On the Steinberg character of a semisimple p-adic group

ON THE STEINBERG CHARACTER OF A SEMISIMPLE p-ADIC GROUP

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Dedicated to Robert Steinberg on the occasion of his 90-th birthday

1. Introduction

1.1. Let K be a nonarchimedean local field and let $\underline{K}$ be a maximal unramified field extension of K. Let $\mathcal{O}$ (resp. $\underline{\mathcal{O}}$) be the ring of integers of K (resp. $\underline{K}$) and let $\mathfrak{p}$ (resp. $\underline{\mathfrak{p}}$) be the maximal ideal of $\mathcal{O}$ (resp. $\underline{\mathcal{O}}$). Let $\underline{K}^* = \underline{K} - \{0\}$. We write $\mathcal{O}/\mathfrak{p} = F_q$, a finite field with q elements, of characteristic p.

Let G be a semisimple almost simple algebraic group defined and split over K with a given $\mathcal{O}$-structure compatible with the K-structure.

If V is an admissible representation of $G(K)$ of finite length, we denote by ϕ_V the character of V in the sense of Harish-Chandra, viewed as a $\mathbf{C}$-valued function on the set $G(K)_{rs} := G_{rs} \cap G(K)$. (Here G_{rs} is the set of regular semisimple elements of G and $\mathbf{C}$ is the field of complex numbers.)

In this paper we study the restriction of the function ϕ_V to:

(a) a certain subset $G(K)_{vr}$ of $G(K)_{rs}$, that is to the set of very regular elements in $G(K)$ (see 1.2), in the case where V is the Steinberg representation of $G(K)$ and

(b) a certain subset $G(K)_{svr}$ of $G(K)_{vr}$, that is to the set of split very regular elements in $G(K)$ (see 1.2), in the case where V is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup.

In case (a) we show that $\phi_V(g)$ with $g \in G(K)_{rs}$ is of the form $\pm q^n$ with $n \in \{0, -1, -2, \dots\}$ (see Corollary 3.4) with more precise information when $g \in G(K)_{svr}$ (see Theorem 2.2) or when $g \in G(K)_{cvr}$ (see Theorem 3.2); in case (b) we show (with some restriction on characteristic) that $\phi_V(g)$ with $G(K)_{svr}$ can be expressed as a trace of a certain element of an affine Hecke algebra in an irreducible module (see Theorem 4.3).

Note that the Steinberg representation $\mathbf{S}$ is an irreducible admissible representation of $G(K)$ with a one dimensional subspace invariant under an Iwahori

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subgroup on which the affine Hecke algebra acts through the "sign" representation, see [MA], [S]. This is a p-adic analogue of the Steinberg representation [St] of a reductive group over F_q. In [R], it is proved that $\phi_{\mathbf{S}}(g) \neq 0$ for any $g \in G(K)_{rs}$.

1.2. Let $g \in G_{rs} \cap G(\underline{K})$. Let $T' = T'_g$ be the maximal torus of G that contains g. We say that g is very regular (resp. compact very regular) if T' is split over $\underline{K}$ and for any root α with respect to T' viewed as a homomorphism $T'(\underline{K}) \to \underline{K}^*$ we have

$\alpha(g) \notin (1 + \underline{\mathfrak{p}})$ (resp. $\alpha(g) \in \underline{\mathcal{O}}$, $\alpha(g) \notin (1 + \underline{\mathfrak{p}})$).

Let $G(\underline{K})_{vr}$ (resp. $G(\underline{K})_{cvr}$) be the set of elements in $G(\underline{K})$ which are very regular (resp. compact very regular). We write $G(K)_{vr} = G(\underline{K})_{vr} \cap G(K)$, $G(K)_{cvr} = G(\underline{K})_{cvr} \cap G(K)$. Let $G(K)_{svr}$ be the set of all $g \in G(K)_{vr}$ such that T'_g is split over K.

1.3. *Notation.* Let $K^* = K - \{0\}$ and let $v : K^* \to \mathbf{Z}$ be the unique (surjective) homomorphism such that $v(\mathfrak{p}^n - \mathfrak{p}^{n+1}) = n$ for any $n \in \mathbf{N}$. For $a \in K^*$ we set $|a| = q^{-v(a)}$.

We fix a maximal torus T of G defined and split over K. Let Y (resp. X) be the group of cocharacters (resp. characters) of the algebraic group T. Let $\langle , \rangle : Y \times X \to \mathbf{Z}$ be the obvious pairing. Let $R \subset X$ be the set of roots of G with respect to T, let R^+ be a set of positive roots for R and let Π be the set of simple roots of R determined by R^+. We write $\Pi = \{\alpha_i; i \in I_0\}$. Let $R^- = R - R^+$. Let Y^+ (resp. Y^{++}) be the set of all $y \in Y$ such that $\langle y, \alpha \rangle \geq 0$ (resp. $\langle y, \alpha \rangle > 0$) for all $\alpha \in R^+$ We define $2\rho \in X$ by $2\rho = \sum_{a \in R^+} \alpha$.

We have canonically $T(K) = K^* \otimes Y$; we define a homomorphism $\chi : T(K) \to Y$ by $\chi(\lambda \otimes y) = v(\lambda) y$ for any $\lambda \in K^*, y \in Y$. For any $y \in Y$ we set $T(K)_y = \chi^{-1}(y)$. For $y \in Y$ let $T(K)^{\spadesuit}_y = T(K)_y \cap G(K)_{svr}$. Note that if $y \in Y^{++}$ then $T(K)^{\spadesuit}_y = T(K)_y$.

For each $\alpha \in R$ let U_α be the corresponding root subgroup of G.

Let $G(K)'$ be the derived subgroup of $G(K)$.

2. Calculation of $\phi_{\mathbf{S}}$ on $G_(K)_{svr}$

2.1. Let $\mathcal{W} \subset \mathrm{Aut}(T)$ be the Weyl group of G regarded as a Coxeter group; for $i \in I_0$ let s_i be the simple reflection in $\mathcal{W}$ determined by α_i. We can also view $\mathcal{W}$ as a subgroup of $\mathrm{Aut}(Y)$ or $\mathrm{Aut}(X)$. Let $w = w_0$ be the longest element of $\mathcal{W}$. For any $J \subset I_0$ let $\mathcal{W}_J$ be the subgroup of $\mathcal{W}$ generated by $\{s_i; i \in J\}$ and let R_J be the set of $\alpha \in R$ such that $\alpha = w(\alpha_i)$ for some $w \in \mathcal{W}_J, i \in J$. Let $R_J^+ = R_J \cap R^+$, $R_J^- = R_J - R_J^+$.

Let $\mathfrak{g}$ be the Lie algebra of G; let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of T. For any $J \subset I_0$ let $\mathfrak{l}_J$ be the Lie subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{t}$ and by the root spaces corresponding to roots in R_J; let $\mathfrak{n}_J$ be the Lie subalgebra of $\mathfrak{g}$ spanned by the root spaces corresponding to roots in $R^+ - R_J^+$.

According to [C1], ϕ is an alternating sum of characters of representations induced from one dimensional representations of various parabolic subgroups of G

defined over K. From this one can deduce that, if $t \in T(K) \cap G(K)_{rs}$, then

$$\phi_{\mathbf{S}}(t) = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in {}^J\mathcal{W}} \delta_J(w(t))^{1/2} D_{I,J}(w(t))^{-1/2}$$

where for any $J \subset I$ and $t' \in T(K) \cap G(K)_{rs}$ we set

$$D_{I,J}(t') = |\det(1 - \mathrm{Ad}(t')|_{\mathfrak{g}/\mathfrak{l}_J})|,$$

$$\delta_J(t') = |\det(\mathrm{Ad}(t')|_{\mathfrak{n}_J})|,$$

and ${}^J\mathcal{W}$ is a set of representatives for the cosets $\mathcal{W}_J \backslash \mathcal{W}$. (It will be convenient to assume that ${}^J\mathcal{W}$ is the set of representatives of minimal length for the cosets $\mathcal{W}_J \backslash \mathcal{W}$.) Here for a real number $a \geq 0$ we denote by $a^{1/2}$ or $\sqrt{a}$ the ≥ 0 square root of a. We have the following result. (We write ϕ instead of $\phi_{\mathbf{S}}$.)

Theorem 2.2. *Let $y \in Y^+$ and let $t \in T(K)^{\spadesuit}_y$. Then $\phi(t) = q^{-\langle y, 2\rho \rangle}$.*

2.3. More generally let $t \in T(K)^{\spadesuit}_y$ where $y \in Y$. By a standard property of Weyl chambers there exists $w \in \mathcal{W}$ such that $w(y) \in Y^+$. Let $t_1 = w(t)$. Then the theorem is applicable to t_1 and we have $\phi(t) = \phi(t_1) = q^{-\langle w(y), 2\rho \rangle}$.

2.4. Let $y' = w_0(y), t' = w_0(t)$. We have $\phi_{\mathbf{S}}(t) = \phi_{\mathbf{S}}(t')$, $t' \in T(K)^{\spadesuit}_{y'}$, $-y' \in Y^+$. We show:

(a) if $\beta \in R^+$ then $v(1 - \beta(t'))) = v((\beta(t'))$; if $\beta \in R^-$ then $v(1 - \beta(t'))) = 0$.
Assume first that $\beta \in R^+$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) < 0$ (since $\langle y', \beta \rangle \neq 0$, $\langle y', \beta \rangle \leq 0$) hence $v(1 - \beta(t'))) = v((\beta(t'))$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1 - \beta(t'))) = 0 = v((\beta(t'))$ as required.

Assume next that $\beta \in R^-$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) > 0$ (since $\langle y', \beta \rangle \neq 0$, $\langle y', \beta \rangle \geq 0$) hence $v(1 - \beta(t'))) = 0$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1 - \beta(t'))) = 0$ as required.

For any $w \in \mathcal{W}, J \subset I$ we have:

$$D_{I,J}(w(t')) = \prod_{\alpha \in R - R_J} q^{-v(1-\alpha(w(t')))}$$

$$= \prod_{\alpha \in R - R_J; w^{-1}\alpha \in R^+} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R - R_J; w^{-1}\alpha \in R^+} q^{-\langle y', w^{-1}\alpha \rangle},$$

$$\delta_J(w(t')) = \prod_{\alpha \in R^+ - R_J^+} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^+ - R_J^+} q^{-\langle y', w^{-1}\alpha \rangle},$$

$$D_I(t') = \prod_{\alpha \in R^+} q^{-\langle y', \alpha \rangle}.$$

(We have used (a) with $\beta = w^{-1}(\alpha)$.) We see that

$$\phi(t) = \phi(t') = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in {}^J\mathcal{W}} \sqrt{q}^{-\langle y', x_{w,J} \rangle}$$

where for $w \in {}^J\mathcal{W}$ we have

$$\begin{aligned} x_{w,J} &= \sum_{\alpha \in R^+ - R^+_J} w^{-1}\alpha - \sum_{\alpha \in R - R_J; w^{-1}\alpha \in R^+} w^{-1}\alpha \\ &= \sum_{\alpha \in R^+ - R^+_J; w^{-1}(\alpha) \in R^-} w^{-1}\alpha - \sum_{\alpha \in R^- - R^-_J; w^{-1}(\alpha) \in R^+} w^{-1}\alpha \\ &= 2 \sum_{\alpha \in R^+ - R^+_J; w^{-1}\alpha \in R^-} w^{-1}\alpha \in X. \end{aligned}$$

For $w \in {}^J\mathcal{W}$ we have $\alpha \in R^+_J \implies w^{-1}\alpha \in R^+$ hence

$$\sum_{\alpha \in R^+ - R^+_J; w^{-1}\alpha \in R^-} w^{-1}\alpha = \sum_{\alpha \in R^+; w^{-1}\alpha \in R^-} w^{-1}\alpha$$

so that $x_{w,J} = x_w$ where

$$x_w = 2 \sum_{\alpha \in R^+; w^{-1}\alpha \in R^-} w^{-1}\alpha \in X.$$

Thus we have

$$\phi(t) = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in {}^J\mathcal{W}} \sqrt{q}^{-\langle y', x_w \rangle} = \sum_{w \in \mathcal{W}} c_w \sqrt{q}^{-\langle y', x_w \rangle}$$

where for $w \in \mathcal{W}$ we set

$$c_w = \sum_{J \subset I; w \in {}^J\mathcal{W}} (-1)^{\sharp J}.$$

For $w \in \mathcal{W}$ let $\mathcal{L}(w) = \{i \in I; s_i w > w\}$ where $<$ is the standard partial order on $\mathcal{W}$. For $J \subset I$ we have $w \in {}^J\mathcal{W}$ if and only if $J \subset \mathcal{L}(w)$. Thus,

$$c_w = \sum_{J \subset \mathcal{L}(w)} (-1)^{\sharp J}$$

and this is 0 unless $\mathcal{L}(w) = \emptyset$ (that is $w = w_0$) when $c_w = 1$. Note also that $x_{w_0} = -4\rho$. Thus we have

$$\phi(t) = c_{w_0} \sqrt{q}^{-\langle y', x_{w_0} \rangle} = q^{\langle y', 2\rho \rangle} = q^{-\langle y, 2\rho \rangle}.$$

Theorem 2.2 is proved.

2.5. Assume now that $\tau \in T(K)$ satisfies the following condition: for any $\alpha \in R$ we have $\alpha(\tau)-1 \in \mathfrak{p}-\{0\}$ so that $\alpha(\tau)-1 \in \mathfrak{p}^{n_\alpha}-\mathfrak{p}^{n_\alpha+1}$ for a well defined integer $n_\alpha \geq 1$. Note that $n_{-\alpha}=n_\alpha$ and $v(1-\alpha(\tau))=n_\alpha \geq 1$ for all $\alpha \in R$. Hence

$$\phi(\tau)=\sum_{J\subset I}(-1)^{\sharp J}\sum_{w\in {}^J\mathcal{W}} q^{\sum_{\alpha\in R} n_\alpha/2-\sum_{\alpha\in R_J} n_{w^{-1}(\alpha)}/2}.$$

Thus,

$$\phi(\tau)=\sharp(\mathcal{W})q^{\sum_{\alpha\in R} n_\alpha/2}+\text{strictly smaller powers of } q.$$

In the case where K is the field of power series over F_q, the leading term $\sharp(W)q^{\sum_{\alpha\in R} n_\alpha/2}$ is equal to $\sharp(\mathcal{W})q^m$ where m is the dimension of the "variety" of Iwahori subgroups of $G(\underline{K})$ that contain the topologically unipotent element τ (see [KL2]).

3. Calculation of $\phi_{\mathbf{S}}$ on $G_(K)_{vr}$

3.1. We will again write ϕ instead of $\phi_{\mathbf{S}}$. In this section we assume that we are given $\gamma \in G(K)_{vr}$. Let $T'=T'_\gamma$. Note that T' is defined over K; let A' be the largest K-split torus of T'. For any parabolic subgroup P of G defined over K such that $\gamma \in P$ we set $\delta_P(\gamma)=|\det(\mathrm{Ad}(\gamma)|_{\mathfrak{n}})|$ where $\mathfrak{n}$ is the Lie algebra of the unipotent radical of P.

Let $\mathcal{X}$ be the set of all pairs (P,A) where P is a parabolic subgroup of G defined over K and A is the unique maximal K-split torus in the centre of some Levi subgroup of P defined over K; then that Levi subgroup is uniquely determined by A and is denoted by M_A. Let $\mathcal{X}'=\{(P,A)\in\mathcal{X}; A\subset A'\}$. According to Harish-Chandra [H] we have

(a) $$\phi(\gamma)=(-1)^{\dim T}\sum_{(P,A)\in\mathcal{X}'}(-1)^{\dim A}\delta_P(\gamma)^{1/2}D_{G/M_A}(\gamma)^{-1/2}$$

where $D_{G/M_A}(\gamma)=|\det(1-\mathrm{Ad}(\gamma)|_{\mathfrak{g}/\mathfrak{l}})|$ (we denote by $\mathfrak{l}$ the Lie algebra of M_A).

Theorem 3.2. *Assume in addition that $\gamma\in G(K)_{cvr}$. Then* $\phi(\gamma)=(-1)^{\dim T-\dim A'}$.

From our assumptions we see that for any $(P,A)\in\mathcal{X}'$ we have $\delta_P(\gamma)=1=D_{G/M_A}(\gamma)$. Hence 3.1(a) becomes

$$\phi(\gamma)=(-1)^{\dim T}\sum_{(P,A)\in\mathcal{X}'}(-1)^{\dim A}.$$

Let $\mathcal{Y}$ be the group of cocharacters of A' and let $\mathfrak{H}=\mathcal{Y}\otimes\mathbf{R}$. The real vector space $\mathfrak{H}$ can be partitioned into facets $F_{P,A}$ indexed by $(P,A)\in\mathcal{X}'$ such that $F_{P,A}$ is homeomorphic to $\mathbf{R}^{\dim A}$. Note that the Euler characteristic with compact support of $F_{P,A}$ is $(-1)^{\dim A}$ and the Euler characteristic with compact support of $\mathfrak{H}$ is $(-1)^{\dim_{\mathbf{R}}\mathfrak{H}}=(-1)^{\dim A'}$. Using the additivity of the Euler characteristic with compact support we see that $\sum_{(P,A)\in\mathcal{X}'}(-1)^{\dim A}=(-1)^{\dim A'}$. Thus, $\phi(\gamma)=(-1)^{\dim T-\dim A'}$, as required. $\square$

3.3. In the setup of 3.1 let P_γ be the parabolic subgroup of G associated to γ as in [C2]. Note that P_γ is defined over K. The following result can be deduced by combining Theorem 3.2 with the results in [C2] and with Proposition 2 of [R].

Corollary 3.4. *We have* $\phi(\gamma) = (-1)^{\dim T - \dim A'} \delta_{P_\gamma}(\gamma)$.

4. Iwahori spherical representations: split elements

4.1. Let B be the subgroup of $G(K)$ generated by $U_\alpha(\mathcal{O}), (\alpha \in R^+)$, $U_\alpha(\mathfrak{p}), (\alpha \in R^-)$ and $T(K)_0$. (The subgroups $U_\alpha(\mathcal{O}), U_\alpha(\mathfrak{p})$ of U_α are defined by the $\mathcal{O}$-structure of G. We have $B \in \mathcal{B}$ where $\mathcal{B}$ is the set of Iwahori subgroups of $G(K)$. Note that $B \subset G(K)'$. For any $\alpha \in R$ we choose an isomorphism $x_\alpha : K \xrightarrow{\sim} U_\alpha(K)$ (the restriction of an isomorphism of algebraic groups from the additive group to U_α) which carries $\mathcal{O}$ onto $U_\alpha(\mathcal{O})$ and $\mathfrak{p}$ onto $U_\alpha(\mathfrak{p})$. We set $W := Y \cdot \mathcal{W}$ with Y normal in W (recall that $\mathcal{W}$ acts naturally on Y). Let Y' be the subgroup of Y generated by the coroots. Then $W' := Y' \cdot \mathcal{W}$ is naturally a subgroup of W. According to [IM], W is an extended Coxeter group (the semidirect product of the Coxeter group W' with the finite abelian group Y/Y') with length function

$$l(yw) = \sum_{\alpha \in R^+; w^{-1}(\alpha) \in R^+} ||\langle y, \alpha \rangle|| + \sum_{\alpha \in R^+; w^{-1}(\alpha) \in R^-} ||\langle y, \alpha \rangle - 1||$$

where $||a|| = a$ if $a \ge 0$, $||a|| = -a$ if $a < 0$. According to [IM], the set of double cosets $B \backslash G(K)/B$ is in bijection with W; to yw (where $y \in Y, w \in \mathcal{W}$) corresponds the double coset Ω_{yw} containing $T(K)_y \dot{w}$ (here $\dot{w}$ is an element in $G(\mathcal{O})$ which normalizes $T(K)_0$ and acts on it in the same way as w); moreover, $\sharp(\Omega_{yw}/B) = \sharp(B \backslash \Omega_{yw}) = q^{l(yw)}$ for any $y \in Y$, $w \in \mathcal{W}$. For example, if $y \in Y^{++}$ then $l(y) = \langle y, 2\rho \rangle$.

Let H be the algebra of B-biinvariant functions $G(K) \to \mathbf{C}$ with compact support with respect to convolution (we use the Haar measure dg on $G(K)$ for which $vol(B) = 1$). For y, w as above let $\mathfrak{T}_{yw} \in H$ be the characteristic function of Ω_{yw}. Then the functions $\mathfrak{T}_{\underline{w}}$, $\underline{w} \in W$, form a $\mathbf{C}$-basis of H and according to [IM] we have

$\mathfrak{T}_{\underline{w}} \mathfrak{T}_{\underline{w}'} = \mathfrak{T}_{\underline{w}\underline{w}'}$ if $\underline{w}, \underline{w}' \in W$ satisfy $l(\underline{w}\underline{w}') = l(\underline{w}) + l(\underline{w}')$,
$(\mathfrak{T}_{\underline{w}} + 1)(\mathfrak{T}_{\underline{w}} - q) = 0$ if $\underline{w} \in W', l(\underline{w}) = 1$.

In other words, H is what now one calls the Iwahori-Hecke algebra of the (extended) Coxeter group W with parameter q.

4.2. Let $\mathcal{C}_0^\infty(G(K))$ be the vector space of locally constant functions with compact support from $G(K)$ to $\mathbf{C}$. Let (V, σ) be an irreducible admissible representation of $G(K)$ such that the space V^B of B-invariant vectors in V is nonzero. If $f \in \mathcal{C}_0^\infty(G(K))$ then there is a well defined linear map $\sigma_f : V \to V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g)\sigma(g)(x)dg$. This linear map has finite rank hence it has a well defined trace $\mathrm{tr}(\sigma_f) \in \mathbf{C}$. From the definitions we see that for $f, f' \in \mathcal{C}_0^\infty(G(K))$ we have $\sigma_{f*f'} = \sigma_f \sigma_{f'} : V \to V$ where $*$ denotes convolution.

If $f \in H$, then σ_f maps V into V^B and $\mathrm{tr}(\sigma_f) = \mathrm{tr}(\sigma_f|_{V^B})$. (Recall that $\dim V^B < \infty$.) We see that the maps $\sigma_f|_{V^B}$ define a (unital) H-module structure on V^B. It is known [BO] that the H-module V^B is irreducible. Moreover for $\underline{w} \in W$ we have $\mathrm{tr}(\sigma_{\mathfrak{T}_{\underline{w}}}) = \mathrm{tr}(\mathfrak{T}_{\underline{w}})$ where the trace in the right side is taken in the H-module V^B. We have the following result.

Theorem 4.3. *Assume that K has characteristic zero and that p is sufficiently large. Let $y \in Y^+$ and let $t \in T(K)_y^\spadesuit$. We have*

$$\phi_V(t) = q^{-\langle y, 2\rho\rangle}\mathrm{tr}(\mathfrak{T}_y)$$

where the trace in the right side is taken in the irreducible H-module V^B.

An equivalent statement is that

$$\phi_V(t) = \mathrm{tr}(\sigma_{\mathfrak{T}_y})/vol(\Omega_y).$$

(Recall that $\mathfrak{T}_y$ in the right hand side is the characteristic function of $\Omega_y = BT(K)_yB$.)

The assumption on characteristic in the theorem is needed only to be able to use a result from [AK], see 5.1(†). We expect that the theorem holds without that assumption.

In the case where $y = 0$ the theorem becomes:

(a) *If $t \in T(K) \cap G_{cvr}$ then $\phi_V(t) = \dim(V^B)$.*

As pointed out to us by R. Bezrukavnikov and S. Varma, in the special case where $y \in Y^{++}$, Theorem 4.3 can be deduced from results in [C2].

4.4. In the case where $V = \mathbf{S}$, see 1.1, for any $y \in Y^+$, $\mathfrak{T}_y$ acts on the one dimensional vector space V^B as the identity map so that $\phi_V(t) = q^{-\langle y,2\rho\rangle}$; we thus recover Theorem 2.2 (which holds without assumption on the characteristic).

5. Proof of Theorem 4.3

5.1. Let $B = B_0, B_1, B_2, \cdots$ be the strictly decreasing Moy-Prasad filtration of B. In [MP], this is a sequence associated to a point x in the building such that $B = G_{x,0}$. Note that each B_i/B_{i+1} is abelian. Let $T_n := T(K) \cap B_n$. Applying Corollary 12.11 in [AK] to ϕ_V, we have

(†) ϕ_V is constant on the $\mathrm{Ad}(G)$-orbit ${}^G(tT_1)$ of tT_1.

Lemma 5.2. *Let $n \geq 1$. For any $t' \in T(K)_y^\spadesuit$ and $z \in B_n$, there exist $g \in B_n$, $t'' \in T_n$ and $z' \in B_{n+1}$ such that $\mathrm{Ad}(g)(t'z) = t't''z'$.*

Proof. Let $Z = \{\alpha \in R \mid U_\alpha \cap B_n \supsetneq U_\alpha \cap B_{n+1}\}$. If $Z = \emptyset$, $B_n = T_nB_{n+1}$. Hence, $z = t''z'$ for some $t'' \in T_n$ and $z' \in B_{n+1}$ and one can take $g = 1$. If $Z \neq \emptyset$, there are $a_\alpha \in K$, $\alpha \in Z$ such that $x_\alpha(a_\alpha) \in B_n$ and $z \equiv \prod_{\alpha\in Z} x_\alpha(a_\alpha)$ (mod T_nB_{n+1}). Such a_α can be chosen independent of the order of $\prod$ since

B_n/T_nB_{n+1} is abelian. Take $g = \prod_{\alpha \in Z} x_\alpha((1-\alpha(t'^{-1}))^{-1}a_\alpha)$. Then, we have $t'^{-1}gt'g^{-1} \equiv z^{-1} \pmod{T_nB_{n+1}}$. Moreover, since $y \in Y^+$, we have $|1-\alpha(t'^{-1})| \geq 1$ and thus $g \in B_n$. (We argue as in 2.4(a). Assume first that $\alpha \in R^+$. If $v(\alpha(t'^{-1})) \neq 0$ then $v(\alpha(t'^{-1})) < 0$ (since $\langle y, \alpha\rangle \neq 0$, $\langle y, \alpha \rangle \geq 0$) hence $v(1 - \alpha(t'^{-1}))) = v((\alpha(t'^{-1})) < 0$ and $|1-\alpha(t'^{-1})| > 1$. If $v(\alpha(t'^{-1})) = 0$ then $\alpha(t'^{-1}) - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1-\alpha(t'^{-1}))) = 0$ and $|1-\alpha(t'^{-1})| = 1$ as required. Assume next that $\alpha \in R^-$. If $v(\alpha(t'^{-1})) \neq 0$ then $v(\alpha(t'^{-1})) > 0$ (since $\langle y, \alpha\rangle \neq 0$, $\langle y, \alpha\rangle \leq 0$) hence $v(1-\alpha(t'^{-1}))) = 0$ and $|1-\alpha(t'^{-1})| = 1$ as required. If $v(\alpha(t'^{-1})) = 0$ then $\alpha(t'^{-1}) - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1-\alpha(t'^{-1}))) = 0$ and $|1-\alpha(t'^{-1})| = 1$ as required.)

Writing $\mathrm{Ad}(g)(t'z) = t' \cdot (t'^{-1}gt'g^{-1}) \cdot (gzg^{-1})$, we observe that $gzg^{-1} \equiv z \pmod{B_{n+1}}$ and $t'^{-1}gt'g^{-1}z \in T_nB_{n+1}$. Hence $\mathrm{Ad}(g)(t'z)$ can be written as $t't''z'$ with $t'' \in T_n$ and $z' \in B_{n+1}$. □

Lemma 5.3. $B_1tB_1 \subset {}^G(tT_1)$.

Proof. It is enough to show that $tB_1 \subset {}^G(tT_1)$. Let $t_0z_1 \in tB_1$ with $t_0 = t$ and $z_1 \in B_1$. We will construct inductively sequences $g_1, g_2 \cdots, t_1, t_2 \cdots$ and $z_1, z_2, \cdots$ such that $\mathrm{Ad}(g_k \cdots g_2g_1)(t_0z_1) = \mathrm{Ad}(g_k)(t_0t_1\cdots t_{k-1}z_k) = (t_0t_1\cdots t_k)z_{k+1}$ with $g_i \in B_i$, $t_i \in T_i$ and $z_i \in B_i$.

Applying Lemma 5.2 to $n = 1$, $t' = t_0$ and $z = z_1$, we find $t_1 \in T_1$ and $z_2 \in B_2$ such that $g_1t_0z_1g_1^{-1} = t_0t_1z_2$ with $t_1 \in T_1$ and $z_2 \in B_2$. Suppose we found $g_i \in B_i$, $z_{i+1} \in B_{i+1}$ and $t_i \in T_i$ for $i = 1, \cdots k$ where $k \geq 1$. Applying Lemma 5.2 to $n = k+1$, $t' = t_0t_1\cdots t_k$ and $z = z_{k+1}$, we find $g_{k+1} \in B_{k+1}$, $t_{k+1} \in T_{k+1}$ and $z_{k+2} \in B_{k+2}$ so that $g_{k+1}t_0t_1\cdots t_kz_{k+1}g_{k+1}^{-1} = \mathrm{Ad}(g_{k+1}\cdots g_2g_1)(t_0z_1) = t_0t_1t_2\cdots t_{k+1}z_{k+2}$. (To apply Lemma 5.2 we note that $t' \in T(K)_y^\spadesuit$ since $t_0 \in T(K)_y^\spadesuit$ and $t_1 \cdots t_k \in T_1$ so that for any $\alpha \in R$ we have $\alpha(t_1\cdots t_k) \in 1+\mathfrak{p}$.) Taking $g \in B_1$ be the limit of $g_k \cdots g_2g_1$ as $k \to \infty$, we have $\mathrm{Ad}(g)(t_0z_1) \in tT_1$. □

5.4. Continuing with the proof of Theorem 4.3, we note that by Lemma 5.3 and 5.1(†), for the characteristic function f_t of B_1tB_1 we have
(∗)

$$\mathrm{tr}(\sigma_{f_t}) = \int_G f_t(g)\phi_V(g)\,dg = \int_{B_1tB_1} \phi_V(t)\,dg = vol(B_1tB_1)\phi_V(t).$$

Thus it remains to show that

$$\mathrm{tr}(\sigma_{f_t})/vol(B_1tB_1) = \mathrm{tr}(\sigma_{\mathfrak{T}_y})/vol(BtB).$$

Since B_1 is normalized by B, B acts on V^{B_1}. Moreover, since V is irreducible and $V^B \neq 0$, B acts trivially on V^{B_1} (otherwise, there would exist a nonzero subspace of V on which B acts through a nontrivial character of B/B_1; since $V^B \neq 0$ we see that (V, σ) would have two distinct cuspidal supports, a contradiction). Thus we have $V^{B_1} = V^B$. Since σ_{f_t} and $\sigma_{\mathfrak{T}_y}$ have image contained in $V^{B_1} = V^B$, it is enough to show that

(a) $$\mathrm{tr}(\sigma_{f_t}|_{V^B})/vol(B_1tB_1) = \mathrm{tr}(\sigma_{\mathfrak{T}_y}|_{V^B}))/vol(BtB).$$

We can find a finite subset L of $T(K)_0$ such that $BtB = \sqcup_{\tau\in L} B_1tB_1\tau$. It follows that

(b) $vol(BtB) = vol(B_1tB_1)\sharp(L)$

and $\sigma_{\mathfrak{T}_y} = \sum_{\tau\in L}\sigma_{f_t}\sigma(\tau)$ as linear maps $V\to V$. Restricting this equality to V^B and using the fact that $\sigma(\tau)$ acts as identity on V^B we obtain

(c) $\sigma_{\mathfrak{T}_y}|_{V^B} = \sharp(L)\sigma_{f_t}|_{V^B}$

as linear maps $V^B\to V^B$. Clearly, (a) follows from (b) and (c). This completes the proof of Theorem 4.3.

The following result will not be used in the rest of the paper.

Proposition 5.5. *If $y\in Y^{++}$ and $t\in T(K)_y$ then $BtB\subset {}^G T(K)_y$.*

Proof. It is enough to show that $tz\subset {}^G T(K)_y$ for any $z\in B$. We can write $z = t_0z'$ where $t_0\in T(K)_0, z'\in B_1$. We have $tz = tt_0z'$ where $tt_0\in T(K)_y = T(K)_y^\spadesuit$ (here we use that $y\in Y^{++}$). Using Lemma 5.3 we have $tt_0z'\in {}^G(tt_0T_1)\subset {}^G T(K)_y$. This completes the proof. □

5.6. In the remainder of this section we assume that G is adjoint. In this case the irreducible representations (V,σ) as in 4.2 (up to isomorphism) are known to be in bijection with the irreducible finite dimensional representations of the Hecke algebra H (see [BO]) by $(V,\sigma)\mapsto V^B$. The irreducible finite dimensional representations of H have been classified in [KL1] in terms of geometric data. Moreover in [L] an algorithm to compute the dimensions of the (generalized) weight spaces of the action of the commutative semigroup $\{\mathfrak{T}_y; y\in Y^+\}$ on any tempered H module is given. In particular the right hand side of the equality in Theorem 4.3 (hence also $\phi_V(t)$ in that Theorem) is computable when V is tempered.

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