On Some Partitions of a Flag Manifold

ON SOME PARTITIONS OF A FLAG MANIFOLD

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Introduction

Let G be a connected reductive group over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Let $\mathbf{W}$ be the Weyl group of G. Let $\underline{\mathbf{W}}$ be the set of conjugacy classes in $\mathbf{W}$. The main purpose of this paper is to give a (partly conjectural) definition of a surjective map from $\underline{\mathbf{W}}$ to the set of unipotent classes in G (see 1.2(b)). When $p = 0$, a map in the opposite direction was defined in [KL, 9.1] and we expect that it is a one sided inverse of the map in the present paper. The (conjectural) definition of our map is based on the study of certain subvarieties $\mathcal{B}_g^w$ (see below) of the flag manifold $\mathcal{B}$ of G indexed by a unipotent element $g \in G$ and an element $w \in \mathbf{W}$.

Note that $\mathbf{W}$ naturally indexes ($w \mapsto \mathcal{O}_w$) the orbits of G acting on $\mathcal{B} \times \mathcal{B}$ by simultaneous conjugation on the two factors. For $g \in G$ we set $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$. The varieties $\mathcal{B}_g$ play an important role in representation theory and their geometry has been studied extensively. More generally for $g \in G$ and $w \in \mathbf{W}$ we set

$$\mathcal{B}_g^w = \{B \in \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}.$$

Note that $\mathcal{B}_g^1 = \mathcal{B}_g$ and that for fixed g, $(\mathcal{B}_g^w)_{w \in \mathbf{W}}$ form a partition of the flag manifold $\mathcal{B}$.

For fixed w, the varieties $\mathcal{B}_g^w$ ($g \in G$) appear as fibres of a map to G which was introduced in [L3] as part of the definition of character sheaves. Earlier, the varieties $\mathcal{B}_g^w$ for g regular semisimple appeared in [L1] (a precursor of [L3]) where it was shown that from their topology (for $\mathbf{k} = \mathbf{C}$) one can extract nontrivial information about the character table of the corresponding group over a finite field.

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1. The sets $\mathbf{S}_g$

1.1. We fix a prime number l invertible in $\mathbf{k}$. Let $g \in G$ and $w \in \mathbf{W}$. For $i, j \in \mathbf{Z}$ let $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j$ be the subquotient of pure weight j of the l-adic cohomology space

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$H^i_c(\mathcal{B}^w_g, \bar{\mathbf{Q}}_l)$. The centralizer $Z(g)$ of g in G acts on $\mathcal{B}^w_g$ by conjugation and this induces an action of the group of components $\bar{Z}(g)$ on $H^i_c(\mathcal{B}^w_g, \bar{\mathbf{Q}}_l)$ and on each $H^i_c(\mathcal{B}^w_g, \bar{\mathbf{Q}}_l)_j$. For $z \in \bar{Z}(g)$ we set

$$\Xi^w_{g,z} = \sum_{i,j \in \mathbf{Z}} (-1)^i \text{tr}(z, H^i_c(\mathcal{B}^w_g, \bar{\mathbf{Q}}_l)_j) v^j \in \mathbf{Z}[v]$$

where v is an indeterminate; the fact that this belongs to $\mathbf{Z}[v]$ and is independent of the choice of l is proved by an argument similar to that in the proof of [DL, 3.3].

Let $l : \mathbf{W} \to \mathbf{N}$ be the standard length function. The simple reflections $s \in \mathbf{W}$ (that is the elements of length 1 of $\mathbf{W}$) are numbered as $s_1, s_2, \dots$. Let w_0 be the element of maximal length in $\mathbf{W}$.

Let $\mathcal{H}$ be the Iwahori-Hecke algebra of $\mathbf{W}$ with parameter v^2 (see [GP, 4.4.1]; in the definition in *loc.cit.* we take $A = \mathbf{Z}[v, v^{-1}], a_s = b_s = v^2$). Let $(T_w)_{w \in \mathbf{W}}$ be the standard basis of $\mathcal{H}$ (see [GP, 4.4.3, 4.4.6]). For $w \in \mathbf{W}$ let $\hat{T}_w = v^{-2l(w)} T_w$. If $s_{i_1} s_{i_2} \dots s_{i_t}$ is a reduced expression for $w \in \mathbf{W}$ we write also $\hat{T}_w = \hat{T}_{i_1 i_2 \dots i_t}$.

For any $g \in G, z \in \bar{Z}(g)$ we set

$$\Pi_{g,z} = \sum_{w \in \mathbf{W}} \Xi^w_{g,z} \hat{T}_w \in \mathcal{H}.$$

The following result can be proved along the lines of the proof of [DL, Theorem 1.6] (we replace the Frobenius map in that proof by conjugation by g); alternatively, for g unipotent, we may use 1.5(a).

(a) $\Pi_{g,z}$ *belongs to the centre of the algebra* $\mathcal{H}$.

According to [GP, 8.2.6, 7.1.7], an element $c = \sum_{w \in \mathbf{W}} c_w \hat{T}_w$ ($c_w \in \mathbf{Z}[v, v^{-1}]$) in the centre of $\mathcal{H}$ is uniquely determined by the coefficients $c_w (w \in \mathbf{W}_{min})$ and we have $c_w = c_{w'}$ if $w, w' \in \mathbf{W}_{min}$ are conjugate in $\mathbf{W}$; here $\mathbf{W}_{min}$ is the set of elements of $\mathbf{W}$ which have minimal length in their conjugacy class. This applies in particular to $c = \Pi_{g,z}$, see (a). For any $C \in \underline{\mathbf{W}}$ we set $\Xi^C_{g,z} = \Xi^w_{g,z}$ where w is any element of $C \cap \mathbf{W}_{min}$.

Note that if $g = 1$ then $\Pi_{g,1} = (\sum_w v^{2l(w)} 1$. If g is regular unipotent then $\Pi_{g,1} = \sum_{w \in \mathbf{W}} v^{2l(w)} \hat{T}_w$. If $G = PGL_3(\mathbf{k})$ and $g \in G$ is regular semisimple then $\Pi_{g,1} = 6 + 3(v^2 - 1)(\hat{T}_1 + \hat{T}_2) + (v^2 - 1)^2 (\hat{T}_{12} + \hat{T}_{21}) + (v^6 - 1)\hat{T}_{121}$; if $g \in G$ is a transvection then $\Pi_{g,1} = (2v^2 + 1) + v^4(\hat{T}_1 + \hat{T}_2) + v^6 \hat{T}_{121}$.

For $g \in G$ let $cl(g)$ be the G-conjugacy class of g; let $\overline{cl(g)}$ be the closure of $cl(g)$. Let $\mathbf{S}_g$ be the set of all $C \in \underline{\mathbf{W}}$ such that $\Xi^C_{g,1} \neq 0$ and $\Xi^C_{g',1} = 0$ for any $g' \in \overline{cl(g)} - cl(g)$. If $\mathcal{C}$ is a conjugacy class in G we shall also write $\mathbf{S}_{\mathcal{C}}$ instead of $\mathbf{S}_g$ where $g \in \mathcal{C}$.

We describe the set $\mathbf{S}_g$ and the values $\Xi^C_{g,1}$ for $C \in \mathbf{S}_g$ for various G of low rank and various unipotent elements g in G. We denote by u_n a unipotent element of G such that $\dim \mathcal{B}_{u_n} = n$. The conjugacy class of $w \in \mathbf{W}$ is denoted by (w).

G of type A_1.

$$\mathbf{S}_{u_1}=(1),\mathbf{S}_{u_0}=(s_1);\Xi^1_{u_1,1}=1+v^2,\Xi^{s_1}_{u_0,1}=v^2.$$

G of type A_2.

$$\mathbf{S}_{u_3}=(1),\mathbf{S}_{u_1}=(s_1),\mathbf{S}_{u_0}=(s_1s_2).$$

$$\Xi^1_{u_3,1}=1+2v^2+2v^4+v^6,\Xi^{(s_1)}_{u_1,1}=v^4,\Xi^{(s_1s_2)}_{u_0,1}=v^4.$$

G of type B_2, $p\neq 2$. (The simple reflection corresponding to the long root is denoted by s_1.)

$$\mathbf{S}_{u_4}=(1),\mathbf{S}_{u_2}=(s_1),\mathbf{S}_{u_1}=\{(s_2),(s_1s_2s_1s_2)\},\mathbf{S}_{u_0}=(s_1s_2).$$

$$\Xi^1_{u_4,1}=(1+v^2)^2(1+v^4),\Xi^{(s_1)}_{u_2,1}=v^4(1+v^2),\Xi^{(s_2)}_{u_1,1}=2v^4,$$
$$\Xi^{(s_1s_2s_1s_2)}_{u_1,1}=v^6(v^2-1),\Xi^{(s_1s_2)}_{u_0,1}=v^4.$$

G of type B_2, $p=2$. (u_2' denotes a transvection; u_2'' denotes a unipotent element with $\dim\mathcal{B}_{u_2''}=2$ which is not conjugate to u_2'.)

$$\mathbf{S}_{u_4}=(1),\mathbf{S}_{u_2'}=(s_1),\mathbf{S}(u_2'')=(s_2),\mathbf{S}_{u_1}=(s_1s_2s_1s_2),\mathbf{S}_{u_0}=(s_1s_2).$$

$$\Xi^1_{u_4,1}=(1+v^2)^2(1+v^4),\Xi^{(s_1)}_{u_2',1}=v^4(1+v^2),\Xi^{(s_2)}_{u_2'',1}=v^4(1+v^2),$$
$$\Xi^{(s_1s_2s_1s_2)}_{u_1,1}=v^8,\Xi^{(s_1s_2)}_{u_0,1}=v^4.$$

G of type G_2, $p\neq 2,3$. (The simple reflection corresponding to the long root is denoted by s_2.)

$$\mathbf{S}_{u_6}=(1),\mathbf{S}_{u_3}=(s_2),\mathbf{S}_{u_2}=\{(s_1),(s_1s_2s_1s_2s_1s_2)\},\mathbf{S}_{u_1}=(s_1s_2s_1s_2),$$
$$\mathbf{S}_{u_0}=(s_1s_2).$$

$$\Xi^1_{u_6,1}=(1+v^2)^2(1+v^4+v^8),\Xi^{(s_2)}_{u_3,1}=v^6(1+v^2),\Xi^{(s_1)}_{u_2,1}=v^4(1+v^2),$$
$$\Xi^{(s_1s_2s_1s_2s_1s_2)}_{u_2,1}=v^8(v^4-1),\Xi^{(s_1s_2s_1s_2)}_{u_1,1}=2v^8,\Xi^{(s_1s_2)}_{u_0,1}=v^4.$$

G is of type A_3. (The simple reflections are s_1,s_2,s_3 with $s_1s_3=s_3s_1$).

$$\mathbf{S}_{u_6}=(1),\mathbf{S}_{u_3}=(s_1),\mathbf{S}_{u_2}=(s_1s_3),\mathbf{S}_{u_1}=(s_1s_2),\mathbf{S}_{u_0}=(s_1s_2s_3).$$

$$\Xi^1_{u_6,1}=(1+v^2)(1+v^2+v^4)(1+v^2+v^4+v^6),\Xi^{(s_1)}_{u_3,1}=v^6+v^8,$$
$$\Xi^{(s_1s_3)}_{u_2,1}=v^6+v^8,\Xi^{(s_1s_2)}_{u_1,1}=v^6,\Xi^{(s_1s_2s_3)}_{u_0,1}=v^6.$$

G of type B_3, $p \neq 2$. (The simple reflection corresponding to the short root is denoted by s_3 and $(s_1s_3)^2=1$.)

$$\mathbf{S}_{u_9}=(1), \mathbf{S}_{u_5}=(s_1), \mathbf{S}_{u_4}=\{(s_3),(s_2s_3s_2s_3)\}, \mathbf{S}_{u_3}=\{(s_1s_3),(w_0)\},$$
$$\mathbf{S}_{u_2}=(s_1s_2), \mathbf{S}_{u_1}=\{(s_2s_3),(s_2s_3s_1s_2s_3)\}, \mathbf{S}_{u_0}=(s_1s_2s_3).$$

$$\Xi^1_{u_9,1}=(1+v^2)^3(1+v^4)(1+v^4+v^8), \Xi^{(s_1)}_{u_5,1}=v^8(1+v^2)^2,$$
$$\Xi^{(s_2s_3s_2s_3)}_{u_4,1}=v^8(1+v^2)(v^4-1), \Xi^{(s_3)}_{u_4,1}=2v^6(1+v^2)^2,$$
$$\Xi^{(s_1s_3)}_{u_3,1}=v^8(1+v^2), \Xi^{(w_0)}_{u_3,1}=v^{14}(v^4-1), \Xi^{(s_1s_2)}_{u_2,1}=2v^8,$$
$$\Xi^{(s_2s_3)}_{u_1,1}=2v^6, \Xi^{(s_2s_3s_1s_2s_3)}_{u_1,1}=v^8(v^2-1), \Xi^{(s_1s_2s_3)}_{u_0,1}=v^6.$$

G of type C_3, $p \neq 2$. (The simple reflection corresponding to the long root is denoted by s_3 and $(s_1s_3)^2=1$; u_2'' denotes a unipotent element which is regular inside a Levi subgroup of type C_2; u_2' denotes a unipotent element with $\dim \mathcal{B}_{u_2''}=2$ which is not conjugate to u_2''.)

$$\mathbf{S}_{u_9}=(1), \mathbf{S}_{u_6}=(s_3), \mathbf{S}_{u_4}=\{(s_1),(s_2s_3s_2s_3)\}, \mathbf{S}_{u_3}=\{(s_1s_3),(w_0)\},$$
$$\mathbf{S}_{u_2'}=(s_1s_2), \mathbf{S}_{u_2''}=(s_2s_3), \mathbf{S}_{u_1}=(s_2s_3s_1s_2s_3), \mathbf{S}_{u_0}=(s_1s_2s_3).$$

$$\Xi^1_{u_9,1}=(1+v^2)^3(1+v^4)(1+v^4+v^8), \Xi^{(s_3)}_{u_6,1}=v^6(1+v^2)^2(1+v^4),$$
$$\Xi^{(s_2s_3s_2s_3)}_{u_4,1}=v^{10}(v^4-1), \Xi^{(s_1)}_{u_4,1}=2v^8(1+v^2),$$
$$\Xi^{(s_1s_3)}_{u_3,1}=v^8(1+v^2), \Xi^{(w_0)}_{u_3,1}=v^{14}(v^4-1), \Xi^{(s_1s_2)}_{u_2',1}=v^6(1+v^2),$$
$$\Xi^{(s_2s_3)}_{u_2'',1}=v^6(1+v^2), \Xi^{(s_2s_3s_1s_2s_3)}_{u_1,1}=v^{10}, \Xi^{(s_1s_2s_3)}_{u_0,1}=v^6.$$

1.2. We expect that the following property of G holds:

(a) $$\underline{\mathbf{W}}=\sqcup_u \mathbf{S}_u$$

(u runs over a set of representatives for the unipotent classes in G).

The equality $\mathbf{W}=\cup_u \mathbf{S}_u$ is clear since for a regular unipotent u and any w we have $\Xi^w_{u,1}=v^{2l(w)}$. Note that (a) holds for G of rank ≤ 3 if p is not a bad prime for G (see 1.1). We will show elsewhere that (a) holds for G of type A_n (any p) and of type B_n, C_n, D_n ($p \neq 2$). When G is simple of exceptional type, (a) should follow by computing the product of some known (large) matrices using 1.5(a).

Assuming that (a) holds we define a surjective map from $\underline{\mathbf{W}}$ to the set of unipotent classes in G by

(b) $$C \mapsto \mathcal{C}$$

where $C \in \underline{\mathbf{W}}$ and $\mathcal{C}$ is the unique unipotent class in G such that $C \in \mathbf{S}_u$ for $u \in \mathcal{C}$.

We expect that when $p = 0$ we have

(c) $$c_u \in \mathbf{S}_u$$

where for any unipotent element $u \in G$, c_u denotes the conjugacy class in $\mathbf{W}$ associated to u in [KL, 9.1]. Note that (c) holds for G of rank ≤ 3 (see 1.1). (We have used the computations of the map in [KL, 9.1] given in [KL, §9], [S1], [S2].)

1.3. Assume that $G = Sp_{2n}(\mathbf{k})$ and $p \neq 2$. The Weyl group $\mathbf{W}$ can be identified in the standard way with the subgroup of the symmetric group S_{2n} consisting of all permutations of $[1, 2n]$ which commute with the involution $i \mapsto 2n + 1 - i$. We say that two elements of $\underline{\mathbf{W}}$ are equivalent if they are contained in the same conjugacy class of S_{2n}. The set of equivalence classes in $\underline{\mathbf{W}}$ is in bijection with the set of partitions of $2n$ in which every odd part appears an even number of times (to $C \in \underline{\mathbf{W}}$ we attach the partition which has a part j for every j-cycle of an element of C viewed as a permutation of $[1, 2n]$). The same set of partitions of $2n$ indexes the set of unipotent classes of G. Thus we obtain a bijection between the set of equivalence classes in $\underline{\mathbf{W}}$ and the set of unipotent classes of G. In other words we obtain a surjective map ϕ from $\underline{\mathbf{W}}$ to the set of unipotent classes of G whose fibres are the equivalence classes in $\underline{\mathbf{W}}$. We will show elsewhere that for any unipotent class $\mathcal{C}$ in G we have $\phi^{-1}(\mathcal{C}) = \mathbf{S}_u$ where $u \in \mathcal{C}$.

1.4. Recall that the set of unipotent elements in G can be partitioned into "special pieces" (see [L5]) where each special piece is a union of unipotent classes exactly one of which is "special". Thus the special pieces can be indexed by the set of isomorphism classes of special representations of $\mathbf{W}$ which depends only on $\mathbf{W}$ as a Coxeter group (not on the underlying root system). For each special piece σ of G we consider the subset $\mathbf{S}_\sigma := \sqcup_{\mathcal{C} \subset \sigma} \mathbf{S}_{\mathcal{C}}$ of $\underline{\mathbf{W}}$ (here $\mathcal{C}$ runs over the unipotent classes contained in σ). We expect that each such subset $\mathbf{S}_\sigma$ depends only on the Coxeter group structure of $\mathbf{W}$ (not on the underlying root system). As evidence for this we note that the subsets $\mathbf{S}_\sigma$ for G of type B_3 are the same as the subsets $\mathbf{S}_\sigma$ for G of type C_3. These subsets are as follows:

$$\{1\}, \{(s_1), (s_3), (s_2s_3s_2s_3)\}, \{(s_1s_3), (w_0)\}, \{(s_1s_2)\},$$
$$\{(s_2s_3), (s_2s_3s_1s_2s_3)\}, \{(s_1s_2s_3)\}.$$

1.5. Let $g \in G$ be a unipotent element and let $z \in \bar{Z}(g)$, $w \in W$. We show how the polynomial $\Xi^w_{g,z}$ can be computed using information from representation theory. We may assume that $p > 1$ and that $\mathbf{k}$ is the algebraic closure of the finite field $\mathbf{F}_p$. We choose an $\mathbf{F}_p$ split rational structure on G with Frobenius map $F_0 : G \to G$. We may assume that $g \in G^{F_0}$. Let $q = p^m$ where $m \ge 1$ is sufficiently divisible. In particular $F := F_0^m$ acts trivially on $\bar{Z}(g)$ hence $cl(g)^F$ is a union of G^F-conjugacy classes naturally indexed by the conjugacy classes in $\bar{Z}(g)$; in particular the G^F-conjugacy class of g corresponds to $1 \in \bar{Z}(g)$. Let g_z be an element of the G^F-conjugacy class in $cl(g)^F$ corresponding to the $\bar{Z}(g)$-conjugacy class of $z \in \bar{Z}(g)$. The set $\mathcal{B}^w_{g_z}$ is F-stable. We first compute the number of fixed points $|(\mathcal{B}^w_{g_z})^F|$.

Let $\mathcal{H}_q = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}[v,v^{-1}]} \mathcal{H}$ where $\bar{\mathbf{Q}}_l$ is regarded as a $\mathbf{Z}[v, v^{-1}]$-algebra with v acting as multiplication by $\sqrt{q}$. We write T_w instead of $1 \otimes T_w$. Let $\mathrm{Irr}\mathbf{W}$ be a set of representatives for the isomorphism classes of irreducible $\mathbf{W}$-modules over $\bar{\mathbf{Q}}_l$. For any $E \in \mathrm{Irr}\mathbf{W}$ let E_q be the irreducible $\mathcal{H}_q$-module corresponding naturally to E. Let $\mathcal{F}$ be the vector space of functions $\mathcal{B}^F \to \bar{\mathbf{Q}}_l$. We regard $\mathcal{F}$ as a G^F-module by $\gamma : f \mapsto f'$, $f'(B) = f(\gamma^{-1}B\gamma)$ for all $B \in \mathcal{B}^F$. We identify $\mathcal{H}_q$ with the algebra of all endomorphisms of $\mathcal{F}$ which commute with the G^F-action, by identifying T_w with the endomorphism $f \mapsto f'$ where $f'(B) = \sum_{B' \in \mathcal{B}^F; (B,B') \in \mathcal{O}_w} f(B)$ for all $B \in \mathcal{B}^F$. As a module over $\bar{\mathbf{Q}}_l[G^F] \otimes \mathcal{H}_q$ we have canonically $\mathcal{F} = \oplus_{E \in \mathrm{Irr}\mathbf{W}} \rho_E \otimes E_q$ where ρ_E is an irreducible G^F-module. Hence if $\gamma \in G^F$ and $w \in \mathbf{W}$ we have $\mathrm{tr}(\gamma T_w, \mathcal{F}) = \sum_{E \in \mathrm{Irr}\mathbf{W}} \mathrm{tr}(\gamma, \rho_E)\mathrm{tr}(T_w, E_q)$. From the definition we have $\mathrm{tr}(\gamma T_w, \mathcal{F}) = |\{B \in \mathcal{B}^F; (B, \gamma B \gamma^{-1}) \in \mathcal{O}_w\}| = |(\mathcal{B}^w_\gamma)^F|$. Taking $\gamma = g_z$ we obtain

(a)
$$|(\mathcal{B}^w_{g_z})^F| = \sum_{E \in \mathrm{Irr}\mathbf{W}} \mathrm{tr}(g_z, \rho_E)\mathrm{tr}(T_w, E_q).$$

The quantity $\mathrm{tr}(g_z, \rho_E)$ can be computed explicitly, by the method of [L4], in terms of generalized Green functions and of the entries of the non-abelian Fourier transform matrices [L2]; in particular it is a polynomial with rational coefficients in $\sqrt{q}$. The quantity $\mathrm{tr}(T_w, E_q)$ can be also computed explicitly (see [GP], Ch.10,11); it is a polynomial with integer coefficients in $\sqrt{q}$. Thus $|(\mathcal{B}^w_{g_z})^F|$ is an explicitly computable polynomial with rational coefficients in $\sqrt{q}$. Substituting here $\sqrt{q}$ by v we obtain the polynomial $\Xi^w_{g,z}$. This argument shows also that $\Xi^w_{g,z}$ is independent of p (note that the pairs (g, z) up to conjugacy may be parametrized by a set independent of p).

This is how the various $\Xi^w_{g,z}$ in 1.1 were computed, except in type A_1, A_2, B_2 where they were computed directly from the definitions. (For type B_3, C_3 we have used the computation of Green functions in [Sh]; for type G_2 we have used directly [CR] for the character of ρ_E at unipotent elements.)

1.6. In this section we assume that G is simply connected. Let $\tilde{G} = G(\mathbf{k}((\epsilon)))$ where ϵ is an indeterminate. Let $\tilde{\mathcal{B}}$ be the set of Iwahori subgroups of $\tilde{G}$. Let $\tilde{\mathbf{W}}$ the affine Weyl group attached to $\tilde{G}$. Note that $\tilde{\mathbf{W}}$ naturally indexes ($w \mapsto \mathcal{O}_w$) the orbits of $\tilde{G}$ acting on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ by simultaneous conjugation on the two factors. For $g \in \tilde{G}$ and $w \in \tilde{\mathbf{W}}$ we set

$$\tilde{\mathcal{B}}^w_g = \{B \in \tilde{\mathcal{B}}; (B, gBg^{-1}) \in \mathcal{O}_w\}.$$

By analogy with [KL, §3] we expect that when g is regular semisimple, $\tilde{\mathcal{B}}_g^w$ has a natural structure of a locally finite union of algebraic varieties over $\mathbf{k}$ of bounded dimension and that, moreover, if g is also elliptic, then $\tilde{\mathcal{B}}_g^w$ has a natural structure of algebraic variety over $\mathbf{k}$. It would follow that for g elliptic and $w \in \tilde{\mathbf{W}}$,

$$\Xi_g^w = \sum_{i,j \in \mathbf{Z}} (-1)^i \dim H_c^i(\tilde{\mathcal{B}}_g^w, \bar{\mathbf{Q}}_l)_j v^j \in \mathbf{Z}[v]$$

is well defined; one can then show that the formal sum $\sum_{w \in \tilde{\mathbf{W}}} \Xi_g^w \hat{T}_w$ is central in the completion of the affine Hecke algebra consisting of all formal sums $\sum_{w \in \tilde{\mathbf{W}}} a_w \hat{T}_w$ $(a_w \in \mathbf{Q}(v))$ that is, it commutes with any $\hat{T}_w$. (Here $\hat{T}_w$ is defined as in 1.1 and the completion of the affine Hecke algebra is regarded as a bimodule over the actual affine Hecke algebra in the natural way.)

2. The sets $\mathbf{s}_g$

2.1. In this section we assume that G is adjoint and p is not a bad prime for G. For $g \in G, z \in \bar{Z}(g), w \in \mathbf{W}$ we set

$$\xi_{g,z}^w = \Xi_{g,z}^w|_{v=1} = \sum_{i \in \mathbf{Z}} (-1)^i \mathrm{tr}(z, H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)) \in \mathbf{Z}.$$

This integer is independent of l. For any $g \in G, z \in \bar{Z}(g)$ we set

$$\pi_{g,z} = \sum_{w \in \mathbf{W}} \xi_{g,z}^w w \in \mathbf{Z}[W].$$

This is the specialization of $\Pi_{g,z}$ for $v = 1$. Hence from 2(a) we see that $\pi_{g,z}$ is in the centre of the ring $\mathbf{Z}[\mathbf{W}]$. Thus for any $C \in \underline{\mathbf{W}}$ we can set $\xi_{g,z}^C = \xi_{g,z}^w$ where w is any element of C. For $g \in G$ let $\mathbf{s}_g$ be the set of all $C \in \underline{\mathbf{W}}$ such that $\xi_{g,z}^C \neq 0$ for some $z \in \bar{Z}(g)$ and $\xi_{g',z'}^C = 0$ for any $g' \in \overline{cl(g)} - cl(g)$ and any $z' \in \bar{Z}(g')$. We describe the set $\mathbf{s}_g$ and the values $\xi_{g,z}^C = 0$ for $C \in \mathbf{s}_g$, $z \in \bar{Z}(g)$, for various G of low rank and various unipotent elements g in G. We use the notation in 1.1. Moreover in the case where $\bar{Z}(g) \neq \{1\}$ we denote by z_n an element of order n in $\bar{Z}(g)$.

G of type A_1.

$$\mathbf{s}_{u_1} = (1), \mathbf{s}_{u_0} = (s_1); \xi_{u_1,1}^1 = 2, \xi_{u_0,1}^{s_1} = 1.$$

G of type A_2.

$$\mathbf{s}_{u_3} = (1), \mathbf{s}_{u_1} = (s_1), \mathbf{s}_{u_0} = (s_1 s_2).$$

$$\xi_{u_3,1}^1 = 6, \xi_{u_1,1}^{(s_1)} = 1, \xi_{u_0,1}^{(s_1 s_2)} = 1.$$

G of type B_2.

$$\mathbf{s}_{u_4}=(1),\mathbf{s}_{u_2}=(s_1),\mathbf{s}_{u_1}=\{(s_2),(s_1s_2s_1s_2)\},\mathbf{s}_{u_0}=(s_1s_2).$$

$$\xi^1_{u_4,1}=8,\xi^{(s_1)}_{u_2,1}=2,\xi^{(s_2)}_{u_1,1}=2,\xi^{(s_1s_2s_1s_2)}_{u_1,1}=0,$$
$$\xi^{(s_2)}_{u_1,z_2}=0,\xi^{(s_1s_2s_1s_2)}_{u_1,z_2}=2,\xi^{(s_1s_2)}_{u_0,1}=1.$$

G of type G_2.

$$\mathbf{s}_{u_6}=(1),\mathbf{s}_{u_3}=(s_2),\mathbf{s}_{u_2}=(s_1),\mathbf{s}_{u_1}=\{(s_1s_2s_1s_2s_1s_2),(s_1s_2s_1s_2)\},\mathbf{s}_{u_0}=(s_1s_2).$$

$$\xi^1_{u_6,1}=12,\xi^{(s_2)}_{u_3,1}=2,\xi^{(s_1)}_{u_2,1}=2,\xi^{(s_1s_2s_1s_2s_1s_2)}_{u_1,1}=-3,\xi^{(s_1s_2s_1s_2s_1s_2)}_{u_1,z_2}=3,$$
$$\xi^{(s_1s_2s_1s_2s_1s_2)}_{u_1,z_3}=0,\xi^{(s_1s_2s_1s_2)}_{u_1,1}=2,\xi^{(s_1s_2s_1s_2)}_{u_1,z_2}=0,\xi^{(s_1s_2s_1s_2)}_{u_1,z_3}=2,\xi^{(s_1s_2)}_{u_0,1}=1.$$

G of type B_3.

$$\mathbf{s}_{u_9}=(1),\mathbf{s}_{u_5}=(s_1),\mathbf{s}_{u_4}=\{(s_3),(s_2s_3s_2s_3)\},\mathbf{s}_{u_3}=(s_1s_3),$$
$$\mathbf{s}_{u_2}=\{(s_1s_2),(w_0)\},\mathbf{s}_{u_1}=\{(s_2s_3),(s_2s_3s_1s_2s_3)\},\mathbf{s}_{u_0}=(s_1s_2s_3).$$

$$\xi^1_{u_9,1}=48,\xi^{(s_1)}_{u_5,1}=4,\xi^{(s_2s_3s_2s_3)}_{u_4,1}=0,\xi^{(s_2s_3s_2s_3)}_{u_4,z_2}=4,\xi^{(s_3)}_{u_4,1}=8,$$
$$\xi^{(s_3)}_{u_4,1}=0,\xi^{(s_1s_3)}_{u_3,1}=2,\xi^{(w_0)}_{u_2,1}=0,\xi^{(w_0)}_{u_2,z_2}=6$$
$$\xi^{(s_1s_2)}_{u_2,1}=2,\xi^{(s_1s_2)}_{u_2,z_2}=0,\xi^{(s_2s_3)}_{u_1,1}=2,\xi^{(s_2s_3)}_{u_1,z_2}=0,$$
$$\xi^{(s_2s_3s_1s_2s_3)}_{u_1,1}=0,\xi^{(s_2s_3s_1s_2s_3)}_{u_1,z_2}=2,\xi^{(s_1s_2s_3)}_{u_0,1}=1.$$

G of type C_3.

$$\mathbf{s}_{u_9}=(1),\mathbf{s}_{u_6}=(s_3),\mathbf{s}_{u_4}=\{(s_1),(s_2s_3s_2s_3)\},\mathbf{s}_{u_3}=(s_1s_3),$$
$$\mathbf{s}_{u'_2}=(s_1s_2),\mathbf{s}_{u''_2}=(s_2s_3),\mathbf{s}_{u_1}=\{(s_2s_3s_1s_2s_3),w_0\}\mathbf{s}_{u_0})=(s_1s_2s_3).$$

$$\xi^1_{u_9,1}=48,\xi^{(s_3)}_{u_6,1}=8,\xi^{(s_2s_3s_2s_3)}_{u_4,1}=0,\xi^{(s_2s_3s_2s_3)}_{u_4,z_2}=4,$$
$$\xi^{(s_1)}_{u_4,1}=4,\xi^{(s_1)}_{u_4,1}=0,\xi^{(s_1s_3)}_{u_3,1}=2,\xi^{(s_1s_2)}_{u'_2,1}=2,\xi^{(s_2s_3)}_{u''_2,1}=2,$$
$$\xi^{(s_2s_3s_1s_2s_3)}_{u_1,1}=1,\xi^{(s_2s_3s_1s_2s_3)}_{u_1,z_2}=1,\xi^{(w_0)}_{u_1,1}=-3,\xi^{(w_0)}_{u_1,z_2}=3,\xi^{(s_1s_2s_3)}_{u_0,1}=1.$$

2.2. For any unipotent element $u \in G$ let n_u be the number of isomorphism classes of irreducible representations of $\bar{Z}(u)$ which appear in the Springer correspondence for G. Consider the following properties of G:

(a) $$\underline{\mathbf{W}} = \sqcup_u \mathbf{s}_u$$

(u runs over a set of representatives for the unipotent classes in G); for any unipotent element $u \in G$,

(b) $$|\mathbf{s}_u| = n_u.$$

The equality $\mathbf{W} = \cup_u \mathbf{s}_u$ is clear since for a regular unipotent u and any w we have $\xi_{u,1}^w = 1$. Note that (a),(b) hold in the examples in 2.1. We will show elsewhere that (a),(b) hold if G is of type A. We expect that (a),(b) hold in general.

Consider also the following property of G: for any $g \in G$, $w \in \mathbf{W}$,

(c) $\xi_{g,1}^w$ is equal to the trace of w on the Springer representation of $\mathbf{W}$ on $\oplus_i H^{2i}(\mathcal{B}_g, \bar{\mathbf{Q}}_l)$.

Again (c) holds if G is of type A and in the examples in 2.1; we expect that it holds in general. Note that in (c) one can ask whether for any z, $\xi_{g,z}^w$ is equal to the trace of wz on the Springer representation of $\mathbf{W} \times \bar{Z}(g)$ on $\oplus_i H^{2i}(\mathcal{B}_g, \bar{\mathbf{Q}}_l)$; but such an equality is not true in general for $z \neq 1$ (for example for G of type B_2).

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