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FAMILIES AND SPRINGER'S CORRESPONDENCE

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Introduction

0.1. Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic p. Let W be the Weyl group of G; let $\text{Irr}W$ be a set of representatives for the isomorphism classes of irreducible representations of W over $\bar{\mathbf{Q}}_l$, an algebraic closure of the field of l-adic numbers (l is a fixed prime number $\neq p$).

Now $\text{Irr}W$ is partitioned into subsets called *families* as in [L1, Sec.9], [L3, 4.2]. Moreover to each family $\mathcal{F}$ in $\text{Irr}W$, a certain set $\mathbf{X}_{\mathcal{F}}$, a pairing $\{,\}:\mathbf{X}_{\mathcal{F}}\times\mathbf{X}_{\mathcal{F}}\to\bar{\mathbf{Q}}_l$, and an imbedding $\mathcal{F}\to\mathbf{X}_{\mathcal{F}}$ was canonically attached in [L1],[L3, Ch.4]. (The set $\mathbf{X}_{\mathcal{F}}$ with the pairing $\{,\}$, which can be viewed as a nonabelian analogue of a symplectic vector space, plays a key role in the classification of unipotent representations of a finite Chevalley group [L3] and in that of unipotent character sheaves on G). In [L1],[L3] it is shown that $\mathbf{X}_{\mathcal{F}}=M(\mathcal{G}_{\mathcal{F}})$ where $\mathcal{G}_{\mathcal{F}}$ is a certain finite group associated to $\mathcal{F}$ and, for any finite group Γ, $M(\Gamma)$ is the set of all pairs (g,ρ) where g is an element of Γ defined up to conjugacy and ρ is an irreducible representation over $\bar{\mathbf{Q}}_l$ (up to isomorphism) of the centralizer of g in Γ; moreover $\{,\}$ is given by the "nonabelian Fourier transform matrix" of [L1, Sec.4] for $\mathcal{G}_{\mathcal{F}}$.

In the remainder of this paper we assume that p is not a bad prime for G. In this case a uniform definition of the group $\mathcal{G}_{\mathcal{F}}$ was proposed in [L3, 13.1] in terms of special unipotent classes in G and the Springer correspondence, but the fact that this leads to a group isomorphic to $\mathcal{G}_{\mathcal{F}}$ as defined in [L3, Ch.4] was stated in [L3, (13.1.3)] without proof. One of the aims of this paper is to supply the missing proof.

To state the results of this paper we need some definitions. For $E\in\text{Irr}W$ let $a_E\in\mathbf{N},b_E\in\mathbf{N}$ be as in [L3, 4.1]. As noted in [L2], for $E\in\text{Irr}W$ we have

(a) $a_E\le b_E$;

we say that E is *special* if $a_E=b_E$.

For $g\in G$ let $Z_G(g)$ or $Z(g)$ be the centralizer of g in G and let $A_G(g)$ or $A(g)$ be the group of connected components of $Z(g)$. Let C be a unipotent conjugacy class in G and let $u\in C$. Let $\mathcal{B}_u$ be the variety of Borel subgroups of G that contain

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u; this is a nonempty variety of dimension, say, e_C. The conjugation action of $Z(u)$ on $\mathcal{B}_u$ induces an action of $A(u)$ on $\mathbf{S}_u := H^{2e_C}(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$. Now W acts on $\mathbf{S}_u$ by Springer's representation [Spr]; however here we adopt the definition of the W-action on $\mathbf{S}_u$ given in [L4] which differs from Springer's original definition by tensoring by sign. The W-action on $\mathbf{S}_u$ commutes with the $A(u)$-action. Hence we have canonically $\mathbf{S}_u = \oplus_{E\in\text{Irr}W} E \otimes \mathcal{V}_E$ (as $W \times A(u)$-modules) where $\mathcal{V}_E$ are finite dimensional $\bar{\mathbf{Q}}_l$-vector spaces with $A(u)$-action. Let $\text{Irr}_C W = \{E \in \text{Irr}W; \mathcal{V}_E \neq 0\}$; this set does not depend on the choice of u in C. By [Spr], the sets $\text{Irr}_C W$ (for C variable) form a partition of $\text{Irr}W$; also, if $E \in \text{Irr}_C W$ then $\mathcal{V}_E$ is an irreducible $A(u)$-module and, if $E \neq E'$ in $\text{Irr}_C W$, then the $A(u)$-modules $\mathcal{V}_E, \mathcal{V}_{E'}$ are not isomorphic. By [BM] we have

(b) $e_C \le b_E$ for any $E \in \text{Irr}_C W$

and the equality $b_E = e_C$ holds for exactly one $E \in \text{Irr}_C W$ which we denote by E_C (for this E, $\mathcal{V}_E$ is the unit representation of $A(u)$).

Following [L3, (13.1.1)] we say that C is *special* if E_C is special. (This concept was introduced in [L2, Sec.9] although the word "special" was not used there.) From (b) we see that C is special if and only if $a_{E_C} = e_C$.

Now assume that C is special. We denote by $\mathcal{F} \subset \text{Irr}W$ the family that contains E_C. (Note that $C \mapsto \mathcal{F}$ is a bijection from the set of special unipotent classes in G to the set of families in $\text{Irr}W$.) We set $\text{Irr}^*_C W = \{E \in \text{Irr}_C W; E \in \mathcal{F}\}$ and

$$\mathcal{K}(u) = \{a \in A(u); a \text{ acts trivially on } \mathcal{V}_E \text{ for any } E \in \text{Irr}^*_C W\}.$$

This is a normal subgroup of $A(u)$. We set $\bar{A}(u) = A(u)/\mathcal{K}(u)$, a quotient group of $A(u)$. Now, for any $E \in \text{Irr}^*_C W$, $\mathcal{V}_E$ is naturally an (irreducible) $\bar{A}(u)$-module. Another definition of $\bar{A}_u$ is given in [L3, (13.1.1)]. In that definition $\text{Irr}^*_C W$ is replaced by $\{E \in \text{Irr}_C W; a_E = e_C\}$ and $\mathcal{K}(u), \bar{A}(u)$ are defined as above but in terms of this modified $\text{Irr}^*_C W$. However the two definitions are equivalent in view of the following result.

Proposition 0.2. *Assume that C is special. Let $E \in \text{Irr}_C W$.*

(a) We have $a_E \le e_C$.

(b) We have $a_E = e_C$ if and only if $E \in \mathcal{F}$.

This follows from [L8, 10.9]. Note that (a) was stated without proof in [L3, (13.1.2)] (the proof I had in mind at the time of [L3] was combinatorial).

0.3. The following result is equivalent to a result stated without proof in [L3, (13.1.3)].

Theorem 0.4. *Let C be a special unipotent class of G, let $u \in C$ and let $\mathcal{F}$ be the family that contains E_C. Then we have canonically $\mathbf{X}_{\mathcal{F}} = M(\bar{A}(u))$ so that the pairing $\{,\}$ on $\mathbf{X}_{\mathcal{F}}$ coincides with the pairing $\{,\}$ on $M(\bar{A}(u))$. Hence $\mathcal{G}_{\mathcal{F}}$ can be taken to be $\bar{A}(u)$.*

This is equivalent to the corresponding statement in the case where G is adjoint, which reduces immediately to the case where G is adjoint simple. It is then enough to prove the theorem for one G in each isogeny class of semisimple, almost simple algebraic groups; this will be done in §3 after some combinatorial preliminaries in §1,§2. The proof uses the explicit description of the Springer correspondence: for type A_n, G_2 in [Spr]; for type B_n, C_n, D_n in [S1] (as an algorithm) and in [L4] (by a closed formula); for type F_4 in [S2]; for type E_n in [AL],[Sp1].

An immediate consequence of (the proof of) Theorem 0.4 is the following result which answers a question of R. Bezrukavnikov.

Corollary 0.5. *In the setup of 0.4 let $E \in \mathrm{Irr}_C^* W$ and let $\mathcal{V}_E$ be the corresponding $A(u)$-module viewed as an (irreducible) $\bar{A}(u)$-module. The image of E under the canonical imbedding $\mathcal{F} \to \mathbf{X}_{\mathcal{F}} = M(\bar{A}(u))$ is represented by the pair $(1, \mathcal{V}_E) \in M(\bar{A}(u))$. Conversely, if $E \in \mathcal{F}$ and the image of E under $\mathcal{F} \to \mathbf{X}_{\mathcal{F}} = M(\bar{A}(u))$ is represented by the pair $(1, \rho) \in M(\bar{A}(u))$ where ρ is an irreducible representation of $\bar{A}(u)$, then $E \in \mathrm{Irr}_C^* W$ and $\rho \cong \mathcal{V}_E$.*

0.6. Corollary 0.5 has the following interpretation. Let Y be a (unipotent) character sheaf on G whose restriction to the regular semisimple elements is $\neq 0$; assume that in the usual parametrization of unipotent character sheaves by $\sqcup_{\mathcal{F}'} \mathbf{X}_{\mathcal{F}'}$, Y corresponds to $(1, \rho) \in M(\bar{A}(u))$ where C is the special unipotent class corresponding to a family $\mathcal{F}$, $u \in C$ and ρ is an irreducible representation of $\bar{A}(u)$. Then $Y|_C$ is (up to shift) the irreducible local system on C defined by ρ.

A parametrization of unipotent character sheaves on G in terms of restrictions to various conjugacy classes of G is outlined in §4.

0.7. Notation. If A, B are subsets of $\mathbf{N}$ we denote by $A \dot{\cup} B$ the union of A and B regarded as a multiset (each element of $A \cap B$ appears twice). For any set $\mathcal{X}$, we denote by $\mathcal{P}(\mathcal{X})$ the set of subsets of $\mathcal{X}$ viewed as an F_2-vector space with sum given by the symmetric difference. If $\mathcal{X} \neq \emptyset$ we note that $\{\emptyset, \mathcal{X}\}$ is a line in $\mathcal{P}(\mathcal{X})$ and we set $\bar{\mathcal{P}}(\mathcal{X}) = \mathcal{P}(\mathcal{X})/\{\emptyset, \mathcal{X}\}$, $\mathcal{P}_{ev}(\mathcal{X}) = \{L \in \mathcal{P}(\mathcal{X}); |L| = 0 \mod 2\}$; let $\bar{\mathcal{P}}_{ev}(\mathcal{X})$ be the image of $\mathcal{P}_{ev}(\mathcal{X})$ under the obvious map $\mathcal{P}(\mathcal{X}) \to \bar{\mathcal{P}}(\mathcal{X})$ (thus $\bar{\mathcal{P}}_{ev}(\mathcal{X}) = \bar{\mathcal{P}}(\mathcal{X})$ if $|\mathcal{X}|$ is odd and $\bar{\mathcal{P}}_{ev}(\mathcal{X})$ is a hyperplane in $\bar{\mathcal{P}}(\mathcal{X})$ if $|\mathcal{X}|$ is even). Now if $\mathcal{X} \neq \emptyset$, the assignment $L, L' \mapsto |L \cap L'| \mod 2$ defines a symplectic form on $\mathcal{P}_{ev}(\mathcal{X})$ which induces a nondegenerate symplectic form $(,)$ on $\bar{\mathcal{P}}_{ev}(\mathcal{X})$ via the obvious linear map $\mathcal{P}_{ev}(\mathcal{X}) \to \bar{\mathcal{P}}_{ev}(\mathcal{X})$.

For $g \in G$ let g_s (resp. g_ω) be the semisimple (resp. unipotent) part of g.

For $z \in (1/2)\mathbf{Z}$ we set $\lfloor z \rfloor = z$ if $z \in \mathbf{Z}$ and $\lfloor z \rfloor = z - (1/2)$ if $z \in \mathbf{Z} + (1/2)$.

Erratum to [L3]. On page 86, line -6 delete: "$b' < b$" and on line -4 before "In the language..." insert: "The array above is regarded as identical to the array obtained by interchanging its two rows."

On page 343, line -5, after "respect to M" insert: "and where the group $\mathcal{G}_{\mathcal{F}}$ defined in terms of (u', M) is isomorphic to the group $\mathcal{G}_{\mathcal{F}}$ defined in terms of (u, G)".

Erratum to [L4]. In the definition of A_α, B_α in [L4, 11.5], the condition $I \in \alpha$ should be replaced by $I \in \alpha'$ and the condition $I \in \alpha'$ should be replaced by $I \in \alpha$.

1. Combinatorics

1.1. Let N be an even integer ≥ 0. Let $a := (a_0, a_1, a_2, \dots, a_N) \in \mathbf{N}^{N+1}$ be such that $a_0 \le a_1 \le a_2 \le \dots \le a_N$, $a_0 < a_2 < a_4 < \dots$, $a_1 < a_3 < a_5 < \dots$. Let $\mathcal{J} = \{i \in [0, N]; a_i \text{ appears exactly once in } a\}$. We have $\mathcal{J} = \{i_0, i_1, \dots, i_{2M}\}$ where $M \in \mathbf{N}$ and $i_0 < i_1 < \dots < i_{2M}$ satisfy $i_s = s \mod 2$ for $s \in [0, 2M]$. Hence for any $s \in [0, 2M - 1]$ we have $i_{s+1} = i_s + 2m_s + 1$ for some $m_s \in \mathbf{N}$. Let $\mathcal{E}$ be the set of $b := (b_0, b_1, b_2, \dots, b_N) \in \mathbf{N}^{N+1}$ such that $b_0 < b_2 < b_4 < \dots$, $b_1 < b_3 < b_5 < \dots$ and such that $[b] = [a]$ (we denote by $[b], [a]$ the multisets $\{b_0, b_1, \dots, b_N\}$, $\{a_0, a_1, \dots, a_N\}$). We have $a \in \mathcal{E}$. For $b \in \mathcal{E}$ we set

$$\hat{b} = (\hat{b}_0, \hat{b}_1, \hat{b}_2, \dots, \hat{b}_N) = (b_0, b_1+1, b_2+1, b_3+2, b_4+2, \dots, b_{N-1}+(N/2), b_N+(N/2)).$$

Let $[\hat{b}]$ be the multiset $\{\hat{b}_0, \hat{b}_1, \hat{b}_2, \dots, \hat{b}_N\}$.

For $s \in \{1, 3, \dots, 2M - 1\}$ we define $a^{\{s\}} = (a_0^{\{s\}}, a_1^{\{s\}}, a_2^{\{s\}}, \dots, a_N^{\{s\}}) \in \mathcal{E}$ by

$$(a_{i_s}^{\{s\}}, a_{i_s+1}^{\{s\}}, a_{i_s+2}^{\{s\}}, a_{i_s+3}^{\{s\}}, \dots, a_{i_s+2m_s}^{\{s\}}, a_{i_s+2m_s+1}^{\{s\}})$$
$$= (a_{i_s+1}, a_{i_s}, a_{i_s+3}, a_{i_s+2}, \dots, a_{i_s+2m_s+1}, a_{i_s+2m_s})$$

and $a_i^{\{s\}} = a_i$ if $i \in [0, N] - [i_s, i_{s+1}]$. More generally for $X \subset \{1, 3, \dots, 2M - 1\}$ we define $a^X = (a_0^X, a_1^X, a_2^X, \dots, a_N^X) \in \mathcal{E}$ by $a_i^X = a_i^{\{s\}}$ if $s \in X$, $i \in [i_s, i_{s+1}]$, and $a_i^X = a_i$ for all other $i \in [0, N]$. Note that $[\widehat{a^X}] = [\hat{a}]$. Conversely, we have the following result.

Lemma 1.2. *Let $b \in \mathcal{E}$ be such that $[\hat{b}] = [\hat{a}]$. There exists $X \subset \{1, 3, \dots, 2M-1\}$ such that $b = a^X$.*

The proof is given in 1.3-1.5.

1.3. We argue by induction on M. We have

$$a = (y_1 = y_1 < y_2 = y_2 < \dots < y_r = y_r < a_{i_0} < \dots)$$

for some r. Since $[b] = [a]$, we must have

$$(b_0, b_2, b_4, \dots) = (y_1, y_2, \dots, y_r, \dots), (b_1, b_3, b_5, \dots) = (y_1, y_2, \dots, y_r, \dots).$$

Thus,

(a) $b_i = a_i$ for $i < i_0$.

We have $a = (\dots > a_{2M} > y'_1 = y'_1 < y'_2 = y'_2 < \dots < y'_{r'} = y'_{r'})$ for some r'. Since $[b] = [a]$, we must have

$$(b_0, b_2, b_4, \dots) = (\dots, y'_1, y'_2, \dots, y'_{r'}), (b_1, b_3, b_5, \dots) = (\dots, y'_1, y'_2, \dots, y'_{r'}).$$

Thus,

(b) $b_i = a_i$ for $i > i_{2M}$.

If $M = 0$ we see that $b = a$ and there is nothing further to prove. In the rest of the proof we assume that $M \ge 1$.

1.4. From 1.3 we see that

$(a_0, a_1, a_2, \dots, a_{i_{2M}}) = (\dots, a_{i_{2M-1}} < x_1 = x_1 < x_2 = x_2 < \dots < x_q = x_q < a_{i_{2M}})$

(for some q) has the same entries as $(b_0, b_1, b_2, \dots, b_{i_{2M}}$ (in some order). Hence the pair

$$(\dots, b_{i_{2M}-5}, b_{i_{2M}-3}, b_{i_{2M}-1}), (\dots, b_{i_{2M}-4}, b_{i_{2M}-2}, b_{i_{2M}})$$

must have one of the following four forms.
$(\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q), (\dots, x_1, x_2, \dots, x_q, a_{i_{2M}}),$
$(\dots, x_1, x_2, \dots, x_q, a_{i_{2M}}), (\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q),$
$(\dots, x_1, x_2, \dots, x_q), (\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q, a_{i_{2M}}),$
$(\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q, a_{i_{2M}}), (\dots, x_1, x_2, \dots, x_q).$
Hence $(\dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}})$ must have one of the following four forms.
(I) $(\dots, a_{i_{2M-1}}, x_1, x_1, x_2, x_2, \dots, x_q, x_q, a_{i_{2M}}),$
(II) $(\dots, x_1, a_{i_{2M-1}}, x_2, x_1, x_3, x_2, \dots, x_q, x_{q-1}, a_{i_{2M}}, x_q),$
(III) $(\dots, a_{i_{2M-1}}, z, x_1, x_1, x_2, x_2, \dots, x_q, x_q, a_{i_{2M}}),$
(IV) $(\dots, a_{i_{2M-1}}, z', x_1, z'', x_2, x_1, x_3, x_2, \dots, x_q, x_{q-1}, a_{i_{2M}}, x_q),$
where $a_{i_{2M-1}} > z$, $a_{i_{2M-1}} > z'' \ge z'$ and all entries in $\dots$ are $< a_{i_{2M-1}}$. Correspondingly, $(\dots, \hat{b}_{i_{2M}-2}, \hat{b}_{i_{2M}-1}, \hat{b}_{i_{2M}})$ must have one of the following four forms.
(I) $(\dots, a_{i_{2M-1}} + h - q, x_1 + h - q, x_1 + h - q + 1, x_2 + h - q + 1, x_2 + h - q + 2, \dots, x_q + h - 1, x_q + h, a_{i_{2M}} + h),$
(II) $(\dots, x_1 + h - q, a_{i_{2M-1}} + h - q, x_2 + h - q + 1, x_1 + h - q + 1, x_3 + h - q + 2, x_2 + h - q + 1, \dots, x_q + h - 1, x_{q-1} + h - 1, a_{i_{2M}} + h, x_q + h),$
(III) $(\dots, a_{i_{2M-1}} + h - q - 1, z + h - q, x_1 + h - q, x_1 + h - q + 1, x_2 + h - q + 1, x_2 + h - q + 2, \dots, x_q + h - 1, x_q + h, a_{i_{2M}} + h),$
(IV) $(\dots, a_{i_{2M-1}} + h - q - 1, z' + h - q - 1, x_1 + h - q, z'' + h - q, x_2 + h - q + 1, x_1 + h - q + 1, x_3 + h - q + 2, x_2 + h - q + 1, \dots, x_q + h - 1, x_{q-1} + h - 1, a_{i_{2M}} + h, x_q + h)$
where $h = i_{2M}/2$ and in case (III) and (IV), $a_{i_{2M-1}} + h - q$ is not an entry of $(\dots, \hat{b}_{i_{2M}-2}, \hat{b}_{i_{2M}-1}, \hat{b}_{i_{2M}})$.

Since $(\dots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}})$ is given by (I) we see that $a_{i_{2M-1}} + h - q$ is an entry of $(\dots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}})$. Using 1.3(b) we see that
$\{\dots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}}\} = (\dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}})$
as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus

we have either

(a) $$(b_{i_{2M-1}}, b_{i_{2M-1}+1}, \dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}}) = (a_{i_{2M-1}}, a_{i_{2M-1}+1}, \dots, a_{i_{2M}-2}, a_{i_{2M}-1}, a_{i_{2M}})$$

or

(b) $$(b_{i_{2M-1}}, b_{i_{2M-1}+1}, \dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}}) = (a_{i_{2M-1}+1}, a_{i_{2M-1}}, a_{i_{2M-1}+3}, a_{i_{2M-1}+2}, \dots, a_{i_{2M}}, a_{i_{2M}-1}).$$

1.5. Let $a'=(a_0,a_1,a_2,\dots,a_{i_{2M-1}-1})$, $b'=(b_0,b_1,b_2,\dots,b_{i_{2M-1}-1})$,
$\hat a'=(a_0,a_1+1,a_2+1,a_3+2,a_4+2,\dots,a_{i_{2M-1}-1}+(i_{2M-1}-1)/2)$,
$\hat b'=(b_0,b_1+1,b_2+1,b_3+2,b_4+2,\dots,b_{i_{2M-1}-1}+(i_{2M-1}-1)/2)$,
From $[\hat b]=[\hat a]$ and 1.3(b), 1.4(a),(b) we see that the multiset formed by the entries of $\hat a'$ coincides with the multiset formed by the entries of $\hat b'$. Using the induction hypothesis we see that there exists $X'\subset\{1,3,\dots,2M-3\}$ such that $b'=a'^{X'}$ where $a'^{X'}$ is defined in terms of a',X' in the same way as a^X was defined (see 1.1) in terms of a,X. We set $X=X'$ if we are in case 1.4(a) and $X=X'\cup\{2M-1\}$ if we are in case 1.4(b). Then we have $b=a^X$ (see 1.4(a),(b)), as required. This completes the proof of Lemma 1.2.

1.6. We shall use the notation of 1.1. Let $\mathfrak{T}$ be the set of all unordered pairs $(\mathfrak{A},\mathfrak{B})$ where $\mathfrak{A},\mathfrak{B}$ are subsets of $\{0,1,2,\dots\}$ and $\mathfrak{A}\dot\cup\mathfrak{B}=(a_0,a_1,a_2,\dots,a_N)$ as multisets. For example, setting $\mathfrak{A}_\emptyset=(a_0,a_2,a_4,\dots,a_N)$, $\mathfrak{B}_\emptyset=(a_1,a_3,\dots,a_{N-1})$, we have $(\mathfrak{A}_\emptyset,\mathfrak{B}_\emptyset)\in\mathfrak{T}$. For any subset $\mathfrak{a}$ of $\mathcal{J}$ we consider

$$\mathfrak{A}_\mathfrak{a}=((\mathcal{J}-\mathfrak{a})\cap\mathfrak{A}_\emptyset)\cup(\mathfrak{a}\cap\mathfrak{B}_\emptyset)\cup(\mathfrak{A}_\emptyset\cap\mathfrak{B}_\emptyset),$$

$$\mathfrak{B}_\mathfrak{a}=((\mathcal{J}-\mathfrak{a})\cap\mathfrak{B}_\emptyset)\cup(\mathfrak{a}\cap\mathfrak{A}_\emptyset)\cup(\mathfrak{A}_\emptyset\cap\mathfrak{B}_\emptyset).$$

Then $(\mathfrak{A}_\mathfrak{a},\mathfrak{B}_\mathfrak{a})\in\mathfrak{T}$ and the map $\mathfrak{a}\mapsto(\mathfrak{A}_\mathfrak{a},\mathfrak{B}_\mathfrak{a})$ induces a bijection $\bar{\mathcal{P}}(\mathcal{J})\leftrightarrow\mathfrak{T}$. (Note that if $\mathfrak{a}=\emptyset$ then $(\mathfrak{A}_\mathfrak{a},\mathfrak{B}_\mathfrak{a})$ agrees with the earlier definition of $(\mathfrak{A}_\emptyset,\mathfrak{B}_\emptyset)$.)

Let $\mathfrak{T}'$ be the set of all $(\mathfrak{A},\mathfrak{B})\in\mathfrak{T}$ such that $|\mathfrak{A}|=|\mathfrak{A}_\emptyset|,|\mathfrak{B}|=|\mathfrak{B}_\emptyset|$.

Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}_{ev}(\mathcal{J})$ spanned by the 2-element subsets

$$\{a_{i_0},a_{i_1}\},\{a_{i_2},a_{i_3}\},\dots,\{a_{i_{2M-2}},a_{i_{2M-1}}\}$$

of $\mathcal{J}$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}_{ev}(\mathcal{J})$ spanned by the 2-element subsets

$$\{a_{i_1},a_{i_2}\},\{a_{i_3},a_{i_4}\},\dots,\{a_{i_{2M-1}},a_{i_{2M}}\}$$

of $\mathcal{J}$.

Let $\bar{\mathcal{P}}(\mathcal{J})_0$ (resp. $\bar{\mathcal{P}}(\mathcal{J})_1$) be the image of $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the obvious map $\mathcal{P}(\mathcal{J})\to\bar{\mathcal{P}}(\mathcal{J})$. Note that

(a) *$\bar{\mathcal{P}}(\mathcal{J})_0$ and $\bar{\mathcal{P}}(\mathcal{J})_1$ are opposed Lagrangian subspaces of the symplectic vector space $\bar{\mathcal{P}}(\mathcal{J}),(,)$, (see 0.7); hence $(,)$ defines an identification $\bar{\mathcal{P}}(\mathcal{J})_0=\bar{\mathcal{P}}(\mathcal{J})_1^*$* where $\bar{\mathcal{P}}(\mathcal{J})_1^*$ is the vector space dual to $\bar{\mathcal{P}}(\mathcal{J})_1$.

Let $\mathfrak{T}_0$ (resp. $\mathfrak{T}_1$) be the subset of $\mathfrak{T}$ corresponding to $\bar{\mathcal{P}}(\mathcal{J})_0$ (resp. $\bar{\mathcal{P}}(\mathcal{J})_1$) under the bijection $\bar{\mathcal{P}}(\mathcal{J})\leftrightarrow\mathfrak{T}$. Note that $\mathfrak{T}_0\subset\mathfrak{T}'$, $\mathfrak{T}_1\subset\mathfrak{T}'$, $|\mathfrak{T}_0|=|\mathfrak{T}_1|=2^M$.

For any $X\subset\{1,3,\dots,2M-1\}$ we set $\mathfrak{a}_X=\cup_{s\in X}\{a_{i_s},a_{i_{s+1}}\}\in\mathcal{P}(\mathcal{J})$. Then $(\mathfrak{A}_{\mathfrak{a}_X},\mathfrak{B}_{\mathfrak{a}_X})\in\mathfrak{T}_1$ is related to a^X in 1.1 as follows:
$\mathfrak{A}_{\mathfrak{a}_X}=\{a^X_0,a^X_2,a^X_4,\dots,a^X_N\}$, $\mathfrak{B}_{\mathfrak{a}_X}=\{a^X_1,a^X_3,\dots,a^X_{N-1}\}$.

1.7. We shall use the notation of 1.1. Let T be the set of all ordered pairs (A,B) where A is a subset of $\{0,1,2,\dots\}$, B is a subset of $\{1,2,3,\dots\}$, A contains no consecutive integers, B contains no consecutive integers, and $A\dot\cup B=(\hat a_0,\hat a_1,\hat a_2,\dots,\hat a_N)$ as multisets. For example, setting $A_\emptyset=(\hat a_0,\hat a_2,\hat a_4,\dots,\hat a_N)$, $B_\emptyset=(\hat a_1,\hat a_3,\dots,\hat a_{N-1})$, we have $(A_\emptyset,B_\emptyset)\in T$.

For any $(A,B)\in T$ we define (A^-,B^-) as follows: A^- consists of $x_0<x_1-1<x_2-2<\dots<x_p-p$ where $x_0<x_1<\dots<x_p$ are the elements of A; B^- consists of $y_1-1<y_2-2<\dots<y_q-q$ where $y_1<y_2<\dots<y_q$ are the elements of B.

We can enumerate the elements of T as in [L4, 11.5]. Let J be the set of all $c\in\mathbf{N}$ such that c appears exactly once in the sequence

$$(\hat a_0,\hat a_1,\hat a_2,\dots,\hat a_N)=(a_0,a_1+1,a_2+1,a_3+2,a_4+2,\dots,a_{N-1}+(N/2),a_N+(N/2)).$$

A nonempty subset I of J is said to be an interval if it is of the form $\{i,i+1,i+2,\dots,j\}$ with $i-1\notin J, j+1\notin J$ and with $i\ne 0$. Let $\mathcal{I}$ be the set of intervals of J. For any $s\in\{1,3,\dots,2M-1\}$, the set $I_s:=\{\hat a_{i_s},\hat a_{i_s+1},\hat a_{i_s+2},\dots,\hat a_{i_s+2m_s+1}\}$ is either a single interval or a union of intervals $I_s^1\sqcup I_s^2\sqcup\dots\sqcup I_s^{t_s}$ ($t_s\ge 2$) where $\hat a_{i_s}\in I_s^1$, $\hat a_{i_s+2m_s+1}\in I_s^{t_s}$, $|I_s^1|,|I_s^{t_s}|$ are odd, $|I_s^h|$ are even for $h\in[2,t_s-1]$ and any element in I_s^e is $<$ than any element in $I_s^{e'}$ for $e<e'$. Let $\mathcal{I}_s$ be the set of all $I\in\mathcal{I}$ such that $I\subset I_s$. We have a partition $\mathcal{I}=\sqcup_{s\in\{1,3,\dots,2M-1\}}\mathcal{I}_s$. Let H be the set of elements of $c\in J$ such that $c<a_{i_1}$ (that is such that c does not belong to any interval). For any subset $\alpha\subset\mathcal{I}$ we consider

$$A_\alpha=\cup_{I\in\mathcal{I}-\alpha}(I\cap A_\emptyset)\cup\cup_{I\in\alpha}(I\cap B_\emptyset)\cup(H\cap A_\emptyset)\cup(A_\emptyset\cap B_\emptyset),$$

$$B_\alpha=\cup_{I\in\mathcal{I}-\alpha}(I\cap B_\emptyset)\cup\cup_{I\in\alpha}(I\cap A_\emptyset)\cup(H\cap B_\emptyset)\cup(A_\emptyset\cap B_\emptyset).$$

Then $(A_\alpha,B_\alpha)\in T$ and the map $\alpha\mapsto(A_\alpha,B_\alpha)$ is a bijection $\mathcal{P}(\mathcal{I})\leftrightarrow T$. (Note that if $\alpha=\emptyset$ then (A_α,B_α) agrees with the earlier definition of $(A_\emptyset,B_\emptyset)$.)

Let $T'=\{(A,B)\in T;|A|=|A_\emptyset|,|B|=|B_\emptyset|\}$, $T_1=\{(A,B)\in T';A^-\dot\cup B^-=A_\emptyset^-\dot\cup B_\emptyset^-\}$. Let $\mathcal{P}(\mathcal{I})'$ (resp. $\mathcal{P}(\mathcal{I})_1$) be the subset of $\mathcal{P}(\mathcal{I})$ corresponding to T' (resp. T_1) under the bijection $\mathcal{P}(\mathcal{I})\leftrightarrow T$.

Now let X be a subset of $\{1,3,\dots,2M-1\}$. Let $\alpha_X=\cup_{s\in X}\mathcal{I}_s\in\mathcal{P}(\mathcal{I})$. From the definitions we see that

(a) $A^-_{\alpha_X}=\mathfrak{A}_{\mathfrak{a}_X}$, $B^-_{\alpha_X}=\mathfrak{B}_{\mathfrak{a}_X}$

(notation of 1.6). In particular we have $(A_{\alpha_X},B_{\alpha_X})\in T_1$. Thus $|T_1|\ge 2^M$. Using Lemma 1.2 we see that

(b) $|T_1|=2^M$ *and* T_1 *consists of the pairs* $(A_{\alpha_X},B_{\alpha_X})$ *with* $X\subset\{1,3,\dots,2M-1\}$.

Using (a),(b) we deduce:

(c) *The map* $T_1\to\mathfrak{T}_1$ *given by* $(A,B)\mapsto(A^-,B^-)$ *is a bijection.*

2. Combinatorics (continued)

2.1. Let $N\in\mathbf{N}$. Let

$$a:=(a_0,a_1,a_2,\dots,a_N)\in\mathbf{N}^{N+1}$$

be such that $a_0 \le a_1 \le a_2 \le \dots \le a_N$, $a_0 < a_2 < a_4 < \dots$, $a_1 < a_3 < a_5 < \dots$ and such that the set $\mathcal{J} := \{i \in [0,N]; a_i \text{ appears exactly once in } a\}$ is nonempty. Now $\mathcal{J}$ consists of $\mu+1$ elements $i_0 < i_1 < \dots < i_\mu$ where $\mu \in \mathbf{N}$, $\mu = N \mod 2$. We have $i_s = s \mod 2$ for $s \in [0,\mu]$. Hence for any $s \in [0,\mu-1]$ we have $i_{s+1} = i_s + 2m_s + 1$ for some $m_s \in \mathbf{N}$. Let $\mathcal{E}$ be the set of $b := (b_0, b_1, b_2, \dots, b_N) \in \mathbf{N}^{N+1}$ such that $b_0 < b_2 < b_4 < \dots$, $b_1 < b_3 < b_5 < \dots$ and such that $[b] = [a]$ (we denote by $[b], [a]$ the multisets $\{b_0, b_1, \dots, b_N\}$, $\{a_0, a_1, \dots, a_N\}$). We have $a \in \mathcal{E}$. For $b \in \mathcal{E}$ we set

$$\mathring{b} = (\mathring{b}_0, \mathring{b}_1, \mathring{b}_2, \dots, \mathring{b}_N) = (b_0, b_1, b_2+1, b_3+1, b_4+2, b_5+2, \dots) \in \mathbf{N}^{N+1}.$$

Let $[\mathring{b}]$ be the multiset $\{\mathring{b}_0, \mathring{b}_1, \mathring{b}_2, \dots, \mathring{b}_N\}$. For any $s \in [0,\mu-1] \in 2\mathbf{N}$ we define $a^{\{s\}} = (a_0^{\{s\}}, a_1^{\{s\}}, a_2^{\{s\}}, \dots, a_N^{\{s\}}) \in \mathcal{E}$ by

$$\begin{aligned}&(a_{i_s}^{\{s\}}, a_{i_s+1}^{\{s\}}, a_{i_s+2}^{\{s\}}, a_{i_s+3}^{\{s\}}, \dots, a_{i_s+2m_s}^{\{s\}}, a_{i_s+2m_s+1}^{\{s\}})\\ &= (a_{i_s+1}, a_{i_s}, a_{i_s+3}, a_{i_s+2}, \dots, a_{i_s+2m_s+1}, a_{i_s+2m_s})\end{aligned}$$

and $a_i^{\{s\}} = a_i$ if $i \in [0,N] - [i_s, i_{s+1}]$. More generally for a subset X of $[0,\mu-1] \cap 2\mathbf{N}$ we define $a^X = (a_0^X, a_1^X, a_2^X, \dots, a_N^X) \in \mathcal{E}$ by $a_i^X = a_i^{\{s\}}$ if $s \in X$, $i \in [i_s, i_{s+1}]$, and $a_i^X = a_i$ for all other $i \in [0,N]$. Note that $[\mathring{a}^X] = [\mathring{a}]$. Conversely, we have the following result.

Lemma 2.2. *Let $b \in \mathcal{E}$ be such that $[\mathring{b}] = [\mathring{a}]$. Then there exists $X \subset [0,\mu-1] \cap 2\mathbf{N}$ such that $b = a^X$.*

The proof is given in 2.3-2.5.

2.3. We argue by induction on μ. By the argument in 1.3 we have

(a) $b_i = a_i$ for $i < i_0$,

(b) $b_i = a_i$ for $i > i_\mu$.

If $\mu = 0$ we see that $b = a$ and there is nothing further to prove. In the rest of the proof we assume that $\mu \ge 1$.

2.4. From 2.3 we see that $(a_{i_0}, a_{i_0+1}, \dots, a_N) = (a_{i_0} < x_1 = x_1 < x_2 = x_2 < \dots < x_p = x_p < a_{i_1} < \dots)$ (for some p) has the same entries as $(b_{i_0}, b_{i_0+1}, \dots, b_N)$ (in some order). Hence the pair $(b_{i_0}, b_{i_0+2}, b_{i_0+4}, \dots), (b_{i_0+1}, b_{i_0+3}, b_{i_0+5}, \dots)$ must have one of the following four forms.

$(a_{i_0}, x_1, x_2, \dots, x_p, \dots), (x_1, x_2, \dots, x_p, a_{i_1}, \dots)$,

$(x_1, x_2, \dots, x_p, a_{i_1}, \dots), (a_{i_0}, x_1, x_2, \dots, x_p, \dots)$,

$(a_{i_0}, x_1, x_2, \dots, x_p, a_{i_1}, \dots), (x_1, x_2, \dots, x_p, \dots)$,

$(x_1, x_2, \dots, x_p, \dots), (a_{i_0}, x_1, x_2, \dots, x_p, a_{i_1}, \dots)$.

Hence $(b_{i_0}, b_{i_0+1}, b_{i_0+2}, \dots, b_N)$ must have one of the following four forms.

(I) $(a_{i_0}, x_1, x_1, x_2, x_2, \dots, x_p, x_p, a_{i_1}, \dots)$,

(II) $(x_1, a_{i_0}, x_2, x_1, x_3, x_2, \dots, x_p, x_{p-1}, a_{i_1}, x_p, \dots)$,
(III) $(a_{i_0}, x_1, x_1, x_2, x_2, \dots, x_p, x_p, z, a_{i_1}, \dots)$,
(IV) $(x_1, a_{i_0}, x_2, x_1, x_3, x_2, \dots, x_p, x_{p-1}, z', x_p, z'', a_{i_1}, \dots)$
where $a_{i_1} < z$, $a_{i_1} < z' \le z''$ and all entries in $\dots$ are $> a_{i_1}$. Correspondingly, $(\mathring{b}_{i_0}, \mathring{b}_{i_0+1}, \mathring{b}_{i_0+2}, \dots, \mathring{b}_N)$ must have one of the following four forms.

(I) $(a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \dots, x_p + h + p - 1, x_p + h + p, a_{i_1} + h + p, \dots)$,

(II) $(x_1 + h, a_{i_0} + h, x_2 + h + 1, x_1 + h + 1, x_3 + h + 2, x_2 + h + 2, \dots, x_p + h + p - 1, x_{p-1} + h + p - 1, a_{i_1} + h + p, x_p + h + p, \dots)$,

(III) $(a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \dots, x_p + h + p - 1, x_p + h + p, z + p, a_{i_1} + h + p + 1, \dots)$,

$$(x_1 + h, a_{i_0} + h, x_2 + h + 1, x_1 + h + 1, x_3 + h + 2, x_2 + h + 2, \dots, x_p + h + p - 1,$$
(IV)
$$x_{p-1} + h + p - 1, z' + h + p, x_p + h + p, z'' + h + p + 1, a_{i_1} + h + p + 1, \dots)$$

where $h = i_0/2$ and in case (III) and (IV) $a_{i_1} + h + p$ is not an entry of $(\mathring{b}_{i_0}, \mathring{b}_{i_0+1}, \mathring{b}_{i_0+2}, \dots)$.

Since $(\mathring{a}_{i_0}, \mathring{a}_{i_0+1}, \mathring{a}_{i_0+2}, \dots)$ is given by (I) we see that $a_{i_1} + h + p$ is an entry of $(\mathring{a}_{i_0}, \mathring{a}_{i_0+1}, \mathring{a}_{i_0+2}, \dots)$. Using 2.3 we see that

$$\{\mathring{a}_{i_0}, \mathring{a}_{i_0+1}, \mathring{a}_{i_0+2}, \dots\} = \{\mathring{b}_{i_0}, \mathring{b}_{i_0+1}, \mathring{b}_{i_0+2}, \dots\}$$

as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus

we have either

(a) $$(b_{i_0}, b_{i_0+1}, b_{i_0+2}, \dots, b_{i_1}) = (a_{i_0}, a_{i_0+1}, a_{i_0+2}, \dots, a_{i_1})$$

or

(b) $$(b_{i_0}, b_{i_0+1}, b_{i_0+2}, \dots, b_{i_1}) = (a_{i_0+1}, a_{i_0}, a_{i_0+3}, a_{i_0+2}, \dots, a_{i_1}, a_{i_1-1}).$$

. From 2.3 and (a),(b) we see that if $\mu = 1$ then Lemma 2.2 holds. Thus in the rest of the proof we can assume that $\mu \ge 2$.

2.5. Let $a' = (a_{i_1+1}, a_{i_1+2}, \dots, a_N)$, $b' = (b_{i_1+1}, b_{i_1+2}, \dots, b_N)$,

$$\mathring{a}' = (a_{i_1+1}, a_{i_1+2}, a_{i_1+3} + 1, a_{i_1+4} + 1, a_{i_1+5} + 2, a_{i_1+6} + 2, \dots),$$

$$\mathring{b}' = (b_{i_1+1}, b_{i_1+2}, b_{i_1+3} + 1, b_{i_1+4} + 1, b_{i_1+5} + 2, b_{i_1+6} + 2, \dots).$$

From $[\overset{\circ}{b}]=[\overset{\circ}{a}]$ and 2.3(a),2.4(a),(b) we see that the multiset formed by the entries of $\overset{\circ}{a}'$ coincides with the multiset formed by the entries of $\overset{\circ}{b}'$. Using the induction hypothesis we see that there exists $X'\subset[2,\mu-1]\cap 2\mathbf{N}$ such that $b'=a'^{X'}$ where $a'^{X'}$ is defined in terms of a',X' in the same way as a^X (see 2.1) was defined in terms of a,X. We set $X=X'$ if we are in case 2.4(a) and $X=\{0\}\cup X'$ if we are in case 2.4(b). Then we have $b=a^X$ (see 2.4(a),(b)), as required. This completes the proof of Lemma 2.2.

2.6. We shall use the notation of 2.1. Let $\mathfrak{T}$ be the set of all unordered pairs $(\mathfrak{A},\mathfrak{B})$ where $\mathfrak{A},\mathfrak{B}$ are subsets of $\{0,1,2,\dots\}$ and $\mathfrak{A}\dot\cup\mathfrak{B}=(a_0,a_1,a_2,\dots,a_N)$ as multisets. For example, setting $\mathfrak{A}_\emptyset=\{a_i;i\in[0,N]\cap 2\mathbf{N}\}$, $\mathfrak{B}_\emptyset=\{a_i;i\in[0,N]\cap(2\mathbf{N}+1)\}$, we have $(\mathfrak{A}_\emptyset,\mathfrak{B}_\emptyset)\in\mathfrak{T}$. For any subset $\mathfrak{a}$ of $\mathcal{J}$ we consider

$$\mathfrak{A}_\mathfrak{a}=((\mathcal{J}-\mathfrak{a})\cap\mathfrak{A}_\emptyset)\cup(\mathfrak{a}\cap\mathfrak{B}_\emptyset)\cup(\mathfrak{A}_\emptyset\cap\mathfrak{B}_\emptyset),$$

$$\mathfrak{B}_\mathfrak{a}=((\mathcal{J}-\mathfrak{a})\cap\mathfrak{B}_\emptyset)\cup(\mathfrak{a}\cap\mathfrak{A}_\emptyset)\cup(\mathfrak{A}_\emptyset\cap\mathfrak{B}_\emptyset).$$

Then $(\mathfrak{A}_\mathfrak{a},\mathfrak{B}_\mathfrak{a})=(\mathfrak{A}_{\mathcal{J}-\mathfrak{a}},\mathfrak{A}_{\mathcal{J}-\mathfrak{a}})\in\mathfrak{T}$ and the map $\mathfrak{a}\mapsto(\mathfrak{A}_\mathfrak{a},\mathfrak{B}_\mathfrak{a})$ induces a bijection $\bar{\mathcal{P}}(\mathcal{J})\leftrightarrow\mathfrak{T}$. (Note that if $\mathfrak{a}=\emptyset$ then $(\mathfrak{A}_\mathfrak{a},\mathfrak{B}_\mathfrak{a})$ agrees with the earlier definition of $(\mathfrak{A}_\emptyset,\mathfrak{B}_\emptyset)$.)

Let $\mathfrak{T}'$ be the set of all $(\mathfrak{A},\mathfrak{B})\in\mathfrak{T}$ such that $|\mathfrak{A}|=|\mathfrak{A}_\emptyset|,|\mathfrak{B}|=|\mathfrak{B}_\emptyset|$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}(\mathcal{J})$ spanned by the following 2-element subsets of $\mathcal{J}$:

$$\{a_{i_1},a_{i_2}\},\{a_{i_3},a_{i_4}\},\dots,\{a_{i_{\mu-2}},a_{i_{\mu-1}}\}(\text{if } N \text{ is odd})$$

or

$$\{a_{i_1},a_{i_2}\},\{a_{i_3},a_{i_4}\},\dots,\{a_{i_{\mu-1}},a_{i_\mu}\}(\text{if } N \text{ is even}).$$

Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}(\mathcal{J})$ spanned by the following 2-element subsets of $\mathcal{J}$:

$$\{a_{i_0},a_{i_1}\},\{a_{i_2},a_{i_3}\},\dots,\{a_{i_{\mu-1}},a_{i_\mu}\}(\text{if } N \text{ is odd})$$

or

$$\{a_{i_0},a_{i_1}\},\{a_{i_2},a_{i_3}\},\dots,\{a_{i_{\mu-2}},a_{i_{\mu-1}}\}(\text{if } N \text{ is even}).$$

Let $\bar{\mathcal{P}}(\mathcal{J})_0$ (resp. $\bar{\mathcal{P}}(\mathcal{J})_1$) be the image of $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the obvious map $\mathcal{P}(\mathcal{J})\to\bar{\mathcal{P}}(\mathcal{J})$.

Note that

(a) *$\bar{\mathcal{P}}(\mathcal{J})_0$ and $\bar{\mathcal{P}}(\mathcal{J})_1$ are opposed Lagrangian subspaces of the symplectic vector space $\bar{\mathcal{P}}_{ev}(\mathcal{J}),(,)$, (see 0.7); hence $(,)$ defines an identification $\bar{\mathcal{P}}(\mathcal{J})_1=\bar{\mathcal{P}}(\mathcal{J})_0^*$* where $\bar{\mathcal{P}}(\mathcal{J})_0^*$ is the vector space dual to $\bar{\mathcal{P}}(\mathcal{J})_0$.

Let $\mathfrak{T}_0$ (resp. $\mathfrak{T}_1$) be the subset of $\mathfrak{T}$ corresponding to $\bar{\mathcal{P}}(\mathcal{J})_0$ (resp. $\bar{\mathcal{P}}(\mathcal{J})_1$) under the bijection $\bar{\mathcal{P}}(\mathcal{J})\leftrightarrow\mathfrak{T}$. Note that $\mathfrak{T}_0\subset\mathfrak{T}',\mathfrak{T}_1\subset\mathfrak{T}'$, $|\mathfrak{T}_0|=|\mathfrak{T}_1|=2^{\lfloor\mu/2\rfloor}$.

For any $X\subset[0,\mu-1]\cap 2\mathbf{N}$ we set $\mathfrak{a}_X=\cup_{s\in X}\{a_{i_s},a_{i_{s+1}}\}\in\mathcal{P}(\mathcal{J})$. Then $(\mathfrak{A}_{\mathfrak{a}_X},\mathfrak{B}_{\mathfrak{a}_X})$ is related to a^X in 2.1 as follows:

$$\mathfrak{A}_{\mathfrak{a}_X}=\{a_i^X;i\in[0,N]\cap 2\mathbf{N}\},\mathfrak{B}_{\mathfrak{a}_X}=\{a_i^X;i\in[0,N]\cap(2\mathbf{N}+1)\}.$$

2.7. We shall use the notation of 2.1. Let T be the set of all unordered pairs (A,B) where A is a subset of $\{0,1,2,\dots\}$, B is a subset of $\{1,2,3,\dots\}$, A contains no consecutive integers, B contains no consecutive integers, and $A\dot{\cup}B=(\overset{\circ}{a}_0,\overset{\circ}{a}_1,\overset{\circ}{a}_2,\dots,\overset{\circ}{a}_N)$ as multisets. For example, setting

$$A_\emptyset=\{\overset{\circ}{a}_i;i\in[0,N]\cap 2\mathbf{N}\},B_\emptyset=(\overset{\circ}{a}_i;i\in[0,N]\cap(2\mathbf{N}+1)\},$$

we have $(A_\emptyset,B_\emptyset)\in T$.

For any $(A,B)\in T$ we define (A^-,B^-) as follows: A^- consists of $x_1<x_2-1<x_3-2<\dots<x_p-p+1$ where $x_1<x_2<\dots<x_p$ are the elements of A; B^- consists of $y_1<y_2-1<\dots<y_q-q+1$ where $y_1<y_2<\dots<y_q$ are the elements of B.

We can enumerate the elements of T as in [L4, 11.5]. Let J be the set of all $c\in\mathbf{N}$ such that c appears exactly once in the sequence

$$(\overset{\circ}{a}_0,\overset{\circ}{a}_1,\overset{\circ}{a}_2,\dots,\overset{\circ}{a}_N)=(a_0,a_1,a_2+1,a_3+1,a_4+2,a_5+2,\dots).$$

A nonempty subset I of J is said to be an interval if it is of the form $\{i,i+1,i+2,\dots,j\}$ with $i-1\notin J,j+1\notin J$. Let $\mathcal{I}$ be the set of intervals of J. For any $s\in[0,\mu-1]\cap 2\mathbf{N}$, the set $I_s:=\{\overset{\circ}{a}_{i_s},\overset{\circ}{a}_{i_s+1},\overset{\circ}{a}_{i_s+2},\dots,\overset{\circ}{a}_{i_s+2m_s+1}\}$ is either a single interval or a union of intervals $I_s^1\sqcup I_s^2\sqcup\dots\sqcup I_s^{t_s}$ $(t_s\ge 2)$ where $\overset{\circ}{a}_{i_s}\in I_s^1$, $\overset{\circ}{a}_{i_s+2m_s+1}\in I_s^{t_s}$, $|I_s^1|,|I_s^{t_s}|$ are odd, $|I_s^h|$ are even for $h\in[2,t_s-1]$ and any element in I_s^e is $<$ than any element in $I_s^{e'}$ for $e<e'$. Let $\mathcal{I}_s$ be the set of all $I\in\mathcal{I}$ such that $I\subset I_s$. We have a partition $\mathcal{I}=\sqcup_{s\in[0,\mu-1]\cap 2\mathbf{N}}\mathcal{I}_s$. For any subset $\alpha\subset\mathcal{I}$ we consider

$$A_\alpha=\cup_{I\in\mathcal{I}-\alpha}(I\cap A_\emptyset)\cup\cup_{I\in\alpha}(I\cap B_\emptyset)\cup(A_\emptyset\cap B_\emptyset),$$

$$B_\alpha=\cup_{I\in\mathcal{I}-\alpha}(I\cap B_\emptyset)\cup\cup_{I\in\alpha}(I\cap A_\emptyset)\cup(A_\emptyset\cap B_\emptyset).$$

Then $(A_\alpha,B_\alpha)\in T$ and the map $\alpha\mapsto(A_\alpha,B_\alpha)$ is a bijection $\bar{\mathcal{P}}(\mathcal{I})\leftrightarrow T$. (Note that if $\alpha=\emptyset$ then (A_α,B_α) agrees with the earlier definition of $(A_\emptyset,B_\emptyset)$.)

Let $T'=\{(A,B)\in T;|A|=|A_\emptyset|,|B|=|B_\emptyset|\}$, $T_1=\{(A,B)\in T';A^-\dot{\cup}B^-=A_\emptyset^-\dot{\cup}B_\emptyset^-\}$. Let $\bar{\mathcal{P}}(\mathcal{I})'$ (resp. $\bar{\mathcal{P}}(\mathcal{I})_1$) be the subset of $\bar{\mathcal{P}}(\mathcal{I})$ corresponding to T' (resp. T_1) under the bijection $\bar{\mathcal{P}}(\mathcal{I})\leftrightarrow T$.

Now let X be a subset of $[0,\mu-1]\cap 2\mathbf{N}$. Let $\alpha_X=\cup_{s\in X}\mathcal{I}_s\in\mathcal{P}(\mathcal{I})$. From the definitions we see that

(a) $A^-_{\alpha_X}=\mathfrak{A}_{\mathfrak{a}_X}$, $B^-_{\alpha_X}=\mathfrak{B}_{\mathfrak{a}_X}$

(notation of 2.6). In particular we have $(A_{\alpha_X},B_{\alpha_X})\in T_1$. Thus $|T_1|\ge 2^{\lfloor\mu/2\rfloor}$. Using Lemma 2.2 we see that

(b) $|T_1|=2^{\lfloor\mu/2\rfloor}$ *and* T_1 *consists of the pairs* $(A_{\alpha_X},B_{\alpha_X})$ *with* $X\subset[0,\mu-1]\cap 2\mathbf{N}$.

Using (a),(b) we deduce:

(c) *The map* $T_1\to\mathfrak{T}_1$ *given by* $(A,B)\mapsto(A^-,B^-)$ *is a bijection.*

3. Proof of Theorem 0.4 and of Corollary 0.5

3.1. If G is simple adjoint of type A_n, $n \geq 1$, then 0.4 and 0.5 are obvious: we have $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.2. Assume that $G = Sp_{2n}(\mathbf{k})$ where $n \geq 2$. Let N be a sufficiently large even integer. Now $u : \mathbf{k}^{2n} \to \mathbf{k}^{2n}$ has i_e Jordan blocks of size e ($e = 1, 2, 3, \dots$). Here $i_1, i_3, i_5, \dots$ are even. Let $\Delta = \{e \in \{2, 4, 6, \dots\}; i_e \geq 1\}$. Then $A(u)$ can be identified in the standard way with $\mathcal{P}(\Delta)$. Hence the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the F_2-vector space dual to $\mathcal{P}(\Delta)$) may be also canonically identified with $\mathcal{P}(\Delta)$ itself (so that the basis given by the one element subsets of Δ is self-dual).

To the partition $1i_1 + 2i_2 + 3i_3 + \dots$ of $2n$ we associate a pair (A, B) as in [L4, 11.6] (with $N, 2m$ replaced by $2n, N$). We have $A = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \dots, \hat{a}_N)$, $B = (\hat{a}_1, \hat{a}_3, \dots, \hat{a}_{N-1})$, where $\hat{a}_0 \leq \hat{a}_1 \leq \hat{a}_2 \leq \dots \leq \hat{a}_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \dots \leq a_N$ as in 1.1. (Here we use that C is special.) Now the definitions and results in §1 are applicable. As in [L3, 4.5] the family $\mathcal{F}$ is in canonical bijection with $\mathfrak{T}'$ in 1.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \dots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of Δ in increasing order $e_1 < e_2 < \dots < e_{f'}$; then $f = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [L4, 11.6]. Using this bijection we see that $A(u)$ and $\hat{A}(u)$ are identified with the F_2-vector space $\mathcal{P}(\mathcal{I})$ with basis given by the one element subsets of $\mathcal{I}$. Let $\pi : \mathcal{P}(\mathcal{I}) \to \mathcal{P}(\mathcal{I})_1^*$ (with $\mathcal{P}(\mathcal{I})_1^*$ as in 1.7(c)) be the (surjective) F_2-linear map which to $X \subset \mathcal{I}$ associates the linear form $L \mapsto |X \cap L| \mod 2$ on $\mathcal{P}(\mathcal{I})_1$. We will show that

(a) $\ker \pi = \mathcal{K}(u)$ ($\mathcal{K}(u)$ as in 0.1).

We identify $\mathrm{Irr}_C W$ with T' (see 1.7) via the restriction of the bijection in [L4, (12.2.4)] (we also use the description of the Springer correspondence in [L4, 12.3]). Under this identification the subset $\mathrm{Irr}_C^* W$ of $\mathrm{Irr}_C W$ becomes the subset T_1 (see 1.7) of T'. Via the identification $\mathcal{P}(\mathcal{I})' \leftrightarrow T'$ in 1.7 and $\hat{A}(u) \leftrightarrow \mathcal{P}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from T' to $\hat{A}(u)$ becomes the obvious imbedding $\mathcal{P}(\mathcal{I})' \to \mathcal{P}(\mathcal{I})$ (we use again [L4, 12.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in \mathcal{P}(\mathcal{I})$ such that for any $L \in \mathcal{P}(\mathcal{I})_1$ we have $|X \cap L| = 0 \mod 2$. Thus, (a) holds.

Using (a) we have canonically $\bar{A}(u) = \mathcal{P}(\mathcal{I})_1^*$ via π. We define an F_2-linear map $\mathcal{P}(\mathcal{I})_1 \to \bar{\mathcal{P}}(\mathcal{J})_1$ (see 1.6) by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in \{1, 3, \dots, 2M-1\}$ (I_s as in 1.7). This is an isomorphism; it corresponds to the bijection 1.7(c) under the identification $T_1 \leftrightarrow \mathcal{P}(\mathcal{I})_1$ in 1.7 and the identification $\mathfrak{T}_1 \leftrightarrow \bar{\mathcal{P}}(\mathcal{J})_1$ in 1.6. Hence we can identify $\mathcal{P}(\mathcal{I})_1^*$ with $\bar{\mathcal{P}}(\mathcal{J})_1^*$ and with $\bar{\mathcal{P}}(\mathcal{J})_0$ (see 1.6(a)). We obtain an identification $\bar{A}(u) = \bar{\mathcal{P}}(\mathcal{J})_0$.

By [L3, 4.5] we have $\mathbf{X}_{\mathcal{F}} = \bar{\mathcal{P}}(\mathcal{J})$. Using 1.6(a) we see that $\bar{\mathcal{P}}(\mathcal{J}) = M(\bar{\mathcal{P}}(\mathcal{J})_0) = M(\bar{A}(u))$ canonically so that 0.4 holds in our case. From the arguments above we see that in our case 0.5 follows from 1.7(c).

3.3. Assume that $G = SO_n(\mathbf{k})$ where $n \geq 7$. Let N be a sufficiently large integer such that $N = n \mod 2$. Now $u : \mathbf{k}^n \to \mathbf{k}^n$ has i_e Jordan blocks of size e $(e = 1, 2, 3, \dots)$. Here $i_2, i_4, i_6, \dots$ are even. Let $\Delta = \{e \in \{1, 3, 5, \dots\}; i_e \ge 1\}$. If $\Delta = \emptyset$ then $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$ and $\mathcal{G}_{\mathcal{F}} = \{1\}$ so that the result is trivial.

In the remainder of this subsection we assume that $\Delta \ne \emptyset$. Then $A(u)$ can be identified in the standard way with the F_2-subspace $\mathcal{P}_{ev}(\Delta)$ of $\mathcal{P}(\Delta)$ and the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the F_2-vector space dual to $A(u)$) becomes $\bar{\mathcal{P}}(\Delta)$; the obvious pairing $A(u) \times \hat{A}(u) \to F_2$ is induced by the inner product $L, L' \mapsto |L \cap L'| \mod 2$ on $\mathcal{P}(\Delta)$.

To the partition $1i_1 + 2i_2 + 3i_3 + \dots$ of n we associate a pair (A, B) as in [L4, 11.7] (with N, M replaced by n, N). We have $A = \{\mathring{a}_i; i \in [0, N] \cap 2\mathbf{N}\}$, $B = (\mathring{a}_i; i \in [0, N] \cap (2\mathbf{N} + 1)\}$ where $\mathring{a}_0 \le \mathring{a}_1 \le \mathring{a}_2 \le \dots \le \mathring{a}_N$ is obtained from a sequence $a_0 \le a_1 \le a_2 \le \dots \le a_N$ as in 2.1. (Here we use that C is special.) Now the definitions and results in §2 are applicable. As in [L3, 4.5] (if N is even) or [L3, 4.6] (if N is odd) the family $\mathcal{F}$ is in canonical bijection with $\mathfrak{T}'$ in 2.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \dots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of Δ in increasing order $e_1 < e_2 < \dots < e_{f'}$; then $f = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [L4, 11.7]. Using this bijection we see that $A(u)$ is identified with $\mathcal{P}_{ev}(\mathcal{I})$ and $\hat{A}(u)$ is identified with $\bar{\mathcal{P}}(\mathcal{I})$. For any $X \in \mathcal{P}_{ev}(\mathcal{I})$, the assignment $L \mapsto |X \cap L| \mod 2$ can be viewed as an element of $\bar{\mathcal{P}}(\mathcal{I})_1^*$ (the dual space of $\bar{\mathcal{P}}(\mathcal{I})_1$ in 2.7 which by 2.7(b) is an F_2-vector space of dimension $2^{\lfloor \mu/2 \rfloor}$). This induces a (surjective) F_2-linear map $\pi : \mathcal{P}_{ev}(\mathcal{I}) \to \bar{\mathcal{P}}(\mathcal{I})_1^*$. We will show that

(a) $\ker \pi = \mathcal{K}(u)$ ($\mathcal{K}(u)$ as in 0.1).

We identify $\mathrm{Irr}_C W$ with T' (see 2.7) via the restriction of the bijection in [L4, (13.2.5)] if N is odd or [L4, (13.2.6)] if N is even (we also use the description of the Springer correspondence in [L4, 13.3]). Under this identification the subset $\mathrm{Irr}_C^* W$ of $\mathrm{Irr}_C W$ becomes the subset T_1 (see 2.7) of T'. Via the identification $\bar{\mathcal{P}}(\mathcal{I})' \leftrightarrow T'$ in 2.7 and $\hat{A}(u) \leftrightarrow \bar{\mathcal{P}}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from T' to $\hat{A}(u)$ becomes the obvious imbedding $\bar{\mathcal{P}}(\mathcal{I})_0 \to \bar{\mathcal{P}}(\mathcal{I})$ (we use again [L4, 13.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in \mathcal{P}_{ev}(\mathcal{I})$ such that for any $L \in \mathcal{P}(\mathcal{I})$ representing a vector in $\bar{\mathcal{P}}(\mathcal{I})_1$ we have $|X \cap L| = 0 \mod 2$. Thus, (a) holds.

Using (a) we have canonically $\bar{A}(u) = \bar{\mathcal{P}}(\mathcal{I})_1^*$ via π. We have an F_2-linear map $\bar{\mathcal{P}}(\mathcal{I})_1 \to \bar{\mathcal{P}}(\mathcal{J})_0$ (see 2.6) induced by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in [0, \mu - 1] \cap 2\mathbf{N}$ (I_s as in 2.7). This is an isomorphism; it corresponds to the bijection 2.7(c) under the identification $T_1 \leftrightarrow \bar{\mathcal{P}}(\mathcal{I})_1$ in 2.7 and the identification $\mathfrak{T}_1 \leftrightarrow \bar{\mathcal{P}}(\mathcal{J})_0$ in 2.6. Hence we can identify $\bar{\mathcal{P}}(\mathcal{I})_1^*$ with $\bar{\mathcal{P}}(\mathcal{J})_0^*$ and with $\bar{\mathcal{P}}(\mathcal{J})_1$ (see 2.6(a)). We obtain an identification $\bar{A}(u) = \bar{\mathcal{P}}(\mathcal{J})_1$.

By [L3, 4.6] we have $\mathbf{X}_{\mathcal{F}} = \bar{\mathcal{P}}_{ev}(\mathcal{J})$. Using 2.6(a) we see that $\bar{\mathcal{P}}(\mathcal{J}) = M(\bar{\mathcal{P}}(\mathcal{J})_1) = M(\bar{A}(u))$ canonically so that 0.4 holds in our case. From the arguments above we see that in our case 0.5 follows from 2.7(c).

3.4. In 3.5-3.9 we consider the case where G is simple adjoint of exceptional type.

In each case we list the elements of the set $\mathrm{Irr}_C W$ for each special unipotent class C of G; the elements of $\mathrm{Irr}_C W - \mathrm{Irr}^* CW$ are enclosed in $[\,]$. (The notation for the various C is as in [Sp2]; the notation for the objects of $\mathrm{Irr}W$ is as in [Sp2] (for type E_n) and as in [L3, 4.10] for type F_4.) In each case the structure of $A(u), \bar{A}(u)$ (for $u \in C$) is indicated; here S_n denotes the symmetric group in n letters. The order in which we list the objects in $\mathrm{Irr}_C W$ corresponds to the following order of the irreducible representations of $A(u) = S_n$:

$1, \epsilon$ $(n = 2)$; $1, r, \epsilon$ $(n = 3, G \neq G_2)$; $1, r$ $(n = 3, G = G_2)$; $1, \lambda^1, \lambda^2, \sigma$ $(n = 4)$;

$1, \nu, \lambda^1, \nu', \lambda^2, \lambda^3$ $(n = 5)$

(notation of [L3, 4.3]). Now 0.4 and 0.5 follow in our case from the tables in 3.5-3.9 and the definitions in [L3, 4.8-4.13]. (In those tables S_n is the symmetric group in n letters.)

3.5. Assume that G is of type E_8.

$\mathrm{Irr}_{E_8} W = \{1_0\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{E_8(a_1)} W = \{8_1\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{E_8(a_2)} W = \{35_2\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{E_7 A_1} W = \{112_3, 28_8\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{D_8} W = \{210_4, 160_7\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{E_7(a_1)A_1} W = \{560_5, [50_8]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{E_7(a_1)} W = \{567_6\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{D_8(a_1)} W = \{700_6, 300_8\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{E_7(a_2)A_1} W = \{1400_7, 1008_9, 56_{19}\}$; $A(u) = S_3, \bar{A}(u) = S_3$

$\mathrm{Irr}_{A_8} W = \{1400_8, 1575_{10}, 350_{14}\}$; $A(u) = S_3, \bar{A}(u) = S_3$

$\mathrm{Irr}_{D_7(a_1)} W = \{3240_9, [1050_{10}]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{D_8(a_3)} W = \{2240_{10}, [175_{12}], 840_{13}\}$; $A(u) = S_3, \bar{A}(u) = S_2$

$\mathrm{Irr}_{D_6 A_1} W = \{2268_{10}, 1296_{13}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{E_6(a_1)A_1} W = \{4096_{11}, 4096_{12}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{E_6} W = \{525_{12}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{D_7(a_2)} W = \{4200_{12}, 3360_{13}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{E_6(a_1)} W = \{2800_{13}, 2100_{16}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{D_5 A_2} W = \{4536_{13}, [840_{14}]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{D_6(a_1)A_1} W = \{6075_{14}, [700_{16}]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{A_6 A_1} W = \{2835_{14}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{A_6} W = \{4200_{15}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{D_6(a_1)} W = \{5600_{15}, 2400_{17}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{2A_4} W = \{4480_{16}, 4536_{18}, 5670_{18}, 1400_{20}, 1680_{22}, 70_{32}\}$; $A(u) = S_5, \bar{A}(u) = S_5$

$\mathrm{Irr}_{D_5} W = \{2100_{20}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{(A_5 A_1)''} W = \{5600_{21}, 2400_{23}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{D_4 A_2} W = \{4200_{15}, [168_{24}]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{A_4 A_2 A_1} W = \{2835_{22}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{A_4 A_2} W = \{4536_{23}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{D_5(a_1)} W = \{2800_{25}, 2100_{28}\}$; $A(u) = S_2, \bar{A}(u) = S_2$

$\mathrm{Irr}_{A_4 2A_1} W = \{4200_{24}, 3360_{25}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{D_4} W = \{525_{36}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_4 A_1} W = \{4096_{26}, 4096_{27}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{A_4} W = \{2268_{30}, 1296_{33}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{D_4(a_1) A_2} = \{2240_{28}, 840_{31}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{A_3 A_2} W = \{3240_{31}, [972_{32}]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_4(a_1) A_1} W = \{1400_{32}, 1575_{34}, 350_{38}\}$; $A(u) = S_3, \bar{A}(u) = S_3$
$\mathrm{Irr}_{D_4(a_1)} W = \{1400_{37}, 1008_{39}, 56_{49}\}$; $A(u) = S_3, \bar{A}(u) = S_3$
$\mathrm{Irr}_{2A_2} W = \{700_{42}, 300_{44}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{A_3} W = \{567_{46}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_2 2A_1} W = \{560_{47}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_2 A_1} W = \{210_{52}, 160_{55}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{A_2} W = \{112_{63}, 28_{68}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{2A_1} W = \{35_{74}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_1} W = \{8_{91}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{\emptyset} W = \{1_{120}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

3.6. Assume that G is adjoint of type E_7.
$\mathrm{Irr}_{E_7} W = \{1_0\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{E_7(a_1)} W = \{7_1\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{E_7(a_2)} W = \{27_2\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_6 A_1} W = \{56_3, 21_6\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{E_6} W = \{21_3\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{E_6(a_1)} W = \{120_4, 105_5\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{D_6(a_1) A_1} W = \{189_5, [15_7]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_6(a_1)} W = \{210_6\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_6} W = \{105_6\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_5 A_1} W = \{168_6\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_5} W = \{189_7\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_6(a_2) A_1} W = \{315_7, 280_9, 35_{13}\}$; $A(u) = S_3, \bar{A}(u) = S_3$
$\mathrm{Irr}_{(A_5 A_1)'} = \{405_8, 189_{10}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{D_5(a_1) A_1} W = \{378_9\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_4 A_2} W = \{210_{10}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_5(a_1)} W = \{420_{10}, 336_{11}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{A_5''} W = \{105_{12}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_4 A_1} W = \{512_{11}, 512_{12}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{D_4} W = \{105_{15}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_4} W = \{420_{13}, 336_{14}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{A_3 A_2 A_1} W = \{210_{13}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{A_3 A_2} W = \{378_{14}, [84_{15}]\}$; $A(u) = S_2, \bar{A}(u) = \{1\}$
$\mathrm{Irr}_{D_4(a_1) A_1} W = \{405_{15}, 189_{17}\}$; $A(u) = S_2, \bar{A}(u) = S_2$
$\mathrm{Irr}_{D_4(a_1)} W = \{315_{16}, 280_{18}, 35_{22}\}$; $A(u) = S_3, \bar{A}(u) = S_3$
$\mathrm{Irr}_{(A_3 A_1)''} W = \{189_{20}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{2A_2}W=\{168_{21}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_23A_1}W=\{105_{21}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_3}W=\{210_{21}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_22A_1}W=\{189_{22}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_2A_1}W=\{120_{25},105_{26}\}$; $A(u)=S_2,\bar A(u)=S_2$
$\mathrm{Irr}_{3A_1''}W=\{21_{36}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_2}W=\{56_{30},21_{33}\}$; $A(u)=S_2,\bar A(u)=S_2$
$\mathrm{Irr}_{2A_1}W=\{27_{37}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_1}W=\{7_{46}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{\emptyset}W=\{1_{63}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$

3.7. Assume that G is adjoint of type E_6.
$\mathrm{Irr}_{E_6}W=\{1_0\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{E_6(a_1)}W=\{6_1\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{D_5}W=\{20_2\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_5A_1}W=\{30_3,15_5\}$; $A(u)=S_2,\bar A(u)=S_2$
$\mathrm{Irr}_{D_5(a_1)}W=\{64_4\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_4A_1}W=\{60_5\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_4}W=\{81_6\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{D_4}W=\{24_6\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{D_4(a_1)}W=\{80_7,90_8,20_{10}\}$; $A(u)=S_3,\bar A(u)=S_3$
$\mathrm{Irr}_{2A_2}W=\{24_{12}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_3}W=\{81_{10}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_22A_1}W=\{60_{11}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_2A_1}w=\{64_{13}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_2}W=\{30_{15},15_{17}\}$; $A(u)=S_2,\bar A(u)=S_2$
$\mathrm{Irr}_{2A_1}W=\{20_{20}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_1}W=\{6_{25}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{\emptyset}W=\{1_{36}\}$; $A(u)=\{1\},\bar A(u)=\{1\}$

3.8. Assume that G is of type F_4.
$\mathrm{Irr}_{F_4}W=\{1_1\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{F_4(a_1)}W=\{4_2,2_3\}$; $A(u)=S_2,\bar A(u)=S_2$
$\mathrm{Irr}_{F_4(a_2)}W=\{9_1\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{B_3}W=\{8_1\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{C_3}W=\{8_3\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{F_4(a_3)}W=\{12_1,9_3,6_2,1_3\}$; $A(u)=S_4,\bar A(u)=S_4$
$\mathrm{Irr}_{\tilde A_2}W=\{8_2\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_2}W=\{8_4,[1_2]\}$; $A(u)=S_2,\bar A(u)=\{1\}$
$\mathrm{Irr}_{A_1\tilde A_1}W=\{9_4\}$; $A(u)=\{1\},\bar A(u)=\{1\}$
$\mathrm{Irr}_{\tilde A_1}W=\{4_5,2_2\}$; $A(u)=S_2,\bar A(u)=S_2$
$\mathrm{Irr}_{\emptyset}W=\{1_4\}$; $A(u)=\{1\},\bar A(u)=\{1\}$

3.9. Assume that G is of type G_2.

$\mathrm{Irr}_{G_2}W$ is the unit representation; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\mathrm{Irr}_{G_2(a_1)}W$ consists of the reflection representation and the one dimensional representation on which the reflection with respect to a long (resp.short) simple coroot acts nontrivially (resp. trivially); $A(u) = S_3, \bar{A}(u) = S_3$

$\mathrm{Irr}_{\emptyset}W = \{\mathrm{sgn}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

3.10. This completes the proof of Theorem 0.4 and that of Corollary 0.5.

We note that the definition of $\mathcal{G}_{\mathcal{F}}$ given in [L3] (for type C_n, B_n) is $\bar{\mathcal{P}}(\mathcal{J})_1$ (in the setup of 3.2) and $\bar{\mathcal{P}}(\mathcal{J})_0$ (in the setup of 3.3) which is noncanonically isomorphic to $\bar{A}(u)$, unlike the definition adopted here that is, $\bar{\mathcal{P}}(\mathcal{J})_0$ (in the setup of 3.2) and $\bar{\mathcal{P}}(\mathcal{J})_1$ (in the setup of 3.3) which makes $\mathcal{G}_{\mathcal{F}}$ canonically isomorphic to $\bar{A}(u)$.

4. Character sheaves

4.1. Let $\hat{G}$ be a set of representatives for the isomorphism classes of character sheaves on G. For any conjugacy class D in G let $D_\omega := \{g_\omega; g \in D\}$, a unipotent class in G. For any unipotent class C in G let $\mathcal{S}_C$ be the set of conjugacy classes D of G such that $D_\omega = C$. It is likely that the following property holds.

(a) *Let $K \in \hat{G}$. There exists a unique unipotent class C of G such that*

-for any $D \in \mathcal{S}_C$, $K|_D$ is a local system (up to shift);

-for some $D \in \mathcal{S}_C$, we have $K|_D \neq 0$;

-for any unipotent class C' of G such that $\dim C' \geq \dim C$, *$C' \neq C$ and any $D \in \mathcal{S}_{C'}$ we have $K|_D = 0$.*

We say that C is the unipotent support of K.

(The uniqueness part is obvious.) Note that [L8, 10.7] provides some support (no pun intended) for (a).

We shall now try to make (a) more precise in the case where $K \in \hat{G}^{un}$, the subset of $\hat{G}$ consisting of unipotent character sheaves (that is $\hat{G}^{un} = \hat{G}_{\bar{\mathbf{Q}}_l}$ with the notation of [L7, 4.2]). As in [L7, 4.6] we have a partition $\hat{G}^{un} = \sqcup_{\mathcal{F}} \hat{G}^{un}_{\mathcal{F}}$ where $\mathcal{F}$ runs over the families of W.

In the remainder of this section we fix a family $\mathcal{F}$ of W and we denote by C the special unipotent class of G such that $E_C \in \mathcal{F}$, see 0.1; let $u \in C$. Let $\Gamma = \bar{A}(u)$ and let $Z(u) \xrightarrow{j'} A(u) \xrightarrow{h} \Gamma$ be the obvious (surjective) homomorphisms; let $j = hj' : Z(u) \to \Gamma$. Let $[\Gamma]$ be the set of conjugacy classes in $A(u)$. For $D \in \mathcal{S}_C$ let $\phi(D)$ be the conjugacy class of $j(g_s)$ in Γ where $g \in D$ is such that $g_\omega = u$; clearly such g exists and is unique up to $Z(u)$-conjugacy so that the conjugacy class of $j(g_s)$ is independent of the choice of g. Thus we get a (surjective) map $\phi : \mathcal{S}_C \to [\Gamma]$. For $\gamma \in [\Gamma]$ we set $\mathcal{S}_{C,\gamma} = \phi^{-1}(\gamma)$. We now select for each $\gamma \in [\Gamma]$ an element $x_\gamma \in \gamma$ and we denote by $\mathrm{Irr}Z_\Gamma(x_\gamma)$ a set of representatives for the isomorphism classes of irreducible representations of $Z_\Gamma(x_\gamma) := \{y \in \Gamma; yx_\gamma = x_\gamma y\}$ (over $\bar{\mathbf{Q}}_l$). Let $D \in \mathcal{S}_{C,\gamma}$, $\mathcal{E} \in \mathrm{Irr}Z_\Gamma(x_\gamma)$. We can find $g \in D$ such that $g_\omega = u, j(g_s) = x_\gamma$ (and another choice for such g

must be of the form bgb^{-1} where $b \in Z(u)$, $j(b) \in Z_\Gamma(x_\gamma)$). Let $\mathcal{E}^D$ be the G-equivariant local system on D whose stalk at $g_1 \in D$ is $\{z \in G; zgz^{-1} = g_1\} \times \mathcal{E}$ modulo the equivalence relation $(z,e) \sim (zh^{-1}, j(h)e)$ for all $h \in Z(g)$. If g is changed to $g_1 = bgb^{-1}$ (b as above) then $\mathcal{E}^D$ is changed to the G-equivariant local system $\mathcal{E}_1^D$ on D whose stalk at $g' \in D$ is $\{z' \in G; z'g_1z'^{-1} = g'\} \times \mathcal{E}$ modulo the equivalence relation $(z',e') \sim (z'h'^{-1}, j(h')e)$ for all $h' \in Z(g_1)$. We have an isomorphism of local systems $\mathcal{E}^D \xrightarrow{\sim} \mathcal{E}_1^D$ which for any $g' \in D$ maps the stalk of $\mathcal{E}^D$ at g' to the stalk of $\mathcal{E}_1^D$ at g' by the rule $(z,e) \mapsto (zb^{-1}, j(b)e)$. (We have $zb^{-1}g_1bz^{-1} = zgz^{-1} = g'$.) This is compatible with the equivalence relations. Thus the isomorphism class of the local system $\mathcal{E}^D$ does not depend on the choice of g.

The properties (b),(c) below appear to be true ([] denotes a shift).

(b) *Let $K \in \hat{G}^{un}_{\mathcal{F}}$. There exists a unique $\gamma \in [\Gamma]$ and a unique $\mathcal{E} \in \mathrm{Irr} Z_\Gamma(x_\gamma)$ such that*

(i) if $D \in \mathcal{S}_{C,\gamma}$, we have $K|_D \cong \mathcal{E}^D[\,]$;

(ii) if $D \in \mathcal{S}_{C,\gamma'}$ with $\gamma' \in [\Gamma] - \{\gamma\}$, we have $K|_D = 0$;

(iii) for any unipotent class C' of G such that $\dim C' \ge \dim C$, $C' \ne C$ *and any $D \in \mathcal{S}_{C'}$ we have $K|_D = 0$.*

(c) *$K \mapsto (\gamma, \mathcal{E})$ in (b) defines a bijection $\hat{G}^{un}_{\mathcal{F}} \xrightarrow{\sim} M(\Gamma)$.*

Note that (b)(iii) follows from [L8, 10.7], at least if p is sufficiently large or 0.

In the case where G is of type E_8 and $\mathcal{F}$ contains the irreducible representation of degree 4480 (so that $\Gamma = S_5$), (b)(i),(b)(ii),(c) have been already stated (without proof) in [L7, 4.7].

For any finite dimensional representation E of W (over $\bar{\mathbf{Q}}_l$) let $\underline{E}$ be the intersection cohomology complex on G with coefficients in the local system with monodromy given by the W-module E on the open set of regular semisimple elements. We have an imbedding $\mathcal{F} \to \hat{G}^{un}_{\mathcal{F}}$, $E \mapsto \underline{E}[\,]$. Composing this imbedding with the map $\hat{G}^{un}_{\mathcal{F}} \xrightarrow{\sim} M(\Gamma)$ in (c) (which we assume to hold) we obtain an imbedding $\mathcal{F} \to M(\Gamma)$. We expect that:

(d) *The imbedding $\mathcal{F} \to M(\Gamma)$ defined above coincides with the imbedding $\mathcal{F} \to M(\Gamma)$ in* [L3, Sec.4].

Note that 0.6 can be regarded as evidence for the validity of (b),(c),(d). Further evidence is given in 4.2-4.5.

4.2. Assume that G is simply connected. Let $D \in \mathcal{S}_C$. Let s be a semisimple element of G such that $su \in D$. Let C_0 be the conjugacy class of u in $Z(s)$. Let W' be the Weyl group of $Z(s)$ regarded as a subgroup of W. For any finite dimensional W'-module E' over $\bar{\mathbf{Q}}_l$ let $\underline{\underline{E'}}$ be the intersection cohomology complex on $Z(s)$ defined in terms of $Z(s), E'$ in the same way as $\underline{E}$ was defined in terms of G, E. Using [L5, (8.8.4)] and the W-equivariance of the isomorphism in *loc.cit.* we see that:

(a) $\underline{E}|_{sC_0} \cong (\underline{\underline{E|_{W'}}})|_{sC_0}[\,]$.

Now, if $K \in \hat{G}^{un}_{\mathcal{F}}$ is of the form $\underline{E}[\,]$ for some $E \in \mathcal{F}$ then the computation of $K|_D$

is reduced by (a) to the computation of $\underline{\underline{E'}}|_{sC_0}$ for any irreducible W'-module E' such that $(E' : E_{W'}) > 0$ (here $(E' : E_{W'})$ is the multiplicity of E' in $E|_{W'}$). If for such E' we define a unipotent class $\mathcal{C}_{E'}$ of $Z(s)$ by $E' \in \mathrm{Irr}_{\mathcal{C}_{E'}} W'$ then, by a known property of $\underline{\underline{E'}}$, we have (with notation of 0.1 with G replaced by $Z(s)$):

(b) if $C_0 = \mathcal{C}_{E'}$ then $\underline{\underline{E'}}|_{sC_0}[\,]$ is the irreducible $Z(s)$-equivariant local system corresponding to $\mathcal{V}_{E'}$;

(c) if $C_0 \ne \mathcal{C}_{E'}$ and $\dim C_0 \ge \dim \mathcal{C}_{E'}$ then $\underline{\underline{E'}}|_{sC_0} = 0$.

We say that D is E-negligible if for any $E' \in \mathrm{Irr}W'$ such that $(E' : E|_{W'}) > 0$ we have $\dim C_0 > \dim \mathcal{C}_{E'}$.

(d) We say that D is E-relevant if

-there is a unique $E'_0 \in \mathrm{Irr}W'$ such that $(E'_0 : E|_{W'}) = 1$ and $\mathcal{C}_{E'} = C_0$ (we then write $E_! = E'_0$);

-for any $E' \in \mathrm{Irr}W'$ such that $(E' : E|_{W'}) > 0, E' \ne E_!$ we have $\dim C_0 > \dim \mathcal{C}_{E'}$.

It is likely that D is always E-negligible or E-relevant. If D is E-negligible then $\underline{E}|_{sC_0} = 0$ (hence $K|_D = 0$); if D is E-relevant then $\underline{E}|_{sC_0}$ (hence $K|_D$) can be explicitly computed using (b),(c).

In the remainder of this subsection we assume in addition that G is almost simple of exceptional type and that C is a distinguished unipotent class. In these cases one can verify that D is E-negligible or E-relevant for any $E \in \mathcal{F}$ hence $K|_D$ can be explicitly computed and we can check that 4.1(b) holds. Moreover, we can compute $K|_D$ for any $K \in \hat{G}^{un}_{\mathcal{F}}$ (not necessarily of form $\underline{E}[\,]$) using an appropriate analogue of (a) (coming again from [L5, (8.8.4)]) and the appropriate analogues of (b),(c) (given in [L4]). We see that 4.1(b) holds again. Moreover we see that 4.1(c),(d) hold in these cases.

4.3. In this subsection we assume that G is of type E_8 and C is distinguished. In this subsection we indicate for each $D \in \mathcal{S}_C$ the set $\mathcal{F}_D = \{E \in \mathcal{F}; D \text{ is } E-$ relevant$\}$ and we describe the map $E \mapsto E_!$ (see 4.2(d)). (Note that if $E \in \mathcal{F} - \mathcal{F}_D$, D is E-negligible.) The notation is as in [Sp2]. We denote by g_i an element of order i of $A(u)$ (except that if $A(u) = S_5$, g_2 denotes a transposition and we denote by g'_2 an element of $A(u)$ whose centralizer has order 8). For each g_i we denote by $\dot{g}_i$ a semisimple element of $Z(u)$ that represents g_i; similarly when $A(u) = S_5$, we denote by $\dot{g}'_2$ a semisimple element of $Z(u)$ that represents g'_2. We write $\mathcal{F}_{g_i}$ (resp. $\mathcal{F}_{g'_2}$) instead of $\mathcal{F}_D$ where D is the G-conjugacy class of $u\dot{g}_i$ (resp. of $u\dot{g}'_2$). We write $\mathcal{H}_{g_i}$ (resp. $\mathcal{H}_{g'_2}$) for the set of all $E_! \in \mathrm{Irr}W'$ where E runs through $\mathcal{F}_{g_i}$ (resp. $\mathcal{F}_{g'_2}$); here $W' \subset W$ is the Weyl group of $Z(\dot{g}_i)$ (resp. $Z(\dot{g}'_2)$) and $E_!$ is as in 4.2(d). We write C_{g_i} (resp. $C_{g'_2}$) for the conjugacy class of u in $Z(\dot{g}_i)$ (resp. $Z(\dot{g}'_2)$).

Assume that C is the regular unipotent class. Then $A(u) = \{1\}$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{1_0\}$.

Assume that C is the subregular unipotent class. Then $A(u) = \{1\}$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{8_1\}$.

Assume that $C=E_8(a_2)$. Then $A(u)=\{1\}$, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{35_2\}$.

Assume that $C=E_7A_1$. Then $A(u)=S_2$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type E_7A_1, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{112_3,28_8\}$, $\mathcal{F}_{g_2}=\{84_4\}$, $\mathcal{H}_{g_2}=\{1_0\}$, $C_{g_2}=$ regular unipotent class.

Assume that $C=D_8$. Then $A(u)=S_2$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type D_8, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{210_4,160_7\}$, $\mathcal{F}_{g_2}=\{50_8\}$, $\mathcal{H}_{g_2}=\{1\}$, $C_{g_2}=$ regular unipotent class.

Assume that $C=E_7(a_1)A_1$. Then $A(u)=S_2$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type E_7A_1, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{560_5\}$, $\mathcal{F}_{g_2}=\{560_5\}$, $\mathcal{H}_{g_2}=\{7_1\boxtimes 1\}$, $C_{g_2}=$ subregular unipotent class in E_7 factor times regular unipotent class in A_1 factor.

Assume that $C=D_8(a_1)$. Then $A(u)=S_2$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type D_8, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{700_6,300_8\}$, $\mathcal{F}_{g_2}=\{400_7\}$, $\mathcal{H}_{g_2}=\{$reflection repres.$\}$, $C_{g_2}=$ subregular unipotent class.

Assume that $C=E_7(a_2)A_1$. Then $A(u)=S_3$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type E_7A_1, $Z(\dot g_3)$ is of type E_6A_2, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{1400_7,1008_9,56_{19}\}$, $\mathcal{F}_{g_2}=\{1344_8\}$, $\mathcal{H}_{g_2}=\{27_2\boxtimes 1\}$, $C_{g_2}=$ subsubregular unipotent class in E_7-factor times regular unipotent class in A_1 factor, $\mathcal{F}_{g_3}=\{448_9\}$, $\mathcal{H}_{g_3}=\{1\}$, $C_{g_3}=$ regular unipotent class.

Assume that $C=A_8$. Then $A(u)=S_3$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type D_8, $Z(\dot g_3)$ is of type A_8, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{1400_8,1575_{10},350_{14}\}$, $\mathcal{F}_{g_2}=\{1050_{10}\}$, $\mathcal{H}_{g_2}=\{28-$dimensional repres.$\}$, $C_{g_2}=$ unipotent class with Jordan blocks of size $5,11$, $\mathcal{F}_{g_3}=\{175_{12}\}$, $\mathcal{H}_{g_3}=\{1\}$, $C_{g_3}=$ regular unipotent class.

Assume that $C=D_8(a_3)$. Then $A(u)=S_3$, $Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type D_8, $Z(\dot g_3)$ is of type E_6A_2, $\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{2240_{10},840_{13}\}$, $\mathcal{F}_{g_2}=\{1400_{11}\}$, $\mathcal{H}_{g_2}=\{56-$dimensional repres.$\}$, $C_{g_2}=$ unipotent class with Jordan blocks of size $7,9$, $\mathcal{F}_{g_3}=\{2240_{10}\}$, $\mathcal{H}_{g_3}=\{6_1\boxtimes 1\}$, $C_{g_3}=$ subregular unipotent class in E_6-factor times regular unipotent class in A_1 factor.

Assume that $C=2A_4$. Then $A(u)=S_5$,

$Z(\dot g_1)=G$, $Z(\dot g_2)$ is of type E_7A_1, $Z(\dot g_2')$ is of type D_8,

$Z(\dot g_3)$ is of type E_6A_2, $Z(\dot g_4)$ is of type D_5A_3, $Z(\dot g_5)$ is of type A_4A_4,

$Z(\dot g_6)$ is of type $A_5A_2A_1$,

$\mathcal{F}_{g_1}=\mathcal{H}_{g_1}=\{4480_{16},4536_{18},5670_{18},1400_{20},1680_{22},70_{32}\}$,

$\mathcal{F}_{g_2}=\{7168_{17},5600_{19},448_{25}\}$, $\mathcal{H}_{g_2}=\{315_7\otimes 1,280_9\otimes 1,35_{13}\otimes 1\}$,

$C_{g_2}=D_6(a_1)A_1$ in E_7-factor times regular unipotent class in A_1-factor,

$\mathcal{F}_{g_2'}=\{4200_{18},2688_{20}\epsilon'',168_{24}\}$,

$\mathcal{H}_{g_2'}=\{$ repres. with symbol $(2<5;0<3),(2<3;0<5),(0<1,4<5)\}$,

$C_{g_2'}=$unipotent class with Jordan blocks of sizes $1,3,5,7$,

$\mathcal{F}_{g_3}=\{3150_{18},1134_{20}\}$, $\mathcal{H}_{g_3}=\{30_3\boxtimes 1,15_5\boxtimes 1\}$,

$C_{g_3}=A_5A_1$ in E_6-factor times regular unipotent class in A_2-factor,

$\mathcal{F}_{g_4}=\{1344_{19}\}$, $\mathcal{H}_{g_4}=\{5-$dimensional repres.$\}$,

$C_{g_4}=$subregular unipotent class in D_5-factor times regular unipotent class in A_3-factor,

$\mathcal{F}_{g_5}=\{420_{20}\}$, $\mathcal{H}_{g_5}=\{1\}$, $C_{g_5}=$regular unipotent class,

$\mathcal{F}_{g_6} = \{2016_{19}\}$, $\mathcal{H}_{g_6} = \{1\}$, C_{g_6} =regular unipotent class.

In each case the i-th member of a list $\mathcal{F}_?$ and the i-th member of the corresponding list $\mathcal{H}_?$ are related by the map $E \mapsto E_!$. Note that the members of the list $\mathcal{F}_{g'_2}$ (when $C = 2A_4$) are not all in the same family. But in all cases, the members of the list $\mathcal{F}_g$ form exactly the subset of IrrW' corresponding to the unipotent class C_g under Springer's correspondence for $Z(\dot{g})$; thus they can be indexed by certain irreducible representations of the group of components of the centralizer of u in $Z(\dot{g})$ modulo the centre of $Z(\dot{g})$. (Here g is g_i or g'_2.) From this one recovers the imbedding $\mathcal{F} \to M(\bar{A}(u))$ in geometric terms.

4.4. In this subsection we assume that $G = Sp_4(\mathbf{k})$ and that $\mathcal{F}$ is the family in Irr$W$ containing the reflection representation so that C is the subregular unipotent class in G. Let D be the conjugacy class in G containing sv where s is semisimple with $Z(s) \cong SL_2(\mathbf{k}) \times SL_2(\mathbf{k})$ and v is a regular unipotent element of $Z(s)$ so that $v \in C$. Let D' be a conjugacy class in G containing $s'v'$ where s' is semisimple with $Z(s') \cong GL_2(\mathbf{k})$ and v' is a regular unipotent element of $Z(s')$ so that $v' \in C$. In this case $\hat{G}^{un}_{\mathcal{F}}$ consists of four character sheaves K_1, K_2, K_3, K_4, the last one being cuspidal. They can be characterized as follows.

$K_1|_C = \bar{\mathbf{Q}}_l[\,]$, $K_1|_D = 0$, $K_1|_{D'} = \bar{\mathbf{Q}}_l[\,]$;
$K_2|_C = \mathcal{L}[\,]$, $K_2|_D = 0$, $K_2|_{D'} = \bar{\mathbf{Q}}_l[\,]$;
$K_3|_C = 0$, $K_3|_D = \bar{\mathbf{Q}}_l[\,]$, $K_3|_{D'} = 0$;
$K_4|_C = 0$, $K_4|_D = \mathcal{L}'[\,]$, $K_4|_{D'} = 0$.

Here $\mathcal{L}$ is a notrivial G-equivariant local system of rank 1 on C, $\mathcal{L}'$ is the inverse image of $\mathcal{L}$ under the obvious map $D \to C$. We see that 4.1(b) holds for all $K \in \hat{G}^{un}_{\mathcal{F}}$ and 4.1(c),(d) hold.

4.5. Assume that $\mathcal{F}$ is the family containing the unit representation of W. Then C is the regular unipotent class of G and $\hat{G}^{un}_{\mathcal{F}}$ consists of a single character sheaf, namely $\bar{\mathbf{Q}}_l[\,]$. Clearly, 4.1(b),(c),(d) hold in this case.

Next we assume that $\mathcal{F}$ is the family containing the sign representation of W. Then $C = \{1\}$ and $\hat{G}^{un}_{\mathcal{F}}$ consists of a single character sheaf, namely $K = \underline{\text{sgn}}[\,]$. Note that for any semisimple class D of G we have $K|_D = \bar{\mathbf{Q}}_l[\,]$ so that 4.1(b),(c),(d) hold in this case.

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