4-15: Hydrodynamic Whirl Stability Chart for $L/D = 0.075$ using Cheng and Pan's [7] speed-parameter format.
Figure 4-16: LubePack vs. SPECTRES: Static Load Capacity $L/D = 0.075$

Figure 4-17: LubePack vs. SPECTRES: Static Attitude Angle $L/D = 0.075$
racy of SPECTRES in the $\bar{M}$ stability charts of Piekos et al. [48]. Each curve in this figure represents constant threshold eccentricity ratio which may be interpreted as a measure of loading at threshold of whirl instability. The horizontal axis may be considered a “speed axis” in light of the previous discussion of the bearing number, $\Lambda$. Therefore, the $\epsilon$ and $\Lambda$ values depicted in Figure 4-18 are threshold operational variables for a bearing of a given design, determined by $\bar{M}$ and $L/D$. The chart itself is for a fixed $L/D$, for example, $L/D = 0.075$ in Figure 4-18. Strictly speaking, $\bar{M}$ is also a function of speed since it depends on the nominal bearing clearance which may vary as the rotor deforms under centripetal acceleration at high $\omega$. However, as a first approximation, the deformation of the rotor may be neglected. In this case, $\bar{M}$ may then be viewed as a design parameter representing a non-dimensional rotor mass that is fixed for a given rotor [47].

Continuing on in this framework of Piekos [47], a horizontal line drawn across Figure 4-18 denotes a particular rotor design described by the $L/D$ of the chart and the value of constant $\bar{M}$. All intersections of $\epsilon$ curves with this horizontal line will describe a locus of $\epsilon - \Lambda$ pairs where the given system is at the threshold of whirl instability. Using direct simulation of the rotor-dynamic system, Piekos showed that this $\epsilon - \Lambda$ threshold boundary represents the minimum eccentricity ratios necessary for stability of the rotor whirl mode for the corresponding values of bearing number. Using a high-order eigenvalue analysis, this result will be demonstrated in the next section.

To apply this minimum eccentricity ratio stability criterion for whirl to the “$\bar{M}$” stability chart of Piekos, consider the point labeled “A” in Figure 4-18. Point A represents a rotor with $L/D = 0.075$ and $\bar{M} \approx 0.045$ operating neutrally stable with respect to whirl with $\epsilon = 0.85$ and $\Lambda \approx 0.3$. If one attempted to operate a rotor with a slightly higher $\bar{M}$ at the same values of $\epsilon$ and $\Lambda$, whirl onset would occur since this rotor has a larger minimum eccentricity ratio requirement for the given value of $\Lambda$. Therefore, bearing designs with $\bar{M}$ above point A will be unstable with respect to whirl at $\epsilon = 0.85$ and $\Lambda = 0.3$ while bearings with $\bar{M}$ below point A will be stable under the same operating conditions. This is denoted by the “Unstable” and “Stable”
One might expect that the LubePack calculations would have accuracy problems at higher $L/D$ when it may become necessary to increase the axial truncation index beyond a single mode. Surprisingly, Figure 4-19 demonstrates that this is not the case - the LubePack results are again in excellent agreement with those of SPECTRES. This is likely due to the fact that even at $L/D = 0.2$, the short bearing effects described earlier are still substantial; all multiple-scale behavior is still confined to the circumferential direction and the axial pressure distributions are approximately parabolic. Because the DRIE techniques used in the MIT Micro-Engine will likely limit designs to $L/D$ values less than 0.2, the most important implication of this result is that LubePack should provide accurate calculations over the entire micro-bearing design regime.

A crucial difference between the stability calculations of SPECTRES and LubePack is that hydrodynamic whirl stability calculations of SPECTRES were accomplished through direct simulation of the dynamical system. That is, the rotor was displaced slightly from its equilibrium position and the coupled rotor/film system was marched...
forward in time to see whether or not the resulting disturbances decayed or grew in time. It is clear that the analytical Jacobian formulation of LubePack provides a substantial advantage in linear stability calculations of low $L/D$, plain, cylindrical, gas journal bearings compared to direct simulation.

First of all, to simulate the unsteady rotor/film system requires converging the solution for the film PDE at every time step. If the time-stepping routine requires evaluation of the system at intermediate points in the time interval then the film PDE must be converged more than once for each time step. Furthermore, since the hydrodynamic stability boundary occurs at relatively high eccentricity ratio, many Fourier modes will need to be retained to maintain accuracy. This computational burden is compounded by the fact that one will need an increasingly large number of simulation time steps as the system approaches bifurcation since the growth rate of disturbances approaches zero by definition of neutral stability. At the very least, enough time steps must be used to allow for damping of initial start-up transients before measuring growth or decay rates.

On the other hand, because LubePack utilizes the available static design charts of
the bearing $L/D$ class along with a high-order closed form expression for the system Jacobian matrix to determine stability, the film PDE is \textit{never} solved during stability testing. For a given $\epsilon$, $\Lambda$ operational point, the cost of stepping the bifurcation parameter across the neutral stability threshold is simply that associated with numerical evaluation of the analytical Jacobian formula and application of Routh's criterion at each step. From a practical standpoint, this means that the stability plane for this class of bearing can be accurately blackened in a timely fashion (see Figure 4-20) by a modest PC system.

This economy of computation is the result of being able to express the entire Galerkin coefficient system in terms of the Euler beta function. Consequently, every integral generated by the orthogonality relations which would normally require symbolic evaluation using the standard Mathematica integration rules is discharged with a single Beta function evaluation. In effect, the integrals are evaluated as quickly as the Mathematica kernel can recognize their identity.

Figure 4-21 gives the whirl onset frequency normalized by the rotor rotational speed ("whirl ratio") for the stability curves shown in Figure 4-20. In Figure 4-21, the eccentricity ratios are identical to those used in Figure 4-20 with the largest eccentricity ratio associated with the leftmost curve of the graph. For the full-Sommerfeld, short bearing theory, it is well known that the autonomous dynamical system is always unstable with respect to whirl which occurs at a whirl ratio of exactly 0.5. Note that the whirl frequency curves tend to the incompressible whirl ratio of 0.5 in the low bearing number limit. Depression of the threshold whirl frequency ratio below 0.5 is a classical characteristic of compressibility effects in the film.

In terms of the more physical operational parameters of $\epsilon$ and $\Lambda$, the stability boundary implied by Figure 4-20 is shown in Figure 4-22. This format for presenting the stability boundary is interesting since it shows that stable operation can only be achieved along a high $\epsilon$ corridor. Figure 4-22 also shows that this stability corridor has no upper limit on $\Lambda$. In fact, operation of a journal bearing should become easier in the sense that the stability corridor widens as $\Lambda$ increases - assuming that one can apply sufficient loading to hold a stable eccentricity as the bearing stiffens with
Figure 4-20: Hydrodynamic Stability Chart for $L/D = 0.075$ using the $\bar{M}$ format of Piekos et al. [48].

Figure 4-21: Onset Whirl Frequency Chart for $L/D = 0.075$ corresponding to the $\epsilon$ values of Figure 4-20.
Figure 4-22: Theoretical stability boundary for hydrodynamic runs for \( L/D = 0.075 \), \( \tilde{M} \approx 10.6 \).

increasing \( \Lambda \). The ratio of the onset whirl frequency to the rotor rotational speed along this stability boundary is shown in Figure 4-23. Again, note that the response frequency ratio tends to the incompressible value of 0.5 as \( \Lambda \) tends to zero.

4.8 Generalized Stability Analysis with LubePack

4.8.1 Global Perspective of Eigenvalue Spectrum

Bifurcation analysis may be performed in the complex plane for a given \( L/D \), fixed \( \Lambda \), fixed \( \tilde{M} \) and varying \( \epsilon \). The complete root locus under such conditions is shown in Figure 4-24. This figure was constructed for \( L/D = 0.075 \), \( \Lambda = 4.0 \), \( \tilde{M} = 0.95 \), and \( \epsilon \) increased from 0.05 to 0.984 in steps of 0.0005. The global view of Figure 4-24 is dominated by the film eigenvalues which are coupled to the rotor eigenvalues by the unsteady, compressible gas film. The “noisy” region between \(-50\) and \(-100\) on the horizontal axis of Figure 4-24 is due to the high-order modes at low eccentricity ratio. Under these conditions, the coefficients of the highest modes in the Galerkin
expansion are virtually zero, resulting in spurious eigenvalues. As $\epsilon$ is increased, the higher modes become significant and the relative noise in eigenvalue spectrum drops.

The ability of LubePack to generate graphs such as Figure 4-24 is important because it allows one to search all modes in the rotor/film system for instability rather than just the rotor whirl modes treated in conventional analysis. By extracting the eigenvalues from the Jacobian matrix, even the latent response frequencies of the system are evident and one may vary bifurcation parameters to check for characteristic roots crossing into the right half of the complex plane.

4.8.2 Rotor Mechanical Modes: Hopf Bifurcations

As a start to this more general stability analysis, consider Figure 4-25 which shows a close-up view in the vicinity of the origin for a root locus of the type shown in Figure 4-24 with $\Lambda = 0.1$ and the corresponding macro-rig $\bar{M} = 10.17$. Since the root locus is symmetric with respect to the real axis, negative frequencies are omitted in the plot. Note that $\bar{M}$ will depend on $\omega$ since the nominal clearance, $c$, will decrease
as the rotor expands radially under centrifugal loading at high speeds. Consequently, calculations at different bearing numbers will, in general, require different values of $\bar{M}$. The $\epsilon$ mesh for Figures 4-25, 4-26 and 4-27 is identical to that described above for Figure 4-24.

Although the system is highly coupled for large $\epsilon$, eigenvalue trajectories can be identified for the various state variable of the system by considering the associated eigenvectors [30]. In Figure 4-25, the trajectory with the imaginary axis crossing, labeled “Whirl,” is associated with the rotor whirl mode while the trajectory in the left half plane, denoted by “Radial,” is paired with the eccentricity ratio state variable. Eccentricity ratios corresponding to points along each trajectory are marked for solutions of different resolution. The large, solid markers are for Galerkin solutions with one axial mode and 24 circumferential modes. The open markers are for lower resolution solutions utilizing one axial mode and 16 circumferential modes.

The imaginary axis crossing of the whirl eigenvalue in Figure 4-25 occurs at the whirl threshold eccentricity ratio of $\epsilon \approx 0.945$ at $\Lambda = 0.1$ in Figure 4-22. The whirl
Figure 4-25: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for macro-bearing test rig with \( \epsilon \) increasing from 0.05 to 0.995 in increments of 0.0005 (\( \Lambda = 0.1, \bar{M} = 10.17, \epsilon_{rad} \approx 0.99035 \)).

Figure 4-26: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for macro-bearing test rig with \( \epsilon \) increasing from 0.05 to 0.995 in increments of 0.0005 (\( \Lambda = 2.0, \bar{M} = 2.85, \epsilon_{rad} \approx 0.972 \)).
ratio of this crossing in Figure 4-25 occurs at a frequency ratio of approximately 0.17 which corresponds to the \( \Lambda = 0.1 \) ordinate of the threshold whirl ratio curve shown in Figure 4-23. On the other hand, the \( \epsilon \) trajectory in Figure 4-25 becomes more highly damped as \( \epsilon \) is increased and never crosses into the right half-plane, indicating stability for this state variable.

In the SPECTRES orbit simulations of Piekos, it was noted [49] that under certain conditions, in addition to the subsynchronous, hydrodynamic whirl mode, there would often exist a high frequency oscillation as well. This second frequency was “high” in the sense that it was observed to exceed half the synchronous whirl rate, suggesting origins other than that of the rotor whirl mode. Further analysis revealed that the frequency in question was associated with the radial position of the rotor. Moreover, this radial frequency could exhibit growth and therefore could potentially be a new class of instability associated with the eccentricity ratio state variable of the dynamical system.

More recently, using a low-order Galerkin truncation that maintained one axial mode and only four circumferential modes, Jacobson [30] raised the possibility that the eigenvalue trajectory associated with the radial position of the rotor may be capable of producing a second Hopf bifurcation in addition to the bifurcation associated with the rotor whirl. In light of the current discussion, it is clear that LubePack provides the necessary technology for a high-order investigation of the observations of Piekos [49] and Jacobson [30].

### 4.8.3 High Order Calculation of the Radial Stability Criterion for Plain, Cylindrical, Gas Journal Bearings

As \( \Lambda \) is increased to 2.0 as in Figure 4-26, the whirl trajectory still produces a Hopf bifurcation as expected but now the eccentricity state variable produces a second Hopf bifurcation at higher \( \epsilon \) relative to the whirl threshold boundary. As \( \epsilon \) is increased, the whirl trajectory emerges into the stable regime as expected from Figure 4-22. However, the \( \epsilon \) mode undergoes increased damping initially but the damping subsequently
falls as the higher $\varepsilon$ values stiffen the bearing in the radial direction. Eventually, an eccentricity ratio above that of the whirl stability boundary is reached, $\varepsilon_{\text{rad}}$, beyond which the rotor becomes unstable with respect to radial disturbances. The response frequency of the radial disturbance normalized by the rotor speed is denoted by the symbol $\nu$.

For these conditions, this instability is “super-synchronous” in the sense that the rotor rotational speed is less than the $\varepsilon$ state variable response frequency as denoted by the imaginary axis crossing of approximately 1.5 in Figure 4-26. Figures 4-25, 4-26, and 4-27 show that the response frequency of the radial instability drops with increasing bearing number. In fact, by the time $\Lambda = 4.0$ is reached in Figure 4-27, the radial response frequency has dropped to little more than the synchronous speed. Increases in $\Lambda$ beyond this were not considered since the design speed of the macro-bearing test rig corresponds to only $\Lambda = 3.6$. However, it is possible that the instability frequency ratio can drop below 1.0, indicating a subsynchronous mode. It is notable that the Galerkin solutions with 16 circumferential modes are nearly identical to the 24 mode solution, indicating adequate grid convergence for the computations even at the extremely high eccentricity ratios investigated.

Figure 4-28 shows speed-parameter stability contours for the radial instability. This is analogous to the $\Gamma$ chart for whirl instability shown in Figure 4-15. Figure 4-29 depicts the radial instability boundary in terms of the $\bar{M}$ stability parameter developed by Piekos et al. [48] for the study of the hydrodynamic whirl instability. As with the $\bar{M}$ whirl charts, a horizontal line across Figure 4-29 can be considered as representing a fixed bearing design. The intersections of this horizontal line with the $\varepsilon$ contours for the radial mode denote threshold $\varepsilon - \Lambda$ pairs where the radial rotor mode is neutrally stable. However, in contrast to the whirl boundary, the preceding eigenvalue analysis showed that the eccentricity ratios of this neutral stability curve represent maximum values of $\varepsilon$ for stability.

It is notable that $\bar{M}$ contours for the radial instability have an ordering with respect to $\varepsilon$ that is contrary to the analogous curves for the whirl instability. The radial stability chart of Figure 4-29 shows that the highest values of $\varepsilon$ produce the
lowest values of threshold $\bar{M}$ for a given $\Lambda$. This is in contrast to the $\bar{M}$ whirl instability curves of Figure 4-20 which indicate that the highest values of $\epsilon$ produce the highest threshold $\bar{M}$ results for fixed $\Lambda$. Consequently, for the bearing design operating at threshold conditions denoted by point "B" in Figure 4-29, a bearing with higher $\bar{M}$ operating at the same values of $\epsilon, \Lambda$ would be unstable since it would possess a lower maximum threshold eccentricity ratio for the given bearing number.

The response frequency ratio chart corresponding to the radial instability boundary of Figure 4-29 is given in Figure 4-30. One must keep in mind that the frequency ratio in this figure is not related to whirl of the rotor but rather vibrations of the rotor in the radial direction. As is evident from the figure, the response frequency ratio at the threshold of instability is quite high because the bearing generates massive amounts of stiffness at high eccentricity ratio where the instability occurs. Additional insight into the trends of Figure 4-30 is gained by recalling that the radial instability has decreasing threshold $\bar{M}$ for increasing $\epsilon$. Decreasing $\bar{M}$ may be thought of as decreasing the rotor mass. For a given bearing stiffness, one would expect to observe the highest response frequencies at the lowest values of rotor mass. This is in fact observed in Figure 4-30 where the highest eccentricity ratios corresponding to the lowest rotor masses in Figure 4-29 produce the highest response frequency ratios in Figure 4-30.

The high-order eigenvalue analysis of LubePack allows construction of composite radial/whirl stability and frequency response ratio charts as given in Figures 4-31 and 4-32. In these figures, the whirl data were taken from the previously presented whirl charts. In Figure 4-31, it is seen that for a given $\bar{M}$, a stability margin on eccentricity ratio is established since the radial instability occurs at relatively high $\epsilon$ while the whirl instability occurs at lower $\epsilon$. The difference between these two stability boundaries defines an operational corridor in $\epsilon, \Lambda$ space within which all devices with this class of bearings must operate to satisfy both stability criteria.

Figure 4-32 illustrates the disparity between the response frequency ratios (response frequency normalized by rotor rotational frequency) of the whirl and radial instabilities. This large separation of responses in frequency space should make ex-
Figure 4-27: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for macro-bearing test rig with $\epsilon$ increasing from 0.05 to 0.995 in increments of 0.0005 ($\Lambda = 4.0$, $\bar{M} = 0.948$, $\epsilon_{\text{rad}} \approx 0.9627$).

Figure 4-28: LubePack speed-parameter stability contours for Hopf bifurcation in $\epsilon$ state variable ($L/D = 0.075$).
Figure 4-29: LubePack $M$ stability contours for Hopf bifurcation in $\epsilon$ state variable ($L/D = 0.075$).

Figure 4-30: LubePack threshold frequency response ratios for Hopf bifurcation in $\epsilon$ state variable ($L/D = 0.075$).
experimental recognition of each mode fairly straightforward.

4.8.4 Practical Implications of the Radial Stability Boundary

To summarize the impact of this radial instability, it is convenient to plot the stability boundary in the physical ($\epsilon$-$\Lambda$) space along with the whirl stability boundary of Figure 4-22. Figure 4-33 shows how the radial instability acts to narrow the operating corridor of any device using this class of journal bearing. For a given speed, care must be taken to load the journal bearing to achieve a sufficiently high $\epsilon$ in order to produce stable operation with respect to rotor whirl. However, one should not load the bearing excessively so that the radial stability threshold is crossed. At low to moderate speeds, the radial stability boundary can be neglected since the $\epsilon$ excursions necessary for onset are so high that they are virtually indistinguishable from a wall collision. In a real bearing, such conditions would probably lead to a collapse of the lubrication film. At higher speeds, when the film has the load capacity to support
high eccentricity ratio operation, the radial instability boundary drops so that the stable corridor of operation is significantly reduced.

The practical implications of the radial instability boundary are tangible when one considers running a device with residual unbalance due to manufacturing processes. For the macro-rig generalized stability map shown in Figure 4-33, the stability corridor would be further reduced if the rotor had a moderate static unbalance of 4% (expressed as a fraction of the nominal bearing clearance). Near full speed, $\Lambda \approx 3.0$, the radial stability boundary drops to approximately 96%. Because the radial instability amplifies $\varepsilon$-disturbances of the sort generated by unbalance, prudence would dictate that the device be run at eccentricity ratios less than $\varepsilon \approx 0.92$. In reality, it would be desirable to maintain a margin of safety so that perhaps one would not want to exceed $\varepsilon \approx 0.90$. Given that the whirl stability boundary occurs at $\varepsilon \approx 0.84$, the effective stability corridor has indeed been reduced significantly.

Because previous research in gas bearings has focussed solely on the whirl insta-
Figure 4-33: Macro-rig radial and whirl stability boundaries based upon generalized stability analysis with LubePack.
bility which requires a minimum eccentricity ratio for stability, it is easy to get the impression from the literature that high-ε corresponds to high levels of damping. This is true, but only for the whirl mode of the rotor. In reality, high-ε operation means low or even negative damping for the radial mode. Consequently, an operator of a device using this class of hydrodynamic gas bearing should be mindful of the risks of over-loading the system.

4.8.5 Comments on Previous Gas Bearing Stability Work

Note that the radial instability suggested by the results of Piekos [49] and Jacobson [30] and calculated to high-order with LubePack in the previous section is absent from the existing gas bearing literature. However, upon further consideration, there are several probable reasons for this omission. First of all, the whirl instability is the first to be encountered as ε is increased while the radial instability is characterized by high eccentricity ratios. It could be that prior researchers did not fully explore the operational space that was stable with respect to whirl for the existence of other instabilities.

Secondly, as mentioned earlier, past analysis dealt with the nonlinearities of the system by employing approximations that restricted accurate computations to relatively low eccentricity ratios where the radial instability did not exist. Such schemes were often not conducive to accurate formulation of the Jacobian matrix of the fully-coupled, unsteady dynamical system. As a result, an efficient, high-order, analysis of the eigenvalue spectrum of the dynamical system was impractical.

For example, Jacobson’s results [30] relied on early work from the Micro-Engine Project in which a low-order Galerkin calculation with a Jacobian based upon numerical differentiation was developed. Derivation of the equations was done using standard symbolic manipulation algorithms rather than highly optimized routines such as those in LubePack. Consequently, the computational “brick wall” of Figures 4-2 and 4-3 was encountered and it was only practical to retain four circumferential modes in the Galerkin expansion along with one axial mode. At this low-order, accuracy begins to break down at approximately $\epsilon = 0.8$ so that the ability to detect a bifurcation at
\( \epsilon = 0.97 \) is questionable.

To examine this issue, Galerkin calculations were performed with LubePack using the reduced accuracy of Jacobson’s \( n = 1, m = 4 \) approximation. Figures 4-34, 4-35, and 4-36 represent the low-order approximations of Jacobson analogous to the high-order LubePack results shown in Figures 4-25, 4-26, and 4-27. The low-order calculations are plotted on the same scale as the high-order results for comparison.

As can be seen from Figures 4-34, and 4-35, this low-order Galerkin calculation fails to detect the radial bifurcation. In Figure 4-36, the low-order calculation does actually detect the second bifurcation outside the plotted range, but the error is substantial compared to the high-order LubePack calculations shown in Figure 4-27.

In light of this, the low-order scheme is seen to be capable of detecting the radial bifurcation at lower values of \( \tilde{M} \). For a quantitative accuracy comparison of the low and high-order results, Figure 4-37 shows a radial bifurcation detected with the low-order scheme. The eccentricity ratio at bifurcation is \( \epsilon \approx 0.987 \). The frequency ratio at bifurcation is approximately 1.5. The analogous high-order plot is given in Figure 4-38. The bifurcation occurs at \( \epsilon \approx 0.966 \) with a frequency ratio.
Figure 4-35: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for Galerkin truncation of $n = 1, m = 4$ with $\epsilon$ increasing from 0.05 to 0.995 in increments of 0.0005 ($\Lambda = 2.0, \bar{M} = 2.85$).

Figure 4-36: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for Galerkin truncation of $n = 1, m = 4$ with $\epsilon$ increasing from 0.05 to 0.995 in increments of 0.0005 ($\Lambda = 4.0, \bar{M} = 0.95$).
of 0.88. Thus, the largest discrepancy is in the frequency content of the instability rather than the onset eccentricity ratio. It is worthwhile to note that if one were to construct the composite stability map of Figure 4-33 with the low-order scheme, the radial stability onset eccentricity ratio would increase as $\Lambda$ was decreased. Below $\Lambda = 4.5$, this would imply a radial stability boundary that is nearly coincident with the bearing wall. Consequently, the low-order calculation represents an overestimate of the operating stability margin between the radial and whirl hydrodynamic stability boundaries. Finally, if one wanted to use the radial stability calculations to interpret the frequency content of experimental rotor-dynamic measurements, use of the higher order LubePack calculation would be necessary, as the low-order scheme incurs significant errors in the bifurcation frequency.

The high frequency content associated with the radial direction observed in the highly accurate direct simulations of Piekos [49] was most likely due to the radial Hopf bifurcation suggested by Jacobson [30] and subsequently computed to high-order in the current investigation. Mapping the radial bifurcation by direct simulation would require many time steps at very high $\epsilon$ where many modes would need to be retained to get the necessary accuracy. Since the film gradients are very steep at high $\epsilon$, there would be a substantial increase in computational cost for simulating at $\epsilon = 0.97$ to detect a radial instability versus operating at $\epsilon = 0.9$ to detect the whirl boundary. Nevertheless, subsequent simulations of Piekos [50] have in fact confirmed the growth of radial disturbances as predicted by LubePack.

Lastly, it should be noted that with the exception of Piekos [48], [47], there is no research in the unique, extremely low $L/D$ bearing design space needed by MEMS devices such as the MIT Micro-Engine. Due to the narrow stability corridor shown in Figure 4-22, such micro-bearings must operate at very high eccentricity ratios. The conventional bearings in existing literature simply do not operate at these extreme values of $\epsilon$. In fact, bearing designers strive to ensure that the gas bearings operate at moderate eccentricity ratios to allow for more tolerance of surface defects and possible contamination [28]. The DRIE fabrication constraints do not afford micro-bearing designers this luxury.
Figure 4-37: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for Galerkin truncation of $n = 1, m = 4$ with increasing eccentricity ratio ($\Lambda = 4.5, \bar{M} = 1.7, \varepsilon_{rad} \approx 0.987$).

Figure 4-38: Magnified view of eigenvalue root locus of the type shown in Figure 4-24 for Galerkin truncation of $n = 1, m = 24$ with increasing eccentricity ratio ($\Lambda = 4.5, \bar{M} = 1.7, \varepsilon_{rad} \approx 0.966$).
4.9 Conclusions

The analysis described in this chapter showed that use of standard symbolic manipulation packages to implement Cheng and Pan’s Galerkin scheme [7] led to intractable calculations that demand memory at exponential rates. It was shown that the Galerkin system of coefficient equations for plain, cylindrical, gas journal bearings could be expressed entirely in terms of the Euler beta function. This was exploited in the development of a new symbolic integration package, LubePack, specially designed for Galerkin solution of this class of gas journal bearings.

It was shown that LubePack enables the user to perform computations not possible with the standard routines of commercial packages for deriving the Galerkin system, allowing formulation of high-order calculations on small PC systems. LubePack calculations were shown to be in excellent agreement with the most accurate results in the literature [48] for steady state properties as well as hydrodynamic stability characteristics. Because LubePack uses symbolic differentiation to develop an analytical formula for the system Jacobian matrix, calculation of bearing stability properties is expedited while maintaining accuracy. This fact was demonstrated by using LubePack to virtually exhaust the design space for the static and whirl stability characteristics of the Micro-Engine class of plain, cylindrical, gas journal bearings.

Finally, this Jacobian calculation enables one to perform generalized stability studies on the entire eigenvalue spectrum of high-order approximations for gas journal bearings. This capability was exploited in performing a high-order analysis used to compute the radial hydrodynamic instability suggested by direct simulations [49] and low-order Galerkin analysis [30]. The high-order results showed that the radial instability significantly narrows but does not close the stability corridor for a bearing of this class. Thus, the high eccentricity ratios required for stable operation of such bearings at high speeds are still possible.
Chapter 5

Hydrodynamic Operation: Experiment

5.1 Introduction

Experiments have played an indispensable role throughout the development of gas bearing technology. The work of experimentalists has challenged theoreticians to explain observed phenomena, served as a benchmark for the validation of calculations, developed operational protocols for real machines, and demonstrated the practicality and merit of new bearing designs. Not surprisingly, the bearings tested in the laboratory have mirrored the machines of the era. In particular, bearing clearances were often limited by the ability of the current state-of-the-art in fabrication technology to limit effects of surface roughness, rotor non-circularity, and axial taper. The bearing length-to-diameter ratios were mostly dictated by the load capacity requirements of the rotating machinery to be supported.

Since the 1950's and 1960's when the bulk of gas bearing technology was developed, much has changed. The previous chapter on theoretical hydrodynamic calculations demonstrates the impact of the synergism between advanced symbolic manipulation languages and the modest computing capabilities of widely available PC systems.

The MIT Micro-Engine Project is a striking example of how machines have changed since the advent of gas bearings. For this class of machine, the oil and conventional
cutting tools of the machine shop have been replaced by the chemical reactants and particle beams of the cleanroom. In a similar vein, these MEMS devices require the design and testing of gas bearings well outside the realm of previous gas bearing research. Specifically, the low journal bearing $L/D$ values are substantially below those investigated in past work.

5.2 Comparison of LubePack Calculations and Experimental Results

The calculations presented in the previous chapter were performed for a balanced rotor. The rotor used for hydrodynamic experiments on the macro-bearing test rig had a static unbalance equal to 0.7% of the nominal bearing clearance. Because the macro-rig rotor was well-balanced for the hydrodynamic runs, one would not expect deviations from the LubePack calculations as a result of rotor unbalance.

Due to the strong dependence of film reaction force on eccentricity ratio depicted in Figure 4-16, high eccentricity ratio operation implies that the dominant forces exerted on the journal will be hydrodynamic. Consequently, when the macro-bearing test rig previously described is operated with the rotor loaded to high eccentricity ratios, the resulting behavior should resemble that predicted by LubePack.

5.2.1 Steady Operation

Figure 4-22 illustrates some of the points elucidated by Piekos et al. [48]. Specifically, for a given bearing number, if the rotor is side-loaded so that the eccentricity ratio lies to the right of the stability boundary shown in Figure 4-22, hydrodynamic operation of the bearing which is stable with respect to whirl should be possible. With whirl stability established, one may examine the relationship between load parameter, eccentricity ratio, and bearing number quantified by Figures 4-12 and 4-13.

To check this assertion, tests were conducted using the macro-scale test rig shown in Figure 2-5. The rotor speeds, lubrication temperature and pressure, and rotor
position were measured with the instrumentation previously described. Although the
test rig has the capability to apply side-loads using the externally pressurized aft
pressure chambers shown in Figure 2-17, steady-state bearing behavior was studied
using the gravity loading provided by the tilt table upon which the rig is mounted.
This allowed more precise control over the amount of side-load applied since issues
of chamber "cross-talk" due to leakage flows could be avoided. Moreover, the aft
pressure chambers were hydraulically joined and short-circuited with external tubing
to the inter-row pressure on the forward side of the journal. This provided a uniform
ambient pressure on both the forward and aft ends of the bearing and prevented
inadvertent side-loading due to unequal chamber pressures. Using this scheme, it
was found that steady operation of the journal at high eccentricity ratios was indeed
possible.

The static properties depicted in Figures 4-12 and 4-13 are plotted in terms of
the operational parameters \( c \) and \( \Lambda \) for the fixed values of \( \zeta \) in Figures 5-1 and 5-2.
Physically, this corresponds to setting the tilt table angle to establish a gravity load
and observing the relationship between eccentricity ratio and bearing number as the
rotational speed of the rotor is varied. Figure 5-1 shows that when the tilt table angle
is adjusted to set the load parameter to the values shown, for \( 0.0 \leq \Lambda \leq 0.35 \), one
would expect to see eccentricity ratios of 0.95 and greater.

Figure 5-2 shows that for the loads and bearing number range of Figure 5-1,
the attitude angle quickly becomes depressed below the ideal value of 90° which
characterizes the incompressible, full Sommerfeld, short bearing limit. In fact, for
bearing numbers greater than 0.1, one would expect to see attitude angles of less
than 50° for the specified conditions.

Finally, the asymptotic behavior for the high \( \Lambda \) limit of Figures 5-1 and 5-2 is not
surprising since it is well-known that finite length hydrodynamic gas bearing solutions
have the same high speed asymptote as the infinite length case due to the dominance
of circumferential Couette terms over axial Poiseuille terms in the governing equations
as \( \Lambda \to \infty \) [28].

Figures 5-3 and 5-4 show steady hydrodynamic data from the macro-bearing test
Figure 5-1: LubePack calculations of eccentricity ratio versus bearing number at constant load parameter for hydrodynamic runs.

Figure 5-2: LubePack calculations of attitude angle versus bearing number at constant load parameter for hydrodynamic runs.
rig for the $\epsilon$ and $\alpha$ dependencies on $\Lambda$ for $\zeta = 0.0253$. Comparison of these experimental results with the theoretical calculations of Figures 5-1 and 5-2 shows a key difference between the behavior of real gas lubrication films and the ideal solutions. During the steady state macro-rig experiments, it was noted that at low $\Lambda$, decreasing the bearing number further would cause a collapse of the lubrication film which was soon followed by unlubricated rubbing of the rotor and bearing.

Even for light loading, the bearing loses load capacity as $\Lambda$ is decreased. This is accompanied by an increase in $\epsilon$ and a decrease in $\alpha$. In terms of Figure 4-1, the rotor settles into the "bottom" of the bearing where $\alpha \approx 0$ before the film collapses with lower $\Lambda$. These empirical results are shown in Figures 5-3 and 5-4. In Figure 5-3, the eccentricity ratio is seen to increase at a higher rate than expected as $\Lambda \rightarrow 0$. Figure 5-4 shows a more marked departure from the theoretical calculations of LubePack where a sudden drop in $\alpha$ is observed to occur at $\Lambda \approx 0.17$ as the film undergoes collapse. This is in contrast to the theoretical calculations which would predict behavior approaching the ideal full Sommerfeld short bearing solutions or so-called "2$\pi$" theory in which the attitude angle approaches 90° as $\Lambda \rightarrow 0$ (see Figure 5-2).

At higher bearing numbers, the experimental results are in better agreement with theory. For example Figures 5-5 and 5-6 are for steady, hydrodynamic operation with $\zeta = 0.0429$ in contrast to the lighter loading conditions of $\zeta = 0.0253$ used in Figures 5-3 and 5-4. Heavier loading allows steady operation out to a bearing number range of $0.25 \leq \Lambda \leq 0.35$. For these higher speeds, no collapse of the lubrication film is observed. In fact, the experimental results shadow the LubePack calculations although there is a discrepancy which is larger than the uncertainty in the macro-rig measurements. Reasons for this offset will be discussed below.

The collapse of the lubrication film at low speed is likely due to the fact that the experimental bearing has many non-ideal effects such as surface roughness, liner non-circularity, and film disruptions caused by radial probe holes that are not modeled in the calculations. The theory assumes that one can always maintain an unbroken lubrication film even for very small $\Lambda$ under any load. Clearly, this is an unreasonable
Figure 5-3: LubePack calculations vs. experiment: eccentricity ratio vs. bearing number for $\zeta = 0.0253$.

Figure 5-4: LubePack calculations vs. experiment: attitude angle vs. bearing number for $\zeta = 0.0253$. 
Figure 5-5: Eccentricity ratio vs. bearing number: $\zeta = 0.0429$

Figure 5-6: Attitude Angle vs. bearing number: $\zeta = 0.0429$
Figure 5-7: Comparison between experiment and LubePack: Eccentricity ratio versus bearing number for hydrodynamic runs.

assumption for real bearings with non-zero loading.

Figures 5-7 and 5-8 show the experimental observations compared to the steady state results from LubePack. In Figure 5-7 it is seen that the experimentally observed eccentricity ratios do indeed range from 0.95 and above as expected. There is a substantial discrepancy between the computed results and the uncertainty bounds (given in the appendices) of the experimental measurements but it is worth noting that the observed eccentricity ratios are within 5 percent of the predicted values.

Furthermore, it can be seen that the data in Figure 5-7 are exhibiting trends consistent with the behavior of a hydrodynamic journal bearing. For example, as the load parameter is increased from 0.00549 to 0.0548, each $\epsilon$ vs. $\lambda$ curve at constant $\zeta$ shifts upward and to the right as expected. This means that higher values of $\epsilon$ are required for a given value of $\lambda$ as loading is increased. Also, for a fixed load, $\zeta$, Figure 5-7 shows that increasing $\lambda$ decreases $\epsilon$. In other words, as the rotational speed of the rotor is increased, the gas bearing will hydrodynamically "stiffen" and force the rotor to a lower eccentricity ratio for the case of a constant load.
The relationship between steady state attitude angle and bearing number observed during macro-scale experiments is shown in Figure 5-8. Like the static eccentricity ratio measurements of Figure 5-7, there are significant discrepancies between the theoretical calculations and the experimental measurements. However, once again the observed attitude angles are within the expected range for $0.1 \leq \Lambda \leq 0.35$.

Also, the ordering of the experimental data sets are consistent with changes expected from a hydrodynamic bearing experiencing changes in loading. For example, the lowest attitude angles are for the data sets with the heaviest loading, while the highest attitude angles occur for more lightly loaded bearings. This is consistent with an increase in compressibility effects due to increased loading that would depress the attitude angles below the ideal incompressible, short bearing solution of $90^\circ$. For fixed load, the data sets of Figure 5-8 show an increase in attitude angle with an increase in $\Lambda$. This is to be expected since at fixed load, the bearing will stiffen with larger $\Lambda$ and the eccentricity ratio will drop so that the attitude angle should
simultaneously increase. A complete set of plots for the experimental observations and comparison to LubePack for the data sets used to generate Figures 5-7 and 5-8 along with measurement uncertainties is given in the appendices.

5.2.2 Whirl Stability Results

For the experimental data sets shown in Figures 5-7 and 5-8, the rotor was nominally in a state of static equilibrium. That is, any oscillations in rotor position were small relative to the mean rotor position; no large-scale whirling was present. For hydrodynamic behavior, one would expect to see the onset of whirl when the eccentricity ratio drops below the critical $E$ boundary shown in Figure 4-22. In the case of a fixed load parameter, $\zeta$, increasing the bearing number, $A$, should decrease $E$ until the stability boundary is crossed, triggering the onset of whirl.

To test for this phenomenon, a fixed gravity side-load was applied to the macro-bearing test rig using the supporting tilt-table shown in Figure 2-5. The $E - A$ relationship was then explored by varying the rotor rotational speed to change $A$ while measuring the corresponding eccentricity ratio. Finally, any incidents of large-scale whirl were noted along with the corresponding values of $E$ and $A$.

During the experiments, whirl onset was found to occur as the bearing stiffened under larger $A$ and fixed $\zeta$. As the stability boundary was approached in this fashion, $E$ was found to decrease according to the trends shown in Figure 5-7. This process was repeated for various load parameters and the $E - A$ pairs corresponding the the onset of whirl are shown in Figure 5-9. Such whirl onset was characterized by the rotor whirling around the bearing liner accompanied by a large drop in $\omega$, signifying substantial energy dissipation due to loss of lubrication.

As expected, a narrow, high eccentricity ratio corridor of stable operation similar to that shown in Figure 4-22 was observed. This whirl stability boundary was computed with LubePack and is plotted as a solid curve in Figure 5-9 for comparison to experimental results. Much care had to be taken to gather the data in Figure 5-9 as it was discovered that small external disturbances could initiate a loss in stability that would otherwise not have occurred. For example, vibrations from dropping a tool on
Figure 5-9: Experimental and computed stability boundaries for hydrodynamic runs.

the floor or abruptly varying an operational parameter of the test rig such as $\omega$ or $\zeta$ could force the system into whirl. Whenever possible, incidents of whirl that were the result of such noticeable exogenous factors were not included in the data shown in Figure 5-9.

Although there is a $2 - 4\%$ discrepancy between the experimental data and theoretical stability boundary of Figure 5-9, the experimental results do confirm the key qualitative feature of the hydrodynamic stability calculations: the existence of a high eccentricity ratio corridor of stable operation.

Figure 5-10 gives a typical power spectrum from a snapshot of a radial probe time series for steady, hydrodynamic operation. The sharp, synchronous peak due to light rotor unbalance of $0.7\%$ is at approximately $38$ Hz. Note that there is a very broad peak at somewhat less than half of the rotor rotational speed. In contrast, Figure 5-11 shows the onset of a typical episode of whirl instability. Again, the synchronous peak is well-defined at $38$ Hz as in Figure 5-10, but now there is a dominant sub-synchronous peak at slightly less than half the rotor speed. As previously discussed,
this is indicative of the classical hydrodynamic whirl instability.

Figure 5-12 shows a “spectrogram” computed using MATLAB. The spectrogram is formed by computing the power spectrum of a windowed portion of time series data from a radial displacement probe in the macro-bearing test rig during hydrodynamic operation at fixed load. This window is then stepped along through the entire time series with a 50% overlap. The sequence of spectra that result are ordered in time and plotted as a grayscale image with time as the abscissa and response frequency as the ordinate. Similar to a “waterfall” chart, maximum intensity is denoted by white while the areas of low power are denoted by black. This format is convenient since it provides a quick overview of the rotor-dynamic spectral content for an entire experiment.

In Figure 5-12, there are two non-physical signals present. First of all, the familiar 60 Hz line noise is visible as is a sub-harmonic at 30 Hz. In addition, there is another signal which is constant at approximately 67 Hz due to external electrical interference. Both signals are non-physical in the sense that they are not related to the rotor-
dynamics of the experiment - both persist in the acquisition system when the rotor is motionless before and after experiments. Of more interest is the synchronous signal which shows the rotor rotational speed increasing from 48 Hz at \( t = 0 \) seconds to a peak value of 82 Hz at \( t = 138 \) seconds before decelerating to 52 Hz at the end of the data set. In the vicinity of \( t = 130 \) a sub-synchronous signal is seen to emerge at 31.5 Hz for a whirl ratio of 0.42. This corresponds to the onset of whirl instability which occurs when \( \omega \) is increased under constant loading. It is also interesting to note that in the diffuse regions to the left and right of this episode of whirl onset the sub-synchronous response appears to be linked to the rotor speed. This is again consistent with hydrodynamic theory which relates the whirl frequency at onset to the rotor speed; for hydrodynamic operation, the whirl frequency should be dependent on \( \omega \) for a fixed load.

The whirl ratios for a series of whirl onset incidents are shown plotted versus bearing number in Figure 5-13 with the Figure 4-23 LubePack predictions for whirl frequency along the computed stability boundary of Figure 4-22. As is evident from
Figure 5-12: Spectrogram of radial probe signal at whirl onset during hydrodynamic operation. Geometry is from macro-rig (Table 4.1). $\Lambda = 0.048$, $\zeta = 0.0253$.

Figure 5-13, the experimental observations are consistent with hydrodynamic operation as described by LubePack.

5.2.3 High Speed Hydrodynamic Operation

To take advantage of the hydrodynamic stability corridor predicted by the LubePack results of Figure 4-22 and the experimental verification provided by the data of Figure 5-9, a series of high speed runs on the macro-bearing test rig were attempted at high eccentricity ratio.

Figure 5-14 shows that this strategy was indeed successful in achieving a bearing number of approximately $\Lambda \approx 1.8$. The top rotational speed reached during these speed runs was approximately 45.7 kRPM, representing 68% of the dimensional design speed of 67 kRPM which would correspond to a bearing number of approximately 3.6. It is worth noting that as $\omega$ increases, $\Lambda$ increases not just through the increased rotational speed, but through a decrease in the nominal bearing clearance due to
Figure 5-13: Comparison between experimentally observed whirl ratios during hydrodynamic operation at whirl onset and the LubePack calculations of Figure 4-23.

rotor growth from centrifugal loading. This “pancaking” of the rotor also decreases \( \tilde{M} \) dramatically so that at \( \Lambda = 3.6 \) the \( \tilde{M} \) value of the macro-rig matches that of the various micro-devices in MIT Micro-Engine Project to provide dynamic similitude between the macro- and micro- scales. All of these factors add up to increase the available operating eccentricity ratio margin above the hydrodynamic whirl limit.

### 5.2.4 Axial Equilibrium During Hydrodynamic Runs

Ironically, despite the fact that these hydrodynamic runs were to test the hydrodynamic stability limits of the journal bearing, the primary difficulty encountered was maintaining axial equilibrium of the rotor. In particular, as the main turbine pressure was increased to achieve larger \( \omega \), pressure increases were incurred on the forward side of the rotor. Physically, this resulted in the tendency of the rotor to “sink” toward the thrust surface on the aft axial cover plate shown in Figure 2-17. Consequently, the speed runs of Figure 5-14 were limited by the rotor rubbing on the aft cover plate.
at high speed rather than the hydrodynamic instability of the journal bearing.

Many elaborate countermeasures were employed to alleviate this axial equilibrium problem with varying degrees of success. First of all, in order to provide hydrostatic loading on the aft side of the rotor in excess of that provided by the aft thrust bearing, the center exhaust hole of the aft thrust bearing (see Figures 2-17 and 2-18) was isolated from the outer exhaust port which was vented to atmospheric pressure. The center exhaust port was then pressurized with an additional external pressure source and regulator in order to provide more axial control authority during high speed operation. In a similar vein, unused axial probe ports on the aft cover plate (see Figures 2-17 and 2-18) were plumbed with pressure fittings and connected to an external pressure source for increased axial load capacity.

To further assist in control of axial equilibrium of the rotor, the main turbine was exhausted to a high volume vacuum source rather than ambient pressure. With this scheme, it was even possible to spin the rotor by “sucking” down the exhaust pressure with the main turbine inlet at ambient pressure rather than “blowing” high pressure into the main turbine supply. Thus, the absolute pressure levels on the forward side of the rotor were reduced below those of the exclusively pressure-fed scheme so that the net axial load in the aft direction was decreased for all rotor speeds.

Finally, the supply pressure of the aft thrust bearing was maximized to help maintain a non-zero aft thrust clearance. During high speed operation, the forward thrust bearing pressure could be adjusted downward but it was necessary to take care not to collapse the lubrication film in the forward thrust clearance. This was likely due to the fact that the thrust bearings have effective hydrostatic compensation schemes of the sort discussed in the “Thrust Bearing” chapter while the build-up of pressure on the forward side of the rotor due to the turbomachinery resembles an uncompensated, “dead load.” An uncompensated load would not increase if the rotor were perturbed in the forward direction so that any such excursions would be difficult to control and collapse of the lubrication film might occur.

Figure 5-15 shows the aft axial clearance of the rotor during a high speed, hydrodynamic run to 45.2 kRPM. The nominal axial clearance measured at the start
of the run was approximately 1.9 mils. Thus, the rotor could move a total of 3.8 mils between the thrust plates before striking one of the thrust surfaces. As seen in Figure 5-15, the mean aft axial clearance falls steadily as the rotor speed is monotonically increased from approximately 20 kRPM at the start of the run to a final value of just over 45 kRPM. The figure shows axial “corrections” that were made primarily using the pressurized aft center exhaust hole to push the rotor upward toward the forward thrust plate as the rotor speed was increased. The journal was in stable hydrodynamic operation when the rotor struck the aft thrust surface which then initiated friction in the journal as well.

At the time of the final high speed, hydrodynamic runs, thrust plate crashes from earlier runs had caused sufficient wear of the chrome oxide coating on the aft cover plate to increase the nominal axial rotor clearance from 0.8 to 2.0 mils. During the final high speed run, a loss of axial equilibrium caused the rotor to strike the aft cover plate which not only wore off the remaining ceramic coating, but literally welded the
Figure 5-15: Aft axial thrust clearance (mils) during high speed run to 45.2 kRPM. Speed increases from left to right.

aft thrust bearing restrictor holes shut as the kinetic energy of the 40+ kRPM rotor was released as heat.

5.2.5 Possible Reasons for Discrepancies between Theory and Experiment

There are several reasons one would expect the experimental results to deviate from the hydrodynamic calculations. First of all, because the removable journal bearing liner was designed to slip-fit with the stator plate, it was necessary to install the liner using shims so that deviations from non-circularity could be minimized. Figure 2-13 shows the residual non-circularity despite extensive efforts to correct non-uniformity with shims. These distortions of the liner could have an impact on sensitive measurements such as the stability boundary or even the static parameters for hydrodynamic operation. To analyze these effects, it would be necessary to use a full numerical approach such as that of Piekos [48] which is well suited for handling non-ideal bearings.
Another reason for discrepancies between experiment and theory is the fact that the radial displacement probes that are spaced every 90° in the bearing liner surface were designed to be recessed below the bearing coating to prevent contact with the rotor. The depth of the recess cavity is 0.003 – 0.005 inches deep and was determined by the accuracy to which the probes could be epoxied into place. The width of the liner probe hole is 0.11 inches, or about 35% of the bearing length. Since the nominal bearing clearance is approximately 0.004 inches, the radial probe recess cavities represent a sizeable disruption for the lubrication film. With only 0.1 inches on either side of the probe hole, it is likely that the probe cavities equilibrate to approximately ambient pressure. Consequently, it may be more accurate to approximate the system as a segmented, 4-pad bearing.

With regard to the scatter in experimental data from the stability experiments, the importance of limiting external disturbances to prevent a premature, forced, onset of whirl is crucial. This is especially important when the experiment is located in a busy lab where there are many opportunities for sources of vibrations. Every effort was made to take note when a large external disturbance forced whirl onset, but it is humanly impossible to detect very small vibrations. Possibilities for elimination of this problem would be to either mount the experimental apparatus on a vibration isolation pad or install accelerometers that could be correlated to the rotor-dynamic data of the experiments and would thus identify any incidents of forced whirl onset.

It is noteworthy that the radial hydrodynamic instability barrier predicted by LubePack’s generalized stability analysis was not observed during the macro-rig tests. There are several reasons why one would not expect to see this behavior during the hydrodynamic experiments conducted. As previously discussed, best control of the hydrodynamic tests was obtained using the tilt table to apply a fixed gravity load to the journal bearing while varying the speed rather than loading the bearing with the side-pressure scheme. This was primarily to de-couple the side-loading scheme from issues of axial equilibrium as well as eliminating fluctuations in the system supply pressure as demand for pressurized air changed during the experiment. This reduced the complexity of the experiments to reasonable levels but also limited the available
side-load to the rotor weight so that a limited range of bearing numbers was accessible. Within the available range of $0.0 \leq \Lambda \leq 0.4$, the radial instability predicted by LubePack occurs within 2% – 3% normalized clearance of the bearing wall. It was precisely within this range that the gradual collapse of the lubrication film was observed to occur. During this film collapse, the rotor behavior was steady; no telltale unsteadiness was observed before unlubricated rubbing in the journal was initiated. In the absence of bearing imperfections such as roughness, non-circularity and radial probe ports, it might be possible to maintain lubrication to extremely high $\epsilon$ and observe the radial instability at low $\Lambda$. Unfortunately, the macro-bearing test rig did not satisfy these ideal specifications.

At higher speeds, the eccentricity ratios for operation were significantly below the predicted radial instability boundary but still in compliance with the whirl instability criterion. Consequently, one would not expect to see the radial instability during these runs as well. Nevertheless, it is crucial to note that the whirl instability calculated with LubePack was observed. Due to the heavy coupling of the state variables of the compressible, unsteady, gas film/rotor dynamical system at high $\epsilon$, all eigenvalues of the Jacobian are coupled and must be computed simultaneously. Therefore, it is unlikely that the LubePack whirl bifurcation is quantitatively accurate while the accompanying calculation of the eccentricity bifurcation is qualitatively incorrect. For such a coupled system, the accuracy of computed whirl eigenvalues depends on the accuracy of the eccentricity eigenvalues and vice-versa. In light of this, it is probably prudent to take into account the theoretical radial instability boundary while operating devices with this class of bearings rather than applying arbitrarily large loads that would be permissible under consideration of only the classical whirl stability criterion.

### 5.3 Conclusions

The macro-bearing rig results presented in this chapter provide the first well-instrumented experiments performed on hydrodynamic gas bearings in the unique, low-aspect ratio
regime required by Micro-Engine technology. In fact, the experiments of this thesis are conducted on gas bearings with $L/D$ ratios an order of magnitude lower than those of previous studies. The data produced by these experiments demonstrate behavior consistent with predominantly hydrodynamic operation for both static as well as stability properties. Reasons for discrepancies between experiment and theory are discussed. Most importantly, the existence of the high eccentricity, stability corridor predicted by hydrodynamic theory is demonstrated in the experiments. It is noteworthy that the eccentricity ratios used in this work ($\epsilon > 0.90$) to achieve high speed operation are far in excess of those used in traditional gas bearings [27] which are normally designed to run at eccentricity ratios half this size. These extreme running conditions were dictated by the very low bearing $L/D$ ratios required by the DRIE manufacturing processes used for the Micro-Engine bearings.

The limiting factor in the high speed operation of the macro-bearing test rig was found to be maintenance of axial thrust clearances of the rotor rather than instability in the journal bearing. This is important because it implies that higher rotational speeds could be achieved by engineering more effective thrust bearings rather than implementing substantial design changes in the journal bearing.

The new, radial instability predicted by LubePack was not observed during the experiments conducted. Experimental constraints imposed by practical considerations were not conducive to exploring the operational regimes where this instability was likely to be encountered. However, because the whirl instability was observed reasonably close to LubePack expectations and it is known that the state variables of the system are heavily coupled under these conditions, it is not likely that the computed eccentricity bifurcation is erroneous.

Finally, the rotational speeds attained with hydrodynamic operation of the macro-bearing test rig scale to roughly half the target bearing number for the devices of the MIT Micro-Engine Project, demonstrating the applicability of the tested lubrication system to this class of aggressive MEMS technology.
Chapter 6

Hybrid Operation

6.1 Introduction

The literature on plain, cylindrical, gas journal bearings can be divided into the two broad categories of hydrostatic lubrication and hydrodynamic lubrication. As noted earlier, hydrostatic bearings are sometimes termed "externally pressurized" because the film pressures that separate the bearing surfaces are supplied by an external pressure source. The chapter on "Thrust Bearings" gave a detailed account of this type of system. In that chapter, it was shown that hydrostatic bearings must have some sort of compensation mechanism that regulates the fraction of the supply pressure entering the film for a given bearing clearance. The compensation mechanism sets the film entrance pressure of the externally pressurized lubricant in an inverse relation to bearing clearance to provide a restoring force.

On the other hand, hydrodynamic bearings are often described as "self-acting" in the sense that the load capacity is the result of pressures generated by relative motion of the bearing surfaces. The previous chapter on "Hydrodynamic Operation: Theory" included a detailed discussion on the calculation of both steady and unsteady hydrodynamic forces for the case of a plain, cylindrical journal bearing. Strictly speaking, these hydrodynamic forces can only be eliminated by holding the rotor motionless with respect to the surrounding sleeve. However, it was also shown that the self-acting bearing reaction was most significant when the journal was operated
at high eccentricity ratios.

In general, for devices such as the MIT Micro-Engine, both hydrostatic and hydrodynamic forces will be present in the journal lubrication film. When this occurs, the bearing is said to be operating in a “hybrid” mode of operation [28]. There are several reasons why the Micro-Engine journal bearing is capable of operating as a hybrid lubrication system. First of all, because the rotor weight is negligible compared to the hydrodynamic reactions in the journal film, the rotor side-load necessary for satisfying the hydrodynamic whirl stability criterion must be applied using static pressures [24]. Introduction of a pressure source exogenous to the journal bearing raises the possibility of hydrostatic forces. The pressures generated by the turbomachinery of the Micro-Engine at various operating points will also have a substantial impact on the ambient pressures experienced by the journal bearing. Moreover, because of the extremely short bearings dictated by the DRIE techniques used to fabricate the Micro-Engine, one may expect that these ambient pressures will have a significant influence on the fluid mechanics in the journal bearing interior. With these factors in mind, it is conceivable that hydrostatic effects in the Micro-Engine could impact operation of the bearings.

6.2 Journal Bearing Through-Flow for the Incompressible, Lubrication Regime

6.2.1 The Governing Fluid Equation

Consider the case of a plain, cylindrical, journal bearing such as the one depicted in Figure 4-1 whose ends are exposed to different ambient pressures. Under the assumptions of lubrication theory discussed in the chapter on “Hydrodynamic Operation,” the lubricant flow in such a bearing is governed by Equation (4.5). This non-dimensional equation is cast in the so-called “PH” form which is conducive to numerical solution.

For incompressible flows, solutions of Equation (4.5) become accessible to analyt-
ical methods and the PDE is typically cast in a different form that provides more physical insight. In particular, the time derivative term on the right hand side of Equation (4.5), $\frac{\partial \Psi}{\partial t}$, is usually expanded in terms of bearing surface velocities. To see this, replace $\Psi$ with its definition as the product of pressure and film thickness. The equation of state substitution that replaced density with pressure under the time derivative operator must be reversed for the incompressible case so that $\rho$ again appears multiplying $H$ in this term. Since density is constant in an incompressible flow, $\rho$ can be taken outside the time derivative. "H" is a geometric parameter that describes the shape of the lubrication film as defined by the rotor and bearing surfaces. Therefore, time derivatives of $H$ can be interpreted as various velocity components of these surfaces [28].

With this approach, the governing incompressible lubrication equation is given as:

$$
\frac{\partial}{\partial x} \left( \frac{\rho h^3 \partial p}{12 \mu \partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\rho h^3 \partial p}{12 \mu \partial z} \right) = \frac{\partial}{\partial x} \left[ \frac{\rho h (u_a + u_b)}{2} \right] + \frac{\partial}{\partial z} \left[ \frac{\rho h (w_a + w_b)}{2} \right]
$$

$$
+ \rho (v_a + v_b) - \rho u_a \frac{\partial h}{\partial x} - \rho w_a \frac{\partial h}{\partial z} + h \frac{\partial p}{\partial t}.
$$

(6.1)

A nomenclature list is given in Table 6.1.

Consider the case of zero rotor rotation and an externally applied axial pressure gradient established by dissimilar ambient pressures at the ends of the bearing:

$$
p \left( z = \frac{L}{2} \right) = p_u; \quad p \left( z = -\frac{L}{2} \right) = p_l.
$$

(6.2)

In Equation (6.2), the convention is adopted that $p_u > p_l$. If $\omega = 0$, the circumferential Couette velocity, $u_b$, is zero. If there is no rotor translation in the axial direction, the axial Couette velocity, $w_b$, is zero. Furthermore, if the sleeve is stationary, $v_a = u_a = w_a = 0$. Equation (6.1) then simplifies to:

$$
\frac{\partial}{\partial x} \left( \frac{\rho h^3 \partial p}{12 \mu \partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\rho h^3 \partial p}{12 \mu \partial z} \right) = -\rho v_b + h \frac{\partial p}{\partial t}.
$$

(6.3)
<table>
<thead>
<tr>
<th>symbol/definition</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>unwrapped circumferential coordinate</td>
</tr>
<tr>
<td>( u )</td>
<td>film velocity associated with ( x )</td>
</tr>
<tr>
<td>( y )</td>
<td>cross-film coordinate</td>
</tr>
<tr>
<td>( v )</td>
<td>film velocity associated with ( y )</td>
</tr>
<tr>
<td>( z )</td>
<td>axial coordinate</td>
</tr>
<tr>
<td>( w )</td>
<td>film velocity associated with ( z )</td>
</tr>
<tr>
<td>( a )</td>
<td>denotes bearing (sleeve) surface</td>
</tr>
<tr>
<td>( b )</td>
<td>denotes journal (rotor) surface</td>
</tr>
<tr>
<td>( h = c + e \cos \theta )</td>
<td>dimensional film thickness</td>
</tr>
<tr>
<td>( \rho )</td>
<td>lubricant density</td>
</tr>
<tr>
<td>( \mu )</td>
<td>absolute viscosity of lubricant</td>
</tr>
<tr>
<td>( p )</td>
<td>dimensional pressure</td>
</tr>
<tr>
<td>( p_u )</td>
<td>“high” ambient pressure</td>
</tr>
<tr>
<td>( p_l )</td>
<td>“low” ambient pressure</td>
</tr>
<tr>
<td>( t )</td>
<td>dimensional time</td>
</tr>
<tr>
<td>( L )</td>
<td>bearing axial length</td>
</tr>
<tr>
<td>( \omega )</td>
<td>rotor rotational speed</td>
</tr>
</tbody>
</table>

Table 6.1: Nomenclature list for incompressible analysis.

In the context of short bearing theory, the circumferential Poiseuille term of the left hand side of Equation (6.3) may be neglected. Also, for incompressible lubricants, the unsteady density term on the right hand side of Equation (6.3) is identically zero. Under these conditions, Equation (6.3) reduces to:

\[
\frac{\partial}{\partial z} \left( \frac{\rho h^3}{12\mu} \frac{\partial p}{\partial z} \right) = -\rho v_b . \tag{6.4}
\]

Finally, to produce an equation that may be readily integrated, for the case of no conical misalignment, the film thickness as well as \( v_b \) is independent of \( z \). That is, only the planar or "cylindrical" motion of the journal is considered in lieu of "conical" or wobble modes. This yields:

\[
\frac{\partial^2 p}{\partial z^2} = -\frac{12\mu}{h^3} v_b . \tag{6.5}
\]
6.2.2 Solution/Discussion

Integrating Equation (6.5) twice and applying the boundary conditions of Equation (6.2) results in:

\[ p = 6\mu \left( \frac{v_b}{h} \right) \left( \frac{L}{h} \right)^2 \left[ \frac{1}{4} - \left( \frac{z}{L} \right)^2 \right] + (p_u - p_l) \left( \frac{z}{L} \right) + \left( \frac{p_u + p_l}{2} \right) \].

From Equation (6.6) it is clear that the first term on the right hand side is simply the "normal squeeze" term. Note that this contribution to the overall pressure field on the rotor is proportional to the bearing squeeze velocity, \( v_b \). The effect of the hydrostatic pressures \( p_u \) and \( p_l \) is encompassed in the last two terms on the right hand side of Equation (6.6). Specifically, \( (p_u - p_l) \left( \frac{z}{L} \right) \) is just the familiar linear pressure distribution for a Poiseuille flow in a channel. This is a statement of the "fully developed" or non-accelerating nature of flows in the lubrication regime. Note that \( \left( \frac{p_u + p_l}{2} \right) \) is just the average pressure applied to the ends of the bearing. If \( p_u = p_l = p_0 \), the axial Poiseuille flow disappears and the average pressure applied is simply \( p_0 \) as expected. The squeeze term then represents deviations in the bearing pressure from ambient due to rotor translation.

What is noteworthy about this simple result is that the axial through-flow terms in Equation (6.6) are independent of the bearing film thickness, \( h \). This means that the phenomena associated with dissimilar ambient pressures applied to the ends of the bearing are insensitive to eccentricity ratio. In other words, an axial hydrostatic flow in the incompressible lubrication regime does not provide a compensation mechanism. Thus, the bearing will not produce a stiffness component due to the axial through-flow.

Furthermore, since the governing PDE for incompressible analysis is linear, spinning or translating the rotor in a whirl motion produces new pressure contributions that will not couple with the axial through-flow component of the pressure. The result will be simply a linear superposition of the various pressure components. In passing, it should be noted that the compressible lubrication analysis presented in the chapter on "Hydrodynamic Operation" showed that the class of short bearings used in the
Micro-Engine regime has a substantial range of operation which is largely incompressible, thus, establishing the relevance of the above conclusion on axial through-flow compensation.

This lack of compensation for a hydrostatic bearing operating exclusively in the lubrication regime should not be surprising in light of the discussion of compensation mechanisms in the “Thrust Bearings” chapter of this thesis. In that chapter, it was demonstrated that hydrostatic thrust bearings require inertial effects in the fluid film to generate additional pressure losses dependent on film thickness. The integrated effect of these inertial losses provides a bearing reaction that resists changes in the bearing clearance. Consequently, any analysis of a hydrostatic lubrication film that neglects inertial effects will omit all potential compensation mechanisms. To quantify the compensation mechanism associated with the axial through-flow, Breuer [6] suggested consideration of the inertia effects corresponding to the classical entrance length problem formulated in the axial direction.

6.3 Inertial Effects for Hydrostatic Compensation

6.3.1 Physical Description of Hydrostatic, Axial Through-Flow Compensation

To illustrate how an axial through-flow of the sort computed in the above references can produce a compensation mechanism that results in a restoring force on a radially displaced rotor supported by a journal bearing of the Micro-Engine class, one must re-examine the assumptions of lubrication theory. Since the lubrication equations are reduced from the Navier-Stokes equations by neglecting inertia, care must be taken to use a reasonable approximation to the true static pressure before integrating over the rotor surface to obtain the fluid forces for the equations of motion. As a consequence of neglecting inertia, pressure-driven flows are assumed to be “fully developed.” In particular, for a journal bearing through-flow driven by axial pressure gradients as described above, the growth of the side-wall boundary layers is not included in the
Figure 6-1: Schematic for axial journal bearing through-flow with boundary layer growth.

inertialess approximation.

Figure 6-1 depicts a channel flow driven by a pressure difference applied to the inlet and exit boundaries. Such a channel is a good approximation to a quasi-two-dimensional slice at fixed circumferential location of the axial through-flow scenario. The direction of the flow is left to right so that the ambient pressure on the left hand side, $p_u$ is greater than that on the right hand side, $p_l$. The flow is assumed to be initially uniform at velocity $\bar{U}$ and parallel. In general, there will be boundary layer growth on the channel sidewalls in a classical “entrance” flow. The boundary layers of thickness $\delta$ will act to accelerate the inviscid core flow via blockage effects.

As the core accelerates, there will be an associated pressure drop which will change the freestream conditions on the boundary layers, altering their development. Computation of such coupling between the freestream conditions and the boundary layer growth is the essence of interacting boundary layer theory [14]. Some distance downstream at $z = z_i$, the sidewall boundary layers will merge and the flow will cease to
accelerate having achieved a “fully developed” state. Downstream of this point, the velocity profile assumes the classical, non-accelerating, parabolic shape and the flow is governed by the inertia-free lubrication equations.

6.3.2 Previous Work on Axial Through-Flows

Determination of the pressure distribution and velocity field in the entrance region of concentric/eccentric annuli for laminar, axial through-flow has been the topic of extensive investigation for at least 35 years. For example, Sparrow and Lin [61] utilized integral methods to determine the pressure and velocity fields for annular ducts with stationary walls.

In work well-suited for inertial effects in journal bearings, Coney and El-Shaarawi [9] used a finite difference technique to solve the streamwise as well as azimuthal boundary layer equations for the case of concentric annuli with a spinning rotor. This study not only documented the axial boundary layer evolution into the classic Poiseuille profile, but also the development of the azimuthal boundary layers into the well-known Couette solutions. Coney and El-Shaarawi also state conditions under which the effect of rotor spin may be neglected.

El-Shaarawi and Mokheimer [21] compiled a substantial bibliography on studies of such axial through-flows. In addition, these authors conducted a study on through-flow similar to that of reference [21] with the added complication of non-zero rotor eccentricity.

While references [61], [9], and [21] allow for a full, three-dimensional analysis of the boundary layer systems for axial through-flows, there also exists much work on simple, two-dimensional formulations that might be useful for modeling a streamwise azimuthal slice of an annular duct as a channel flow. The entire annulus can then be approximated by a discrete number of these two-dimensional ducts at various azimuthal stations. This approximation is especially applicable to bearings with low $L/D$ ratios since azimuthal flows in short bearings are dominated by pressure-driven flows in the axial direction.

For such two-dimensional flows, Sparrow [60] solved the incompressible integral
boundary layer equations by approximating the boundary layer velocity profile with an analytical function. Substitution of this approximation into the governing equations yielded solvable conditions for the profile coefficients as a function of the streamwise coordinate. This technique is simple and easy to program but the validity of the assumption regarding the velocity profile is not clear without further investigation.

Van Dyke formulated the entrance flow problem in terms of an “upstream” region where inertial effects are important and a “downstream” region where lubrication theory applies [16]. With this formulation, perturbation techniques were applied to each region with matching conditions enforced to join the solutions. This technique is rather cumbersome to implement, especially when attempting to progress to higher order solutions for increased accuracy.

The interacting boundary layer theory (IBLT) presented by Drela [14] is effective for accurate calculation of flows of this type. One such technique from reference [14] solves the integral boundary layer equations using Falkner-Skan closure relations and a viscous-inviscid interaction law. The resulting $3 \times 3$ parabolic system of coupled differential equations is easily marched down the journal with a simple forward-Euler scheme utilizing any one-equation method (such as Thwaite’s method [64]) to bootstrap the calculations at the entrance of the bearing.

6.3.3 Computations for Developing Flow in a Parallel Channel

Given the macro-rig parameters listed in Table 4.1, it can be shown using the semi-empirical scheme of White [68] that for a 2 psi axial pressure difference and a channel width equal to the nominal journal clearance, 40% of the axial bearing length is occupied by the entrance region. In this “entrance region,” an inertial pressure drop of the type discussed in the earlier chapter, “Thrust Bearings,” will occur in addition to the linear pressure drop predicted by lubrication theory in Equation (6.6). Since this amounts to a gross violation of the lubrication equations under conditions routinely observed during macro-rig operation, a more detailed calculation is in order to
augment the earlier analysis on “Hydrodynamic Operation.”

Many authors have addressed the issue of computing entrance length flows using a variety of techniques. One simplification common to many of the approaches is the so-called “integral” approximation to the boundary layer equations. This takes advantage of the fact that the wall-normal pressure gradient is zero throughout the boundary layer so that the pressure field may be fully described by a scalar at each streamwise location. Consequently, in order to describe the streamwise evolution of the boundary layers, one need not satisfy the conservation equations at all points in the flow field. Instead, the governing equations may be satisfied in an integral sense at each streamwise location. Such an approach is used to develop the “integral boundary layer equations [68].”

6.3.4 Computational Results

The 2-D schemes of references [60] and [14] were implemented for computing the entrance length problem for the axial through-flow mechanism described above. This level of simplification was found to be appropriate for two reasons. First of all, since the difference between the bearing and rotor diameters is approximately 0.1%, the standard practice of unwrapping the film and neglecting curvature terms is appropriate. Thus, the annular flow is well approximated by a series of 2\(-D\), axial channels at various circumferential locations.

Secondly, Coney and El-Shaarawi [9] showed that when the square of the axial film entrance Reynolds number divided by the Taylor number is greater than order \(10^1\), the impact of a rotating inner annulus on boundary layer growth is negligible. For the range of typical axial pressure differences and rotational speeds of the macro-bearing test rig, this parameter varies from \(10^2\) to \(10^4\). Therefore, one may expect that the axial through-flow mechanism will be effectively de-coupled from the rotor rotational speed and the analysis for a stationary rotor will suffice.

With the approaches of references [60] and [14], the entrance length problem for the macro-bearing test rig was calculated for various axial pressure differences and rotor eccentricities. The resulting pressure field was numerically integrated over the
Figure 6-2: Hydrostatic force along $\alpha = 0$ line of journal bearing computed using the scheme of Sparrow.

surface of the rotor to yield a net force. Figure 6-2 shows the force versus eccentricity characteristic ($\alpha = 0$) computed in this manner for the macro-bearing rig at an axial pressure difference of 1 psi in the journal perimeter along the low side-pressure chamber and a 2 psi axial pressure difference along the high side-pressure chamber. The net side-load force along the symmetry axis from the high to low side-pressure chambers shown in Figure 2-17 is directed toward the $\alpha = 0$ station. As seen in the figure, the force/displacement characteristic is approximately linear out to moderate eccentricity ratios so that the hydrostatic “stiffness” is relatively constant. Also, there is a net force of approximately 1.9 N at $\epsilon = 0$ due to the side-load from the differential pressures in the aft cover plate chambers. If the high and low chambers had been set equal to some common pressure greater than that at the inter-row station on the forward side of the journal, the hydrostatic force would have been zero at $\epsilon = 0$.

For eccentricities in this linear range, the natural frequency that would be observed for a rotor supported in this manner is simply the square root of the stiffness divided by the rotor mass. A plot of natural frequency as a function of axial pressure
Figure 6-3: Theoretical IBLT relation between axial pressure difference and hydrostatic natural frequency.

difference, \( p_u - p_l \), computed in this fashion is shown in Figure 6-3. In principle, the axial pressure difference could be increased until the entrance length exceeded the bearing length. In this case, the flow never achieves a fully developed state. However, as a practical matter, the axial pressure differences that can be safely tolerated by the macro-rig while maintaining axial equilibrium between the thrust plates limits one to pressure differences less than those required for this condition to occur. Therefore, the calculations were restricted to flows that achieve the fully developed limit within the bearing.

It should be noted that this model does not take into account the possibility that separation regions could exist just downstream of the entrance of the axial through-flow. This would result in additional pressure losses and would distort the axial entrance velocity profile which is assumed to be uniform in the IBLT analysis. In a similar vein, flow non-uniformities due to vorticity in the aft side-pressure chambers are also neglected in the current analysis.
6.4 The Rotor-Dynamics of Hybrid Operation

In addition to the reaction from the external pressure supply, a spinning rotor supported by hydrostatic compensation of the type described above is also subject to the hydrodynamic forces analyzed in previous chapters. Due to the strong dependence of the hydrodynamic reaction on eccentricity ratio, one may expect that the hydrostatic force will be dominant at low to moderate values of $\epsilon$ while the hydrodynamic reaction will characterize bearing load capacity at high $\epsilon$. A bearing characterized by this dichotomy of externally applied and self-generated sources of pressure is termed "hybrid" [28].

6.4.1 The Role of Unbalance

The macro-rig rotor originally had a static unbalance equal to 20% of the nominal bearing clearance as a result of the manufacturing processes used to fabricate the device. The hybrid runs described in this chapter were performed with this level of unbalance. Consequently, the hybrid system is subjected to substantial amounts of forcing at the rotor synchronous speed, producing a wide range of eccentricity ratios during a given experiment.

6.4.2 Hydrostatic Natural Frequency

The macro-bearing test rig was operated under a range of axial pressure differences corresponding to different levels of stiffness from the previously computed axial through-flow mechanism. Analysis of the frequency content of the radial probe signals used to measure rotor position revealed that the system did in fact exhibit a frequency, $\omega_n$, that was directly correlated with the hydrostatic axial pressure difference. This hydrostatic "natural" frequency was insensitive to gravity loading of the rotor and, therefore, the position of the rotor between the thrust plates since axial gravity loading decreased with increasing gravity load on the journal due to tipping of the tilt table. Furthermore, for the range of rotational speeds examined (0 – 6300 RPM), no effect of $\omega$ on $\omega_n$ was observed. This speed independence is consistent with
Figure 6-4: Experimental hydrostatic natural frequency compared to results from integral boundary layer calculations.

the conclusions of reference [9].

Figure 6-4 shows experimental values of $\omega_n$ for a range of axial pressure differences. The upper limit on allowable pressure differences was determined by axial equilibrium issues related to the thrust loads produced by the pressure-driven side-load scheme. The natural frequencies derived from the integral boundary layer calculations described above are shown for comparison. The axial pressure differences used in this hydrostatic natural frequency mapping were obtained by setting the aft pressure chambers shown in Figure 2-21 to a common pressure which was greater than the inter-row pressure on the forward side of the journal. It should be noted that although the rotor speed does affect the inter-row pressure, the $\omega_n$ data were found to be sensitive to axial pressure difference rather than the absolute pressure values.

The data in Figure 6-4 show the largest deviation from the calculations at low pressure differences. It is important to note that the entrance length calculations presented in Figure 6-4 assume perfect circularity of the journal liner. However, from
Figure 2-13, it is clear that this is just a first approximation to the macro-bearing test rig. Piekos [47] has shown that this discrepancy between the calculated and observed natural frequencies can be virtually eliminated for the entire range of pressures considered by assuming a larger film entrance Reynolds number than that implied by boundary layer analysis. One physical justification for this technique of matching theory and experiment is the existence of a larger effective bearing clearance due to liner non-circularity. On a practical level, the agreement between theory and experiment in Figure 6-4 was sufficiently close to allow effective, a priori determination of the axial pressure difference needed to obtain a desired hydrostatic natural frequency. Consequently, by setting the prescribed axial pressure difference to the theoretical level at the beginning of an experiment, very little subsequent adjustment was needed to fine tune $\omega_n$ to the target value.

Figure 6-5 shows the experimentally measured macro-rig hydrostatic natural frequency for the more general case of dissimilar axial pressure differences corresponding to the two pressure chambers in the aft cover-plate shown in Figure 2-17. The axial pressure difference of the high side-pressure chamber, which covers 2/3 of the journal circumference, is denoted by $\Delta P_H$. Similarly, the analogous pressure difference for the low side-pressure chamber is given the symbol $\Delta P_L$. The $\omega_n - \Delta P$ correlation for equal axial pressure differences in the two chambers shown in Figure 6-4 corresponds to the intersection of a line with slope 1 that passes through the point $(0, 0)$ and the pressure contours of Figure 6-5. Because data points away from this symmetry line represent a net side-load on the bearing, attempts to obtain more data outside the contours of Figure 6-5 resulted in rotor crashes against the bearing liner.

6.4.3 Nonlinear Interactions

Figure 6-6 gives the synchronous and hydrostatic frequencies from a typical experiment in hybrid mode where there are substantial axial pressure differences to support the rotor at low eccentricity ratio. The hydrostatic "critical," represented by $\omega_n$, is easily identified and noted at the start of the experiment by varying the pressure regulator controlling axial pressure difference and monitoring a real-time spectrum
Figure 6-5: Experimental hydrostatic natural frequency contours for axial pressure difference combinations of the two aft cover-plate pressure chambers (contour units: Hz).

The synchronous signal, $\omega$, is an indication of forcing due to rotor unbalance. In Figure 6-6, $\omega$ varies with changes in the main turbine supply pressure and is measured with a redundant system of two independent probes over-looking the rotor turbine blades that were described in the “Experimental Facility” chapter. Using these independent measurements of rotor speed, one is able to easily locate $\omega$ in the radial probe spectra.

Figure 6-7 shows the synchronous and hydrostatic signals of Figure 6-6 traced and labeled in black on the corresponding radial probe spectrogram. A radial probe power spectrum taken at $t = 235$ seconds from the spectrogram shown in Figure 6-7 is shown in Figure 6-8. From these perspectives, the complexity and rich spectral content of the underlying nonlinear dynamical system is apparent. However, behind
Figure 6-6: Synchronous and hydrostatic “natural” frequencies extracted from radial probe spectrogram.

Figure 6-7: Synchronous and hydrostatic “natural” frequencies traced on radial probe spectrogram.
Figure 6-8: Power spectrum taken from slice of spectrogram shown in Figure 6-7 at $t = 235$ s.

Figure 6-9: Nonlinear mode interactions derived from synchronous and hydrostatic "natural" frequencies traced on radial probe spectrogram.
the apparent complexity of the spectrogram there is a very simple pattern.

The frequencies in the spectrogram of Figure 6-7 can be generated by the sums and differences of \( \omega, \omega_n \), and their associated harmonics. This nonlinear interaction rule for calculating the set of response frequencies, \( \Phi \), is shown in Equation (6.7) where \( i \) denotes the imaginary unit. Such sum and difference interactions are common in nonlinear systems and were studied in rotor-dynamic responses by Ehrich [17]. In this paper, Ehrich postulated an empirical formula which captures many of the possible interactions but lacks some of the terms contained in the full set of possibilities represented by the Cauchy product:

\[
\Phi = \left( \sum_{p=1}^{\infty} e^{ip\omega_n} \right) \times \left( \sum_{q=1}^{\infty} e^{iq\omega} \right).
\]

(6.7)

Of course, in a physical system, there is a limit to the number of harmonics that are observable.

Graphically, if one selects truncation indices for the harmonics in Equation (6.7) and computes a finite representation of \( \Phi \) for the \( \omega \) and \( \omega_n \) schedules shown in Figure 6-6, Figure 6-7 is obtained. In this figure, the frequencies of \( \Phi \) are overlaid as black curves on the spectrogram of Figure 6-6 and the thoroughness of Equation (6.7) is apparent. The truncation indices for \( \Phi \) were limited to \( p = 3, q = 3 \) in Figure 6-7. Close inspection of the figure reveals that a higher order truncation would be necessary to account for all the features in the spectrogram. Indeed, if one retains 3 additional harmonics in each truncation, all of the coherent modes in the spectrogram may be explained.

The representation of Equation (6.7) is of more than passing interest. In fact, Equation (6.7) is of substantial practical value for trouble-shooting bearing experiments on the micro-scale where the number of observables is inherently low due to lack of instrumentation. If one has only knowledge of the rotor spin rate and is able to acquire the frequency content of the rotor position, then it should be possible to identify the hydrostatic natural frequency by correlating the signal with changes in axial pressure difference. Once \( \omega \) and \( \omega_n \) are in hand, Equation (6.7) can be used to
account for the other frequencies in the spectrogram.

In fact, unlike the empirical formula of reference [17], Equation (6.7) is easily generalized to include an arbitrary number of fundamental frequencies. For example, if it is suspected that the hydrodynamic whirl or radial instabilities may be present, hydrodynamic frequency response calculations of the sort presented in Figure 4-32 can be included in Equation (6.7) to try and account for any unexplained frequencies in the spectrogram. This generalization is given as:

$$\Phi = \left( \sum_{p=1}^{\infty} e^{ip\omega_n} \right) \times \left( \sum_{q=1}^{\infty} e^{iq\omega} \right) \times \left( \sum_{m=1}^{\infty} e^{im\omega_{a}} \right) \times \left( \sum_{n=1}^{\infty} e^{in\omega_{c}} \right)$$

(6.8)

where \(\omega_{a}\) is the dimensional hydrodynamic whirl frequency and \(\omega_{c}\) is the analogous hydrodynamic radial vibration frequency. As mentioned earlier, because \(0 \leq \omega_{a}/\omega \leq 0.5\) and \(\omega_{c}/\omega\) tends to be larger than 0.5, it should be possible to distinguish between these two instabilities without the benefit of rotor-dynamic orbit data or eccentricity measurements; the only requirement is the frequency content of the rotor translation.

In lieu of hydrodynamic instabilities in the frequency content of the rotor position, one could envision an experiment where a pressure side-load was increased until the radial instability frequency appeared. Hydrodynamic calculations could then be used to estimate the likely eccentricity ratio. Similarly, one could start from the same “clean” operational point and decrease the pressure side-load until the hydrodynamic whirl frequency appeared, providing a LubePack-based estimate of \(\epsilon\) once again. Of course, it would be necessary to make sure that the hydrostatic forces were small relative to the potential hydrodynamic force levels, but this can be accomplished by controlling the axial pressure difference across the rotor. Clearly, Equation (6.7) is a useful tool for understanding experiments conducted on the micro-scale with limited instrumentation.

Figure 6-10 shows an example of a “run-through” of the harmonics of the hydrostatic critical, \(\omega_n\). At \(t = 0\) seconds, the synchronous signal is 50 Hz. At the end of the data set (\(t \approx 270\) seconds), \(\omega\) has increased to 150 Hz. The hydrostatic critical remains solid at approximately 41.5 Hz throughout the entire run. At \(t \approx 70\) seconds,
the synchronous speed passes through $2\omega_n$, triggering a second order, subharmonic resonance. When $t \approx 185$ seconds, the third order, subharmonic response is reached at $\omega = 3\omega_n$ amid a flurry of sum and difference interactions of the type described by Equation (6.7).

It is edifying to adopt the viewpoint of an experimentalist operating a gas bearing test rig in order to appreciate the difference between the hybrid operation depicted in Figure 6-10 and the predominantly hydrodynamic case given in Figure 5-12. At $t = 40$ seconds during the hydrodynamic run of Figure 5-12, the experimentalist increases the rotor speed because the waterfall chart on the spectrum analyzer shows "smooth sailing." Simultaneously, the whirl eigenvalues of Figure 4-14 are approaching the imaginary axis as $\Lambda$ is increased at constant $\zeta$. When $t = 130$ seconds, the rotor transitions into an unstable whirl before the experimentalist can react, resulting in an unlubricated crash.

On the other hand, during the hybrid run of Figure 6-10, the experimentalist sees the hydrostatic critical and its harmonics on the spectrum analyzer as soon as the axial pressure difference is established. This spectral signature looks like a set of barriers, so one might expect that something will happen when the rotor speed crosses $\omega_n$ and its multiples. At the very least, even the unexperienced operator has no difficulty distinguishing between the hydrodynamic and hybrid modes of operation.

6.4.4 Resonant Rotor Orbits

The hybrid mode of operation is observed to produce two types of nonlinear resonances. In addition to the subharmonic resonances already mentioned, superharmonic resonances are also present. Figure 6-11 shows subcritical operation of the hybrid system. From $0 \leq t \leq 50$ seconds, $\omega_n \approx 21$ Hz. As $\omega$ is increased from zero at $t = 0$ seconds, $\omega = 0.5\omega_n$ when $t \approx 30$ seconds.

This point corresponds to the rotor orbit shown inside the leftmost inset figure on the spectrogram. This "figure eight" shape represents the trajectory of the rotor geometric center inside the limit circle of the bearing clearance. The coherent figure eight shape results from a "fast" frequency of $\omega_n$ which forms each loop of the total
orbit and the “slow” frequency of $\omega = 0.5 \omega_n$ which is the inverse of the period of the entire orbit.

When $\omega \approx \omega_n$ at $t = 50$ seconds the primary hydrostatic resonance occurs and is characterized by a “teardrop” shape with period equal to $1/\omega$. This orbit is shown in the middle inset figure in the spectrogram of Figure 6-11. For the remainder of the run, $\omega_n$ was increased to track $\omega$ by increasing the axial pressure difference. As a result, the teardrop-shaped orbit persisted until the end of the run.

To examine the orbits corresponding to the subharmonic resonances of the hybrid system, Figure 6-12 displays supercritical operation in the familiar form of the Campbell diagram rather than the spectrogram of Figure 6-11. This format eliminates the parameter of time by using $\omega$ on the horizontal axis for the corresponding power spectra. The synchronous signal now appears as a line of slope 1. The hydrostatic natural frequency is located at approximately 50 Hz and drops slightly due to a decrease in axial pressure difference as the inter-row pressure of the test rig increases with increasing main turbine supply air. At $\omega = 100$ Hz, the synchronous speed is twice $\omega_n$. 

Figure 6-10: Spectrogram of radial probe signal for run-through of hydrostatic critical.
Figure 6-11: Spectrogram of subcritical hybrid operation with rotor orbits.

Figure 6-12: Campbell diagram for run-through of various order resonances with associated rotor orbit.
so that a second order subharmonic resonance occurs. Under these conditions, the orbit assumes a two-lobed beta shape. Unlike the figure eight associated with the second order superharmonic resonance, the “fast” frequency that forms each loop is now $\omega$ while the “slow” frequency associated with the entire orbit is $\omega_n$.

Between the second and third order subharmonic resonances, an interesting phenomena occurs: the rotor orbit intermittently switches back and forth between the two-lobed beta shape and a three-lobed shape which previews the third order orbit. The intermittent nature is apparent from the noisy nature of the Campbell diagram for $100 \leq t \leq 115$ Hz. It appears that the system may have multiple solutions which are marginally stable. Unfortunately, the data leading up to $\omega = 100$ Hz were not captured, so the spectrum cannot be checked for period doubling sequences that may lead to so-called “chaotic” behavior. However, it would be possible to run standard algorithms to see if any positive Liapunov exponents can be detected in the time series from this region.

When the third order subharmonic resonance is reached, $\omega = 3\omega_n$ and a coherent, three-lobed orbit emerges. Beyond this third “pseudo-critical,” another notable event occurs which had a profound impact on the success of the macro-rig testing program: the rotor orbit stabilizes into a perfectly circular whirl whose radius is independent of speed. The significance of this observation will be established in a later section.

Figures 6-11 and 6-12 describe hybrid operation largely in terms of the fundamental frequencies of the system. Figure 6-13 views hybrid operation in terms of response amplitude. For a hybrid run-through of the various nonlinear resonances, Figure 6-13 plots local maxima in the rotor eccentricity ratio versus speed ratio. It is evident that a large resonance occurs at the primary resonance of $\omega = \omega_n$. A smaller resonance appears when the synchronous speed is $3/2$ the hydrostatic natural frequency followed by another large resonance at the second order subharmonic resonance. The third order subharmonic resonance is just visible at the right hand side of the figure.

From Figure 6-13, it is observed that the amplitudes of the resonances decrease with increasing speed ratio. Also note that the amplitude peaks tend to bend leftward, implying the existence of “jump resonances” between multiple solutions in the system.
Figure 6-13: Local extremum measurements in eccentricity ratio for run-through of various resonances during hybrid operation.

This would also suggest a possible hysteresis effect so that traversing a critical from below would be different than crossing that same critical from above. Finally, it should be noted that the amplitude of the system response appears to tend towards $\varepsilon_{\text{max}} \approx 0.4$. The reason for this behavior will be described below.

### 6.4.5 Source of Nonlinearity

The nonlinear behavior of the hybrid system can be explained in terms of a simple, ad-hoc rotor-dynamics model that incorporates the salient features of the inertial effects approximated by IBLT along with LubePack lubrication solutions. In the simplest case, superposition is used to combine the hydrostatic stiffness and the hydrodynamic forces of the lubrication calculations [31, 58]. This may appear unsound at first, but further inspection reveals that this approach is reasonable.

As previously described, the hydrodynamic reaction increases strongly with $\varepsilon$. 

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Figure 6-14 shows LubePack macro-rig solutions for the radial force component of the hydrodynamic system versus $\epsilon$ at various $\Lambda$. The radial force is defined as the component of the bearing reaction that lies along the line joining the bearing and rotor centers. The figure shows that at low speed ($\Lambda = 0.25$), the so-called “direct stiffness” of the hydrodynamic film is virtually zero for $\epsilon \leq 0.8$. Even at the macro-rig design bearing number of $\Lambda = 3.6$, the hydrodynamic radial reaction is small for $\epsilon \leq 0.6$.

The corresponding diagram for the hydrodynamic tangential reaction, the force component normal to the line of centers, is given in Figure 6-15. Comparing Figures 6-14 and 6-15, it is observed that the tangential hydrodynamic reaction dominates the radial reaction for $\epsilon \leq 0.6$. In other words, the force component that makes the rotor tend to whirl is much larger than the component associated with providing whirl stability. This is consistent with the earlier conclusion drawn from Figures 4-7, 4-8, 4-9, 4-10, and 4-11. From these figures, it was observed that for $\epsilon \leq 0.6$, the low $L/D$ bearings required by the Micro-Engine are closely approximated by full Sommerfeld theory. Of course, full Sommerfeld theory predicts zero radial reaction and unconditional instability with respect to whirl. Because the radial reaction is associated with whirl stability for supporting applied loads, one might expect the hydrodynamic whirl stability properties of this bearing to be poor - a fact verified by LubePack calculations.

Figure 6-16 shows the composite radial force from the ad-hoc rotor-dynamics model which combines the effects of Figures 6-2 and 6-14. The hybrid radial force is seen to possess the high-$\epsilon$ singularity of the hydrodynamic system, the low-$\epsilon$ reaction of the inertial effects and the side-load at zero $\epsilon$ from the differential pressure between the aft side-load pressure chambers. Figures 6-15 and 6-16 summarize the steady force field for the ad-hoc hybrid model.

Perhaps the most striking feature of Figure 6-16 is that the hybrid system exhibits the steady force characteristic of a classical model from dynamical systems theory: the “hardening spring.” From this point of view, the support stiffness comes from the inertial effects shown in Figure 6-2 while the “hardening” regime is manifested in
Figure 6-14: LubePack-computed hydrodynamic radial force along $\alpha = 0$ for macro-bearing test rig at various bearing numbers.

Figure 6-15: LubePack-computed hydrodynamic tangential force along $\alpha = 0$ for macro-bearing test rig at various bearing numbers.
Figure 6-16: Composite [31, 58], hybrid radial force from superposition of LubePack hydrodynamic solutions and integral boundary layer results. along $\alpha = 0$ for macro-bearing test rig at various bearing numbers.

the massive hydrodynamic reaction at high $\epsilon$. Thus, the system is distinguished by two different sources of stiffness. Furthermore, due to the large dynamic forcing from the 20% static rotor unbalance, the rotor will cover a wide range of eccentricity ratios during a typical experiment. This produces frequent transitions between the "hard" and "soft" regimes of the hybrid force-displacement characteristic, accentuating the non-linear features of the system.

Ehrich [18] studied the nonlinear response of rotor-dynamic systems mounted on supports characterized by bilinear-linear stiffness. That is, the system under consideration exhibited a force characteristic that could be approximated by two straight lines of differing slope. Among a variety of rich phenomena, Ehrich found that the system was particularly prone to sub-harmonic resonances of the type observed during hybrid operation of the macro-rig. In fact, early in the macro-rig experiments, Ehrich used the model of reference [19] to produce rotor orbits which were qualitatively similar to those presented in Figures 6-11 and 6-12 [19]. Although the stiffnesses and damping rates in this model were prescribed in order to produce consistent results rather than derived from the physics of the system, the agreement was good.
To implement a physics-based model from the IBLT and LubePack results described above, the unsteady or dynamic bearing forces are approximated by the nonlinear reaction of full Sommerfeld theory that may be found in any elementary text on lubrication theory [28]. To allow for the effects of unbalance, the appropriate synchronous forcing is implemented. The resulting equations of motion were then integrated in time numerically using the routines of Mathematica [71].

Figure 6-17 shows the rotor orbits computed using this IBLT/LubePack hybrid bearing rotor-dynamics model outlined under conditions of super/subharmonic resonance similar to those observed in the macro-bearing rig experiments. The results are in qualitative agreement with the empirical evidence shown in Figures 6-11 and 6-12. For example, the familiar “figure eight” orbit corresponding to the superharmonic resonance of order two in Figure 6-11 is reproduced in Figure 6-17. The “tear-drop” of the primary resonance, the “beta” of the second order subharmonic response, the three-lobed orbit produced by the third order subharmonic resonance, and the speed invariant whirl circles observed at higher speed ratios, $\omega/\omega_n$, are also replicated.
The results of Figure 6-17 should not be interpreted as an imprimatur for such ad-hoc modeling, but rather, as evidence that the nonlinear hybrid force characteristic of Figure 6-16 is the source of the nonlinear interactions observed in the macro-rig experiments. Accurate time-domain simulations would require a proper treatment of the compressible, unsteady, fluid equations retaining the necessary inertia terms that were dropped in the Reynolds equation approximation. How much the resulting fluid equations resemble the full Navier-Stokes equations as opposed to the Reynolds equation is another issue.

The significant stiffness of the hybrid system at low $\epsilon$ (Figure 6-16) provided by the through-flow mechanism provides significant radial load capacity. This fact was exploited in the macro-bearing rig experiments which used this effect to "float" the rotor in the journal during preliminary spin-up for tests. Even for hydrodynamic experiments at high eccentricity ratios, this technique was often used to start the bearing with minimal friction before a side-load was used to push the rotor to high $\epsilon$ where hydrodynamic effects are dominant. In this way, the poor stability properties of the low $L/D$, Micro-Engine class of bearings were bypassed in order simplify operation of the test rig.

6.4.6 Speed Limitations of Hybrid Operation: A Review of Larson and Richardson's Theory (1962)

The hybrid mode of operation proved to be ineffective for attaining high speeds relative to those achieved with hydrodynamic operation. The speeds reached with the range of axial pressure differences allowable by the thrust balance of the rotor were limited to $\omega \leq 16$ kRPM, corresponding to approximately six times the hydrostatic natural frequency. This is roughly $1/3$ of the 46 kRPM attained with hydrodynamic operation.

Figure 6-15 gives a clue as to the origins of this speed limitation. As $\omega$ increases, $A$ becomes larger and the whirl component of the bearing reaction becomes significant. Because the hydrostatic mechanism largely provides radial stiffness and no tangential
component to cancel the hydrodynamic whirl force, the bearing is likely to lose stability in a hydrodynamic whirl. In fact, Larson and Richardson [31] concluded that hydrodynamic effects can destabilize hybrid bearings in precisely this manner.

To see this, recall that Larson and Richardson [31] modeled hybrid bearing effects by superimposing the hydrodynamic and hydrostatic mechanisms. This led to two, coupled, second order ODE's describing the rotor motion:

\[
\frac{M}{2} \ddot{y} = F_y + F_{ys} + F_{yh} \tag{6.9}
\]

\[
\frac{M}{2} \ddot{x} = F_x + F_{xs} + F_{xh} \tag{6.10}
\]

where \( F_x \) and \( F_y \) are the projections of the externally applied force along the inertial \( \hat{i}_1 \) and \( \hat{i}_2 \) axes (Figure 4-1), respectively, \( F_{zs}, F_{ys} \) are similar projections of the hydrostatic force, and \( F_{xh}, F_{yh} \) are the analogous components of the hydrodynamic reaction.

For analysis of the hybrid reactions, Larson and Richardson used the customary approximations for small eccentricity ratios and no interaction between the hydrostatic and hydrodynamic forces. For the modern implementation of their theory described below, the first of these assumptions will be discarded since the hydrodynamic LubePack and hydrostatic IBLT calculations are valid for large \( \epsilon \). As previously described, the hybrid de-coupling criterion developed by Coney [9] is easily satisfied by the macro-rig geometry, so hydrodynamic/hydrostatic interaction in the current study is negligible.

For the hydrodynamic force, Larson and Richardson [31] implemented incompressible solutions. With regard to the hydrostatic force due to radial injection of high pressure air, the authors not only modeled the static force/displacement characteristic of the mechanism, but also the time lag associated with the external pressurization system which is relevant when the rotor is in rapid motion. The earlier work of Richardson [57] was used to model this hydrostatic time constant.

Like the "Thrust Bearing" chapter of this thesis, Richardson's work [57] subdi-
vided the hydrostatic system into a compensation mechanism and a lubrication film. Compressible flow, lumped-parameter analysis was used for both components. Two compensation mechanisms were considered: orifice compensation and inherent compensation. For each case, differential equations were derived for the lumped parameter model and the following result was obtained for the inherent mechanism:

\[ F_{xs} = -k_s \left( \frac{\tau_1 D + 1}{\tau_2 D + 1} \right) x \]  \hspace{1cm} (6.11)

where \( \tau_1 \) and \( \tau_2 \) are time constants related to the phase of the hydrostatic force with respect to the rotor position, \( D \) is the differential operator \( d/dt \), and \( k_s \) is the hydrostatic film stiffness. A similar expression is obtained for the \( y \)-component of the hydrostatic reaction.

For periodic rotor motion, inserting the hydrostatic force of Equation (6.4.6) and incompressible hydrodynamic solutions into Equations (6.4.6) and (6.4.6) yields a characteristic equation for the dynamical system. This characteristic equation leads to the following stability criterion:

\[ \left( \frac{1}{r} - 2 \right) k_s + 2 k_1 = 2 \]  \hspace{1cm} (6.12)

where \( r \) is the ratio of rotor speed to system response frequency ("whirl ratio") at onset of whirl, \( k_s \) is the hydrostatic stiffness, \( k_1 \) is the hydrodynamic tangential stiffness and the time lag constants are as described above.

It is noteworthy that Equation (6.4.6) readily provides insight into the stability of hybrid bearing systems. For example, in the absence of hydrostatic stiffness, \( k_s = 0 \) and the threshold whirl ratio is 2.0 indicating a state of pure, "half-speed" hydrodynamic whirl. Similarly, even for non-zero hydrostatic stiffness, increasing the tangential hydrodynamic stiffness, \( k_1 \) destabilizes the bearing by reducing the threshold whirl ratio closer to the minimal value of 2. This is consistent with the observation that the hydrodynamic tangential or "whirl" reaction is the primary instigator of instability in a self-acting bearing. Finally, the highest threshold whirl ratios are obtained by making the hydrostatic stiffness as large as possible relative to
the tangential hydrodynamic stiffness.

Because of the difficulty in calculating the damping properties of a more general system, Larson and Richardson recommended using correlation with experimental data to determine the necessary time constants [31] despite the fact that they had obtained approximations for their system. For a more complicated system such as the macro-bearing test rig, there are many sources of time lags external to the journal bearing itself. These sources of damping include viscous shear in the thrust clearances, aerodynamics of the turbomachinery used to spin the rotor, flows in the tip clearances of the turbine blades, and mechanical compliance of the removable journal bearing liner. Consequently, empirical determination of the system damping properties is necessary.

Using the hydrostatic stiffness from the IBLT analysis and the hydrodynamic tangential stiffness obtained from LubePack results, the macro-rig hybrid stability boundary as defined by Larson and Richardson’s [31] theory (Equation (6.4.6)) is shown in Figure 6-18. For determination of the hydrodynamic properties, the super-critical eccentricity ratio extremum of $\epsilon = 0.40$ for unloaded, hydrostatic operation of a rotor with 20% static unbalance (see Figure 6-13) was used. These results correspond to $\tau_1 - \tau_2 = 0.75$.

From Figure 6-18, it is observed that there exists an extremum in the whirl ratio stability boundary. This is consistent with the calculations of Larson and Richardson [31] who noted the existence of an optimal hydrostatic supply pressure for achieving a maximum stable whirl ratio. If the hydrostatic supply pressure is pushed beyond this optimum, a lower whirl ratio limit results. Note that Figure 6-18 shows that as the hydrostatic supply pressure is decreased, the whirl ratio stability boundary tends to 2.0. This is sensible because in the absence of hydrostatics, the hydrodynamic tangential reaction for a lightly loaded, short bearing will force the system into the classical “half-speed whirl.”

Figure 6-19 compares the theoretical hybrid stability boundary with the experimental results from the macro-bearing test rig. Each experimental data point represents a bearing failure that terminated a macro-rig run in hybrid mode for the case
of uniform axial pressure difference around the circumference of the journal bearing. The static unbalance of the rotor for these experiments was $\delta = 0.2$. During hybrid operation, it was not possible to achieve whirl ratio values ($\omega/\omega_n$) greater than 6.0. This is consistent with the results of Figure 6-18. Furthermore, the experimental data of Figure 6-19 exhibit whirl ratios approaching 2.0 as the axial pressure difference is decreased to zero which is also predicted by theory.

Figure 6-20 expresses the results of Figure 6-19 in terms of dimensional rotor speed. The speed limit of approximately 16 kRPM encountered during hybrid operation of the macro-bearing test rig is in reasonable agreement with the theoretical boundary. Note that this speed limitation is a result of the loss of effectiveness of the hydrostatic mechanism as the axial pressure difference is increased. Consequently, the hydrostatic force is too weak to overcome the destabilizing hydrodynamic forces at high rotor speeds.

### 6.4.7 In-Situ Balancing: The Jeffcott Analogy

In the discussion of Figure 6-12 it was noted that at high values of $\omega/\omega_n$, circular rotor orbits appeared. Figure 6-13 gave an estimate of the size of this circular orbit, indicating an eccentricity ratio span of approximately 0.4. In fact, direct measurement of the speed invariant whirl radius, $\delta$, showed that $2\delta = 0.4c$ mils. To underscore the importance of these observations, the following propositions are enumerated:

1. The lubrication film properties are speed dependent.

2. The circular whirl orbit radius produced by the rotor/film system, $\delta$, is speed independent.

3. Therefore, $\delta$ must be a characteristic of the rotor and not the film.

With this syllogism in hand, it is instructive to recall some facts pertaining to the classical dynamical system of the Jeffcott rotor. The Jeffcott rotor is simply a rotor mounted on a linearly elastic shaft midspan between two identical journal bearings. In the simplest non-autonomous case, the rotor carries a one-plane unbalance. Because
Figure 6-18: Stability boundary for hybrid operation computed using the low-order eigenvalue analysis of Larson and Richardson.

Figure 6-19: Comparison between the stability boundary for hybrid operation computed using the low-order eigenvalue analysis of Larson and Richardson [31] and experimental results from the macro-bearing test rig ($\delta = 0.20$).
the system is linear, a single critical frequency is encountered as the speed is increased. The amplitude of the resonance, $r_u$, is usually normalized by the static unbalance distance, $\delta$, and depends on the damping present in the system. Figure 6-21 shows the resonant response of the Jeffcott system for different levels of damping. The lowest response amplitudes correspond to the most heavily damped systems.

It is important to notice that as $\omega$ is increased above the lone critical speed of the Jeffcott system, the unbalance response approaches a normalized value of 1.0 in an asymptotic fashion. In other words, the whirl amplitude for supercritical speeds will tend to the magnitude of the rotor static unbalance as $\omega$ increases. This "Jeffcott analogy" suggests that the speed invariant whirl circles observed during hybrid whirl operation of the macro-bearing test rig reveal the static unbalance of the rotor. If true, the whirl circles would allow one to use the gas lubrication film to transduce the unbalance in the implementation of an in-situ, trial-weight balancing scheme.
The ability to implement a balancing scheme is of great value to the experimental efforts of the current study for two reasons. First of all, the design of the macro-rig rotor is not conducive to balancing in the conventional, rigid-mount balancing machines commonly used for industrial applications. This is because the oblate rotor lacks a shaft that can be inserted into a chuck on a balancing machine. Furthermore, since the rotor bearing surfaces are protected with a ceramic coating for friction tolerance, a balanced, custom-made chuck would have to be used to secure the rotor without chipping the brittle chrome oxide layers on the rotor. The cost of such a system makes standard balancing machines an unattractive option.

Secondly, during the macro-rig testing program it was discovered that high $\epsilon$ operation was impossible with the macro-rig rotor as manufactured. Any attempt to load the rotor to high eccentricity ratios resulted in unlubricated rubbing of the bearing surfaces. It was hypothesized that the level of residual unbalance from the manufacturing process of the rotor was too great to achieve operation at high $\epsilon$. In effect, the dynamic forcing of the unbalance was too much for the film to support during operation close to the journal liner. This virtually ruled out hydrodynamic operation and the possible exploitation of the high-speed hydrodynamic stability corridor described earlier. A technique to balance the rotor would alleviate this problem.

The Jeffcott analogy was tested using a trial-weight balancing scheme [20]. This technique is standard in industry; however, the in-situ, macro-rig implementation which exploits the inertial properties of the gas lubrication film is unique. Trial masses are “added” by removing material from the rotor $180^\circ$ from the desired location with a ball end-mill to minimize stress concentrations under centrifugal loading. After each trial cut is made, the rotor is re-installed in the rig and the resulting whirl inversion radius is measured in-situ with the radial capacitive displacement probes. With successive cuts, the system of algebraic equations for the magnitude and location of the original unbalance is closed. With the unbalance vector determined, a final cancellation cut is made to eliminate the original manufacturing unbalance along with the cumulative effect of the trial cuts. The entire procedure is described in detail in the appendices.
Figure 6-21: Amplitude and phase relations for classical Jeffcott rotor with unbalance.

Figure 6-22: Unbalance whirl orbits for hydrostatic operation at various levels of unbalance during trial weight balancing operation.
Figure 6-22 shows the whirl inversion orbits corresponding to successive trial cuts. Originally, the static unbalance distance was 0.8 mils, or 20% of the nominal bearing clearance at zero speed. The figure shows that the first trial cut, which must be arbitrarily chosen, exacerbated the original unbalance by increasing the total level to 30%. The next cut reduced the overall level to 19%. A preliminary cancellation cut dropped the aggregate unbalance to 4% while a refinement cut left a 2% residual. After hydrodynamic operation was established with the 2% unbalance, a final refinement cut (not included in Figure 6-22) reduced the unbalance to just 0.7%.

With the successful in-situ balancing of the macro-rig rotor, hydrodynamic operation of the test rig was made possible and the additional cost of custom-made fittings for standard balancing machines was avoided.

6.5 Conclusions

In light of the extensive literature on computation of the axial through-flow mechanism detailed in this chapter, it may seem somewhat surprising that there have been no accounts of experiments exploiting this mechanism. The likely reason for this lies in the unusual nature of the Micro-Engine class of bearings that are the subject of this investigation. For the higher L/D values used in conventional bearings, inertial entrance length effects would compose only a small fraction of total pressure losses for axial flows. In other words, for an axial pressure difference imposed on bearings of normal length, the majority of the driving pressure will be spent in the Poiseuille losses in the lubrication region of the film. As explained earlier, this type of loss lacks a compensation mechanism capable of producing hydrostatic stiffness. Consequently, these inertial effects would be negligible in conventional gas journal bearings. Furthermore, most of the experiments published in the literature utilize test rig designs which expose the ends of the journal bearing to a common ambient pressure, obviating any source of axial pressure difference.

The primary observable in the macro-rig experiments associated with this hydrostatic mechanism is the existence of a natural frequency which is inconsistent with a
film governed by lubrication theory. This natural frequency, \( \omega_n \), was experimentally correlated with the axial pressure difference imposed across the rotor. Using the IBLT calculation scheme of Sparrow [60], the natural frequency due to inertia effects in the film entrance region was computed and compared to experimental results. Agreement was good at the higher pressure ranges explored in the experiments but the error was greater at lower pressure differences. The correlation for \( \omega_n \) versus different pressure combinations in the aft cover-plate side-load chambers was also measured in the macro-rig.

The axial hydrostatic mechanism was found to produce rotor-dynamic behavior consistent with a nonlinear interaction between the force due to inertial pressure losses from the through-flow and the much larger hydrodynamic forces experienced at high eccentricity ratio. Specifically, both the physical system of the macro-rig and the computational model showed the existence of subharmonic and superharmonic resonances. In addition, episodes of rotor inversion, reminiscent of the Jeffcott system, were observed in both experiment and computation.

The hybrid mode of operation was found to be robust for low speeds, making the scheme convenient for starting the bearing with low friction. Unlike the hydrodynamic mode previously described, the hybrid operation of the macro-rig did not require highly accurate eccentricity ratio measurements. This may be very useful for dealing with start-up problems in micro-devices where the observables are inherently low due to lack of instrumentation. Furthermore, it was found that much could be determined about the rotor-dynamics from the frequency content of the rotor whirl. It was shown that a simple, Cauchy product formula could account for the nonlinear interactions between the fundamental frequencies of the system. Again, this approach may be useful for trouble-shooting micro-systems where it may be impossible to get calibrated measurements but the whirl frequency may be detected.

Attempts to achieve high speeds (relative to those attained with hydrodynamic operation) with the hybrid mode were unsuccessful. However, the rotor inversion phenomena of the hybrid mode were incorporated into an in-situ, trial weight balancing scheme which reduced the manufactured static unbalance of the rotor from 20% of
the nominal bearing clearance to 0.7%. The capability to balance this unusual rotor design in-situ circumvented the need for custom-built fittings for conventional balancing machines and was sufficiently effective to allow high-ε, high-speed, hydrodynamic operation.
Chapter 7

Summary/Conclusions

7.1 Computation

In order to examine hydrodynamic effects in the class of bearing required by the MIT Micro-Engine and associated devices, this thesis presented a computational study of the rotor/lubrication film dynamical system. These gas journal bearings are a challenging theoretical problem because the low $L/D$ ratios dictated by fabrication constraints place the hydrodynamic stability corridor at very high eccentricity ratio. However, at high $e$, accurate resolution of the multiple-scale behavior exhibited by the gas lubrication film places a heavy burden on any potential computational scheme. Moreover, in order to be effective, any such computational study must possess not only accuracy but also efficiency so that the Micro-Engine design space can be thoroughly characterized. A survey of the existing literature shows that it is difficult to attain both efficiency and accuracy: schemes requiring few CPU cycles tend to implement simplifying assumptions that hinder accuracy while the most accurate schemes usually make the fewest simplifying assumptions, increasing the computational cost.

7.1.1 Accurate, Efficient Hydrodynamic Solutions: LubePack

With these requirements in mind, it was shown in Chapter 4 that although Cheng and Pan's Galerkin scheme [7] has the potential for a highly efficient representation of
hydrodynamic properties, formulation of the governing equations requires symbolic
evaluation of a complicated system of integrals. It was this system of integrals that
restricted Cheng and Pan to a low-order solution, inadequate for describing the Micro-
Engine bearings. It was also shown that extending the order of the Galerkin scheme
using the standard symbolic integration routines of commercial symbolic manipula-
tion packages demands memory at exponential rates, making high-order solution with
such routines prohibitively expensive. In response to this issue, this thesis presented
LubePack, a set of highly optimized transformation rules for symbolic evaluation of
the integrals encountered in the Galerkin formulation. It was then demonstrated that
the efficiency attained with LubePack enables high-order solution of the fully-coupled,
nonlinear dynamical system comprised by the rotor and the compressible, unsteady
gas lubrication film using only a modest PC.

The system of equations produced by LubePack was used to construct a closed-
form analytical formula for the Jacobian of the dynamical system, allowing efficient
formulation of the hydrodynamic stability problem. This provides accuracy by obviat-
ing the need for numerical differentiation during evaluation of the Jacobian. Because
the symbolic differentiation embodied in the LubePack Jacobian formula is executed
once and for all, an economy of computation is achieved. This efficiency is signifi-
cant, especially when one considers the cost of locating stability boundaries through
direct simulation of the fully-coupled dynamical system. The LubePack solutions
were shown to be in excellent agreement with the most accurate results in the litera-
ture [48] for both steady properties as well as hydrodynamic whirl stability solutions.
To demonstrate the efficiency of the symbolic Jacobian formula, comprehensive de-
sign charts were constructed for both steady and whirl properties of the Micro-Engine
class of gas journal bearing.

7.1.2 Generalized Stability Analysis: The Radial Instability

The complete set of eigenvalues obtained from the Jacobian formula allows a gener-
alized stability analysis of the dynamical system. This type of analysis was used to
detect a new type of hydrodynamic instability related to the radial mechanical mode
of the rotor. While the classical whirl stability establishes a minimum stable eccentricity ratio, the radial instability results in a new hydrodynamic stability criterion which establishes a maximum stable $\epsilon$. Thus, the radial instability narrows the stability corridor already established by the well-known whirl boundary. At low bearing number, the radial instability is located at extremely high eccentricity ratio where film collapse is likely to be the primary failure mode. However, at higher bearing numbers, the radial stability boundary drops substantially, reducing the hydrodynamic stability margin. Consequently, although high-$\epsilon$ operation is associated with heavy damping of the whirl mode, one must be mindful that it is precisely in this region where radial disturbances, such as those due to rotor unbalance, are amplified. Finally, it is notable that the radial instability boundary predicted by LubePack calculations has been confirmed in the pseudo-spectral simulations of Piekos [49].

7.2 Experiment

7.2.1 High $\epsilon$, High Speed, Hydrodynamic Operation

The macro-bearing test rig was used to demonstrate stable, hydrodynamic operation of the Micro-Engine class of gas journal bearing. With $L/D = 0.075$, these experiments represent the first tests of a gas journal bearing design with such an extremely low aspect ratio. The high eccentricity ratios demanded for stable hydrodynamic operation of such short bearings were achieved during macro-rig experiments. The static, high-$\epsilon$ properties of the macro-bearing rig were found to be in qualitative agreement with hydrodynamic theory. Discrepancies with theory are likely due to non-ideal effects such as liner non-circularity and lubrication film disruptions from instrumentation ports. For low bearing numbers where repeated bearing failures could be tolerated, the hydrodynamic whirl stability boundary of the macro-rig was measured. Agreement between the observed whirl boundary and LubePack calculations was within 4%. High speed operation of 46 kRPM was attained with the macro-rig resulting in a bearing number of approximately 1.8. This $\Lambda$-value is roughly mid-
way in the operational range of the devices associated with the MIT Micro-Engine Project. Finally, the limiting factor in the high-speed runs was found to be the thrust equilibrium of the rotor rather than instability of the journal bearing. Consequently, more effective thrust bearings should enable even higher speeds.

7.2.2 Hybrid Operation: Axial Through-Flow Effects

The low $L/D$ ratio characteristic of the Micro-Engine class of bearing was shown to enable a novel hybrid mode of operation in the form of axial through-flow hydrostatics. The macro-rig experiments demonstrated that by setting an axial pressure difference across the rotor, a hydrostatic support stiffness was created due to inertial pressure losses associated with the induced axial flows. The natural frequency resulting from this hydrostatic mechanism was experimentally correlated to axial pressure difference. Reasonable agreement between the macro-rig experiments and calculations from an integral boundary layer model of the axial inertial effects was demonstrated. Although this type of flow-field has been previously calculated [9], this thesis represents the first experimental observation of a rotor supported by this mechanism. The pressure-natural frequency surface for non-uniform macro-rig aft sump pressures was also mapped.

7.2.3 Hybrid Operation: Nonlinear Interactions

During hybrid operation, the macro-rig exhibited superharmonic and subharmonic resonances consistent with the nonlinear force characteristic of the combined hydrostatic/hydrodynamic model developed by Larson and Richardson [31]. Due to the low $L/D$ ratio, the axial through-flow hydrostatics dominate at low eccentricity ratios while hydrodynamic forces are the most important at high-$\epsilon$. The rotor orbits characteristic of the various nonlinear resonances were shown to be consistent with a simple, nonlinear oscillator combining the LubePack and IBLT results for the hybrid system. This model utilizes the analysis of Larson and Richardson [31] and is similar to that developed by Ehrich [19] and Savoulides [58]. The bearing failures that
occurred during hybrid operation were shown to be consistent with the low-order theory of Larson and Richardson [31] using bearing stiffnesses obtained by IBLT and LubePack calculations.

7.2.4 Hybrid Operation: In-Situ Balancing

Between the higher order nonlinear resonances, the macro-rig was shown to exhibit circular, “rotor inversion” orbits that reveal the level of static unbalance in the rotor. This phenomenon was exploited with an in-situ, trial weight balancing scheme that was used to reduce the original static unbalance from 20% to 0.7%. This reduction in unbalance was crucial to achieving the high eccentricity ratios necessary for hydrodynamic operation.

7.2.5 Trouble-Shooting with Minimal Information

Experimental rotor-dynamics observations were characterized in terms of spectra obtained from radial displacement probes. It was shown that the complex pattern of sum and difference frequencies studied by Ehrich [17] and observed in the macro-rig experiments could be easily expressed as a simple, Cauchy product formula derived from two fundamental frequencies: the rotor synchronous speed and the hydrostatic natural frequency. This technique facilitates identification of spectral features in experimental data and allows one to trouble-shoot rotor crashes with a limited amount of information. Thus, it is shown that substantial information can be extracted from the frequency content of the rotor motion. Such an approach is useful for diagnosing problems in MEMS devices where the number of observables is inherently low.

7.3 Conclusions

Two modes of operation have been demonstrated for the class of bearing studied in this thesis: hydrodynamic and hybrid. The hydrodynamic mode requires precision, high eccentricity ratio operation which is problematic on a micro-device where rotor
position information may not be available. However, it is notable that the highest
speeds attained on the macro-rig utilized this strategy. The hybrid mode allows
operation with a grossly unbalanced rotor and requires only the specification of the
axial pressure difference across the rotor. Thus, the hybrid mode is easily implemented
in a MEMS system since the required pressures can be set external to the device. The
rotor-dynamic spectral signatures of these two operational strategies are strikingly
different, so if one can obtain the frequency content of the rotor motion, it should be
possible to diagnose which mode is occurring in a given micro-device experiment.

The experimental results and computations presented in this thesis are applicable
to devices that require high-speed rotating machinery but are constrained to low
aspect ratio bearings. For example, other MEMS devices that are fabricated with
DRIE techniques are good candidates for this class of bearing. In addition, macro-
scale machines that have low load requirements and limited space for bearings could
also benefit from this technology. Finally, it should be noted that LubePack is a
general scheme which can be used to analyze both short as well as long bearings and
is therefore useful for journal bearing design in a wide range of machines.

The defining aspect of these bearings is the extremely low $L/D$ ratio of 0.075 which
is imposed by the DRIE fabrication techniques used for Micro-Engine technology. To
put this into perspective, the $L/D$ ratio of a nickel is approximately 0.074. Clearly,
the gas journal bearings studied in this thesis are as challenging and unique as the
MEMS technologies that depend upon them for viability. By demonstrating the axial
through-flow hybrid mode, this thesis has shown that the ground rules for operation of
these bearings are radically different than those of conventional gas bearings. Thus,
the experiments described in this document are not merely an extension of earlier
research, but rather a foundation for the implementation of high speed gas bearings
in MEMS devices.
Appendix A

Experimental Data: Static Thrust Bearing Test Results
Figure A-1: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 5$ psig).

Figure A-2: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 10$ psig).
Figure A-3: Load-Displacement characteristic for macro-bearing rig thrust bearings \((P_{ATB} = 15 \text{ psig})\).

Figure A-4: Load-Displacement characteristic for macro-bearing rig thrust bearings \((P_{ATB} = 20 \text{ psig})\).
Figure A-5: Load-Displacement characteristic for macro-bearing rig thrust bearings
\( P_{ATB} = 25 \text{ psig} \).

Figure A-6: Load-Displacement characteristic for macro-bearing rig thrust bearings
\( P_{ATB} = 30 \text{ psig} \).
Figure A-7: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 35 \text{ psig}$).

Figure A-8: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 40 \text{ psig}$).
Figure A-9: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 45$ psig).

Figure A-10: Load-Displacement characteristic for macro-bearing rig thrust bearings ($P_{ATB} = 50$ psig).
Appendix B

Derivation of LubePack Symbolic Integration Schemes
B.1 Fundamental Recursion Formulae

The process of constructing a general formula for the Galerkin film integrals begins with the following fundamental recursion formulae:

\[ I_{0}^{i,j}|_{a} = \int_{a}^{b} \sin^{i} \theta \cos^{j} \theta \, d\theta = -\frac{\sin^{i-1} \theta \cos^{j+1} \theta}{i+j} \bigg|_{a}^{b} + \frac{i-1}{i+j} \int_{a}^{b} \sin^{i-2} \theta \cos^{j} \theta \, d\theta \quad (B.1) \]

\[ I_{0}^{i,j}|_{a} = \int_{a}^{b} \sin^{i} \theta \cos^{j} \theta \, d\theta = \frac{\sin^{i+1} \theta \cos^{j-1} \theta}{i+j} \bigg|_{a}^{b} + \frac{j-1}{i+j} \int_{a}^{b} \sin^{i} \theta \cos^{j-2} \theta \, d\theta \quad (B.2) \]

\[ I_{0}^{0,j}|_{a} = \int_{a}^{b} \cos^{j} \theta \, d\theta = \frac{\sin \theta \cos^{j-1} \theta}{j} \bigg|_{a}^{b} + \frac{j-1}{j} \int_{a}^{b} \cos^{j-2} \theta \, d\theta \quad (B.3) \]

\[ I_{0}^{i,0}|_{a} = \int_{a}^{b} \sin^{i} \theta \, d\theta = -\frac{\sin^{i-1} \theta \cos \theta}{i} \bigg|_{a}^{b} + \frac{i-1}{i} \int_{a}^{b} \sin^{i-2} \theta \, d\theta \quad (B.4) \]

B.2 Case 1: \( i \) even \( \Rightarrow \quad i = 2m \)

Begin by recursively applying Equation B.1 to the journal bearing integral given in Equation 4.18. The first four recursions yield:

\[ I_{0}^{i,j}|_{a} = I_{0}^{2m,j}|_{a} = -\frac{\sin^{2m-1} \theta \cos^{j+1} \theta}{2m+j} \bigg|_{a}^{b} + \frac{2m-1}{2m+j} I_{0}^{2m-2,j}|_{a}^{b} \]

\[ I_{0}^{2m-2,j}|_{a} = -\frac{\sin^{2m-3} \theta \cos^{j+1} \theta}{2m-2+j} \bigg|_{a}^{b} + \frac{2m-3}{2m-2+j} I_{0}^{2m-4,j}|_{a}^{b} \]

\[ I_{0}^{2m-4,j}|_{a} = -\frac{\sin^{2m-5} \theta \cos^{j+1} \theta}{2m-4+j} \bigg|_{a}^{b} + \frac{2m-5}{2m-4+j} I_{0}^{2m-6,j}|_{a}^{b} \]

\[ I_{0}^{2m-6,j}|_{a} = -\frac{\sin^{2m-7} \theta \cos^{j+1} \theta}{2m-6+j} \bigg|_{a}^{b} + \frac{2m-7}{2m-6+j} I_{0}^{2m-8,j}|_{a}^{b} \]

The process terminates after \( m \) recursions to yield:
\[ I_0^{2J} |_a^b = - \frac{\sin \theta \cos^{j+1} \theta |_a^b}{2 + j} + 2 - \frac{1}{2 + j} I_0^{0j} |_a^b \]

Folding the first four recursions into one expression, and multiplying out gives:

\[
I_0^{2m-j} |_a^b = - \frac{\sin^{2m-1} \theta \cos^{j+1} \theta |_a^b}{2m + j} - \frac{\sin^{2m-3} \theta \cos^{j+1} \theta |_a^b}{2m - 2 + j} - \frac{\sin^{2m-5} \theta \cos^{j+1} \theta |_a^b}{2m - 4 + j} - \frac{\sin^{2m-7} \theta \cos^{j+1} \theta |_a^b}{2m - 6 + j} + \frac{2m - 1}{2m + j} \cdot \frac{2m - 2 + j}{2m - 2 + j} \cdot \frac{2m - 4 + j}{2m - 4 + j} \cdot \frac{2m - 6 + j}{2m - 6 + j} \]

Note that Equation B.5 is composed of a sum of terms representing evaluated journal bearing integrals and exactly one term containing an un-evaluated journal bearing integral, namely, the last term. The alert reader will notice that this final term in Equation B.5 will produce an Euler Beta function expression when recursions are exhausted. That is, when the recursive process is continued until the term containing the un-evaluated journal bearing integral, denoted by \( T_0^{2m-j} |_a^b \), is discharged, the result will be in terms of the Euler Beta function. For example, in Equation B.5, we have:

\[
T_0^{2m-8J} |_a^b = T_0^{2m-8J} |_a^b \cdot \frac{2m - 1}{2m + j} \cdot \frac{2m - 3}{2m - 2 + j} \cdot \frac{2m - 5}{2m - 4 + j} \cdot \frac{2m - 7}{2m - 6 + j} \]

Carrying this process out to completion yields:

\[
T_0^{0j} |_a^b = \left( \frac{2m - 1}{2m + j} \right) \cdot \left( \frac{2m - 3}{2m - 2 + j} \right) \cdots \left( \frac{2m - (2m - 1)}{2m - 2(m - 1) + j} \right) \cdot I_0^{0j} |_a^b
\]

\[
= (2m - 1)!! \cdot \frac{1}{2m + j} \cdot \frac{1}{2m - 2 + j} \cdots \frac{1}{2m - 2(m - 1) + j} \cdot I_0^{0j} |_a^b
\]

Furthermore, we specialize to Case 1A: \( j \) even \( \Rightarrow j = 2n \) For this sub-case of \( i \)
even, \( j \) even, we may write:

\[
I_0^{0,j\mid \lambda} = \frac{2n - 1}{2n} \cdot I_0^{0,2n\mid \lambda} - 2n \cdot I_0^{0,2n - 2\mid \lambda} \]

\[
= \left( \frac{2n - 1}{2n} \right) \cdot \left( \frac{2n - 3}{2n - 2} \right) \cdot I_0^{0,2n - 4\mid \lambda} \]

\[
= \left( \frac{2n - 1}{2n} \right) \cdot \left( \frac{2n - 3}{2n - 2} \right) \cdot \left( \frac{2n - 5}{2n - 4} \right) \cdot I_0^{0,2n - 6\mid \lambda} \]

\[
= \left( \frac{2n - 1}{2n} \right) \cdot \left( \frac{2n - 3}{2n - 2} \right) \cdots \left( \frac{2n - (2n - 1)}{2n - 2(n - 1)} \right) \cdot I_0^{0,0\mid \lambda} \]

\[
= (2n - 1)!! \cdot \frac{1}{2n} \cdot \left( \frac{1}{2n - 2} \right) \cdots \left( \frac{1}{2n - 2(n - 1)} \right) \cdot (b - a) \]

\[
= (2n - 1)!! \cdot \frac{1}{n} \cdot \frac{1}{n - 1} \cdots \frac{1}{1} \cdot (b - a) \]

\[
= \frac{(2n - 1)!!}{2^n} \cdot \frac{(b - a)}{n!} \quad (B.8)
\]

Substituting this result into Equation B.7 yields:

\[
T_0^{0,2n\mid \lambda} = (2m - 1)!! \cdot \frac{1}{(2m + 2n)} \cdot \frac{1}{(2m - 2 + 2n)} \cdots \frac{1}{(2m - 2(m - 1) + 2n)} \cdot \frac{(2n - 1)!!}{2^n} \cdot \frac{(b - a)}{n!} \]

\[
= \frac{(2m - 1)!!}{2^m} \cdot \frac{1}{(m + n)} \cdot \frac{1}{(m - 1 + n)} \cdots \frac{1}{(m - (m - 1) + n)} \cdot \frac{(2n - 1)!!}{2^n} \cdot \frac{(b - a)}{n!} \quad (B.9)
\]

Recall the identity:

\[
\Gamma(x + r) = (x + r - 1) \cdot (x + r - 2) \cdots (x + 1) \cdot \Gamma(x + 1) \quad (B.10)
\]

With \( r \to m + 1 \) and \( x \to n \) we have:

\[
\Gamma(n + m + 1) = (n + m) \cdot (n + m - 1) \cdots (n + 1) \cdot \Gamma(n + 1) \quad (B.11)
\]

This yields:

\[
\frac{\Gamma(n + 1)}{\Gamma(n + m + 1)} = \frac{1}{(m + n)} \cdot \frac{1}{(m - 1 + n)} \cdots \frac{1}{(m - (m - 1) + n)} \quad (B.12)
\]
Substituting this result into Equation B.9 produces:

\[
T_{a}^{0,2n} = \frac{(2m - 1)!!}{2^{m} \cdot \Gamma(n + m + 1)} \cdot \frac{(2n - 1)!!}{2^{n} \cdot n!} \cdot \frac{(b - a)}{\pi \Gamma(n + m + 1)}
\]

Recall the definition of the Euler Beta Function, \( B(x, y) \):

\[
B(x, y) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad x > 0, \ y > 0 \quad (B.14)
\]

Inserting Equation B.14 into Equation B.13 with \( x \to m + \frac{1}{2}, \ y \to n + \frac{1}{2} \), and \( x + y \to m + n + 1 \) gives:

\[
T_{a}^{0,2n} = \frac{1}{\pi} B\left( m + \frac{1}{2}, n + \frac{1}{2} \right) \quad (B.15)
\]

Thus, our conjecture regarding the last term in Equation B.5 is confirmed. Equation B.15 is the basis for the last term in Equation 4.19 as well as the first lines of Equations 4.20 and 4.21.

Now, turning attention to the terms in Equation B.5 that contain evaluated journal bearing integrals, we re-write:

\[
(2m - 1) \cdot (2m - 3) \cdot (2m - 5) = \frac{(2m - 1) \cdot (2m - 3) \cdot (2m - 5) \cdot \ldots \cdot (2m - (2m - 1))}{(2m - 7) \cdot (2m - 9) \cdot \ldots \cdot (2m - (2m - 1))}
\]

Also, we have:

\[
(2m - 7) \cdot (2m - 9) \cdot \ldots \cdot (2m - (2m - 1)) = [2(m - 3) - 1] \cdot [2(m - 3) - 3] \cdot \ldots \cdot [1]
\]

Combining these last two expressions, we have:

\[
(2m - 1) \cdot (2m - 3) \cdot (2m - 5) = \frac{(2m - 1)!!}{[2(m - 3) - 1]!!} = 2^{3} \cdot \frac{\Gamma\left( m + \frac{1}{2} \right)}{\pi \Gamma\left( m - 3 + \frac{1}{2} \right)} \quad (B.16)
\]
For the case of $j$ even, $j = 2n$, consider:

$$(2m + j) \cdot (2m - 2 + j) \cdot (2m - 4 + j) \cdot (2m - 6 + j)$$

$$= (2m + 2n) \cdot (2m - 2 + 2n) \cdot (2m - 4 + 2n) \cdot (2m - 6 + 2n)$$

$$= 2^4 \cdot (m + n) \cdot (m + n - 1) \cdot (m + n - 2) \cdot (m + n - 3) \quad (B.17)$$

Recall:

$$\frac{\Gamma(n + m + 1)}{\Gamma(n + 1)} = (m + n) \cdot (m - 1 + n) \ldots \cdot (m - (m - 1) + n) \quad (B.18)$$

This implies that:

$$(m + n) \cdot (m + n - 1) \cdot (m + n - 2) \cdot (m + n - 3) = \frac{\Gamma(n + m + 1)}{\Gamma(n + (m - 4) + 1)} \quad (B.19)$$

Combining Equations B.16 and B.19 yields:

$$\frac{(2m - 1) \cdot (2m - 3) \cdot (2m - 5)}{(2m + 2n) \cdot (2m - 2 + 2n) \cdot (2m - 4 + 2n) \cdot (2m - 6 + 2n)} = \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma(n + (m - 4) + 1)}{2 \cdot \Gamma \left( m - 3 + \frac{1}{2} \right) \cdot \Gamma(n + m + 1)} \quad (B.20)$$

Repeating this technique, we have:

$$\frac{(2m - 1) \cdot (2m - 3)}{(2m + 2n) \cdot (2m - 2 + 2n) \cdot (2m - 4 + 2n)} = \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma(n + (m - 3) + 1)}{2 \cdot \Gamma \left( m - 2 + \frac{1}{2} \right) \cdot \Gamma(n + m + 1)} \quad (B.21)$$

$$\frac{2m - 1}{2m + 2n} \cdot \frac{1}{2m - 2 + 2n} = \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma(n + (m - 2) + 1)}{2 \cdot \Gamma \left( m - 1 + \frac{1}{2} \right) \cdot \Gamma(n + m + 1)} \quad (B.22)$$

$$\frac{1}{2m + 2n} = \frac{\Gamma(n + (m - 1) + 1)}{2^1 \cdot \Gamma(n + m + 1)} \quad (B.23)$$

Inserting Equations B.20, B.21, B.22, and B.23 into equation B.5 gives:

$$f_{0}^{2m, 2n} = \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma(n + (m - 1) + 1)}{2^1 \cdot \Gamma \left( m - 0 + \frac{1}{2} \right) \cdot \Gamma(n + m + 1)} \cdot \sin^{2m-1} \theta \cos^{2n+1} \theta_{a}^{b}$$
Now, considering Equation B.24 and the process that led to Equation B.15, the formula for the general expression is obvious:

\[
I_{0}^{2m,2n} = - \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma \left( n + (m - 2) + 1 \right)}{2^{1} \cdot \Gamma \left( m - 1 + \frac{1}{2} \right) \cdot \Gamma \left( n + m + 1 \right)} \cdot \sin^{2m-3} \theta \cos^{2n+1} \theta |_{a}^{b}
\]

\[
- \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma \left( n + (m - 3) + 1 \right)}{2^{1} \cdot \Gamma \left( m - 2 + \frac{1}{2} \right) \cdot \Gamma \left( n + m + 1 \right)} \cdot \sin^{2m-5} \theta \cos^{2n+1} \theta |_{a}^{b}
\]

\[
- \frac{\Gamma \left( m + \frac{1}{2} \right) \cdot \Gamma \left( n + (m - 4) + 1 \right)}{2^{1} \cdot \Gamma \left( m - 3 + \frac{1}{2} \right) \cdot \Gamma \left( n + m + 1 \right)} \cdot \sin^{2m-7} \theta \cos^{2n+1} \theta |_{a}^{b}
\]

\[
+ I_{0}^{2m-8,2n} |_{a}^{b} \cdot \frac{2m - 1}{2m + 2n} \cdot \frac{2m - 3}{2m - 2 + 2n} \cdot \frac{2m - 5}{2m - 4 + 2n} \cdot \frac{2m - 7}{2m - 6 + 2n}
\]

\[
\text{(B.24)}
\]

This is precisely Equation 4.19. Similar expressions may be found for the remaining cases of \( i \) even and \( j \) odd, \( i \) odd and \( j \) even, as well as \( i \) odd and \( j \) odd. Note that depending on the limits of integration, the integrals involving odd powers of trigonometric functions may not be trivial as was the case in Equations 4.20 and 4.21.

With this approach, the LubePack transformation rules for symbolic integration of the lubrication film are as follows:

\[
c1 = \{ \sin[x_-]^{i-} \cos[x_-]^{j-} :> \frac{1}{2\pi} \ast 2 \ast \text{Beta}[\frac{i + 1}{2}, \frac{j + 1}{2}] ; \}
\]

\[
\text{(EvenQ}[i] \land \text{EvenQ}[j]), \sin[x_-]^{i-} \cos[x_-]^{j-} :> 0 ; \}
\]

\[
- \text{(EvenQ}[i] \land \text{EvenQ}[j]), \sin[x_-]^{i-} \cos[x_-]^{j-} :> 0 ;
\]

\[
\text{Sin}[x_-]^{i-} \cos[x_-] :> 0 , \text{Sin}[x_-] \cos[x_-] :> 0 ;
\]

\[
c2 = \{ \cos[x_-]^{j-} :> \frac{1}{2\pi} \ast 2 \ast \text{Beta}[\frac{1}{2}, \frac{j + 1}{2}] ; \}
\]

\[
\text{EvenQ}[j], \cos[x_-]^{j-} :> 0 ;
\]

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\begin{align*}
-\text{EvenQ}[j], \sin[x_-]^i & :> \frac{1}{2\pi} \ast 2 \ast \beta[i+1, \frac{1}{2}]; \\
\text{EvenQ}[i], \sin[x_-]^i & :> 0/; \\
-\text{EvenQ}[i]}; \\
c3 &= \{\cos[x_-] \rightarrow 0, \sin[x_-] \rightarrow 0\}; \\
(B.26)
\end{align*}

\begin{align*}
a1 &= \{\sin[x_-]^i \cos[x_-]^j :> \frac{1}{\pi} \ast \beta[i+1, j+1]}; \\
\text{EvenQ}[i] \land \text{EvenQ}[j], \sin[x_-]^i \cos[x_-]^j & :> \frac{1}{\pi} \ast \beta[i+1, \frac{j-1}{2} + 1]; \\
\text{EvenQ}[i] \land \text{OddQ}[j], \sin[x_-]^i \cos[x_-]^j & :> 0/; \\
\neg((\text{EvenQ}[i] \land \text{EvenQ}[j]) \lor (\text{EvenQ}[i] \land \text{OddQ}[j])); \\
\sin[x_-] \cos[x_-]^i & :> 0, \sin[x_-]^i \cos[x_-] :> 0/; \\
\text{OddQ}[i], \sin[x_-]^i \cos[x_-] & :> \frac{1}{\pi} \ast \beta[i+1, 1]; \\
\text{EvenQ}[i], \sin[x_-] \cos[x_-] & :> 0}; \\
a2 &= \{\cos[x_-]^i :> \frac{1}{\pi} \ast \beta[j+1, \frac{1}{2}]}; \\
\text{EvenQ}[j], \cos[x_-]^i & :> \frac{1}{\pi} \ast \beta[j+1, \frac{j-1}{2}]; \\
\text{OddQ}[j], \sin[x_-]^i & :> \frac{1}{\pi} \ast \beta[i+1, 
\text{EvenQ}[i], \sin[x_-]^i & :> 0/; \\
\text{OddQ}[i]}; \\
a3 &= \{\cos[x_-] \rightarrow \frac{2}{\pi}, \sin[x_-] \rightarrow 0\}; \\
(B.27)
\end{align*}

Using similar techniques, the recursive system of partial fraction identities documented by Booker[5] may be implemented in the Mathematica language to facilitate symbolic integration of the rotor equations of motion via the following transformation rules:

\begin{align*}
s\text{rep} &= \{\sin[\theta]^i :> (1 - \cos[\theta]^2)^{\frac{i}{2}}/; \\
\text{EvenQ}[i], \sin[\theta]^i & :> (1 - \cos[\theta]^2)^{\frac{i}{2}} \sin[\theta]/; \text{OddQ}[i]}; \\
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\end{align*}
\[ \text{irep1} = \begin{cases} \frac{\sin[\theta]^i - \cos[\theta]^j}{1 + \epsilon \cos[\theta]} & \rightarrow I_{i,j,1}, \\ \frac{\sin[\theta]^i - \cos[\theta]^j}{1 + \epsilon \cos[\theta]} & \rightarrow I_{i,1,1}, \\ \frac{\sin[\theta] \cos[\theta]^j}{1 + \epsilon \cos[\theta]} & \rightarrow I_{1,j,1} \end{cases}; \]

\[ \text{irep2} = \begin{cases} \frac{\sin[\theta]^i}{1 + \epsilon \cos[\theta]} & \rightarrow I_{i,0,1}, \\ \frac{\cos[\theta]^j}{1 + \epsilon \cos[\theta]} & \rightarrow I_{0,j,1} \end{cases}; \]

\[ \text{irep3} = \begin{cases} \frac{\sin[\theta]}{1 + \epsilon \cos[\theta]} & \rightarrow I_{1,0,1}, \\ \frac{\cos[\theta]}{1 + \epsilon \cos[\theta]} & \rightarrow I_{0,1,1} \end{cases}; \]

\[ \text{irep4} = \{ \frac{1}{1 + \epsilon \cos[\theta]} \rightarrow I_{0,0,1} \}; \]

\[ \text{irep5} = \{ \sin[\theta]^i - \cos[\theta]^j \rightarrow I_{i,j,0}, \sin[\theta]^i - \cos[\theta]^j \rightarrow I_{i,1,0}, \sin[\theta] \cos[\theta]^j \rightarrow I_{1,j,0} \}; \]

\[ \text{irep6} = \{ \sin[\theta] \cos[\theta] \rightarrow I_{1,1,0} \}; \]

\[ \text{irep7} = \{ \sin[\theta]^i \rightarrow I_{i,0,0}, \cos[\theta]^j \rightarrow I_{0,j,0} \}; \]

\[ \text{irep8} = \{ \sin[\theta] \rightarrow I_{1,0,0}, \cos[\theta] \rightarrow I_{0,1,0} \}; \] (B.28)

\[ c1eom = \{ I_{-j-0} := \frac{1}{2\pi} * 2 * \text{Beta}[\frac{i+1}{2}, \frac{j+1}{2}] /; \\ (\text{EvenQ}[i] \& \text{EvenQ}[j]), I_{-j-0} := 0 \}; \]

\[ \text{EvenQ}[j], I_{0,j-0} := 0 \}; \]

\[ c2eom = \{ I_{0-j-0} := \frac{1}{2\pi} * 2 * \text{Beta}[\frac{1}{2}, \frac{j+1}{2}] /; \\ \text{EvenQ}[j], I_{0,j-0} := 0 \}; \]

\[ \text{EvenQ}[j], I_{-j-0} := \frac{1}{2\pi} * 2 * \text{Beta}[\frac{i+1}{2}, \frac{1}{2}] /; \]

\[ \text{EvenQ}[i], I_{-j,0} := 0 \}; \]

\[ c3eom = \{ I_{0,1,0} \rightarrow 0, I_{1,0,0} \rightarrow 0 \}; \]

\[ c4eom = \{ I_{1,0,1} \rightarrow 0, I_{0,0,1} \rightarrow \frac{1}{2\pi \sqrt{1 - \epsilon^2}} \}; \] (B.29)
Appendix C

LubePack Calculations: Plain Cylindrical Journal Bearing Static and Dynamic Design Charts for Micro-Engine Regime
Figure C-1: Load parameter vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of $0.05$, $\frac{L}{D} = 0.075$. 
Figure C.2. Attitude angle vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05.
Figure C-3: Hydrodynamic Stability Chart. \( \delta = 0.075 \).
Figure C-4: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.075$. 

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Figure C.6. Load parameter vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $\delta = 0.080$. 
Figure C-6: Attitude angle vs. bearing number: \( \epsilon = 0.05 \) to \( \epsilon = 0.95 \) in steps of 0.05.
Figure C.7: Hydrodynamic Stability Chart: $\beta = 0.080$. 

\[ c = 0.95, \quad \theta = 0.95, \quad E = 0.9, \quad I = 0.89, \quad E = 0.88, \quad \ldots \]
Figure C-8: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.080$. 
Figure C.9: Load parameter vs. bearing number. $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $\delta = 0.085$. 

![Graph showing load parameter vs. bearing number with various curves denoted by $\epsilon$ values.](image)
Figure C-10: Altitude angle vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05.
Figure C-11: Hydrodynamic Stability Chart: $\frac{b}{d} = 0.085$. 
Figure C-12: Hydrodynamic Whirl Frequency Chart: $\frac{l}{D} = 0.085$. 
0.06 = \frac{a}{y}

Figure C.3: Load parameter vs. bearing number: \( c = 0.05 \) to \( c = 0.95 \) in steps of 0.05.
Figure C-14: Altitude angle vs. bearing number; $e = 0.05$ to $e = 0.95$ in steps of 0.05.
Figure C.15: Hydrodynamic Stability Chart. $\delta = 0.009$. 

- $E = 0.935$
- $E = 0.93$
- $e = 0.925$
- $E = 0.92$
- $e = 0.915$
- $e = 0.91$
- $e = 0.905$
- $\varepsilon = 0.9$
- $e = 0.89$
- $\varepsilon = 0.88$
- $\varepsilon = 0.87$
- $\varepsilon = 0.86$
- $\varepsilon = 0.85$
- $\varepsilon = 0.84$
- $\varepsilon = 0.83$
- $\varepsilon = 0.82$
- $\varepsilon = 0.81$
- $\varepsilon = 0.8$
Figure C-16: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.090$. 

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Figure C-17. Load parameter vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05; $\delta = 0.005$. 
Figure C-18: Attitude angle vs. bearing number. $e = 0.05$ to $e = 0.95$ in steps of 0.05.

$$0.05 \leq e \leq 0.95$$
Figure C-19: Hydrodynamic Stability Chart: $\delta = 0.005$. 

\[ \Lambda \]
Figure C-20: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.095$. 
Figure C.22: Attitude angle vs. bearing number. $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $\delta = 0.100$. 
Figure C.23: Hydrodynamic Stability Chart. \( \varepsilon = 0.100 \).
Figure C-24: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.100$. 
Figure C-25: Load Parameter vs. Bearing number e. Load parameter \( \frac{q}{F} = 0.105 \) for \( e = 0.05 \) to \( e = 0.35 \) in steps of 0.05.
Figure C-26: Attitude angle vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $\beta = 0.105$. 
Figure C-27: Hydrodynamic Stability Chart. $\delta = 0.105$. 

\[ \varepsilon = 0.95, 0.945, 0.94, 0.935, 0.93, 0.925, 0.92, 0.915, 0.91, 0.905, 0.9, 0.89, 0.88, 0.87, 0.86, 0.85, 0.84, 0.83, 0.82, 0.81, 0.8 \]
Figure C-28: Hydrodynamic Whirl Frequency Chart: \( \frac{L}{D} = 0.105 \).
Figure C.29: Load parameter vs. bearing number: \( e = 0.05 \) to \( e = 0.95 \) in steps of 0.05. \( \frac{d}{e} = 0.110 \).
Figure C-3: Attitude angle vs. bearing number. $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $b = 0.110$. 
Figure C.31: Hydrodynamic Stability Chart: $\frac{g}{T} = 0.110$.
Figure C-32: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.110$. 
Figure C-33: Load parameter vs. bearing number \( \epsilon = 0.05 \) to \( \epsilon = 0.95 \) in steps of 0.05, \( e \neq 0.112 \).
Figure C.3: Attitude angle vs. bearing number. $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $b = 0.115$. 
Figure C-35: Hydrodynamic Stability Chart: \( \frac{\sigma}{\nu} = 0.115 \)
Figure C-36: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.115$. 
Figure C-37: Load parameter vs. bearing number: $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of $0.05$, $b = 0.120$. 
Figure C-38: Attitude angle vs. bearing number $e$. Data for $e = 0.05$ to $e = 0.95$ in steps of 0.05.

$$0.05, e = 0.120$$

$0.01, e = 0.60$

$0.1, e = 0.65$

$1.0, e = 0.70$

$1.0, e = 0.75$

$1.0, e = 0.80$

$1.0, e = 0.85$

$1.0, e = 0.90$
Figure C-39: Hydrodynamic Stability Chart: $\delta = 0.120$. 

\begin{align*}
\epsilon &= 0.945 \\
\epsilon &= 0.94 \\
\epsilon &= 0.935 \\
\epsilon &= 0.93 \\
\epsilon &= 0.925 \\
\epsilon &= 0.92 \\
\epsilon &= 0.915 \\
\epsilon &= 0.91 \\
\epsilon &= 0.905 \\
\epsilon &= 0.9 \\
\epsilon &= 0.89 \\
\epsilon &= 0.88 \\
\epsilon &= 0.87 \\
\epsilon &= 0.86 \\
\epsilon &= 0.85 \\
\epsilon &= 0.84 \\
\epsilon &= 0.83 \\
\epsilon &= 0.82 \\
\epsilon &= 0.81 \\
\epsilon &= 0.8 
\end{align*}

\begin{align*}
M &= 0.01 0.05 0.1 0.5 1 \\
\Lambda &= 0.01 0.05 0.1 0.5 1
\end{align*}
Figure C-40: Hydrodynamic Whirl Frequency Chart: $\frac{l}{b} = 0.120$. 
Figure C-41: Load parameter vs. bearing number. $\epsilon = 0.05$ to $\epsilon = 0.95$ in steps of 0.05, $\delta = 0.125$. 

\[ \zeta \] vs. $\Lambda$.
\[ \theta = \frac{\sigma}{\gamma} \]

Figure C.42: Attitude angle vs. bearing number. \( e = 0.05 \) to \( e = 0.95 \) in steps of 0.05.
Figure C-43: Hydrodynamic Stability Chart: $\delta = 0.12\gamma$. 

\[ M \]

\[ V \]

\[ \Lambda \]

\[ 0.1 \quad 0.5 \quad 1 \quad 5 \quad 10 \]

\[ 0.01 \quad 0.05 \quad 0.1 \quad 0.5 \quad 1 \]

\[ e = 0.8 \quad e = 0.83 \quad e = 0.84 \quad e = 0.86 \quad e = 0.88 \quad e = 0.9 \quad e = 0.91 \quad e = 0.92 \quad e = 0.93 \quad e = 0.94 \quad e = 0.95 \quad e = 0.96 \quad e = 0.97 \quad e = 0.98 \quad e = 0.99 \quad e = 1.0 \]
Figure C-44: Hydrodynamic Whirl Frequency Chart: $\frac{L}{D} = 0.125$. 
Appendix D

In-Situ Static Balancing Scheme for Rotor of Macro-Bearing Test Rig
D.1 Methodology

The algorithm discussed in this brief is an adaptation of the three-trial-weight method described by Gunter and Jackson and first published by Blake. The technique is usually used in situations where balancing schemes that necessitate accurate phase measurements of the unbalance are difficult or impossible to implement. For example, single-plane balancing by the influence coefficient method requires one to measure the phase and magnitude of the force transmitted to the bearing pedestal. For in-situ balancing methods, this may not be practical. On the other hand, if the rotor is being spun by an electronic rigid-mount balancing machine, such measurements are readily made and balancing can be achieved using a single run. For the case of the macro bearing test rig, the difficulties of constructing a balanced chuck to hold the rotor in an instrumented spin machine make this option expensive.

An alternative approach for single plane static balancing is to devise an in-situ balancing scheme based upon measurements from non-contact displacement probes and the known rotor-dynamic behavior of the macro rig during hydrostatic operation. Specifically, it has been observed that between higher order subharmonic responses of the rotor/gas bearing system, the geometric center of the rotor whirls in a circle whose diameter is independent of speed and consistent from run to run. That is, as rotor rotational speed is increased above the $n^{th}$ subharmonic response, the rotor orbit departs from the characteristic shape associated with the $n$th pseudo-critical and coalesces into a well-defined whirl circle. At this point, the rotor geometric center translates in a circle about the about the motionless rotor center of mass. The radius of the circle breaks up and assumes the characteristic shape of the $(n + 1)^{th}$ subharmonic response. This chain of events repeats for higher order subharmonic responses. By adding known unbalance masses to the rotor and observing the corresponding rotor-dynamic response as described above, one can deduce the position of the initial rotor unbalance and apply a final correction mass to achieve static balance. Figure D-1 shows the original unbalance mass, $m_u$, located at a radial position $\delta_u$ with respect to axes fixed to the rotor geometric center. The response vector associated with $m_u$
is denoted as $\delta_u$ and given by Equation D.1.

$$\delta_u = \frac{m_u}{M + m_u} \mathbf{r}_u = \frac{U}{M + m_u} \approx \frac{U}{M} \quad (D.1)$$

where $\mathbf{r}_u$ is the radial position of the unbalance mass, $M$ is the rotor mass, and $U$ is the unbalance vector. Equation D.1 shows the equivalence of physical coordinates and the unbalance coordinates chosen for the axes in Figure D-1. Of course, the position of the unbalance is unknown in the real case but is located by $\theta_u$ in Figure D-1 to facilitate sample calculations to follow.

For each trial mass added, $m_i$, the response is simply the sum of the unbalance vectors normalized by the rotor mass. The vector summation gives rise to a sin-
gle scalar equation for the resultant unbalance response in terms of the Cartesian components of the constituent unbalance vectors.

\[ \delta_i^2 = \left( \frac{m_u}{M} r_u \cos \theta_u + \frac{m_i}{M} x_i \right)^2 + \left( \frac{m_u}{M} r_u \sin \theta_u + \frac{m_i}{M} y_i \right)^2 \]  

(D.2)

where \( x_i = r_i \cos \theta_i \) and \( y_i = r_i \sin \theta_i \) are the Cartesian coordinates of the trial mass.

As a practical matter, “adding mass” to the macro rig rotor is accomplished by removing the same amount of mass 180° opposite the specified location. It should be noted that Equation D.2 has the geometric interpretation of a circle of radius \( \delta_i \) centered at \( (-\frac{m_u}{M} x_i, -\frac{m_u}{M} y_i) \). Using this geometric interpretation, it is seen that there is a phase ambiguity in Equation D.2. Since no phase measurements are being made in the macro rig experiments, it is impossible to tell if the observed magnitude of the unbalance response vector leads or lags the known trial mass. Both of these possibilities for the original unbalance phase will result in the observed response. The dashed circle in Figure D-2 is the circle for the original unbalance mass (shown in grey). The heavy circle is the unbalance circle for the added trial mass (shown in black) which is equal in magnitude to the experimentally observed original unbalance. The two thin solid circles represent the two cases for the possible phases of the original unbalance. The intersections of these circles with the circle of the trial mass each give the magnitude of the resultant unbalance vector, denoted by heavy black lines.

Equation D.2 may be generalized to the successive addition of multiple trial masses without removal of the previous trial masses as

\[ \delta_n^2 = \left( \frac{m_u}{M} r_u \cos \theta_u + \sum_{i=1}^{n} \frac{m_i}{M} x_i \right)^2 + \left( \frac{m_u}{M} r_u \sin \theta_u + \sum_{i=1}^{n} \frac{m_i}{M} y_i \right)^2 \]  

(D.3)

Due to the phase ambiguity described above, a graphical interpretation is desirable to guide the algebraic computations based upon Equation D.3. Figure D-3 shows the superposition of the unbalance circles for the original response (\( \delta_0 \)) from Figure D-1 as well as the responses (\( \delta_1, \delta_2 \)) corresponding to the successive addition of two identical trial masses (\( m_1, m_2 \)). The two intersections of the \( \delta_1 \) and \( \delta_2 \) circles represent two
possible locations of the original unbalance. The correct location corresponds to the \( \delta_1-\delta_2 \) intersection lying on the original \( \delta_0 \) response circle.

In the unlikely event that both \( \delta_1-\delta_2 \) intersections are near the \( \delta_0 \) response circle, a third trial mass, \( m_3 \), identical to the first two masses may be added as shown in Figure D-1 to determine the correct location of the original unbalance. This scenario is shown in Figure D-4 where it is seen that the \( \delta_3 \) circle selects the correct location of the original unbalance mass shown in Figure D-1. Once the position of \( m_u \) is determined, a final mass is added to cancel the resultant of all the added unbalances as well as \( m_u r_u \) itself.
D.2 Sample Calculations

To make the above methodology a bit more concrete, sample calculations for construction of the included figures are given below.

D.2.1 Step 0

Spin the rotor under hydrostatic operation between the $3^{rd}$ and $4^{th}$ order subharmonic responses to achieve a rotor inversion whirl circle of $0.8 \text{ mil}$ radius. $M = 523 \text{ g}$ so the unbalance is $0.42 \text{ g-in.}$
D.2.2 Step 1

Add unbalance mass \( m_1 \) as in Figure D-1 such that the added unbalance matches that observed in Step 0. Specifically, add \( m_1 = 0.42 \, \text{g} \) at \( r_1 = 1.0 \, \text{in.} \) Spin the rotor as in Step 0. Experimentally, one would observe a whirl circle with radius equal to the distance between the center of mass and the geometric center of the system. In lieu of experimental data, for the purpose of demonstrating the technique one can directly compute the center of mass of the system using the coordinate system of Figure D-1.

\[
X = \frac{M X_{\text{rotor}} + m_u x_u + m_1 x_1}{M + m_u + m_1} = \frac{0.0 + 0.42(1.0 \cos 22^\circ) + 0.42(1.0)}{523 + 0.42 + 0.42} = 1.545 \times 10^{-3} \, \text{in.}
\]

(D.4)

where \( X \) is the \( x \)-coordinate of the system center of mass, \( X_{\text{rotor}} \) is the \( x \)-coordinate of the rotor design center of mass, and \( x_u \) is the \( x \)-coordinate of the original unbalance.
mass.

Similarly,

\[
Y = \frac{MY_{rotor} + m_u y_u + m_1 y_1}{M + m_u + m_1} = \frac{0.0 + 0.42(1.0 \sin 22^\circ) + 0.42(0.0)}{523 + 0.42 + 0.42} = 3.003 \times 10^{-4} \text{ in.} \tag{D.5}
\]

where \( Y \) is the \( y \)-coordinate of the system center of mass, \( Y_{rotor} \) is the \( y \)-coordinate of the rotor design center of mass, and \( y_u \) is the \( y \)-coordinate of the original unbalance mass. The observed response is then

\[
\delta_1 = \sqrt{X^2 + Y^2} = \sqrt{(1.545 \times 10^{-3})^2 + (3.003 \times 10^{-4})^2} = 1.574 \times 10^{-3} \text{ in.} \tag{D.6}
\]

Or, more directly, from the graphical interpretation we can compute from the unbalance plane

\[
\delta_1 = \sqrt{(0.8 + 0.8 \cos 22^\circ)^2 + (0.0 + 0.8 \sin 22^\circ)^2} = 1.571 \times 10^{-3} \text{ in.} \tag{D.7}
\]

where the discrepancy is due to the approximation that the unbalance masses are small relative to the rotor mass.

**D.2.3 Step 2**

Add unbalance mass \( m_2 = m_1 \) as in Figure D-1. Upon spinning the rotor as in Step 0, the observable would be

\[
\delta_2 = \sqrt{(0.8 + 0.8 \cos 22^\circ + 0.0)^2 + (0.0 + 0.8 \sin 22^\circ + 0.8)^2} = 1.894 \times 10^{-3} \text{ in.} \tag{D.8}
\]

**D.2.4 Step 3**

Add unbalance mass \( m_3 = m_2 = m_1 \) as in Figure D-1. Upon spinning the rotor as in Step 0, the observable would be

\[
\delta_3 = \sqrt{(0.8 + 0.8 \cos 22^\circ + 0.0 - 0.8 \cos 45^\circ)^2 + (0.0 + 0.8 \sin 22^\circ + 0.8 - 0.8 \sin 45^\circ)^2}
\]
= 1.113 \times 10^{-3} \text{ in.} \quad \text{(D.9)}

**D.2.5 Step 4**

Construct the unbalance circles for \( \delta_0, \delta_1, \delta_2, \) and \( \text{delta}_3 \) as in Figure D-4. This allows for solution of the location of the unbalance mass, \( \delta_u \approx 0.8 \text{ mils} \) and \( \theta_u \approx 22^\circ \).

**D.2.6 Step 5**

The center of mass of the system can now be computed directly since the location of the unknown balance is determined. The location of the mass removal to balance the rotor is \( \delta_{bal} = 1.113 \times 10^{-3} \text{ in.} \) at \( \theta_{bal} = 28.68^\circ \). The magnitude of the unbalance removed is \( M\delta_3 \).
Appendix E

Macro-Rig Measurements:
Hydrodynamic Operation with
Comparison to LubePack
Calculations
Figure E-1: Eccentricity ratio vs. bearing number: $\zeta = 0.00548$

Figure E-2: Attitude Angle vs. bearing number: $\zeta = 0.00548$
Figure E-3: Eccentricity ratio vs. bearing number: $\zeta = 0.0162$

Figure E-4: Attitude Angle vs. bearing number: $\zeta = 0.0162$
Figure E-5: Eccentricity ratio vs. bearing number: $\zeta = 0.0253$

Figure E-6: Attitude Angle vs. bearing number: $\zeta = 0.0253$
Figure E-7: Eccentricity ratio vs. bearing number: $\zeta = 0.0350$

Figure E-8: Attitude Angle vs. bearing number: $\zeta = 0.0350$
Figure E-9: Eccentricity ratio vs. bearing number: $\zeta = 0.0429$

Figure E-10: Attitude Angle vs. bearing number: $\zeta = 0.0429$
Figure E-11: Eccentricity ratio vs. bearing number: $\zeta = 0.0494$

Figure E-12: Attitude Angle vs. bearing number: $\zeta = 0.0494$
Figure E-13: Eccentricity ratio vs. bearing number: $\zeta = 0.0548$

Figure E-14: Attitude Angle vs. bearing number: $\zeta = 0.0548$
Appendix F

Design Drawings for
Macro-Bearing Test Rig
NOTES:
1 - USE SHIMS (ITEM 16) AS REQUIRED AT ASSEMBLY TO ACHIEVE BUILD SPECIFICATION OF TOTAL AXIAL CLEARANCES BETWEEN ITEM 1 AND ITEM 3 OR ITEM 4 IN RANGE .002 TO .010.
2 - SAE/MS MALE CONNECTOR (ITEM 19) TO BE CAT (SWAGELOK) NO. SS-500-1-5ST (SWAGELOK)
3 - SAE/MS MALE CONNECTOR (ITEM 20) TO BE CAT (SWAGELOK) NO. SS-1610-1-12ST (SWAGELOK)
4 - SHOULDER SCREW (ITEM 27) TO BE 5/16" DIA SOCKET NO SHOULDER SCREW X 3-1/2" LONG (HOLO-KROME OR EQUIV)
5 - SPARE PARTS TO BE ORDERED AS REQUIRED
6 - INSTALL PROBE AND PLUG ASSEMBLIES (ITEMS 14, 15 & 16) PER BUILD SPECIFICATION, WITH PROBE IMMERSION SHIM CROWN TO PROVIDE IMMERSION DEPTH -.000 TO -.002, (SEE ENLARGED VIEW)
7 - BOND RADIAL PROBE MOUNTS (20), ITEM 11, INTO STATOR, ITEM 2.
Figure F-2: Design drawing for macro-rig forward foundation plate.
Figure F.3: Design drawing for forward cover plate.
Figure F-4: Design drawing for macro-turbostator.

BLADE CONTOUR (SEE TABLE ON SHEET 2)

TYPICAL: 31 PLACES EQ SP IN 360

NOTES:
1. MATERIAL: TYPE 303 AS SPECIFIED IN ASTM A336-63
   (3082 K) AMOULAD (HARDLESS ROCKWELL 95H)

2. WMH ROTOR (ENG NO. 63-203) LOCATED IN PLACE,
   SURFACES AND ARE TO BE GROUND AND
   LAPPED FLAT AND PARALLEL TO ONE ANOTHER
   TO THE SAME THICKNESS OF ROTOR.
Figure F-5: Design coordinates for macro-tot stator.
NOTES:

1. MATERIAL: UNS S17400 (17-4 PH) SOLUTION TREATED AS SPECIFIED BY ASTM A693 (HARDNESS: ROCKWELL RC38)

2. SURFACE AND TO BE GROUND AND LAPPED FLAT TO THE SAME THICKNESS OF STATOR (DRAWING NO. GTL-302) WITHIN +.0002-.0000 OF STATOR.

3. SURFACE TO BE FLAT AND PARALLEL TO SURFACE WITHIN .0002 BELOW SURFACE.

4. SURFACES TO BE FLAT AND PARALLEL TO SURFACE WITHIN .010 BELOW SURFACE.

5. SURFACES AND TO BE PLASMA OR THERMAL SPRAYED CHROME OXIDE. FINISHED DIMENSIONS AS INDICATED ON THE DRAWING. THE COATING SHOULD BE A VICKERS HARDNESS, H, GREATER THAN 1100 (110 HRC). THE SURFACES SHOULD BE LAPPED TO BETTER THAN 0.001 MICROINCH. THE COATING AFTER LAPPING SHOULD BE IN THE RANGE OF 0.004-.006.
Figure F-7: Design coordinates for macro-rig rotor.

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Note: The table contains design coordinates for macro-rig rotor as shown in the figure.
Figure F.8: Design drawing for macro-tire air cover plate.
Figure P.9: Design drawing for macro-rig axial clearance shims.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
GAS TURBINE LABORATORY
60 VASSAR STREET
CAMBRIDGE, MASS. 02139

SCALE: 1/2
MATERIAL: BRASS
FINISH: NO. REQUIRED NEXT ASSEMBLY SIMILAR TO:
NAME: SHIM PLATE
REVISION: B
TITLE: PHASE 1 POWER TURBINE SCALED TEST VEHICLE
DRAWN: DATE: 2/21/96
APPROVED BY:
Figure F-10: Design drawing for macro-turbine pressure probe.
Figure F.11: Design drawing for macro-rig blank probe.

<table>
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<th>DESCRIPTION</th>
<th>DRAWING NUMBER</th>
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<td>A</td>
<td>094 DIA WAS 1.25 DIA, 110 DIA WAS .015 DIA &amp; 050 DIM. WAS .150</td>
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**Dimensions:**
- **.094 DIA X 4.38 DEEP**
- **4-5/8”**
- **.312 DIA**
- **.188 DIA**
- **.200 DIA**
- **.062**
- **.312**
- **3.324**

**Material:**
- **Brass**

**Notes:**
- Fractional Dimensions: +/-.002
- Decimal: +/-.01
- Diam: +/-.005
- Filled Radii: .010 Max.
- Hole Angular Tolerance: +/-.5’
- All Other Angular Tolerance: +/-.30’
- No Burr or Sharp Edges

**Fabricated by:**
- **C. Killam 3/27/96**

**Approved by:**
- **C. Killam 3/27/96**

**References:**
- **FILE NAME: UNLESS OTHERWISE SPECIFIED**
- **SCALE: 2/1**
- **STATIC PRESSURE PROBE, PHASE I POWER TURBINE SCALED TEST VEHICLE**

**Dimensions:**
- **.094 DIA X 4.38 DEEP**
- **4-5/8”**
- **.312 DIA**
- **.188 DIA**
- **.200 DIA**
- **.062**
- **.312**
- **3.324**
NOTE:

1 - PROBE IMMERSION SHIM TO BE MACHINED TO CONDITIONS DESCRIBED ON TEST VEHICLE ASSY DWG NO. GTL-300.
NOTES:

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NOTES:

1 - PROBE IMMERSION SHIM TO BE MACHINED TO CONDITIONS DESCRIBED ON TEST VEHICLE ASSY DWG NO. GTL-300.
**NOTES:**

1. MATERIAL: TYPE 303 AS SPECIFIED BY ASTM 582-83 (A582 M) ANNEALED (HARDNESS ROCKWELL RB92)

2. HOLES DIA NOMINALLY 4.1028, BUT FINISHED TO BE .0082-0086 GREATER THAN OUTSIDE DIA (SURFACE "E") OF ROTOR, DWG NO. GTL-304.

3. APPLY COATING TO SURFACE AS FOLLOWS:
   - PLASMA OR THERMAL SPRAY CHROME OXIDE (FINISHED DIMENSIONS AS INDICATED ON THE DRAWING. HARDNESS SHOULD BE VICKERS HARDNESS HV GREATER THAN 1100 (RN 90, HRC71). THE COATING THICKNESS AFTER LAPPING SHOULD BE IN THE RANGE 0.004"-0.006".)

4. HOLES TO MATCH HOLES IN STATOR (DWG NO. GTL-302)
NOTES:

1 - MATERIAL: TYPE 303 AS SPECIFIED BY ASTM A582-93 (A582M) ANNEALED (HARDNESS ROCKWELL RB92)

2 - INSIDE DIA NOMINALLY 4.1028, BUT FINISHED TO BE GREATER THAN OUTSIDE DIA (SURFACE "E") OF ROTOR, DWG NO. GTL-301.

3 - APPLY COATING TO SURFACE AS FOLLOWS:
   - PLASMA OR THERMAL SPRAY CHROME OXIDE
   - FINISHED DIMENSIONS AS INDICATED ON THE DRAWING.
   - THE COATING SHOULD HAVE VICKERS HARDNESS HV GREATER THAN 1100 (RN 90, HRC 71).
   - THE COATING THICKNESS AFTER LAPPING SHOULD BE IN THE RANGE 0.004"-0.006".

- Inside Dia (Min)
- Inside Dia (Max) x 1.38 Deep
- Holes to match Holes in Stator (DWG NO. GTL-302)

- Surface

- Section
Bibliography


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Contract NONR-3730(00), Task NR 061-131, December 1964.


[52] A. A. Raimondi and J. Boyd. A solution for the finite journal bearing and its


