Balanced Fiber Bundles and GKM Theory

BALANCED FIBER BUNDLES AND GKM THEORY

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Abstract. Let T be a torus and B a compact $T-$manifold. Goresky, Kottwitz, and MacPherson show in [GKM] that if B is (what was subsequently called) a GKM manifold, then there exists a simple combinatorial description of the equivariant cohomology ring $H_T^*(B)$ as a subring of $H_T^*(B^T)$. In this paper we prove an analogue of this result for $T-$equivariant fiber bundles: we show that if M is a $T-$manifold and $\pi\colon M\to B$ a fiber bundle for which π intertwines the two $T-$actions, there is a simple combinatorial description of $H_T^*(M)$ as a subring of $H_T^*(\pi^{-1}(B^T))$. Using this result we obtain fiber bundle analogues of results of [GHZ] on GKM theory for homogeneous spaces.

Contents

1. Introduction

Let $T=(S^1)^n$ be an $n-$dimensional torus and M a compact, connected $T-$manifold. We recall that the equivariant cohomology $H_T^*(M)=H_T^*(M;\mathbb{R})$ of M is defined as the usual cohomology of the quotient $(M\times E)/T$, where E is the total space of the classifying bundle of the group T. Let

$$\pi\colon M\to B \tag{1.1}$$

be a $T-$equivariant fiber bundle. We will assume that the base B is simply connected and that the typical fiber is connected.

Then one gets a fiber bundle

$$(M\times E)/T\to(B\times E)/T \tag{1.2}$$

and a Serre-Leray spectral sequence relating the equivariant cohomology groups of M and B; the E_2-term of this spectral sequence is the product

$$H^*(F)\otimes H^*((B\times E)/T) \tag{1.3}$$

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where F is the fiber of the bundle (1.2) and hence of the bundle (1.1). Thus if the spectral sequence collapses at this stage, one gets an isomorphism of additive cohomology

$$H_T^*(M) \simeq H^*(F) \otimes H_T^*(B) \ . \tag{1.4}$$

However, this isomorphism doesn't say much about how the ring structure of $H_T^*(B)$ and $H_T^*(M)$ are related. One of the main goals of this paper is to address that question. We begin by recalling that one approach for computing the equivariant cohomology ring of a $T-$manifold M is by Kirwan localization. Namely, if $H_T^*(M)$ is torsion-free, the restriction map

$$i^* \colon H_T^*(M) \to H_T^*(M^T)$$

is injective and hence computing $H_T^*(M)$ reduces to computing the image of $H_T^*(M)$ in $H_T^*(M^T)$. If M^T is finite, then

$$H_T^*(M^T) = \bigoplus_{p \in M^T} \mathbb{S}(\mathfrak{t}^*) \ ,$$

with one copy of $H_T^*(pt) \simeq \mathbb{S}(\mathfrak{t}^*)$ for each $p \in M^T$, where $\mathbb{S}(\mathfrak{t}^*)$ is the symmetric algebra of $\mathfrak{t}^*$. Determining where $H_T^*(M)$ sits inside this sum is a challenging problem in combinatorics. However, one class of spaces for which this problem has a simple and elegant solution is the one introduced by Goresky-Kottwitz-MacPherson in their seminal paper [GKM]. These are now known as *GKM spaces*, a $T-$manifold M being "GKM" if

(a) M^T is finite
(b) M is equivariantly formal, i.e.
$$H_T(M) \simeq H(M) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*)$$
as $\mathbb{S}(\mathfrak{t}^*)$ modules.
(c) For every codimension one subtorus $T' \subset T$, the connected components of $M^{T'}$ are either points or two-spheres.

If S is one of the edge two-spheres, then S^T consists of exactly two $T-$fixed points, p and q (the "North" and "South" poles of S). To each GKM space M we attach a graph $\Gamma = \Gamma_M$ with set of vertices $V_\Gamma = M^T$, and edges corresponding to these two-spheres. If M has an invariant almost complex or symplectic structure, then the isotropy representations on tangent spaces at fixed points are complex representations and their weights are well-defined.

These data determine a map

$$\alpha \colon E_\Gamma \to \mathbb{Z}_T^*$$

of oriented edges of Γ into the weight lattice of T. This map assigns to the edge (two-sphere) S, joining p to q and oriented from p to q, the weight of the isotropy representation of T on the tangent space to S at p. The map α is called the *axial function* of the graph Γ. We use it to define a subring $H_\alpha^*(\Gamma)$ of $H_T^*(M^T)$ as follows. Let c be an element of $H_T^*(M^T)$, *i.e.* a function which assigns to each $p \in M^T$ an element $c(p)$ of $H_T^*(pt) = \mathbb{S}(\mathfrak{t}^*)$. Then c is in $H_\alpha^*(\Gamma)$ if and only if for each edge e of Γ with vertices p and q as endpoints, $c(p) \in \mathbb{S}(\mathfrak{t}^*)$ and $c(q) \in \mathbb{S}(\mathfrak{t}^*)$ have the same image in $\mathbb{S}(\mathfrak{t}^*)/\alpha_e\mathbb{S}(\mathfrak{t}^*)$. (Without the invariant almost complex or symplectic structure, the isotropy representations are only real representations and the weights are defined only up to sign; however, that does not change the construction of $H_\alpha^*(\Gamma)$.)

For GKM spaces, a direct consequence of a theorem of Chang and Skjelbred ([CS]) is that $H^*_\alpha(\Gamma)$ is the image of i^* (see [GKM]), and therefore there is an isomorphism of rings

$$H^*_T(M) \simeq H^*_\alpha(\Gamma)\ . \tag{1.5}$$

One of our main results is a generalization of (1.5) for $T-$equivariant fiber bundles

$$\pi\colon M \to B \tag{1.6}$$

for which the total space M is equivariantly formal and the base B is a GKM space. By the Kirwan Theorem the composite map

$$H^*_T(M) \to H^*_T(\pi^{-1}(B^T)) \to H^*_T(M^T)$$

is injective. Hence one has an injective homomorphism of rings

$$H^*_T(M) \to \bigoplus_{p\in B^T} H^*_T(\pi^{-1}(p))\ , \tag{1.7}$$

and so to determine the ring structure of $H^*_T(M)$ it suffices to determine the image of this mapping. This we will do by a GKM type recipe similar to (1.5).

Let $(\Gamma = \Gamma_B, \alpha)$ be the GKM graph associated to B, and for $p \in B^T$ (*i.e.* a vertex of Γ) let $F_p = \pi^{-1}(p)$. If e is an edge of Γ joining the vertices p and q, and T_e is the subtorus of T with Lie algebra $\ker\alpha_e$, then F_p and F_q are isomorphic as T_e-spaces and hence

$$H^*_T(F_p)/\langle\alpha_e\rangle = H^*_T(F_q)/\langle\alpha_e\rangle\ , \tag{1.8}$$

and denoting the ring (1.8) by $\mathcal{R}_e$, we will prove the following generalization of (1.5).

Theorem 1.1. A function

$$c\colon V_\Gamma \to \bigoplus_{p\in B^T} H^*_T(F_p),\ c(p) \in H^*_T(F_p)$$

is in the image of (1.7) if and only if for every edge $e = (p, q)$ of Γ, the images of $c(p)$ and $c(q)$ in $\mathcal{R}_e$ coincide.

One of our main applications of this result will be a fiber bundle version of the main result in [GHZ]. In more detail: In [GHZ] it is shown that if G is a compact semisimple Lie group, T a Cartan subgroup, and K a closed subgroup of G, then the following conditions are equivalent:

(1) The action of T on G/K is GKM;
(2) The Euler characteristic of G/K is non-zero;
(3) K is of maximal rank, *i.e.* $T \subset K$.

Moreover, for homogeneous spaces of the form G/K one has a description, due to Borel, of the equivariant cohomology ring of G/K as a tensor product

$$H^*_T(G/K) = \mathbb{S}(\mathfrak{t}^*)^{W_K} \oplus_{\mathbb{S}(\mathfrak{t}^*)^{W_G}} \mathbb{S}(\mathfrak{t}^*) \tag{1.9}$$

and it is shown in [GHZ] how to reconcile this description with the description (1.5).

Our fiber bundle version of this result will be a a description of the cohomology ring of G/K_1, with $K_1 \subset K$, in terms of the fiber bundle $G/K_1 \to G/K$, a description that will be of Borel type on the fibers and of GKM type on the base. This result will (as we've already shown in special cases in [GSZ]) enable one to interpolate between two (in principle) very different descriptions of the ring $H_T(G/K)$.

The fibrations $G/K_1 \to G/K$ are special cases of a class of fibrations which come up in many other context as well (for instance in the theory of toric varieties) and which for the lack of a better name we will call *balanced fibrations*.

To explain what we mean by this term let F_p and F_q be as in (1.8). Then there is a diffeomorphism $f_e \colon F_p \to F_q$, canonical up to isotopy, which is T_e-invariant but in general not $T-$invariant. We will say that the fibration $M \to B$ is balanced at e if one can twist the T action on F_q to make f_e be $T-$invariant, *i.e.* if one can find an automorphism $\tau_e \colon T \to T$, restricting to the identity on T_e, such that

$$f_e(gx) = \tau_e(g) f_e(x) \tag{1.10}$$

for all $g \in T$ and $x \in F_p$. (Since f_e is unique up to isotopy, this τ_e, if it exists, is unique.)

Suppose now that the T action on M is balanced in the sense that it is balanced at all edges e. Then, denoting by $Aut(F_p)$ the group of isotopy classes of diffeomorphisms of F_p and by $Aut(T)$ the group of automorphisms of T, one gets a homomorphism of the loop group $\pi_1(\Gamma, p)$ into $Aut(F_p) \times Aut(T)$ mapping the loop of edges, $e_1, \ldots, e_k$, to $(f_{e_k} \circ \cdots \circ f_{e_1}, \tau_{e_k} \circ \cdots \circ \tau_{e_1})$. The image of this map we'll denote by W_p and call the *Weyl group of* p. This group acts on $H_T(F_p)$ and, modulo some hypotheses on M which we'll spell out more carefully in section 5, we'll show that there is a canonical imbedding of $H_T(F_p)^{W_p}$ into $H_T(M)$ and that its image generates $H_T(M)$ as a module over $H_T(B)$. More explicitily, we will show that

$$H_T(M) = H_T(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma_B) \tag{1.11}$$

where Γ_B is the GKM graph of B.

A few words about the organization of this paper. In Section2 we will generalize the Chang-Skjelbred theorem to equivariant fiber bundles, and in Section 3 use this result to prove Theorem 1.1. In Section 4 we will describe in more detail the results of [GHZ] alluded to above and show that the fibrations $G/K_1 \to G/K$ are balanced. Then in Section 5 we will verify (1.11) and in Section 6 describe some connections between the results of this paper and results of [GSZ] (where we work out the implications of this theory in much greater detail for the classical flag varieties of type A_n, B_n, C_n, and D_n).

The results of this paper are also related to the results of [GZ2], the topic of which is K-theoretic aspects of GKM theory. (In some work-in-progress we are investigating the implications of these results for GKM fibrations. In particular we are able to show that there is a K-theoretic analogue of the Chang-Skjelbred theorem of Section 2 and that it gives one an effective way of computing the equivariant K-groups of balanced fiber bundles.)

2. The Chang-Skjelbred Theorem for Fiber Bundles

Let $\pi \colon M \to B$ be a $T-$equivariant fiber bundle with M equivariantly formal and B a GKM space. Let $K_i,\ i = 1, \ldots, N$ be the codimension one isotropy groups of B and let $\mathfrak{k}_i$ be the Lie algebra of K_i. Since B is GKM one has the following result.

Lemma 2.1. If K is an isotropy group of B, and $\mathfrak{k}$ is the Lie algebra of K, then

$$\mathfrak{k} = \bigcap_{r=1}^{m} \mathfrak{k}_{i_r}$$

for some multi-index $1 \leqslant i_1 < \ldots < i_m \leqslant N$.

For K a subgroup of T let $X^K = \pi^{-1}(B^K)$, where $B^K \subset B$ denotes the set of points in B fixed by K. We recall ([GS, Section 11.3]) that if A is a finitely generated $\mathbb{S}(\mathfrak{t}^*)-$module, then the annihilator ideal of A, I_A is defined to be

$$I_A = \{f \in \mathbb{S}(\mathfrak{t}^*),\ fA = 0\}\ ,$$

and the support of A is the algebraic variety in $\mathfrak{t} \otimes \mathbb{C}$ associated with this ideal, *i.e.*

$$\mathrm{supp} A = \{x \in \mathfrak{t} \otimes \mathbb{C}, f(x) = 0 \text{ for all } f \in I_A\}\ .$$

Then from the lemma and [GS, Theorem 11.4.1] one gets the following.

Theorem 2.2. The $\mathbb{S}(\mathfrak{t}^*)-$module $H_T^*(M \setminus X^T)$ is supported on the set

$$\bigcup_{i=1}^{N} \mathfrak{k}_i \otimes \mathbb{C} \tag{2.1}$$

By [GS, Section 11.3] there is an exact sequence

$$H_T^k(M \setminus X^T)_c \longrightarrow H_T^k(M) \xrightarrow{i^*} H_T^k(X^T) \longrightarrow H_T^{k+1}(M \setminus X^T)_c \tag{2.2}$$

where $H_T^*(\ \cdot\ )_c$ denotes the equivariant cohomology with compact supports. Therefore since $H_T^*(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module Theorem 2.2 implies the following theorem.

Theorem 2.3. The map i^* is injective and coker(i^*) is supported on $\bigcup_{i=1}^{N} \mathfrak{k}_i \otimes \mathbb{C}$.

As a consequence we get the following corollary.

Corollary 2.4. If e is an element of $H_T^*(X^T)$, there exist non-zero weights $\alpha_1, \ldots, \alpha_r$ such that $\alpha_i = 0$ on some $\mathfrak{k}_j$ and

$$\alpha_1 \cdots \alpha_r e \in i^*(H_T^*(M)) \tag{2.3}$$

The next theorem is a fiber bundle version of the Chang-Skjelbred theorem.

Theorem 2.5. The image of i^* is the ring

$$\bigcap_{i=1}^{N} i_{K_i}^* H_T^*(X^{K_i}) \tag{2.4}$$

where i_{K_i} denotes the inclusion of X^T into X^{K_i}.

Proof. Via the inclusion i^* we can view $H_T^*(M)$ as a submodule of $H_T^*(X^T)$. Let $e_1, \ldots, e_m$ be a basis of $H_T^*(M)$ as a free module over $\mathbb{S}(\mathfrak{t}^*)$. Then by Corollary 2.4 for any $e \in H_T^*(X^T)$ we have

$$\alpha_1 \cdots \alpha_r e = \sum f_i e_i,\ f_i \in \mathbb{S}(\mathfrak{t}^*)\ .$$

Then $e = \sum \frac{f_i}{p} e_i$, where $p = \alpha_1 \cdots \alpha_r$. If f_i and p have a common factor we can eliminate it and write e uniquely as

$$e = \sum \frac{g_i}{p_i} e_i \tag{2.5}$$

with $g_i \in \mathbb{S}(\mathfrak{t}^*)$, p_i a product of a subset of the weights $\alpha_1, \ldots, \alpha_r$ and p_i and g_i relatively prime.

Now suppose that K is an isotropy subgroup of B of codimension one and e is in the image of $H_T^*(X^K)$. By [GS, Theorem 11.4.2] the cokernel of the map $H_T^*(M) \to H_T^*(X^K)$ is supported on the subset $\cup \mathfrak{k}_i \otimes \mathbb{C}$, $\mathfrak{k}_i \neq \mathfrak{k}$ of (2.1), and hence there exists weights $\beta_1, \ldots, \beta_r$, β_i vanishing on some $\mathfrak{k}_j$ but not on $\mathfrak{k}$, such that

$$\beta_i \cdots \beta_s e = \sum f_i e_i \ .$$

Thus the p_i in (2.5), which is a product of a subset of the weights $\alpha_1, \ldots, \alpha_r$, is a product of a subset of weights none of which vanish on $\mathfrak{k}$. Repeating this argument for all the codimension one isotropy groups of B we conclude that the weights in this subset cannot vanish on any of these $\mathfrak{k}$'s, and hence is the empty set, i.e. $p_i = 1$. Then if e is in the intersection (2.4), e is in $H_T^*(M)$. □

3. Fiber Bundles over GKM Spaces

Suppose now that $B = \mathbb{C}P^1$. The action of T on B is effectively an action of a quotient group, T/T_e, where T_e is a codimension one subgroup of T. Moreover B^T consists of two points, p_i, $i = 1, 2$, and X^T consists of the two fibers $\pi^{-1}(p_i) = F_i$. Let $T = T_e \times S^1$. Then S^1 acts freely on $\mathbb{C}P^1 \setminus \{p_1, p_2\}$ and the quotient by S^1 of this action is the interval $(0, 1)$, so one has an isomorphism of T_e spaces

$$(M \setminus X^T)/S^1 = F \times (0, 1) \ , \tag{3.1}$$

where, as T_e-spaces, $F = F_1 = F_2$.

Consider now the long exact sequence (2.2). Since i^* is injective this becomes a short exact sequence

$$0 \to H_T^k(M) \to H_T^k(X^T) \to H_T^{k+1}(M \setminus X^T)_c \to 0 \ . \tag{3.2}$$

Since S^1 acts freely on $M \setminus X^T$ we have

$$H_T^{k+1}(M \setminus X^T)_c = H_{T_1}^{k+1}((M \setminus X^T)/S^1)_c$$

and by fiber integration one gets from (3.1)

$$H_{T_e}^{k+1}((M \setminus X^T)/S^1)_c = H_{T_e}^k(F) \ .$$

Therefore, denoting by r the forgetful map $H_T(F_i) \to H_{T_e}(F)$, the sequence (3.2) becomes

$$0 \to H_T^k(M) \to H_T^k(F_1) \oplus H_T^k(F_2) \to H_{T_e}^k(F) \to 0 \ , \tag{3.3}$$

where the second arrow is the map

$$H_T(X^T) = H_T(F_1) \oplus H_T(F_2) \to H_{T_e}(F)$$

sending $f_1 \oplus f_2$ to $-r(f_1) + r(f_2)$. (The $-r$ in the first term is due to the fact that the fiber integral

$$H_c^{k+1}(F \times (0, 1)) \to H^k(F)$$

depends on the orientation of $(0, 1)$: the standard orientation for $F_2 \times (0, 1) \to F_2$ and the reverse orientation for $F_1 \times (0, 1) \to F_1$.) To summarize, we've proved the following theorem.

Theorem 3.1. For $T-$equivariant fiber bundles over $\mathbb{C}P^1$, the image of the map

$$0 \to H_T^*(M) \to H_T^*(X^T)$$

is the set of pairs $(f_1, f_2) \in H_T^*(F_1) \oplus H_T^*(F_2)$ satisfying $r(f_1) = r(f_2)$.

Theorem 1.1 follows from Theorem 2.5 by applying Theorem 3.1 to all edges of the GKM graph of B.

4. Homogeneous Fibrations

Let G be a compact connected semisimple Lie group, T its Cartan subgroup, and K a closed subgroup of G containing T. Then, as asserted above, G/K is a GKM space. The proof of this consists essentially of describing explicitly the GKM structure of G/K in terms of the Weyl groups of G and K. We first note that for $K = T$, i.e. for the generalized flag variety $M = G/T$, the fixed point set, M^T, is just the orbit of $N(T)$ through the identity coset, p_0, and hence M^T can be identified with $N(T)/T = W_G$. To show that M is GKM it suffices to check the GKM condition p_0. To do so we identify the tangent space $T_{p_0}M$ with $\mathfrak{g}/\mathfrak{t}$ and identify $\mathfrak{g}/\mathfrak{t}$ with the sum of the positive root spaces

$$\mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha \in \Delta_G^+} \mathfrak{g}_\alpha, \ , \tag{4.1}$$

the α's being the weights of the isotropy representation of T on $\mathfrak{g}/\mathfrak{t}$. It then follows from a standard theorem in Lie theory that the weights are pairwise independent and this in turn implies "GKM-ness" at p_0.

To see what the edges of the GKM graph are at p_0 let $\chi_\alpha \colon T \to S^1$ be the character homomorphism

$$\chi_\alpha(t) = \exp i\alpha(t) \ ,$$

let H_α be its kernel, and G_α the semisimple component of the centralizer $C(H_\alpha)$ of H_α in G. Then G_α is either $SU(2)$ or $SO(3)$, and in either case $G_\alpha p_0 \simeq \mathbb{C}P^1$. However since G_α centralizes H_α, $G_\alpha p_0$ is $H_\alpha-$fixed and hence is the connected component of M^{H_α} containing p_0. Thus the oriented edges of the GKM graph of M with initial point p_0 can be identified with the elements of Δ_G^+ and the axial function becomes the function which labels by α the oriented edge $G_\alpha p_0$. Moreover, under the identification $M^T = W_G$, the vertices that are joined to p_0 by these edges are of the form $\sigma_\alpha p_0$, where $\sigma_\alpha \in W_G$ is the reflection which leaves fixed the hyperplane $\ker \alpha \subseteq \mathfrak{t}$ and maps α to $-\alpha$.

Letting $a \in N(T)$ and letting $p = ap_0$ one gets essentially the same description of the GKM graph at p. Namely, denoting this graph by Γ, the following are true.

1. The maps, $a \in N(T) \to ap_0$ and $a \in N(T) \to w \in N(T)/T$, set up a one-one correspondence between the vertices, M^T, of Γ and the elements of W_G;
2. Two vertices, w and w', are on a common edge if and only if $w' = w\sigma_\alpha$ for some $\alpha \in \Delta_G^+$;
3. The edges of Γ containing $p = ap_0$ are in one-one correspondence with elements of Δ_G^+;
4. For $\alpha \in \Delta_G^+$ the stabilizer group of the edge corresponding to α is $aH_\alpha a^{-1}$.

Via the fibration $G/T \to G/K$ one gets essentially the same picture for G/K. Namely let $\Delta_{G,K}^+ = \Delta_G^+ \setminus \Delta_K^+$. Then one has (see [GHZ], Theorem 2.4)

Theorem 4.1. *G/K is a GKM space with GKM graph Γ, where*

1. *The vertices of Γ are in one-one correspondence with the elements of W_G/W_K;*
2. *Two vertices $[w]$ and $[w']$ are on a common edge if and only if $[w'] = [w\sigma_\alpha]$ for some $\alpha \in \Delta_{G,K}^+$;*
3. *The edges of Γ containing the vertex $[w]$ are in one-one correspondence with the roots in $\Delta_{G,K}^+$;*

(4) If α is such a root the the stabilizer group of the $\mathbb{C}P^1$ corresponding to the edge is $aH_\alpha a^{-1}$ where a is a preimage in $N(T)$ of $w \in W_G$.

Remark 4.2. The GKM graph that we have just described is not simple in general, i.e. will in general have more than one edge joining two adjacent vertices. There is, however, a simple sufficient condition for simplicity.

Theorem 4.3. If K is a stabilizer group of an element of $\mathfrak{t}^*$, i.e. if G/K is a coadjoint orbit, then the graph we've constructed above is simple.

Now let K_1 be a closed subgroup of K and consider the fibration

$$G/K_1 \to G/K. \tag{4.2}$$

To show that this is balanced it suffices to show that it is balanced at the edges going out of the identity coset, p_0. However, if e is the edge joining p_0 to $\sigma_\alpha p_0$ and a is the preimage of σ_α in $N(T)$ then conjugation by a maps the fiber, $F_{p_0} = K/K_0$ of (4.2) at p_0 onto the fiber $F_p := aK/aK_0$ of (4.2) at $p = \sigma_\alpha p_0$, and conjugates the action of Γ on F_{p_0} to the twisted action, aTa^{-1}, of T on F_p. Moreover, since a is in the centralizer of H_α, this twisted action, restricted to H_α, coincides with the given action of H_α, i.e. if $T_e = H_\alpha$, conjugation by a is a T_e-equivariant isomorphism of F_{p_0} onto F_p. Hence the fibration (4.2) is balanced.

5. Holonomy for Balanced Bundles

Let M be a $T-$manifold and $\tau\colon T \to T$ an automorphism of T. We will begin our derivation of (1.11) by describing how the equivariant cohomology ring, $H_T(M)$ of M is related to the "$\tau-$twisted" equivariant cohomology ring $H_T(M)^\tau$, i.e. the cohomology ring associated with the action, $(g, m) \to \tau(g)m$.

The effect of this twisting is easiest to describe in terms of the Cartan model, $(\Omega_T(M), d_T)$. We recall that in this model cochains are T invariant polynomial maps

$$p\colon \mathfrak{t} \to \Omega(M), \tag{5.1}$$

and the coboundary operator is given by

$$dp(\xi) = d(p(\xi)) + \iota(\xi_M)p(\xi). \tag{5.2}$$

"Twisting by τ" is then given by the pull-back operation

$$\tau^*\colon \Omega_T \to \Omega_T, \ \ \tau^*p(\xi) = p(\tau(\xi))$$

which converts d_T to the coboundary operator

$$\tau^* d_T(\tau^{-1})^* = d + \iota(\tau^{-1}(\xi)) = (d_T)^{\tau^{-1}}, \tag{5.3}$$

the expression on the right being the coboundary operator associated with the τ^{-1}-twisted action of T on M.

Suppose now that M and N are $T-$manifolds and $f\colon M \to N$ a diffeomorphism which intertwines the $T-$action on M with the $\tau-$twisted action on N. Then the pull-back map $f^*\colon \Omega(N) \to \Omega(M)$ satisfies

$$\iota(\xi_M)f^* = f^*\iota(\tau(\xi)_N).$$

Hence if we extend f^* to the Ω_T's by setting $(f^*p)(\xi) = f^*(p(\xi))$ this extended map satisfies

$$d_T f^* = f^*(d_T)^\tau. \tag{5.4}$$

Thus by (5.3) and (5.4) $\tau^* f^*$ intertwines the d_T operators on $\Omega_T(N)$ and $\Omega_T(M)$ and hence defines an isomorphism on cohomology

$$\tau^\# f^\# \colon H_T(N) \to H_T(M). \tag{5.5}$$

Moreover, for any diffeomorphism $f\colon M \to N$ (not just the f above), the pull-back operation $(f^*p)(\xi) = f^*p(\xi)$ intertwines the τ^* operations, i.e.

$$\tau^* f^* = f^* \tau^*. \tag{5.6}$$

Another property of f^* which we will need below if the following. If T_e is a subgroup of T one has restriction maps

$$\Omega_T \to \Omega_{T_e}, \ \ p \to p_{|_{\mathfrak{t}_e}}$$

and these induce maps in cohomology. If $\tau_{|T_e}$ is the identity it is easily checked that the diagram

$$\begin{array}{ccc} H_T(N) & \xrightarrow{\tau^\# f^\#} & H_T(M) \\ \downarrow & & \downarrow \\ H_{T_e}(N) & \xrightarrow{f^\#} & H_{T_e}(M) \end{array}$$

commutes.

To apply these observations to the fibers of (1.1) we begin by recalling a few elementary facts about holonomy. By equipping M with a $T-$invariant Riemannian metric we get for each $m \in M$ an orthonormal complement in T_mM of the tangent space at m to the fiber of π, *i.e.* an "Ehresman connection." Thus, if p and q are points of B and γ is a curve joining p to q we get, by parallel transport, a diffeomorphism $f_\gamma \colon F_p \to F_q$, where $f_\gamma(m)$ is the terminal point of the horizontal curve in M projecting onto γ and having m as its initial point. Moreover, if γ and γ' are homotopic curves joining p to q, then the diffeomorphisms f_γ and $f_{\gamma'}$ are isotopic, *i.e.* the isotopy class of f_γ depends only on the homotopy class of γ.

Suppose now that the base B is GKM, p and q are adjacent vertices of Γ_B, e is the edge joining them and S the two-sphere corresponding to this edge. We can then choose γ to be a longitudinal line on S joining the South pole p to the North pole q; since this line is unique up to homotopy, we get an intrinsically defined isotopy class of diffeomorphisms of F_p onto F_q. Moreover since the Ehresman connection on M is $T-$invariant and T_e fixes γ, the maps in this istopy class are T_e-invariant. We will decree that the fibration (1.1) is *balanced* if there exists a diffeomorphism f_e in this isotopy class and an automorphism τ_e of T such that f_e intertwines the $T-$action on F_p with the τ_e-twisted action of T on F_q.

It is clear that this τ_e, if it exists, has to be unique and has to restrict to the identity on T_e. Moreover, given a path $\gamma\colon e_1, \dots, e_k$, in $\pi_1(\Gamma, p)$ we have for each i a ring isomorphism

$$\tau_{e_i}^\# f_{e_i}^\# \colon H_T(F_{p_{i+1}}) \to H_T(F_{p_i}), \tag{5.7}$$

p_i and p_{i+1} being the initial and terminal vertices of e_i, and by composing these maps we get a ring automorphism, $\tau_{e_k}^\# f_{e_k}^\# \circ \cdots \circ \tau_{e_1}^\# f_{e_1}^\#$, of $H_T(F_p)$. Moreover, by (5.6) we can rewrite the factors in this product as $\tau_\gamma^\# f_\gamma^\#$ where $\tau_\gamma = \tau_{e_1} \circ \cdots \circ \tau_{e_k}$ and $f_\gamma = f_{e_1} \circ \cdots \circ f_{e_k}$. Thus the map, $\gamma \to \tau_\gamma^\# f_\gamma^\#$ gives one a *holonomy action* of $\pi_1(\Gamma, p)$ on $H_T(F_p)$. Alternatively letting W_p be the image in $Aut(F_p) \times Aut(T)$ of this map we can view this as a holonomy action of W_p on $H_T(F_p)$.

Now let c_p be an element of $H_T(F_p)$ and $\gamma_{p,q}\colon e_1, \dots, e_l$ a path in Γ joining p to q. Then one can parallel transport c_p along γ by the series of maps $\tau_{e_i}^\# f_{e_i}^\#$ to get an element c_q in $H_T(F_q)$ and if c_p is in $H_T(F_p)^{W_p}$, this parallel transport operation doesn't depend on the choice of γ. Moreover if q_1 and q_2 are adjacent vertices and e is the edge joining q_1 to q_2, $c_{q_1} = \tau_e^* f_e^* c_{q_2}$ and hence by the commutative diagram above the images of c_{q_1} and c_{q_2} in the quotient space

$$H_T(F_{q_1})/\langle\alpha_e\rangle = H_T(F_{q_2})/\langle\alpha_e\rangle$$

are the same. In other words by (1.8) the assignment, $q \in Vert(F) \to c_q$ defines a cohomology class in $H_T(M)$ and thus gives us a map

$$H_T(F_p)^{W_p} \to H_T(M) \; . \tag{5.8}$$

By tensoring this map with the map

$$H_\alpha(\Gamma) \xrightarrow{\sim} H_T(B) \xrightarrow{\pi^\#} H_T(M)$$

we get a morphism of rings (1.11):

$$H_T(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma) \to H_T(M) \; .$$

To prove that this map is injective we will assume henceforth that not only is M equivariantly formal as a T space but the F_p's are as well. Apropos of this assumption we note:

(i) Since the fibration, $M \to B$, is balanced, it suffices to assume this just for the "base" fiber, F_{p_0}, above a single $p_0 \in B^T$.

(ii) One consequence of this assumption is that the cohomology groups $H_T^k(F)$ and $H^k(F)$ are non-zero only in even dimensions. Hence, since we are also assuming that this is the case for $H_T(B)$, the Serre-Leray spectral sequence associated with the fibration (1.2) has to collapse at its E_2 stage and hence the right and left hand sides of (1.4) are isomorphic as $\mathbb{S}(\mathfrak{t}^*)$ modules.

(iii) For the homogeneous fibrations in Section 4 this assumption is equivalent to the assumption that the F_p's are GKM. To see this we note that if G/K is equivariantly formal then $(G/K)^T$ has to be non-empty by [GS] theorem 11.4.5 and hence for some $g \in G$, $g^{-1}Tg \subseteq K$. In other words K is of maximal rank and hence by the theorem in [GHZ] that we cited above G/K is GKM.

(iv) If M is a Hamiltonian $T-$space, then the fibers are Hamiltonian spaces as well, hence are equivariantly formal. (Notice that in particular if M is a Hamiltonian GKM space, then the fibers are Hamiltonian GKM spaces.)

(v) One consequence of the fact that $H_T(F)$ is equivariantly formal as a module over $\mathbb{S}(\mathfrak{t}^*)$ is that $H_T(F_p)^W$ is equivariantly formal as a module over $\mathbb{S}(\mathfrak{t}^*)^W$. It is interesting to note that this property of F is a consequence of the equivariant formality of M, *i.e.* doesn't require the assumption that F be equivariantly formal. Namely to prove that $H_T(F)^W$ is equivariantly formal one has to show that there are no torsion elements in $H_T(F)^W$: if $0 \neq p \in \mathbb{S}(\mathfrak{t}^*)^W$ and $c \in H_T(F_p)^W$, then $pc = 0$ implies $c = 0$. Suppose this were not the case. Then the cohomology class $\tilde{c} \in H_T(M)$ obtained from c by parallel transport would satisfy $p\tilde{c} = 0$, contradicting the assumption that M is equivariantly formal.

We next note that, since B is simply connected, the diffeomorphisms $f_\gamma \colon F_p \to F_p$ are (non-equivariantly) isotopic to the identity, so they act trivially on $H(F_p)$, and since F_p is by assumption equivariantly formal,

$$H_T(F_p) \simeq H(F_p) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*) \tag{5.9}$$

as an $\mathbb{S}(\mathfrak{t}^*)$ module. Hence if one chooses elements $c_1(p), \dots, c_N(p)$ of $H_T(F_p)$ whose projections, c_i, in $H(F_p) = H_T(F_p)/\mathbb{S}(\mathfrak{t}^*)$ are a basis of $H(F_p)$, these will be a free set of generators of $H_T(F_p)$ as a module over $\mathbb{S}(\mathfrak{t}^*)$. Moreover, we can average these generators by the action of W_p and by the remark above these averaged generators will have the same projections onto $H(F_p)$. Hence we can assume, without loss of generality, that the $c_i(p)$'s themselves are in $H_T(F_p)^{W_p}$ and by (5.9) generate $H_T(F_p)^{W_p}$ as a module over $\mathbb{S}(\mathfrak{t}^*)^{W_p}$.

If we parallel transport these generators to the fiber over q, we will get a set of generators, $c_1(q), \dots, c_N(q)$ of $H_T(F_q)$, and the maps $q \to c(q)$ define, by Chang-Skjelbred, cohomology classes $\tilde{c}_k$ in $H_T(M)$.

Consider the map

$$\sum (c_i \otimes f_i, q) \to \sum f_i(q) c_i(q) \tag{5.10}$$

of $H(F_p) \otimes_{\mathbb{C}} H_\alpha(\Gamma)$ into $H_T(M)$. Since the $c_i(q)$'s are, for every $q \in Vert(\Gamma)$, a free set of generators of $H_T(F_q)$ as a module over $\mathbb{S}(\mathfrak{t}^*)$, this map is an injection and hence so is the equivariant version of this map: (1.11). To see that injectivity implies surjectivity we note that, if we keep track of bi-degrees, the map (5.10) maps the space

$$\bigoplus_{j+k=i} H^j(F_p) \otimes H^k_\alpha(\Gamma) \tag{5.11}$$

into $H^i_T(M)$. However, by assumption, the Serre-Leray spectral sequence associated with the fibration π collapses at its E_2 stage. The E_2 term of this sequence is

$$H(F_p) \otimes_{\mathbb{C}} H_T(B) \tag{5.12}$$

and the E_∞ term is $H_T(M)$, so by (5.11) and (5.12) the space (5.11) has the same dimension as $H^i_T(M)$ and hence (5.10) is a bijective map of (5.11) onto $H^i_T(M)$.

6. Examples

The results of this paper are closely related to the combinatorial results of our recent article [GSZ]. More explicitly in [GSZ] we develop a GKM theory for fibrations in which the objects involved: the base, the fiber and the total space of the fibration, are GKM graphs. We then formulate, in this context, a combinatorial notion of "balanced," show that one has an analogue of the isomorphism (1.11) and use this fact to define some new combinatorial invariants for the classical flag varieties of type A_n, B_n, C_n, and D_n. In this section we will give a brief account of how these invariants can be defined geometrically by means of the techniques developed above.

Example 1. Let $G = SU(n+1)$, $K = T$, the Cartan subgroup of diagonal matrices in $SU(n+1)$, and $K_1 = S(S^1 \times U(n))$. Then $G/K_1 \simeq \mathbb{C}P^n$, the complex projective space. Let $\mathcal{A}_n = G/K$ be the generic coadjoint orbit of type A_n; then $\mathcal{A}_n \simeq \mathcal{F}l(\mathbb{C}^{n+1})$, the variety of complete flags in $\mathbb{C}^{n+1}$. The fibration

$$\pi \colon \mathcal{A}_n \simeq G/K \to G/K_1 \simeq \mathbb{C}P^n$$

sends a flag $V_\bullet$ to its one-dimensional subspace V_1. The fiber over a line L in $\mathbb{C}P^n$ is $\mathcal{F}l(\mathbb{C}^{n+1}/L) \simeq \mathcal{F}l(\mathbb{C}^n) \simeq SU(n)/T'$, where T' is the Cartan subgroup of diagonal matrices in $SU(n)$. The fibers inherit a $T-$action from $\mathcal{F}l(\mathbb{C}^{n+1})$, but are not

$T-$equivariantly isomorphic. If p and q are fixed points for the $T-$action on $\mathbb{C}P^n$, then the fibers F_p and F_q are T_e-equivariantly isomorphic, where T_e is the subtorus fixing the $\mathbb{C}P^1$ in $\mathbb{C}P^n$ with poles p and q. The Weyl group W_p of the fiber at p is isomorphic to S_{n-1}, the Weyl group of $SU(n)$ and the holonomy action action of W_p on the equivariant cohomology of the fiber is equivalent to the induced action of S_{n-1} on the equivariant cohomology of the flag variety $\mathcal{F}l(\mathbb{C}^n)$.

We can iterate this fibration and construct a tower of fiber bundles

$$\begin{array}{ccccccccc} \mathrm{pt} \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathcal{A}_2 & \longrightarrow \dots \longrightarrow & \mathcal{A}_{n-1} & \longrightarrow & \mathcal{A}_n \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \mathbb{C}P^1 & & \mathbb{C}P^2 & & \mathbb{C}P^{n-1} & & \mathbb{C}P^n \end{array}$$

Using this tower we construct a basis of invariant classes on $\mathcal{A}_n$ by repeatedly applying the isomorphism (1.11). A typical stage in the process is the following. By (1.11) we have

$$H_T(\mathcal{A}_k) = H_T(\mathcal{A}_{k-1})^{W_{k-1}} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_{k-1}}} H_T(\mathbb{C}P^k)$$

Suppose we have constructed a basis of invariant classes on $\mathcal{A}_{k-1}$; this is trivial for $\mathcal{A}_0 =$pt. We use, as a basis for $H_T(\mathbb{C}P^k)$, classes represented by powers of the equivariant symplectic form $\omega-\tau \in \Omega^2_T(\mathbb{C}P^k)$. The pull-backs of these classes to $\mathcal{A}_k$ are invariant under the holonomy action, and the classes given by the isomorphism (1.11) form a basis of the equivariant cohomology of $\mathcal{A}_k$. As shown in [GSZ], this basis consists of classes that are invariant under the corresponding holonomy action. By iterating this process we obtain an $\mathbb{S}(\mathfrak{t}^*)$-basis of $H_T(\mathcal{A}_n)$ consisting of $W-$invariant classes. The combinatorial version of this construction is given in [GSZ, Section 5.1].

Example 2. Let $G = SO(2n+1)$, $K = T$ a maximal torus in G, and $K_1 = SO(2) \times SO(2n-1)$. Then $G/K_1 \simeq Gr_2^+(\mathbb{R}^{2n+1})$, the Grassmannian of oriented two planes in $\mathbb{R}^{2n+1}$. Let $\mathcal{B}_n = G/K$ be the generic coadjoint orbit of type B_n and

$$\pi\colon \mathcal{B}_n \simeq G/K \to G/K_1 \simeq Gr_2^+(\mathbb{R}^{2n+1})$$

the natural projection. Since the fibers are isomorphic to $\mathcal{B}_{n-1}$ (but not isomorphic as $T-$spaces since the $T-$action on the pre-image of the $T-$fixed points of $Gr_2^+(\mathbb{R}^{2n+1})$ changes), we can produce a tower of fiber bundles

$$\begin{array}{ccccccccc} \mathrm{pt} \longrightarrow & \mathbb{C}P^1 & \longrightarrow & \mathcal{B}_2 & \longrightarrow \dots \longrightarrow & \mathcal{B}_{n-1} & \longrightarrow & \mathcal{B}_n \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \mathbb{C}P^1 & & Gr_2^+(\mathbb{R}^5) & & Gr_2^+(\mathbb{R}^{2n-1}) & & Gr_2^+(\mathbb{R}^{2n+1}) \end{array}$$

Since the classes represented by powers of the equivariant symplectic form $\omega - \tau \in \Omega^2_T(Gr_2^+(\mathbb{R}^{2k-1}))$ form a $W-$invariant basis for $H_T(Gr_2^+(\mathbb{R}^{2k-1}))$, we can repeat the same argument of the previous example and produce a basis for $H_T(\mathcal{B}_n)$ consisting of $W-$invariant classes.

Example 3. Let $G = Sp(n)$ be the symplectic group, $K = T$ a maximal torus in G, and $K_1 = S^1 \times Sp(n-1)$; then $G/K_1 \simeq \mathbb{C}P^{2n-1}$. Let $\mathcal{C}_n = G/K$ be the generic coadjoint orbit of type C_n and

$$\pi\colon \mathcal{C}_n \simeq G/K \to G/K_1 \simeq \mathbb{C}P^{2n-1}$$

the natural projection. Then, since the fibers are isomorphic to $\mathcal{C}_{n-1}$, we obtain the following tower of fiber bundles

$$\begin{array}{ccccccccc} \text{pt} & \longrightarrow & \mathbb{C}P^1 & \longrightarrow & \mathcal{C}_2 & \longrightarrow \dots \longrightarrow & \mathcal{C}_{n-1} & \longrightarrow & \mathcal{C}_n \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{C}P^1 & & \mathbb{C}P^3 & & \mathbb{C}P^{2n-3} & & \mathbb{C}P^{2n-1} \end{array}$$

By taking classes represented by powers of the equivariant symplectic form $\omega - \tau \in \Omega^2_T(\mathbb{C}P^{2k-1})$ we obtain a $W-$invariant basis of $H_T(\mathbb{C}P^{2k-1})$, and iterating the same procedure as before, a $W-$invariant basis of $H_T(\mathcal{C}_n)$.

Example 4. Let $G = SO(2n)$, $K = T$ a maximal torus in G, and $K_1 = SO(2) \times SO(2n-2)$. Then $G/K_1 \simeq Gr_2^+(\mathbb{R}^{2n})$, the Grassmannian of oriented two planes in $\mathbb{R}^{2n}$. Let $\mathcal{D}_n = G/K$ be the generic coadjoint orbit of type D_n and

$$\pi\colon \mathcal{D}_n \simeq G/K \to G/K_1 \simeq Gr_2^+(\mathbb{R}^{2n})$$

the natural projection. Then, since the fibers are isomorphic to $\mathcal{D}_{n-1}$, we obtain the following tower of fiber bundles

$$\begin{array}{ccccccccc} \text{pt} & \longrightarrow & \mathbb{C}P^1 \times \mathbb{C}P^1 & \longrightarrow & \mathcal{D}_3 & \longrightarrow \dots \longrightarrow & \mathcal{D}_{n-1} & \longrightarrow & \mathcal{D}_n \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{C}P^1 \times \mathbb{C}P^1 & & Gr_2^+(\mathbb{R}^6) & & Gr_2^+(\mathbb{R}^{2n-2}) & & Gr_2^+(\mathbb{R}^{2n}) \end{array}$$

In [GSZ] we also show how these iterated invariant classes relate to a better known family of classes generating the equivariant cohomology of flag varieties, namely the equivariant Schubert classes.

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