Rational matrix pseudodifferential operators

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Abstract

The skewfield $\mathcal{K}(\partial)$ of rational pseudodifferential operators over a differential field $\mathcal{K}$ is the skewfield of fractions of the algebra of differential operators $\mathcal{K}[\partial]$. In our previous paper we showed that any $H \in \mathcal{K}(\partial)$ has a minimal fractional decomposition $H = AB^{-1}$, where $A, B \in \mathcal{K}[\partial]$, $B \neq 0$, and any common right divisor of A and B is a non-zero element of $\mathcal{K}$. Moreover, any right fractional decomposition of H is obtained by multiplying A and B on the right by the same non-zero element of $\mathcal{K}[\partial]$. In the present paper we study the ring $M_n(\mathcal{K}(\partial))$ of $n \times n$ matrices over the skewfield $\mathcal{K}(\partial)$. We show that similarly, any $H \in M_n(\mathcal{K}(\partial))$ has a minimal fractional decomposition $H = AB^{-1}$, where $A, B \in M_n(\mathcal{K}[\partial])$, B is non-degenerate, and any common right divisor of A and B is an invertible element of the ring $M_n(\mathcal{K}[\partial])$. Moreover, any right fractional decomposition of H is obtained by multiplying A and B on the right by the same non-degenerate element of $M_n(\mathcal{K}[\partial])$. We give several equivalent definitions of the minimal fractional decomposition. These results are applied to the study of maximal isotropicity property, used in the theory of Dirac structures.

1 Introduction

Let $\mathcal{K}$ be a differential field with derivation ∂ and let $\mathcal{K}[\partial]$ be the algebra of differential operators over $\mathcal{K}$. The skewfield $\mathcal{K}(\partial)$ of rational pseudodifferential operators is, by definition, the subskewfield of the skewfield of pseudodifferential operators $\mathcal{K}((\partial^{-1}))$, generated by the subalgebra $\mathcal{K}[\partial]$. In our

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paper [CDSK12] we showed that any rational pseudodifferential operator H has a unique right minimal decomposition $H = AB^{-1}$, where $A, B \in \mathcal{K}[\partial]$, B is a non-zero monic differential operator, and any other right fractional decomposition of H can be obtained by multiplying on the right both A and B by a non-zero differential operator D.

In the present paper we establish a similar result for the ring $M_n(\mathcal{K}(\partial))$ of $n \times n$ matrix rational pseudodifferential operators. Namely we show that any $H \in M_n(\mathcal{K}(\partial))$ has a *right minimal fractional decomposition* $H = AB^{-1}$, where $B \in M_n(\mathcal{K}[\partial])$ is non-degenerate (i.e. has a non-zero Dieudonné determinant $\det(B)$), satisfying one of the following equivalent properties :

(i) $d(B)$ is minimal among all possible right fractional decompositions $H = AB^{-1}$, where $d(B)$ is the order of $\det(B)$;

(ii) A and B are *coprime*, i.e. if $A = A_1 D$ and $B = B_1 D$, with $A_1, B_1, D \in M_n(\mathcal{K}[\partial])$, then D is invertible in $M_n(\mathcal{K}[\partial])$;

(iii) $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$ in any differential field extension of $\mathcal{K}$.

By (i), a right minimal fractional decomposition exists for any $n \times n$ matrix rational pseudodifferential operator H. We prove its uniqueness, namely that all right minimal fractional decompositions can be obtained from each other by multiplication on the right of the numerator and the denominator by an invertible $n \times n$ matrix differential operator D. Moreover, any right fractional decomposition of H can be obtained by multiplying on the right the numerator and the denominator of a minimal right fractional decomposition by the same non-degenerate matrix differential operator.

We derive from these results the following maximal isotropicity property of the minimal fractional decomposition $H = AB^{-1}$, which is important for the theory of Dirac structures [D93], [BDSK09], [DSK12]. Introduce the following bilinear form on the space $\mathcal{K}^n \oplus \mathcal{K}^n$ with values in $\mathcal{K}/\partial\mathcal{K}$:

$$(P_1 \oplus Q_1 | P_2 \oplus Q_2) = \int (P_1.Q_2 + P_2.Q_1),$$

where $\int$ stands for the canonical map $\mathcal{K} \to \mathcal{K}/\partial\mathcal{K}$ and $P.Q$ is the standard dot product. Let A and B be two $n \times n$ matrix differential operators. Define

$$\mathcal{L}_{A,B} = \{B(\partial)P \oplus A(\partial)P | P \in \mathcal{K}^n\}.$$

It is easy to see that, assuming that $\det(B) \neq 0$, the subspace $\mathcal{L}_{A,B}$ of $\mathcal{K}^n \oplus \mathcal{K}^n$ is isotropic if and only if the matrix rational pseudodifferential operator

$H = AB^{-1}$ is skewadjoint. We prove that $\mathcal{L}_{A,B}$ is maximal isotropic if AB^{-1} is a right minimal fractional decomposition of H. Note that $\mathcal{L}_{A,B}$ is independent of the choice of the minimal fractional decomposition due to its uniqueness, mentioned above.

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2 Some preliminaries on rational pseudodifferential operators

Let $\mathcal{K}$ be a differential field of characteristic 0, with a derivation ∂, and let $\mathcal{C} = \mathrm{Ker}\, \partial$ be the subfield of constants. Consider the algebra $\mathcal{K}[\partial]$ (over $\mathcal{C}$) of differential operators. It is a subalgebra of the skewfield $\mathcal{K}((\partial^{-1}))$ of pseudodifferential operators. The subskewfield $\mathcal{K}(\partial)$ of $\mathcal{K}((\partial^{-1}))$, generated by $\mathcal{K}[\partial]$, is called the skewfield of *rational pseudodifferential operators* (see [CDSK12] for details). We have obvious inclusions :

$$\mathcal{K} \subset \mathcal{K}[\partial] \subset \mathcal{K}(\partial) \subset \mathcal{K}((\partial^{-1})).$$

If the derivation acts trivially on $\mathcal{K}$, so that $\mathcal{C} = \mathcal{K}$, letting $\partial = \lambda$, an indeterminate, commuting with elements of $\mathcal{K}$, we obtain inclusions of commutative algebras

$$\mathcal{C} \subset \mathcal{C}[\lambda] \subset \mathcal{C}(\lambda) \subset \mathcal{C}((\lambda^{-1})).$$

It is well known that in many respects the non-commutative algebras $\mathcal{K}[\partial]$ and $\mathcal{K}(\partial)$ "behave" in a very similar way to that of $\mathcal{C}[\lambda]$ and $\mathcal{C}(\lambda)$. Namely, the ring $\mathcal{K}[\partial]$ is right (resp. left) Euclidean, hence any right (resp. left) ideal is principal. Moreover, any two right ideals $A\mathcal{K}[\partial]$ and $B\mathcal{K}[\partial]$ have non-zero intersection $M\mathcal{K}[\partial]$, where $M \neq 0$ is called the least right common multiple of A and B; also $A\mathcal{K}[\partial] + B\mathcal{K}[\partial] = D\mathcal{K}[\partial]$, where D is the greatest right common divisor of A and B. Furthermore, any element H of $\mathcal{K}(\partial)$ has a right fractional decomposition $H = AB^{-1}$, where $B \neq 0$. A right fractional decomposition for which the differential operator B has minimal order is called the minimal fractional decomposition (equivalently, the greatest common divisor of A and B is 1). It is unique up to multiplication of A and B on the right by the same non-zero element of $\mathcal{K}$. Any other fractional decomposition of H is obtained from the minimal one by multiplication of A and B on the right by a non-zero element of $\mathcal{K}[\partial]$. See [CDSK12] for details. Of course all these facts still hold if we replace "right" by "left".

3 The Dieudonné determinant

The Dieudonné determinant of an $n \times n$ matrix pseudodifferential operator $A \in M_n(\mathcal{K}((\partial^{-1})))$ has the form $\det(A) = \det_1(A)\lambda^{d(A)}$ where $\det_1(A) \in \mathcal{K}$, λ is an indeterminate, and $d(A) \in \mathbb{Z}$. It exists and is uniquely defined by the following properties (see [Die43], [Art57]) :

(i) $\det(AB) = \det(A)\det(B)$;

(ii) If A is upper triangular with non-zero diagonal entries $A_{ii} \in \mathcal{K}((\partial^{-1}))$ of degree (or order) $d(A_{ii})$ and leading coefficient $a_i \in \mathcal{K}$, then

$$\det_1(A) = \prod_{i=1}^{n} a_i, \ d(A) = \sum_{i=1}^{n} d(A_{ii}).$$

By definition, $\det(A) = 0$ if one of the A_{ii} is 0.

Note that $\det_1(AB) = \det_1(A)\det_1(B)$. A matrix A whose Dieudonné determinant is non-zero is called *non-degenerate*. In this case the integer $d(A)$ is well defined. It is called the degree of $det(A)$ and of A. Note that $d(AB) = d(A) + d(B)$ if both A and B are non-degenerate.

Lemma 3.1. *(a)Any $A \in M_n(\mathcal{K}[\partial])$ can be written in the form $A = UT$ (resp. TU), where U is an invertible element of $M_n(\mathcal{K}[\partial])$ and $T \in M_n(\mathcal{K}[\partial])$ is upper triangular.*

(b) Any non-degenerate $A \in M_n(\mathcal{K}[\partial])$ can be written in the form $A = U_1 D U_2$, where U_1, U_2 are invertible elements of $M_n(\mathcal{K}[\partial])$ and D is a diagonal $n \times n$ matrix with non-zero entries from $\mathcal{K}[\partial]$.

Proof. Recall that an elementary row (resp. column) operation of a matrix from $M_n(\mathcal{K}[\partial])$ is either a permutation of two of its rows (resp. column), or adding to one row (resp. column) another one, multiplied on the left (resp. right) by an element of $\mathcal{K}[\partial]$. Since the row (resp. column) operations are equivalent to multiplication on the left (reps. right) by the corresponding elementary matrix, the first operation only changes the sign of the determinant and the second does not change it.

In the proof of (a) we may assume that $A \neq 0$, and let j be the minimimal index, for which the j-th column is non-zero. Among all matrices that can be obtained from A by elementary row operations choose the one for which the $(1, j)$-entry is non-zero and has the minimal order. Then, by elementary row operations, using the Euclidean property of $\mathcal{K}[\partial]$, we obtain from A a matrix A_1 such that all entries of the j-th column, except the first one, are

zero. Repeating this process for the $(n-1) \times (n-1)$ submatrix obtained from A_1 by deleting the first row and column, we obtain the decomposition $A = UT$ as in (a).

For the decomposition $A = TU$, we use a similar argument, except that we start from largest j for which the j-th row is non-zero, we perform column operations to have the (j,n)-entry non-zero and of minimal possible order, and then we further make elementary column operations to obtain a matrix A_1 such that all entries of the j-th row are zero, except the last one. The claim follows by induction, after deleting the last row and column.

In order to obtain the decomposition in (b), we use the same argument, except that we choose among all matrices obtained from A by elementary row and column operations the one for which the $(1,1)$-entry is non-zero and has the minimal order (it exists since $\det(A) \neq 0$). $\qquad\square$

Corollary 3.2. *Let $A \in M_n(\mathcal{K}[\partial])$ be a non-degenerate matrix differential operator. Then*

(a) $d(A) \in \mathbb{Z}_+$.

(b) A is an invertible element of the ring $M_n(\mathcal{K}[\partial])$ if and only if $d(A) = 0$.

Remark 3.3. Let $A \in M_n(\mathcal{K}((\partial^{-1})))$ and let A^* be the adjoint matrix pseudodifferential operator. If $det(A) = 0$, then $det(A^*) = 0$. If $det(A) \neq 0$, then $det(A^*) = (-1)^{d(A)} det(A)$. This follows from the obvious fact that A can be brought by elementary row transformations over the skewfield $\mathcal{K}((\partial^{-1}))$ to an upper triangular matrix, and in this case the statement becomes clear.

4   Rational matrix pseudodifferential operators

A matrix $H \in M_n(\mathcal{K}(\partial))$ is called a *rational matrix pseudodifferential operator*. In other words, all the entries of such a matrix have the form $h_{ij} = a_{ij}b_{ij}^{-1}$, $i,j = 1,...,n$, where $a_{ij}, b_{ij} \in \mathcal{K}[\partial]$ and all $b_{ij} \neq 0$. Let $b(\neq 0)$ be the least right common multiple of the b_{ij}'s, so that $b_{ij}.c_{ij} = b$ for some $c_{ij} \neq 0$. Multiplying a_{ij} and b_{ij} on the right by c_{ij}, we obtain $H = A_1 b^{-1}$, where $(A_1)_{ij} = a_{ij}c_{ij}$. In other words H has the right fractional decomposition $H = A_1(b\mathbb{I}_n)^{-1}$. However, among all right fractional decompositions $H = AB^{-1}$, where $A, B \in M_n(\mathcal{K}[\partial])$ and $\det B \neq 0$, this might be not the "best" one.

Definition 4.1. A right fractional decomposition $H = AB^{-1}$, where $A, B \in M_n(\mathcal{K}[\partial])$ and $detB \neq 0$, is called *minimal* if $d(B)$ ($\in \mathbb{Z}_+$) is minimal among all right fractional decompositions of H.

Note that, if $H = AB^{-1}$ is a minimal fractional decomposition, then $0 \leq d(B) \leq d(b)$, where b is the least right common multiple of all the entries of H.

Proposition 4.2. *Let A and B be two non-degenerate $n \times n$ matrix differential operators. Then one can find non-degenerate $n \times n$ matrix differential operators C and D, such that $AC = BD$ (resp. $CA = DB$)*

Proof. By induction on n. We know it is true in the scalar case, see e.g. [CDSK12]. By Lemma 3.1, multiplying on the right by invertible matrices, we may assume that both A and B are upper triangular matrices. Let

$$
A = \left(\begin{array}{c|c} A_1 & U \\ \hline 0 & a \end{array} \right), \quad B = \left(\begin{array}{c|c} B_1 & V \\ \hline 0 & b \end{array} \right),
$$

where $A_1, B_1 \in M_{n-1}(\mathcal{K}[\partial])$ are upper triangular non-degenerate, $U, V \in \mathcal{K}[\partial]^n$, and $a, b \in \mathcal{K}[\partial] \backslash \{0\}$. By the inductive assumption, there exist $C_1, D_1 \in M_{n-1}(\mathcal{K}[\partial])$ non-degenerate, such that $A_1 C_1 = B_1 D_1$, and $c, d \in \mathcal{K}[\partial] \backslash \{0\}$ such that $ac = bd$. Hence, after multiplying on the right A by the block diagonal matrix with C_1 and c on the diagonal, and B by the block diagonal matrix with D_1 and d on the diagonal, we may assume that

$$
A_1 = B_1 \quad \text{and} \quad a = b \,.
$$

Consider the matrix

$$
M = \left(\begin{array}{c|c} A_1 & U - V \\ \hline 0 & 0 \end{array} \right) \in M_n(\mathcal{K}[\partial]) \,.
$$

Viewed over the skewfield $\mathcal{K}(\partial)$, it has a non-zero kernel (since $M : \mathcal{K}(\partial)^n \mapsto \mathcal{K}(\partial)^n$ is not surjective), i.e. there exists a vector $\widetilde{X} = \left(\begin{array}{c} X \\ x \end{array} \right) \in \mathcal{K}(\partial)^n$, where $X \in \mathcal{K}(\partial)^{n-1}$ and $x \in \mathcal{K}(\partial)$, such that $M\widetilde{X} = 0$, i.e.

(4.1) $A_1 X + U x = V x \,.$

Replacing $\widetilde{X}$ by $\widetilde{X}d$, where d is a non-zero common multiple of all the denominators of the entries of $\widetilde{X}$, we may assume that $\widetilde{X} \in \mathcal{K}[\partial]^n$. Note

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also that, since A_1 is non-degenerate, it must be $x \neq 0$. To conclude the proof we just observe that, by (4.1), we have the identity $AE = BF$, where

$$E = \left(\begin{array}{c|c} \mathbb{1}_{n-1} & X \\ \hline 0 & x \end{array} \right), \quad F = \left(\begin{array}{c|c} \mathbb{1}_{n-1} & 0 \\ \hline 0 & x \end{array} \right).$$

$\square$

Remark 4.3. Proposition 4.2 can be derived from Goldie theory (see [MR01, Theorem 2.1.12]), but we opted for a simple direct argument.

Theorem 4.4. *For every matrix differential operators $A, B \in M_n(\mathcal{K}[\partial])$ with $\det(B) \neq 0$, there exist matrices $A_1, B_1, D \in M_n(\mathcal{K}[\partial])$, with $\det B_1 \neq 0, \det D \neq 0$, such that:*

(i) $A = A_1 D,\ B = B_1 D,$

(ii) $\operatorname{Ker} A_1 \cap \operatorname{Ker} B_1 = 0.$

Proof. We will prove the statement by induction on $d(B)$. If $d(B) = 0$, then B is invertible in $M_n(\mathcal{K}[\partial])$ by Corollary 3.2 (and $\operatorname{Ker} B = 0$). In this case the claim holds trivially, taking $D = \mathbb{1}_n$. Clearly, if $P \in M_n(\mathcal{K}[\partial])$ is invertible, then $\operatorname{Ker} A = \operatorname{Ker} PA$. Hence, if P and Q are invertible elements of $M_n(\mathcal{K}[\partial])$, then the statement holds for A and B if and only if it holds for PA and QB. Furthermore, if $R \in M_n(\mathcal{K}[\partial])$ is invertible, replacing D by $R^{-1}D$ we get that the statement holds for A and B if and only if it holds for AR and BR. Therefore, by Lemma 3.1, we may assume, without loss of generality, that A is upper triangular and B is diagonal. If $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$ there is nothing to prove. Let then $F = (f_i)_{i=1}^n$ be a non-zero element of $\operatorname{Ker} A \cap \operatorname{Ker} B$, and let $k \in \{1, \dots, n\}$ be such that $f_k \neq 0, f_{k+1} = \cdots = f_n = 0$. The condition $AF = 0$ gives for $i = 1, \dots, k$,

$$A_{i,1}(\partial)f_1 + \cdots + A_{i,k-1}(\partial)f_{k-1} + A_{ik}(\partial)f_k = 0 \quad \text{in } \mathcal{K}.$$

This implies that there is some $L_i(\partial) \in \mathcal{K}[\partial]$ such that
(4.2)
$$A_{i,1}(\partial) \circ \frac{f_1}{f_k} + \cdots + A_{i,k-1}(\partial) \circ \frac{f_{k-1}}{f_k} + A_{ik}(\partial) = L_i(\partial) \circ \left(\partial - \frac{f_k'}{f_k} \right) \quad \text{in } \mathcal{K}[\partial].$$

Indeed, the LHS above is zero when applied to $f_k \in \mathcal{K}$, hence it must be divisible, on the right, by $\partial - \frac{f_k'}{f_k}$. Similarly, from the condition $BF = 0$ we

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have that $B_{ii}(\partial)f_i = 0$ in K for every $i = 1, \ldots, k$, which implies that there is some $M_i(\partial) \in K[\partial]$ such that

(4.3) $$B_{ii}(\partial) \circ \frac{f_i}{f_k} = M_i(\partial) \circ \left(\partial - \frac{f_k'}{f_k}\right) \quad \text{in } \mathcal{K}[\partial].$$

Let then $A_1, B_1, D \in M_n(\mathcal{K}[\partial])$ be the matrices defined as the matrices $A, B, \mathbb{I}$ with the k-th column replaced, respectively, by the following columns

$$\begin{bmatrix} L_1 \\ \vdots \\ L_{k-1} \\ L_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} M_1 \\ \vdots \\ M_{k-1} \\ M_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -f_1/f_k \\ \vdots \\ -f_{k-1}/f_k \\ \partial - f_k'/f_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It follows from equations (4.2) and (4.3) that $A_1 D = A$ and $B_1 D = B$. Moreover, since $\det D = \lambda$, we have $d(B_1) = d(B) - 1$. The statement follows by the inductive assumption. $\square$

5 Linear closure of a differential field

In this section we define a natural embedding of a differential field in a linearly closed one using the theory of Picard-Vessiot extensions. One may find all relevant definitions and constructions in Chapter 3 of [Mag94].

Recall [DSK11] that a differential field $\mathcal{K}$ is called *linearly closed* if every homogeneous linear differential equation of order $n \geq 1$,

(5.4) $$a_n u^{(n)} + \cdots + a_1 u' + a_0 u = 0,$$

with $a_0, \ldots, a_n$ in $\mathcal{K}$, $a_n \neq 0$, has a non-zero solution $u \in \mathcal{K}$.

It is easy to show that the solutions of equation (5.4) in a differential field $\mathcal{K}$ form a vector space over the field of constant $\mathcal{C}$ of dimension less than or equal to n, and equal to n if $\mathcal{K}$ is linearly closed (see e.g. [DSK11]).

Remark 5.1. In a linearly closed field, it is also true that every inhomogeneous linear differential equation $L(\partial)u = b$ has a solution because the homogeneous differential equation $((1/b)L(\partial)u)' = 0$ has a solution u such that $L(\partial)u \neq 0$ (the solutions of $((1/b)L(\partial)u)' = 0$ form a vector space over the subfield of constants $\mathcal{C}$ of dimension strictly bigger than the one of $KerL$).

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More generally, if $A \in M_n(\mathcal{K}[\partial])$ is a non-degenerate matrix differential operator and $b \in \mathcal{K}^n$, then the inhomogeneous system of linear differential equations in $u = (u_i)_{i=1}^n$,

$$(5.5) \qquad\qquad\qquad A(\partial)u = b \,,$$

admits the affine space (over $\mathcal{C}$) of solutions of dimension less than or equal to $d(A)$, and equal to $d(A)$ if $\mathcal{K}$ is linearly closed. (This follows, for example, from Lemma 3.1(b).)

Definition 5.2. Let $\mathcal{K}$ be a differential field with the subfield of constants $\mathcal{C}$, and let $L \in \mathcal{K}[\partial]$ be a differential operator over $\mathcal{K}$ of order n. A differential field extension $\mathcal{K} \subset \mathcal{L}$ is called a Picard-Vessiot extension with respect to L if there are no new constants in $\mathcal{L}$ and if $\mathcal{L} = \mathcal{K}(y_1, ..., y_n)$, where the y_i are linearly independent solutions over $\mathcal{C}$ of the equation $Ly = 0$.

Proofs of the following two propositions can be found in [Mag94].

Proposition 5.3. *Let $\mathcal{K}$ be a differential field with algebraically closed subfield of constants $\mathcal{C}$ and let L be a differential operator of order n over $\mathcal{K}$. Then there exists a Picard-Vessiot extension of $\mathcal{K}$ with respect to L and it is unique up to isomorphism.*

Proposition 5.4. *If $\mathcal{K} \subset \mathcal{L}$ is an extension of differential fields and $\mathcal{K} \subset \mathcal{E}_i \subset \mathcal{L}$, $i = 1, 2$, are two Picard-Vessiot subextensions of $\mathcal{K}$, then the composite field $\mathcal{E}_1 \mathcal{E}_2$ (i.e. the minimal subfield of $\mathcal{L}$ containing both $\mathcal{E}_1$ and $\mathcal{E}_2$) is a Picard-Vessiot extension of $\mathcal{K}$ as well.*

Definition 5.5. Let $\mathcal{K}$ be a differential field with algebraically closed subfield of constants $\mathcal{C}$. The unique minimal extension $\mathcal{K} \subset \mathcal{L}$ such that

(a) $\mathcal{L}$ is the union of its Picard-Vessiot subextensions of $\mathcal{K}$;

(b) $\mathcal{L}$ contains an isomorphic copy of every Picard-vessiot extension of $\mathcal{K}$,

is called the Picard-Vessiot compositum of $\mathcal{K}$.

It is proved in [Mag94] that the Picard-Vessiot compositum of $\mathcal{K}$ exists, and is unique up to isomorphism.

Definition 5.6. Let $\mathcal{K}$ be a differential field with algebraically closed subfield of constants. Let $\mathcal{K}_0 = \mathcal{K}$ and, for $i \in \mathbb{Z}_+$, let $\mathcal{K}_{i+1}$ be the Picard-Vessiot compositum of $\mathcal{K}_i$. We call $\mathcal{L} = \cup_i \mathcal{K}_i$ the *linear closure* of $\mathcal{K}$ (it is called the successive Picard-Vessiot closure in [Mag94]).

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Remark 5.7. The linear closure is linearly closed.

Remark 5.8. The linear closure of a differential field $\mathcal{K}$ with algebraically closed subfield of constants is the unique, up to isomorphism, minimal linearly closed extension of $\mathcal{K}$ with no new constants. To see this, one needs to show that for any linearly closed extension $\mathcal{L}$ of $\mathcal{K}$ without new constants, one can extend the embedding $\mathcal{K} \hookrightarrow \mathcal{L}$ to an embedding of the Picard-Vessiot compositum of $\mathcal{K}$, $\mathcal{K}_1 \hookrightarrow \mathcal{L}$. By Zorn's lemma one can find a maximal subextension $\mathcal{K} \subset \widetilde{\mathcal{K}} \subset \mathcal{K}_1$ extending the embedding $\mathcal{K} \hookrightarrow \mathcal{L}$. Denote by ϕ the embedding $\widetilde{\mathcal{K}} \hookrightarrow \mathcal{L}$. Suppose that $\widetilde{\mathcal{K}} \subsetneq \mathcal{K}_1$. This means that, by definition of $\mathcal{K}_1$, we have a non-trivial Picard-Vessiot extension $\widetilde{\mathcal{K}} \subset \mathcal{P} \subset \mathcal{K}_1$ for a differential operator L over $\mathcal{K}$. As $\mathcal{L}$ is linearly closed, we can find a Picard-Vessiot extension $\phi(\widetilde{K}) \subset \mathcal{P}_1 \subset \mathcal{L}$ for the same differential operator. By Proposition 5.3, these two Picard-Vessiot extension are isomorphic and one can extend the embedding $\widetilde{\mathcal{K}} \hookrightarrow \mathcal{L}$ to an embedding $\mathcal{P} \hookrightarrow \mathcal{L}$, which is a contradiction.

Lemma 5.9. *Let $\mathcal{K}$ be a differential field with algebraically closed subfield of constants, let $\mathcal{L}$ be its linear closure, and let X be a finite subset of $\mathcal{L}$, not contained in $\mathcal{K}$. Then there is an integer i and a Picard-Vessiot extension $\mathcal{K}_i \subset \mathcal{P} \subset \mathcal{K}_{i+1}$ of $\mathcal{K}_i$ such that $X \subset \mathcal{P}$ but $X \not\subset \mathcal{K}_i$.*

Proof. Take the minimal i, such that $X \subset \mathcal{K}_{i+1}$. Since $\mathcal{K}_{i+1}$ is the Picard-Vessiot compositum of $\mathcal{K}_i$, every element of X lies in a Picard-Vessiot extension of $\mathcal{K}_i$. The claim follows by the fact that the composite of two Picard-Vessiot extension is still a Picard-Vessiot extension (Proposition 5.4). $\qquad\square$

Lemma 5.10. *Let $\mathcal{K} \subset \mathcal{L}$ be a differential field extension, and let $\mathcal{C} \subset \mathcal{D}$ be the corresponding field extension of constants. If $\alpha \in \mathcal{L}$ is algebraic over $\mathcal{C}$, then $\alpha \in \mathcal{D}$ and the minimal monic polynomial for α over $\mathcal{K}$ has coefficients in $\mathcal{C}$.*

Proof. Let $P(x) = x^n + c_1 x^{n-1} + \cdots + c_n \in \mathcal{C}[x]$ be the minimal monic polynomial with coefficients in $\mathcal{C}$ satisfied by α. Letting $x = \alpha$ and applying the derivative ∂ we get $\left(n\alpha^{n-1} + (n-1)c_1\alpha^{n-2} + \cdots + c_n\right)\alpha' = 0$. By minimality of $P(x)$, it must be $\alpha' = 0$, i.e. $\alpha \in \mathcal{D}$.

Similarly, for the second statement, let $Q(x) = x^m + f_1 x^{m-1} + \cdots + f_m \in \mathcal{K}[x]$ be the minimal monic polynomial with coefficients in $\mathcal{K}$ satisfied by α. Letting $x = \alpha$ and applying the derivative ∂ we get $f_1'\alpha^{m-1} + \cdots + f_m' = 0$, which, by minimality of $Q(x)$, implies $f_1, \ldots, f_m \in \mathcal{C}$. $\qquad\square$

Lemma 5.11 (see e.g. [PS03]). *Let $\mathcal{K}$ be a differential field with subfield of constants $\mathcal{C}$. Then elements $f_1, \ldots, f_n \in \mathcal{K}$ are linearly independent over any subfield of $\mathcal{C}$ if and only if their Wronskian is non-zero.*

Lemma 5.12. *(a) Let $\mathcal{K}$ be a differential field with field of constants $\mathcal{C}$, and let $\mathcal{D}$ be an algebraic extension of $\mathcal{C}$. Then $\mathcal{D} \otimes_{\mathcal{C}} \mathcal{K}$ is a differential field with field of constants $\mathcal{D}$.*

(b) Let $\mathcal{K}$ be a differential field with field of constants $\mathcal{C}$, and let $\mathcal{L}$ be a differential field extension of $\mathcal{K}$ with field of constants $\bar{\mathcal{C}}$, the algebraic closure of $\mathcal{C}$. Then, for every algebraic extension $\mathcal{D}$ of $\mathcal{C}$, the differential field $\mathcal{D} \otimes_{\mathcal{C}} \mathcal{K}$ is canonically isomorphic to a differential subfield of $\mathcal{L}$.

Proof. For part (a) we need to prove that every non-zero element $f = \sum_i c_i \otimes f_i \in \mathcal{D} \otimes_{\mathcal{C}} \mathcal{K}$ is invertible. Let $\mathcal{C}[\alpha]$ be a finite extension of $\mathcal{C}$ in $\mathcal{D}$ containing all elements $c_1, \ldots, c_n$, and let $P(x) \in \mathcal{C}[x]$ be the minimal monic polynomial for α over $\mathcal{C}$. By Lemma 5.10, $P(x)$ is an irreducible element of $\mathcal{K}[x]$. Therefore $\mathcal{K}[x]/(P(x))$ is a field, and $f \in \mathcal{C}[\alpha] \otimes_{\mathcal{C}} \mathcal{K} \simeq \mathcal{K}[x]/(P(x))$ is invertible.

Next, we prove part (b). By the universal property of the tensor product, there is a canonical map $\varphi : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{K} \to \mathcal{L}$ given by $\varphi(c \otimes f) = cf$. This is a differential field embedding by part (a). $\square$

Definition 5.13. Let $\mathcal{K}$ be a differential field with subfield of constants $\mathcal{C}$. We know from Lemma 5.12(a) that $\bar{\mathcal{C}} \otimes_{\mathcal{C}} \mathcal{K}$ is a d differential field with subfield of constants $\bar{\mathcal{C}}$. We define the linear closure of $\mathcal{K}$ to be the one of $\bar{\mathcal{C}} \otimes_{\mathcal{C}} \mathcal{K}$.

Recall that the *differential Galois group* $Gal(\mathcal{L}/\mathcal{K})$ of a differential field extension $\mathcal{K} \subset \mathcal{L}$ is defined as the group of automorphisms of $\mathcal{L}$ commuting with ∂ and fixing $\mathcal{K}$. One of the main properties of Picard-Vessiot extensions is the following

Proposition 5.14 ([PS03]). *Let $\mathcal{K}$ be a differential field with algebraically closed subfield of constants $\mathcal{C}$, and let $\mathcal{L}$ be a Picard-Vessiot extension of $\mathcal{K}$. Then, the set of fixed points of the differential Galois group $Gal(\mathcal{L}/\mathcal{K})$ is $\mathcal{K}$.*

6 Minimal fractional decomposition

Given a matrix $A \in M_n(\mathcal{K}[\partial])$, we denote by $\bar{A}$ the same matrix A considered as an endomorphism of $\bar{\mathcal{K}}^n$, where $\bar{\mathcal{K}}$ is the linear closure of $\mathcal{K}$. We have the following possible conditions for a "minimal" fractional decomposition $H = AB^{-1} \in M_n(\mathcal{K}(\partial))$, where $A, B \in M_n(\mathcal{K}[\partial])$ and B is non-degenerate:

(*i*) $d(B)$ is minimal among all possible fractional decompositions of H;

(*ii*) A and B are *coprime*, i.e. if $A = A_1 D$ and $B = B_1 D$, with $A_1, B_1, D \in M_n(\mathcal{K}[\partial])$, then D is invertible in $M_n(\mathcal{K}[\partial])$;

(*iii*) $\operatorname{Ker} \bar{A} \cap \operatorname{Ker} \bar{B} = 0$.

Obviously, condition (*iii*) implies:

(*iii'*) $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$.

Example 6.1. Condition (*iii'*) is weaker than condition (*iii*). Consider, for example, $A = \partial(\partial - 1)$ and $B = \partial - 1$. We have $e^x \in \operatorname{Ker} \bar{A} \cap \bar{B}$, and $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$ unless the differential field $\mathcal{K}$ contains a solution to the equation $u' = u$.

Remark 6.2. Condition (iii) is equivalent to ask that A and B have no common eigenvector with eigenvalue 0 over any differential field extension of $\mathcal{K}$.

Proposition 6.3. *In the "scalar" case $n = 1$, conditions* (*i*), (*ii*) *and* (*iii*) *are equivalent.*

Proof. It follows from [CDSK12] that conditions (*i*) and (*ii*) are equivalent. Moreover, condition (*iii*) implies condition (*ii*) since, if $D \in \mathcal{K}[\partial]$ is not invertible, than it has some root in the linear closure $\bar{\mathcal{K}}$. We are left to prove that condition (*ii*) implies condition (*iii*). Note that, by the Euclidean algorithm, the right greatest common divisor of A and B is independent of the differential field extension of $\mathcal{K}$. Suppose, by contradiction, that $0 \neq f \in \operatorname{Ker} \bar{A} \cap \operatorname{Ker} \bar{B}$, which means that $A = A_1(\partial - \frac{f'}{f})$ and $B = B_1(\partial - \frac{f'}{f})$, for some $A_1, B_1 \in \bar{\mathcal{K}}[\partial]$, so that the right greatest common divisor of A and B is not invertible, contradicting assumption (*ii*). $\square$

Theorem 6.4. (a) *Every $H \in M_n(\mathcal{K}(\partial))$ can be represented as $H = AB^{-1}$, with B non-degenerate, such that* (*iii*) *holds.*

(b) *Conditions* (*i*), (*ii*), *and* (*iii*) *are equivalent. Any fraction which satisfies one of these equivalent conditions is called a minimal fractional decomposition.*

(c) *If $A_0 B_0^{-1}$ is a minimal fractional decomposition of the fraction $H = AB^{-1}$, then one can find a matrix differential operator D such that $A = A_0 D$ and $B = B_0 D$.*

Proposition 6.5. *Theorem 6.4 holds if $\mathcal{K}$ is linearly closed.*

Lemma 6.6. *Assuming that Theorem 6.4(c) holds, let $K = A_1 B_1^{-1}$ be a minimal fractional decomposition, with $A_1, B_1 \in M_k(\mathcal{K}[\partial])$. Let also $V \in \mathcal{K}[\partial]^k$ be such that $AB^{-1}V \in \mathcal{V}[\partial]^\ell$. Then $V = BZ$ for some $Z \in \mathcal{K}[\partial]^k$.*

Proof. After replacing, if necessary, A by AU_1, B by $U_2 BU_1$, and V by $U_2 V$, with U_1 and U_2 invertible elements of $M_n(\mathcal{K}[\partial])$, we can assume by Lemma 3.1 that B is diagonal. If $V = 0$ there is nothing to prove, so let the i-th entry of V be non zero. Consider the matrix $\widetilde{V} \in M_k(\mathcal{K}[\partial])$ be the same as B, with the i-th column replaced by V. Clearly, $\widetilde{V}$ is non-degenerate. By assumption $AB^{-1}\widetilde{V} = K$ lies in $M_n(\mathcal{K}[\partial])$, so that $K\widetilde{V}^{-1}$ is another fractional decomposition for $H = AB^{-1}$. Hence, by Thorem 6.4(c), we have that $\widetilde{V} = B\widetilde{Z}$ for some $\widetilde{Z} \in M_n(\mathcal{K}[\partial])$, so that $V = BZ$, where Z is the i-th column of $\widetilde{Z}$. $\qquad\square$

Proof of Proposition 6.5. Part (a) holds by Theorem 4.4. In part (b), condition (iii) implies condition (ii) since, by assumption, $\mathcal{K}$ is linearly closed. Conversely, let $A, B \in M_n(\mathcal{K}[\partial])$ satisfy condition (ii). By Theorem 4.4 we have $A = A_1 D, B = B_1 D$ with $\operatorname{Ker} A_1 \cap \operatorname{Ker} B_1 = 0$, and by assumption (ii), $D \in M_n(\mathcal{K}[\partial])$ is invertible. Hence, $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$, proving (iii). Furthermore, it is clear that condition (i) implies condition (iii). Indeed if $KerA \cap KerB \neq 0$, then by Theorem 4.4 one can find C, D, E such that $A = CE, B = DE, KerC \cap KerD = 0$ and $d(E) > 0$. Then $AB^{-1} = CD^{-1}$ and $d(D) < d(B)$, contradicting assumption (i). To conclude, we are going to prove, by induction on n, that condition (iii) implies condition (i), and that part (c) holds.

If $n = 1$ the statement holds by Proposition 6.3 and the results in [CDSK12]. Let then $n > 1$ and $A, B \in M_n(\mathcal{K}[\partial])$, with B non degenerate, be such that condition (iii) holds: $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$. Let also $CD^{-1} = AB^{-1}$ be any other fractional decomposition of $H = AB^{-1}$, with $C, D \in M_n(\mathcal{K}[\partial])$, D non degenerate. We need to prove that there exists $T \in M_n(\mathcal{K}[\partial])$ such that $C = AT$ and $D = BT$. (In this case, $d(D) = d(B) + d(T) \geq d(B)$, proving condition (i)).

Firts, note that, if U_i, $i = 1, \ldots, 4$, are invertible elements of $M_n(\mathcal{K}[\partial])$, then $\operatorname{Ker}(U_1 AU_3) \cap \operatorname{Ker}(U_2 BU_3) = 0$, and we have $(U_1 AU_3)(U_2 BU_3)^{-1} = (U_1 CU_4)(U_2 DU_4)^{-1}$. Hence, by Lemma 3.1 we can assume, without loss of generality, that B is diagonal, A, D are upper triangular, and hence $C =$

$AB^{-1}D$ is upper triangula as well. Let then

$$A = \left(\begin{array}{c|c} A_1 & U \\ \hline 0 & a \end{array} \right) \ , \quad B = \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & b \end{array} \right) \ ,$$

$$C = \left(\begin{array}{c|c} C_1 & V \\ \hline 0 & c \end{array} \right) \ , \quad D = \left(\begin{array}{c|c} D_1 & W \\ \hline 0 & d \end{array} \right) \ ,$$

where B_1 is diagonal and A_1, C_1, D_1 are upper triangular $n-1 \times n-1$ matrices with entries in $\mathcal{K}[\partial]$, with B_1 and D_1 non degenerate, U, V, W lie in $\mathcal{K}[\partial]^{n-1}$, and a, b, c, d lie in $\mathcal{K}[\partial]$, with $b, d \neq 0$. By assumption $AB^{-1} = CD^{-1}$, meaning that

(6.6) $\quad A_1 B_1^{-1} = C_1 D_1^{-1} \ , \quad ab^{-1} = cd^{-1} \ , \quad Ub^{-1} = -C_1 D_1^{-1} W d^{-1} + V d^{-1} \ .$

Moreover, the assumption $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$ clearly implies that $\operatorname{Ker} A_1 \cap \operatorname{Ker} B_1 = 0$ (if $X \in \mathcal{K}^{n-1}$ is such that $A_1(X) = B_1(X) = 0$, then $\widetilde{X} = \left(\begin{array}{c} X \\ 0 \end{array} \right) \in \mathcal{K}^n$ lies in $\operatorname{Ker} A \cap \operatorname{Ker} B$). Hence, by the first identity in (6.6) and the inductive assumption, there exists $T_1 \in M_{n-1}(\mathcal{K}[\partial])$ such that

(6.7) $$C_1 = A_1 T_1 \ , \quad D_1 = B_1 T_1 \ .$$

The main problem is that we do not know that $\operatorname{Ker} a \cap \operatorname{Ker} b = 0$ (it is false in general), hence we cannot conclude, yet, that $c = at$ and $d = bt$ for some $t \in \mathcal{K}[\partial]$. Let then ef^{-1} be a minimal fractional decomposition of $ab^{-1} = cd^{-1}$. By the $n = 1$ case we know that there exist $p, q \in \mathcal{K}[\partial]$ such that

(6.8) $$a = ep \ , \quad b = fp \ , \quad c = eq \ , \quad d = fq \ ,$$

and let $k \in \mathcal{K}[\partial]$ be a right greatest common divisor of p ad q, i.e. there exist $s, t, i, j \in \mathcal{K}[\partial]$ such that

(6.9) $$p = ks \ , \quad q = kt \ , \quad si + tj = 1 \ .$$

Eventually we will want to prove that we can choose $k = p$ (i.e. $s = 1$, $i = 1$ and $j = 0$). Using the identities (6.7), (6.8) and (6.9), we can rewrite the third equation in (6.6) as follows

$$Us^{-1} = -A_1 B_1^{-1} W t^{-1} + V t^{-1} \ ,$$

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and multiplying each side of the above equation by each side of the identity $1 - si = tj$, we get

(6.10) $$U + A_1 B_1^{-1} W j s = (U i + V j) s.$$

Since $A_1 B_1^{-1}$ is a minimal fractional decomposition, we get, by the inductive assumption and Lemma 6.6, that there exists $Z \in \mathcal{K}[\partial]^{n-1}$ such that

(6.11) $$W j s = B_1 Z, \quad U + A_1 Z = (U i + V j) s.$$

Let $x \in \mathcal{K}$ be such that $s(x) = 0$. Letting $X = \begin{pmatrix} Z(x) \\ x \end{pmatrix} \in \mathcal{K}^n$, we get

$$A(X) = \begin{pmatrix} A_1 Z(x) + U(x) \\ a(x) \end{pmatrix} = \begin{pmatrix} (U i + V j) s(x) \\ e k s(x) \end{pmatrix} = 0,$$
$$B(X) = \begin{pmatrix} B_1 Z(x) \\ b(x) \end{pmatrix} = \begin{pmatrix} W j s(x) \\ f k s(x) \end{pmatrix} = 0.$$

Hence, since by assumption $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$, it follows that $\operatorname{Ker} s = 0$. Namely, since $\mathcal{K}$ is linearly closed, s is a scalar, that we can shoose to be 1. In conclusion, we get, as we wanted, that $k = p$ and $q = pt$, so that, by (6.8),

(6.12) $$c = at, \quad d = bt.$$

Going back to the third equation in (6.6), we then get

(6.13) $$U t = -A_1 B_1^{-1} W + V.$$

Again, by the inductive assumption on the minimality of $A_1 B_1^{-1}$ and Lemma 6.6, it follows that there exists $Z \in \mathcal{K}[\partial]^{n-1}$ such that

(6.14) $$W = B_1 Z, \quad V = U t + A_1 Z.$$

Hence, letting

$$T = \left(\begin{array}{c|c} T_1 & Z \\ \hline 0 & t \end{array} \right) \in M_n(\mathcal{K}[\partial]),$$

we get that $C = AT$ and $D = BT$, completing the proof. $\square$

Proposition 6.7. *Part (a) of Theorem 6.4 holds, namely if A and B are two $n \times n$ matrix differential operators with B non-degenerate, then we can find $n \times n$ matrix differential operators C, D and E, such that $A = CE$, $B = DE$ and $\operatorname{Ker} \bar{C} \cap \operatorname{Ker} \bar{D} = 0$.*

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Proof. First, assume that the subfield of constants of $\mathcal{K}$ is algebraically closed. By Lemma 3.1, we may assume that is A upper triangular and B is diagonal. Consider a minimal fractional decomposition CD^{-1} of the fraction AB^{-1} in the linear closure of $\mathcal{K}$. By Lemma 3.1(a), we can choose C and D to be upper triangular matrix differential operators. We may assume that all the diagonal entries of D are monic and, using elementary column transformations, that $d(D_{ij}) < d(D_{ii})$ for all $i < j$. Since the linear closure is the union of the iterate Picard-Vessiot compositum of $\mathcal{K}$, all the coefficients of the entries of C and of D lie in some iterate Picard-Vessiot compositum of $\mathcal{K}$. Take i minimal such that $\mathcal{K}_i$ satisfies this property. Assume $i \neq 0$. By Lemma 5.9, all the coefficients of the entries of C and D lie in some Picard-Vessiot subextension $\mathcal{K}_{i-1} \subset \mathcal{P} \subset \mathcal{K}_i$. Pick an automorphism ϕ of this extension. By Theorem 6.4, the fractional decomposition $\phi(C)\phi(D)^{-1} = CD^{-1}$ is still a minimal fractional decomposition because $d(\phi(D)) = d(D)$. So C (resp D) and $\phi(C)$ (resp $\phi(D)$) are equal up to right multiplication by an invertible upper triangular matrix differential operator E. As all the diagonal entries of $\phi(D)$ are monic and $deg(\phi(D)_{ij}) < deg(\phi(D)_{ii})$ for all $i < j$, E has to be the identity matrix. Hence $C = \phi(C)$ and $D = \phi(D)$ for all ϕ. It follows, by Proposition 5.14, that all the coefficients of the entries of C and D actually lie in $\mathcal{K}_{i-1}$, which is a contradiction. So $i = 0$ and all the coefficients of C and D are differential operators over $\mathcal{K}$.

In the general case, one can find C, D and E satisfying the assumptions of the proposition, whose entries are differential operators a priori over $\mathcal{K} \otimes_\mathcal{C} \bar{\mathcal{C}}$. So all the coefficients of the entries of C and D lie in a Galois extension $\mathcal{K} \subset \mathcal{G}$. As the extension of the derivation to an algebraic extension is unique, all automorphisms commute with the derivation. Hence, using the same argument as above with the usual Galois theory, we obtain that the entries of C and D, hence those of E, are actually differential operators over $\mathcal{K}$. $\square$

Proof of Theorem 6.4. By Proposition 6.7, condition (i) implies condition (iii). Let AB^{-1} be a fractional decomposition, satisfying (iii). Then by Proposition 6.5 it satisfies (i) as a fraction of matrix differential operators over the linear closure of $\mathcal{K}$, hence a fortiori over $\mathcal{K}$. The implication $(iii) \Rightarrow (ii)$ is clear by definition of a linearly closed field and $(ii) \Rightarrow (iii)$ follows from Proposition 6.7. Hence part (b) of the theorem holds. If $A_0 B_0^{-1}$ i s a minimal fractional decomposition of the fraction AB^{-1}, then there is a matrix differential operator D over the linear closure of $\mathcal{K}$ such that $A = A_0 D$ and $B = B_0 D$. Since B is non-degenerate, $D = B_0^{-1} B$ is actually a matrix differential operator over $\mathcal{K}$. $\square$

Remark 6.8. We have the following two more equivalent definitions for a minimal fracitonal decomposition $H = AB^{-1}$, with $A, B \in M_n(\mathcal{K}[\partial])$:

(iv) the "Bezout identity" holds: $CA + DB = I$ for some $C, D \in M_n(\mathcal{K}[\partial])$,

(v) A and B have kernels intersecting trivially over any differential field extension of $\mathcal{K}$.

Condition (v) obviously implies condition (iii), and, also, condition (iv) implies condition (v) since the identity matrix has zero kernel over any field extension of $\mathcal{K}$. To prove that (iii) implies (iv), we use the fact that any left ideal of $M_n(\mathcal{K}[\partial])$ is principal (cf. [MR01, Prop.4.10, p.82]). But if $E \in M_n(\mathcal{K}[\partial])$ is a generator of the left ideal generated by A and B, then by condition (iii) we have that $\mathrm{Ker}(\bar{E}) = 0$, and therefore E must be invertible.

7   Maximal isotropicity of $\mathcal{L}_{A,B}$

Let $A, B \in M_n(\mathcal{K}[\partial])$ with $\det B \neq 0$. Recall that the subspace $\mathcal{L}_{A,B} \subset \mathcal{K}^n \oplus \mathcal{K}^n$ (defined in the Introduction) is isotropic if and only if $AB^{-1} \in M_n(\mathcal{K}(\partial))$ is skewadjoint, which in turn is equivalent to the following condition ([DSK12],Proposition 6.5):

$$(7.15) \qquad\qquad\qquad A^*B + B^*A = 0 \,.$$

Hence, $\mathcal{L}_{A,B} \subset \mathcal{K}^n \oplus \mathcal{K}^n$ is maximal isotropic if and only if (7.15) and the following condition hold:

(vi) if $G, H \in \mathcal{K}^n$ are such that $A^*H + B^*G = 0$, then there exists $F \in \mathcal{K}^n$ such that $G = AF$ and $H = BF$.

Theorem 7.1. *Suppose that $A, B \in M_n(\mathcal{K}[\partial])$ with $\det B \neq 0$ satisfy equation (7.15). If AB^{-1} is a minimal fractional decomposition, then $\mathcal{L}_{A,B} \subset \mathcal{K}^n \oplus \mathcal{K}^n$ is a maximal isotropic subspace. Namely, condition (iii) of Section 6 implies condition (vi).*

Proof. First, we prove the statement in the case when the differential field $\mathcal{K}$ is linearly closed. Due to equation (7.15), A maps $\mathrm{Ker}\, B$ to $\mathrm{Ker}\, B^*$. Since, by assumption, $\mathrm{Ker}\, A \cap \mathrm{Ker}\, B = 0$, this map is injective. Moreover, since $\mathrm{Ker}\, B$ and $\mathrm{Ker}\, B^*$ have the same dimension (equal to $d(B)$, by Lemma 3.1(b)), we conclude that we have a bijective map:

$$(7.16) \qquad\qquad\qquad A : \ \mathrm{Ker}\, B \ \xrightarrow{\sim} \mathrm{Ker}\, B^* \,.$$

Let $G, H \in \mathcal{K}^n$ be such that $A^*H + B^*G = 0$. Since $\det B \neq 0$, we have that $B : \mathcal{K}^n \to \mathcal{K}^n$ is surjective (by Lemma 3.1(b)). Hence we can choose $F_1 \in \mathcal{K}^n$ such that $G = BF_1$. Due to equation (7.15), we get

$$B^*AF_1 = -A^*BF_1 = -A^*G = B^*H \, .$$

Hence, $H - AF_1 \in \operatorname{Ker} B^*$, and by (7.16) there exists $F_2 \in \operatorname{Ker} B$ such that $AF_2 = H - AF_1$. So, $H = A(F_1 + F_2)$ and $G = BF_1 = B(F_1 + F_2)$, proving condition (vi).

Next, we prove the claim for a differential field $\mathcal{K}$ with algebraically closed subfield of constants. Since, by assumption, $\operatorname{Ker} \bar{A} \cap \operatorname{Ker} \bar{B} = 0$, we know by the previous result that there is a solution $F \in \mathcal{L}^n$ to the equations $G = AF$ and $H = BF$, where $\mathcal{L}$ is the linear closure of $\mathcal{K}$, and this solution is obviously unique (since two solutions differ by an element in $\operatorname{Ker} \bar{A} \cap \operatorname{Ker} \bar{B}$). We will next use a standard differential Galois theory argument to conclude that this solution F must lie in $\mathcal{K}^n$.

By definition of the linear closure, all the entries of F lie in some iterate Picard-Vessiot compositum of $\mathcal{K}$. Take i minimal such that $\mathcal{K}_i$ satisfies this property. Assume $i \neq 0$. By Lemma lem:5.9, all entries of F lie in some Picard-Vessiot subextension $\mathcal{K}_{i-1} \subset \mathcal{P} \subset \mathcal{K}_i$. As the solution F is unique in the linear closure, it is fixed by all the differential automorphisms of the extension $\mathcal{K}_{i-1} \subset \mathcal{P}$, hence it lies in $\mathcal{K}_{i-1}^n$, which contradicts the minimality of i. In the general case, we know from the previous discussion that there is a unique solution F in $(\mathcal{K} \otimes_\mathcal{C} \bar{\mathcal{C}})^n$. Hence all the entries of F lie in a Galois extension $\mathcal{G}$ of $\mathcal{K}$. We know that there is a unique way to extend a derivation to an algebraic extension, so all algebraic automorphisms of this Galois extension are also differential automorphisms. Hence F is fixed under the action of $Gal(\mathcal{G}/\mathcal{K})$ which means that it lies in $\mathcal{K}^n$. $\qquad\square$

Proposition 7.2. *If a fraction AB^{-1} of matrix differential operators satisfies $A^*B + B^*A = 0$, and $AB^{-1} = (A_0 D)(B_0 D)^{-1}$ with $\operatorname{Ker}\bar{A}_0 \cap \operatorname{Ker}\bar{B}_0 = 0$, then $\mathcal{L}_{A,B}$ is maximal isotropic if and only if D is surjective and $\operatorname{Ker} D^* \cap (\operatorname{Im} A_0{}^* + \operatorname{Im} B_0{}^*) = 0$.*

Proof. Assume that D is surjective and $\operatorname{Ker} D^* \cap (\operatorname{Im} A_0{}^* + \operatorname{Im} B_0{}^*) = 0$. We have $A_0{}^* B_0 + B_0{}^* A_0 = 0$, hence $\mathcal{L}_{A_0, B_0}$ is maximal isotropic, since $A_0 B_0^{-1}$ is a minimal fractional decomposition. Let $f, g \in \mathcal{K}^n$ be such that $A^* f + B^* g = 0$. Since $\operatorname{Ker} D^* \cap (\operatorname{Im} A_0{}^* + \operatorname{Im} B_0{}^*) = 0$, we get that $A_0{}^* f + B_0{}^* g = 0$. By maximal isotropicity of $\mathcal{L}_{A_0, B_0}$, we can find some $h \in \mathcal{K}^n$ such that $f = B_0 h$

and $g = A_0 h$. Since D is surjective, there is $k \in \mathcal{K}^n$ such that $h = Dk$. So $f = Ak$ and $g = Bk$, hence $\mathcal{L}_{A,B}$ is maximal isotropic.

Conversely, assume that $\mathcal{L}_{A,B}$ is maximal isotropic. First, we prove that D is surjective. Take $f \in \mathcal{K}^n$. Multiplying on the left by D^* the equation $A_0{}^* B_0 + B_0{}^* A_0 = 0$ and evaluating it at f, we get that $A^* B_0 f + B^* A_0 f = 0$, hence by maximal isotropicity of $\mathcal{L}_{A,B}$, $B_0 f = Bg$ and $A_0 f = Ag$ for some $g \in \mathcal{K}^n$. Therefore $f - Dg \in Ker A_0 \cap Ker B_0 = 0$, hence $f = Dg$. So D is surjective. Next, take $x \in Ker D^* \cap (Im A_0{}^* + Im B_0{}^*)$. In particular, $x = A_0{}^* g + B_0{}^* h$ for some $g, h \in \mathcal{K}^n$ and $A^* g + B^* h = 0$. By maximal isotropicity of $\mathcal{L}_{A,B}$, we see that $g = Bk$ and $h = Ak$ for some $k \in \mathcal{K}^n$. Multiplying the equation $A_0^* B_0 + B_0^* A_0 = 0$ by Dk on the right, we get that $x = 0$. $\qquad\square$

Remark 7.3. In the linearly closed case, a skewadjoint fraction AB^{-1} is a minimal fractional decomposition if and only if $\mathcal{L}_{A,B}$ is maximal isotropic. Indeed, since $Ker D^* \cap (Im A_0^* + Im B_0^*) = 0$ and $det(B_0^*) = \pm det(B_0) \neq 0$, we see that B_0^* is surjective, hence $Ker D^* = 0$. Therefore $d(D^*) = 0 = d(D)$ and D is invertible. Here we used Corollary 3.2 and Remark 3.3.

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