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Knot homology groups from instantons

P. B. Kronheimer and T. S. Mrowka

Harvard University, Cambridge MA 02138
Massachusetts Institute of Technology, Cambridge MA 02139

1 Introduction

1.1 An observation of some coincidences

For a knot or link $K$ in $S^3$, the Khovanov homology $Kh(K)$ is a bigraded abelian group whose construction can be described in entirely combinatorial terms [16]. If we forget the bigrading, then as abelian groups we have, for example,

$$Kh(\text{unknot}) = \mathbb{Z}^2$$

and

$$Kh(\text{trefoil}) = \mathbb{Z}^4 \oplus \mathbb{Z}/2.$$ 

The second equality holds for both the right- and left-handed trefoils, though the bigrading would distinguish these two cases.

The present paper was motivated in large part by the observation that the group $\mathbb{Z}^4 \oplus \mathbb{Z}/2$ arises in a different context. Pick a basepoint $y_0$ in the complement of the knot or link, and consider the space of all homomorphisms $\rho : \pi_1(S^3 \setminus K, y_0) \to SU(2)$ satisfying the additional constraint that

$$\rho(m) \text{ is conjugate to } \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$ 

for every $m$ in the conjugacy class of a meridian of the link. (There is one such conjugacy class for each component of $K$, once the components are...
oriented. The orientation does not matter here, because the above element of $SU(2)$ is conjugate to its inverse.) Let us write

$$R(K) \subset \text{Hom} \left( \pi_1(S^3\setminus K, y_0), SU(2) \right)$$

for the set of these homomorphisms. Note that we are not defining $R(K)$ as a set of equivalence classes of such homomorphisms under the action of conjugation by $SU(2)$. The observation, then, is the following:

**Observation 1.1.** In the case that $K$ is either the unknot or the trefoil, the Khovanov homology of $K$ is isomorphic to the ordinary homology of $R(K)$, as an abelian group. That is,

$$Kh(K) \cong H_*(R(K)).$$

This observation extends to all the torus knots of type $(2, p)$.

To understand this observation, we can begin with the case of the unknot, where the fundamental group of the complement is $\mathbb{Z}$. After choosing a generator, we have a correspondence between $R(\text{unknot})$ and the conjugacy class of the distinguished element of $SU(2)$ in (1) above. This conjugacy class is a 2-sphere in $SU(2)$, so we can write

$$R(\text{unknot}) = S^2.$$ 

For a non-trivial knot $K$, we always have homomorphisms $\rho$ which factor through the abelianization $H_1(S^3\setminus K) = \mathbb{Z}$, and these are again parametrized by $S^2$. Every other homomorphism has stabilizer $\{\pm 1\} \subset SU(2)$ under the action by conjugation, so its equivalence class contributes a copy of $SU(2)/\{\pm 1\} = RP^3$ to $R(K)$. In the case of the trefoil, for example, there is exactly one such conjugacy class, and so

$$R(\text{trefoil}) = S^2 \amalg RP^3;$$

and the homology of this space is indeed $\mathbb{Z}^4 \oplus \mathbb{Z}/2$, just like the Khovanov homology. This explains why the observation holds for the trefoil, and the case of the $(2, p)$ torus knots is much the same: for larger odd $p$, there are $(p - 1)/2$ copies of $RP^3$ in the $R(K)$. In unpublished work, the above observation has been shown to extend to all 2-bridge knots by Sam Lewallen [26].

The homology of the space $R(K)$, while it is certainly an invariant of the knot or link, should not be expected to behave well or share any of the
more interesting properties of Khovanov homology; no should the coincidence noted above be expected to hold. A better way to proceed is instead to imitate the construction of Floer’s instanton homology for 3-manifolds, by constructing a framework in which \( R(K) \) appears as the set of critical points of a Chern-Simons functional on a space of \( SU(2) \) connections on the complement of the link. One should then construct the Morse homology of this Chern-Simons invariant. In this way, one should associate a finitely-generated abelian group to \( K \) that would coincide with the ordinary homology of \( R(K) \) in the very simplest cases. The main purpose of the present paper is to carry through this construction. The invariant that comes out of this construction is certainly not isomorphic to Khovanov homology for all knots; but it does share some of its formal properties. The definition that we propose is a variant of the orbifold Floer homology considered by Collin and Steer in [5].

In some generality, given a knot or link \( K \) in a 3-manifold \( Y \), we will produce an “instanton Floer homology group” that is an invariant of \( (Y, K) \). These groups will be functorial for oriented cobordisms of pairs. Rather than work only with \( SU(2) \), we will work of much of this paper with a more general compact Lie group \( G \), though in the end it is only for the case of \( SU(N) \) that we are able to construct these invariants.

1.2 Summary of results

The basic construction. Let \( Y \) be a closed oriented 3-manifold, let \( K \subset Y \) be an oriented link, and let \( P \to Y \) be a principal \( U(2) \)-bundle. Let \( K_1, \ldots, K_r \) be the components of \( K \). We will say that \( (Y, K) \) and \( P \) satisfies the non-integrality condition if none of the \( 2^r \) rational cohomology classes

\[
\frac{1}{2}c_1(P) \pm \frac{1}{2}P.D.[K_1] \pm \cdots \pm \frac{1}{2}P.D.[K_r]
\]

is an integer class. When the non-integrality condition holds, we will define a finitely-generated abelian group \( I_*(Y, K, P) \). This group has a canonical \( \mathbb{Z}/2 \) grading, and a relative grading by \( \mathbb{Z}/4 \).

In the case that \( K \) is empty, the group \( I_*(Y, P) \) coincides with the familiar variant of Floer’s instanton homology arising from a \( U(2) \) bundle \( P \to Y \) with odd first Chern class [8]. We recall, in outline, how this group is constructed. One considers the space \( \mathcal{C}(Y, P) \) of all connections in the \( SO(3) \) bundle \( \text{ad}(P) \). This affine space is acted on by the “determinant-1 gauge group”: the group \( \mathcal{G}(Y, P) \) of automorphisms of \( P \) that have determinant 1 everywhere. Inside \( \mathcal{C}(Y, P) \) one has the flat connections: these can be
characterized as critical points of the Chern-Simons functional,

$$\text{CS} : \mathcal{C}(Y, P) \rightarrow \mathbb{R}.$$ 

The Chern-Simons functional descends to a circle-valued function on the quotient space

$$\mathcal{B}(Y, P) = \mathcal{C}(Y, P)/\mathcal{G}(Y, P).$$

The image of the set of critical points in $\mathcal{B}(Y, P)$ is compact, and after perturbing $\text{CS}$ carefully by a term that is invariant under $\mathcal{G}(Y, P)$, one obtains a function whose set of critical points has finite image in this quotient. If $(1/2)c_1(P)$ is not an integral class and the perturbation is small, then the critical points in $\mathcal{C}(Y, P)$ are all irreducible connections. One can arrange also a Morse-type non-degeneracy condition: the Hessian if $\text{CS}$ can be assumed to be non-degenerate in the directions normal to the gauge orbits. The group $\mathbb{I}_*(Y, P)$ is then constructed as the Morse homology of the circle-valued Morse function on $\mathcal{B}(Y, P)$.

In the case that $K$ is non-empty, the construction of $\mathbb{I}_*(Y, K, P)$ mimics the standard construction very closely. The difference is that we start not with the space $\mathcal{C}$ of all smooth connections in $\text{ad}(P)$, but with a space $\mathcal{C}(Y, K, P)$ of connections in the restriction of $\text{ad}(P)$ to $Y \setminus K$ which have a singularity along $K$. This space is acted on by a group $\mathcal{G}(Y, K, P)$ of determinant-one gauge transformations, and we have a quotient space

$$\mathcal{B}(Y, K, P) = \mathcal{C}(Y, K, P)/\mathcal{G}(Y, K, P).$$

In the case that $c_1(P) = 0$, the singularity is such that the flat connections in the quotient space $\mathcal{B}(Y, K, P)$ correspond to conjugacy classes of homomorphisms from the fundamental group of $Y \setminus K$ to $SU(2)$ which have the behavior (1) for meridians of the link. Thus, if we write $\mathcal{C}(Y, K, P) \subset \mathcal{B}(Y, K, P)$ for this set of critical points of the Chern-Simons functional, then we have

$$\mathcal{C}(Y, K, P) = R(Y, K)/SU(2)$$

where $R(Y, K)$ is the set of homomorphisms $\rho : \pi_1(Y \setminus K) \rightarrow SU(2)$ satisfying (1) and $SU(2)$ is acting by conjugation. The non-integrality of the classes (2) is required in order to ensure that there will be no reducible flat connections.

**Application to classical knots.** Because of the non-integrality requirement, the construction of $\mathbb{I}_*$ cannot be applied directly when the 3-manifold
Y has first Betti number zero. In particular, we cannot apply this construction to “classical knots” (knots in $S^3$). However, there is a simple device we can apply. Pick a point $y_0$ in $Y \setminus K$, and form the connected sum at $y_0$ of $Y$ and $T^3$, to obtain a new pair $(Y \# T^3, K)$. Let $P_0$ be the trivial $U(2)$ bundle on $Y$, and let $Q$ be the $U(2)$ bundle on $T^3 = S^1 \times T^2$ whose first Chern class is Poincaré dual to $S^1 \times \{\text{point}\}$. We can form a bundle $P_0 \# Q$ over $Y \# T^3$. This bundle satisfies the non-integral condition, so we define

$$FI_*(Y, K) = \mathbb{I}_s(Y \# T^3, K, P_0 \# Q).$$

and call this the \textit{framed instanton homology} of the pair $(Y, K)$. In the special case that $Y = S^3$, we write

$$FI_*(K) = FI_*(S^3, K).$$

To get a feel for $FI_*(Y, K)$ for knots in $S^3$, and to understand the reason for the word “framed” here, it is first necessary to understand that the adjoint bundle $\text{ad}(Q) \to T^3$ admits only irreducible flat connections, and that these form two orbits under the determinant-one gauge group. (See section 3.1. Under the full gauge group of all automorphisms of $\text{ad}(Q)$, they form a single orbit.) When we form a connected sum, the fundamental group becomes a free product, and we have a general relationship of the form

$$R(Y_0 \# Y_1)/SU(2) = R(Y_0) \times_{SU(2)} R(Y_1). \quad \text{(4)}$$

Applying this to the connected sum $(Y \# T^3, K)$ and recalling (3), we find that the flat connections in the quotient space $\mathcal{B}(Y \# T^3, K, P_0 \# Q)$ form two disjoint copies of the space we called $R(Y, K)$ above: that is,

$$\mathcal{C}(Y \# T^3, K, P_0 \# Q) = R(Y, K) \amalg R(Y, K).$$

Note that, on the right-hand side, we no longer have the quotient of $R(Y, K)$ by $SU(2)$ as we did before at (3). The space $R(Y, K)$ can be thought of as parametrizing isomorphism classes of flat connections on $Y \setminus K$ with a \textit{framing} at the basepoint $y_0$: that is, an isomorphism of the fiber at $y_0$ with $U(2)$.

For a general knot, as long as the set of critical points is non-degenerate in the Morse-Bott sense, there will be a spectral sequence starting at the homology of the critical set, $\mathcal{C}$, and abutting to the framed instanton homology. In the case of the unknot in $S^3$, the spectral sequence is trivial and the
group $FI_*(K)$ is the homology of the two copies of $R(K)$ which comprise $C$. Thus,

$$FI_*(\text{unknot}) = H_*(C) = H_*(R(\text{unknot}) \amalg R(\text{unknot})) = H_*(S^2 \amalg S^2) = \mathbb{Z}^2 \oplus \mathbb{Z}^2.$$  

Noting again Observation 1.1, we can say that $FI_*(\text{unknot})$ is isomorphic to two copies of the Khovanov homology of the unknot. It is natural to ask whether this isomorphism holds for a larger class of knots:

**Question 1.2.** Is there an isomorphism of abelian groups

$$FI_*(K) \cong \text{Kh}(K) \oplus \text{Kh}(K)$$

for all alternating knots?

There is evidence for an affirmative answer to this question for the torus knots of type $(2, p)$, as well as for the (non-alternating) torus knots of type $(3, 4)$ and $(3, 5)$. It also seems likely that the answer is in the affirmative for all alternating knots if we use $\mathbb{Z}/2$ coefficients instead of integer coefficients. Already for the $(4, 5)$ torus knot, however, it is clear from an examination of $R(K)$ that the framed instanton homology $FI_*(K)$ has smaller rank than two copies of the Khovanov homology, so the isomorphism does not extend to all knots.

For a general knot, we expect that $\text{Kh}(K) \oplus \text{Kh}(K)$ is related to $FI_*(K)$ through a spectral sequence. There is a similar spectral sequence (though only with $\mathbb{Z}/2$ coefficients) relating (reduced) Khovanov homology to the Heegaard Floer homology of the branched double cover [34]. The argument of [34] provides a potential model for a similar argument in the case of our framed instanton homology. An important ingredient is to show that there is a long exact sequence relating the framed instanton homologies for $K$, $K_0$ and $K_1$, when $K_0$ and $K_1$ are obtained from a knot or link $K$ by making the two different smoothings of a single crossing. It seems that a proof of such a long exact sequence can be given using the same ideas that were used in [18] to prove a surgery exact sequence for Seiberg-Witten Floer homology, and the authors hope to return to this and other related issues in a future paper.

**Other variations.** Forming a connected sum with $T^3$, as we just did in the definition of $FI_*(Y, K)$, is one way to take an arbitrary pair $(Y, K)$ and
modify it so as to satisfy the non-integrality condition; but of course there are many other ways. Rather than using $T^3$, one can use any pair satisfying the non-integrality condition; and an attractive example that – like the 3-torus – carries isolated flat connections, is the pair $(S^1 \times S^2, L)$, where $L$ is the 3-component link formed from three copies of the $S^1$ factor.

We shall examine this and some other variations of the basic construction in section 4. Amongst these is a “reduced” version of $FI_*(K)$ that appears to bear the same relation to reduced Khovanov homology as $FI_*(K)$ does to the Khovanov homology of $K$. (For this reduced variant, the homology of the unknot is $\mathbb{Z}$.) Another variant arises if, instead of forming a connected sum, we perform 0-surgery on a knot (in $S^3$, for example) and apply the basic construction to the core of the surgery torus in the resulting 3-manifold. The group obtained this way is reminiscent of the “longitude Floer homology” of Eftekhary [12]. It is trivial for the “unknot” in any $Y$ (i.e. a knot that bounds a disk), and is $\mathbb{Z}^4$ for the trefoil in $S^3$.

In another direction, we can alter the construction of $FI_*(K)$ by dividing the relevant configuration space by a slightly larger gauge group, and in this way we obtain a variant of $FI_*(K)$ which we refer to as $\overline{FI}_*(K)$ and which is (roughly speaking) half the size of $FI_*(K)$. For this variant, the appropriate modification of Question 1.2 has only one copy of $Kh(K)$ on the right-hand side. (For example, if $K$ is the unknot, then $\overline{FI}_*(K)$ is the ordinary homology of $S^2$, which coincides with the Khovanov homology.)

**Slice-genus bounds.** A very interesting aspect of Khovanov homology was discovered by Rasmussen [36], who showed how the Khovanov homology of a knot can be used to provide a lower bound for the knot’s slice-genus. An argument with a very similar structure can be constructed using the framed instanton homology $FI_*(K)$. The construction begins by replacing $\mathbb{Z}$ as the coefficient group with a certain system of local coefficients $\Gamma$ on $B(S^3, K)$. In this way we obtain a new group $FI_*(K; \Gamma)$ that is finitely-presented module over the ring $\mathbb{Z}[t^{-1}, t]$ of finite Laurent series in a variable $t$. We shall show that $FI_*(K; \Gamma)$ modulo torsion is essentially independent of the knot $K$: it is always a free module of rank 2. On the other hand, $FI_*(K; \Gamma)$ comes with a descending filtration, and we can define a knot invariant by considering the level in this filtration at which the two generators lie. From this knot invariant, we obtain a lower bound for the slice genus.

Although the formal aspects of this argument are modeled on [36], the actual mechanisms behind the proof are the same ones that were first used in [22] and [19].
Monotonicity and other structure groups. The particular conjugacy class chosen in (1) is a distinguished one. It might seem, at first, that the definition of $\mathbb{I}_s(Y,K,P)$ could be carried out without much change if instead we used any of the non-central conjugacy classes in $SU(2)$ represented by the elements

$$\exp 2\pi i \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda$ in $(0,1/2)$. This is not the case, however, because unless $\lambda = 1/4$ we cannot establish the necessary finiteness results for the spaces of trajectories in the Morse theory that defines $\mathbb{I}_s(Y,K,P)$. The issue is what is usually called “monotonicity” in a similar context in symplectic topology. In general, a Morse theory of Floer type involves a circle-valued Morse function $f$ on an infinite-dimensional space $\mathcal{B}$ whose periods define a map

$$\Delta_f : \pi_1(\mathcal{B}) \to \mathbb{R}.$$ 

The Hessian of $f$ may also have spectral flow, defining another map,

$$\text{sf} : \pi_1(\mathcal{B}) \to \mathbb{Z}.$$ 

The theory is called monotone if these two are proportional. Varying the eigenvalue $\lambda$ varies the periods of the Chern-Simons functional in our theory, and it is only for $\lambda = 1/4$ that we have monotonicity. In the non-monotone case, one can still define a Morse homology group, but it is necessary to use a local system that has a suitable completeness [31].

A related issue is the question of replacing $SU(2)$ by a general compact Lie group, say a simple, simply-connected Lie group $G$. The choice of $\lambda$ above now becomes the choice of an element $\Phi$ of the Lie algebra of $G$, which will determine the leading term in the singularity of the connections that we use. If we wish to construct a Floer homology theory, then the choice of $\Phi$ is constrained again by the monotonicity requirement. It turns out that the monotonicity condition is equivalent to requiring that the adjoint orbit of $\Phi$ is Kähler-Einstein manifold with Einstein constant 1, when equipped with the Kähler metric corresponding to the Kirillov-Kostant-Souriau 2-form. We shall develop quite a lot of the machinery in the context of a general $G$, but in the end we find that it is only for $SU(N)$ that we can satisfy two competing requirements: the first requirement is monotonicity; the second requirement is that we avoid connections with non-trivial stabilizers among the critical points of the perturbed Chern-Simons functional. Using $SU(N)$, we shall construct a variant of $FI_*(K)$ that seems to bear the same relation to the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology [17] as $FI_*(K)$ does to the original Khovanov homology.
1.3 Discussion

Unlike $FI_\ast(K)$, the Khovanov homology of a knot is bigraded. If we write $P_K(q,t)$ for the 2-variable Poincaré polynomial of $Kh(K)$, then $P_K(q,-1)$ is (to within a standard factor) the Jones polynomial of $K$ [16]. Relationships between the Jones polynomial and gauge theory can be traced back to Witten’s reinterpretation of the Jones polynomial as arising from a $(2+1)$-dimensional topological quantum field theory [45]. What we are exploring here is in one higher dimension: a relationship between the Khovanov homology and gauge theory in $3 + 1$ dimensions.

Our definition of $FI_\ast(Y,K)$ seems somewhat unsatisfactory, in that it involves a rather unnatural-looking connected sum with $T^3$. As pointed out above, one can achieve apparently the same effect by replacing $T^3$ here with the pair $(S^1 \times S^2, L)$ where $L$ is a standard 3-stranded link. The unsatisfactory state of affairs is reflected in the fact that we are unable to prove that these two choices would lead to isomorphic homology groups. The authors feel that there should be a more natural construction, involving the Morse theory of the Chern-Simons functional on the “framed” configuration space $\tilde{B}(Y,K) = C(Y,K) / G^0(Y,K)$, where $G^0 \subset G$ is the subgroup consisting of gauge transformations that are 1 at a basepoint $y_0 \in (Y \setminus K)$. The reduced homology theory would be constructed in a similar manner, but using a basepoint $k_0$ on $K$. A related construction, for homology 3-spheres, appears in an algebraic guise in [8, section 7.3.3].

This idea of using the framed configuration space $\tilde{B}(Y,K)$ and dispensing with the connect-sum with $T^3$ is attractive: it would enable one to work with a general $G$ without concern for avoiding reducible solutions. However, it cannot be carried through without overcoming obstacles involving bubbles in the instanton theory: the particular issue is bubbling at the chosen basepoint.

Acknowledgments. The development of these ideas was strongly influenced by the paper of Seidel and Smith [40]. Although gauge theory as such does not appear there, it does not seem to be far below the surface. The first author presented an early version of some of the ideas of the present paper at the Institute for Advanced Study in June 2005, and learned there from Katrin Wehrheim and Chris Woodward that they were pursuing a very similar program (developing from [43] in the context of Lagrangian intersection Floer homology). Ciprian Manolescu and Chris Woodward have described a similar program, also involving Lagrangian intersections, motivated by the idea of using the framed configuration space. The idea of using a 3-stranded...
link in $S^1 \times S^2$ as an alternative to $T^3$ was suggested to us by Paul Seidel and also appears in the work of Wehrheim and Woodward. The authors have been informed that Magnus Jacobsson and Ryszard Rubinsztein independently noticed the coincidence described in Observation 1.1, for various knots [15].

2 Instantons with singularities

For the case that the structure group is $SU(2)$ or $PSU(2)$, instantons with codimension-2 singularities were studied in [21, 22] and related papers. Our purpose here is to review that material and at the same time to generalize some of the constructions to the case of more general compact groups $G$. In the next section, we will be considering cylindrical 4-manifolds and Floer homology; but in this section we begin with the closed case. We find it convenient to work first with the case that $G$ is simple and simply-connected. Thus our discussion here applies directly to $SU(N)$ but not to $U(N)$. Later in this section we will indicate the adjustments necessary to work with other Lie groups, including $U(N)$.

For instantons with codimension-2 singularities and arbitrary structure group, the formulae for the energy of solutions and the dimension of moduli spaces which we examine here are closely related to similar formulae for non-abelian Bogomoln’yi monopoles. See [29, 30, 44], for example.

2.1 Notation and root systems

For use in the rest of the paper, we set down some of the notation we shall use for root systems and related matters. Fix $G$, a compact connected Lie group that is both simple and simply connected. We will fix a maximal torus $T \subset G$ and denote by $t \subset \mathfrak{g}$ its Lie algebra. Inside $t$ is the integer lattice consisting of points $x$ such that $\exp(2\pi x)$ is the identity. The dual lattice is the lattice of weights: the elements in $t^*$ taking integer values on the integer lattice. We denote by $R \subset t^*$ the set of roots. We choose a set of positive roots $R^+ \subset R$, so that $R = R^+ \cup R^-$, with $R^- = -R^+$. The set of simple roots corresponding to this choice of positive roots will be denoted by $\Delta^+ \subset R^+$. We denote by $\rho$ half the sum of the positive roots, sometimes called the Weyl vector,

$$\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta,$$

and we write $\theta$ for the highest root.
We define the Killing form on $\mathfrak{g}$ with a minus sign, as
\[
\langle a, b \rangle = -\text{tr}(\text{ad}(a) \text{ ad}(b)),
\]
so that it is positive definite. The corresponding map $\mathfrak{g}^* \rightarrow \mathfrak{g}$ will be denoted $\alpha \mapsto \alpha^\dagger$, and the inverse map is $\psi \mapsto \psi^\dagger$. If $\alpha$ is a root, we denote by $\alpha^\vee$ the coroot
\[
\alpha^\vee = \frac{2\alpha^\dagger}{\langle \alpha, \alpha \rangle}.
\]
The simple coroots, $\alpha^\vee$ for $\alpha \in \Delta^+$, form an integral basis for the integer lattice in $t$. The fundamental weights are the elements of the dual basis $w_\alpha$, $\alpha \in \Delta^+$ for the lattice of weights. The fundamental Weyl chamber is the cone in $t$ on which all the simple roots are non-negative.\(^2\) This is the cone spanned by the duals of the fundamental weights, $w_\alpha^\dagger \in t$.

The highest root $\theta$ and the corresponding coroot $\theta^\vee$ can be written as positive integer combinations of the simple roots and coroots respectively: that is,
\[
\theta = \sum_{\alpha \in \Delta^+} n_\alpha \alpha \\
\theta^\vee = \sum_{\alpha \in \Delta^+} n^\vee_\alpha \alpha^\vee
\]
for non-negative integers $n_\alpha$ and $n^\vee_\alpha$. The numbers
\[
h = 1 + \sum_{\alpha \in \Delta^+} n_\alpha \\
h^\vee = 1 + \sum_{\alpha \in \Delta^+} n^\vee_\alpha
\]
are the Coxeter number and dual Coxeter number respectively. The squared length of the highest root is equal to $1/h^\vee$:
\[
\langle \theta, \theta \rangle = 1/h^\vee.
\]
(The inner product on $\mathfrak{g}^*$ here is understood to be the dual inner product to the Killing form.) We also record here the relation
\[
\rho = \sum_{\alpha \in \Delta^+} w_\alpha;
\]
\(^2\)Note that our convention is that the fundamental Weyl chamber is closed: it is not the locus where the simple roots are strictly positive.
this and the previous relation have the corollary
\[ 2\langle \rho, \theta \rangle = 1 - \frac{1}{h^\vee}. \tag{8} \]

For each root \( \alpha \), there is a preferred homomorphism
\[ j_\alpha : SU(2) \to G \]
whose derivative maps
\[ dj_\alpha : \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto \alpha^\vee. \]

In the case that \( G \) is simply laced, or if \( \alpha \) is a long root in the non-simply laced case, then the map \( j_\alpha \) is injective and represents a generator of \( \pi_3(G) \).
In particular, this applies when \( \alpha \) is the highest root \( \theta \). Under the adjoint action of \( j_\theta(SU(2)) \), the Lie algebra \( \mathfrak{g} \) decomposes as one copy of the adjoint representation of \( SU(2) \), a number of copies of the defining 2-dimensional representation of \( SU(2) \), and a number of copies of the trivial representation. The pair \((G, j_\theta(SU(2)))\) is 4-connected.

### 2.2 Connections and moduli spaces

Let \( X \) be a closed, connected, oriented, Riemannian 4-manifold, and let \( \Sigma \subset X \) be a smoothly embedded, compact oriented 2-manifold. We do not assume that \( \Sigma \) is connected. We take a principal \( G \)-bundle \( P \to X \), where \( G \) is as above. Such a \( P \) is classified by an element of \( \pi_3(G) \), and hence by a single characteristic number: following [2], we choose to normalize this characteristic number by defining
\[ k = -\frac{1}{(4h^\vee)p_1(\mathfrak{g})[X]}, \]
where \( h^\vee \) is the dual Coxeter number of \( G \). This normalization is chosen as in [2] so that \( k \) takes all values in \( \mathbb{Z} \) as \( P \) ranges through all bundles on \( X \). If the structure group of \( P \) is reduced to the subgroup \( j_\theta(SU(2)) \), the \( k \) coincides with the second Chern number of the corresponding \( SU(2) \) bundle.

Fix \( \Phi \in \mathfrak{t} \) belonging to the fundamental Weyl chamber, and suppose that
\[ \theta(\Phi) < 1 \tag{9} \]
where \( \theta \) is the highest root. This condition is equivalent to saying that
\[ -1 < \alpha(\Phi) < 1 \]
for all roots $\alpha$. This in turn means that an element $U \in \mathfrak{g}$ is fixed by the adjoint action of $\exp(2\pi \Phi)$ if and only if $[U, \Phi] = 0$. In a simply-connected group, the commutant of any element is always connected, and it therefore follows that the subgroup of $G$ which commutes with $\exp(2\pi \Phi)$ coincides with the stabilizer of $\Phi$ under the adjoint action. We call this group $G_\Phi$. We write $\mathfrak{g}_\Phi \subset \mathfrak{g}$ for its Lie algebra, and we let $\mathfrak{o}$ stand for the unique $G$-invariant complement, so that

$$\mathfrak{g} = \mathfrak{g}_\Phi \oplus \mathfrak{o}. \quad (10)$$

The set of roots can be decomposed according to the sign of $\alpha(\Phi)$, as

$$R = R^+(\Phi) \cup R^0(\Phi) \cup R^-(\Phi).$$

Similarly, the set of simple roots decomposes as

$$\Delta^+ = \Delta^+(\Phi) \cup \Delta^0(\Phi),$$

where $\Delta^0(\Phi)$ are the simple roots that vanish on $\Phi$. Knowing $\Delta^0(\Phi)$, we can recover $R^0(\Phi)$ as the set of those positive roots lying in the span of $\Delta^0(\Phi)$. We can decompose the complexification $\mathfrak{o} \otimes \mathbb{C}$ as

$$\mathfrak{o} \otimes \mathbb{C} = \mathfrak{o}^+ \oplus \mathfrak{o}^- \quad (11)$$

where $\mathfrak{o}^+ \subset \mathfrak{g} \otimes \mathbb{C}$ is the sum of the weight spaces for all roots $\alpha$ in $R^+(\Phi)$, and $\mathfrak{o}^-$ is the sum of the weight spaces for roots in $R^-(\Phi)$.

Choose a reduction of the structure group of $P|_\Sigma$ to the subgroup $G_\Phi \subset G$. Extend this arbitrarily to a tubular neighborhood $\nu \supset \Sigma$, so that we have a reduction of $P|_\nu$. If $O \cong G/G_\Phi$ is the adjoint orbit of $\Phi$ in the Lie algebra $\mathfrak{g}$ and if $O_P \subset \mathfrak{g}_P$ is the corresponding subbundle of the adjoint bundle, then such a reduction of structure group to $G_\Phi \subset G$ is determined by giving a section $\varphi$ of $O_P$ over the neighborhood $\nu$. We denote the principal $G_\Phi$-bundle resulting from this reduction by $P_\varphi \subset P|_\nu$. There are corresponding reductions of the associated bundle $\mathfrak{g}_P$ with fiber $\mathfrak{g}$ and its complexification, which we write as

$$\mathfrak{g}_P|_\nu = \mathfrak{g}_\varphi \oplus \mathfrak{o}_\varphi \quad (12)$$

and

$$(\mathfrak{g}_P \otimes \mathbb{C})|_\nu = (\mathfrak{g}_\varphi \otimes \mathbb{C}) \oplus (\mathfrak{o}_\varphi \otimes \mathbb{C})$$

$$= (\mathfrak{g}_\varphi \otimes \mathbb{C}) \oplus \mathfrak{o}^+_\varphi \oplus \mathfrak{o}^-_{\varphi}.$$

We identify $\nu$ diffeomorphically with the disk bundle of the oriented normal bundle to $\Sigma$, we let $i\eta$ be a connection 1-form on the circle bundle.
∂ν, and we extend η by pull-back to the deleted tubular neighborhood ν\Σ. Thus η is a 1-form that coincides with dθ in polar coordinates (r, θ) on each normal disk under the chosen diffeomorphism.

The data ϕ and η together allow us to define the model for our singular connections. We choose a smooth G-connection A₀ on P. We take β(r) to be a cutoff-function equal to 1 in a neighborhood of Σ and supported in the neighborhood ν. Then we define

$$A^φ = A^0 + \beta(r)φ \otimes η$$

(13)
as a connection in P over X\Σ. (The form β(r)φη has been extended by zero to all of X\Σ.) The holonomy of A^φ around a loop r = ε in a normal disk to Σ (oriented with the standard θ coordinate increasing) defines an automorphism of P over ν\Σ which is asymptotically equal to

$$\exp(-2πφ)$$

(14)
when ε is small.

Following [21], we fix a p > 2 and consider a space of connections on P_{X\Σ} defined as

$$C^p(X, Σ, P; φ) = \{ A^φ + a \mid a, \nabla A^φ a ∈ L^p(X\Σ) \}.$$ 

(15)
As in [21, Section 3], the definition of this space of connections can be reformulated to make clear that it depends only on the reduction of structure group defined by φ, and does not otherwise depend on Φ. To do this, extend the radial distance function r as a positive function on X\Σ and define a Banach space W^k_p(X) for k ≥ 1 by taking the completion of the compactly supported smooth functions on X\Σ with respect to norm

$$\|f\|_{W^k_p} = \left\| \frac{1}{r^k} f \right\|_p + \left\| \frac{1}{r^{k-1}} \nabla f \right\|_p + \cdots + \left\| \nabla^k f \right\|_p.$$ 

(For k = 0, we just define W^0_p to be L^p.) The essential point then is that the condition on a|ν that arises from the definition (15) can be equivalently written (using the decomposition (12)) as

$$a|ν ∈ L^p_t(ν; g_φ) ⊕ W^p_1(ν; o_φ).$$

This shows that the space C^p(X, Σ, P, φ) depends only on the decomposition of g_P as g_φ ⊕ o_φ. It is important here that the condition (9) is satisfied: this condition ensures that the eigenvalues of the bundle automorphism (14)
acting on $\phi$ are all different from 1, for these eigenvalues are $\exp(\pm 2\pi i \alpha(\Phi))$ as $\alpha$ runs through $R^+(\Phi)$. This space of connections is acted on by the gauge group

$$\mathcal{G}^p(X, \Sigma, P, \phi) = \{ g \in \text{Aut}(P|_{X\setminus \Sigma}) \mid \nabla_{A^\phi} g, \nabla^2_{A^\phi} g \in L^p(X\setminus \Sigma) \}.$$

The fact that this is a Banach Lie group acting smoothly on $\mathcal{C}^p(X, \Sigma, P, \phi)$ is a consequence of multiplication theorems, such as the continuity of the multiplications $W^p_2 \times W^p_1 \to W^p_1$, just as in [21].

The center of the gauge group $\mathcal{G}^p = \mathcal{G}^p(X, \Sigma, P, \phi)$ is canonically isomorphic to the center $Z(G)$ of $G$, a finite group. This subgroup $Z(\mathcal{G}^p)$ acts trivially on the $\mathcal{C}^p(X, \Sigma, P, \phi)$, so the group that acts effectively is the quotient $\mathcal{G}^p/Z(\mathcal{G}^p)$. Some connections $A$ will have larger stabilizer; but there is an important distinction here that is not present in the most familiar case, when $G = SU(N)$. In the case of $SU(N)$ if the stabilizer of $A$ is larger than $Z(\mathcal{G}^p)$, then the stabilizer has positive dimension, but for other $G$ the stabilizer may be a finite group strictly larger than $G$. To understand this point, recall that the stabilizer of a connection $A$ in the gauge group is isomorphic to the centralizer $C_G(S)$ where $S$ is the set of holonomies around all loops based at some chosen basepoint. So the question of which stabilizers occur is equivalent to the question of which subgroups of $G$ arise as $C_G(H)$ for some $H \subset G$, which we may as well take to be a closed subgroup. In the case of $SU(N)$, the only finite group that arises this way is the center. But for other simple Lie groups $G$, there may be a semi-simple subgroup $H \subset G$ of the same rank as $G$; and in this case the centralizer $C_G(H)$ is isomorphic to the center of $H$, which may be strictly larger than the center of $G$. Examples of this phenomenon include the case where $G = \text{Spin}(2n+1)$ and $H$ is the subgroup $\text{Spin}(2n)$. In this case $C_G(H)$ contains $Z(G)$ as a subgroup of index 2. Another case is $G = G_2$ and $H = SU(3)$: in this case $C_G(H)$ has order 3 while the center of $G$ is trivial.

We reserve the word reducible for connections $A$ whose stabilizer has positive dimension:

**Definition 2.1.** We will say that a connection $A$ is irreducible if its stabilizer in the gauge group is finite.

The homotopy type of $\mathcal{G}^p(X, \Sigma, P, \phi)$ coincides with that of the group of all smooth automorphisms of $P$ which respect the reduction of structure group along $\Sigma$. The bundle $P$ is classified its characteristic number $k$, and the section $\phi$ is determined up to homotopy by the induced map on cohomology,

$$\phi^* : H^2(O_P) \to H^2(\Sigma).$$  \hspace{1cm} (16)
Because the restriction of $P$ to $\Sigma$ is trivial, and because the choice of trivialization is unique up to homotopy, we can also think of the reduction of structure group along $\Sigma$ as being determined simply by specifying the isomorphism class of the principal $G_\Phi$-bundle $P_\varphi \to \Sigma$. In this way, when $\Sigma$ is connected, the classification is by $\pi_1(G_\Phi)$. The inclusion $T \to G_\Phi$ induces a surjection on $\pi_1$, so we can lift to an element of $\pi_1(T)$, which we can reinterpret as an integer lattice point $\xi$ in $t$. Let $Z(G_\Phi) \subset T$ be the center of $G_\Phi$, let $\mathfrak{z}(G_\Phi)$ be its Lie algebra, and let $\Pi$ be the orthogonal projection \[ \Pi : t \to \mathfrak{z}(G_\Phi). \]

We can describe $\mathfrak{z}(G_\Phi)$ as
\[
\mathfrak{z}(G_\Phi) = \bigcap_{\alpha \in S^0(\Phi)} \ker \alpha = \text{span}\{ w_\beta^\dagger \mid \beta \in S^+(\Phi) \}.
\]

The projection $\Pi(\xi)$ may not be an integer lattice point, but the image under $\Pi$ of the integer lattice in $t$ is isomorphic to $\pi_1(G_\Phi)$, and the reduction of structure group is determined up to homotopy by the element
\[ \Pi(\xi) \in \mathfrak{z}(G_\Phi). \]

We give the image of the integer lattice under $\Pi$ a name:

**Definition 2.2.** We write $L(G_\Phi) \subset \mathfrak{z}(G_\Phi)$ for the image under $\Pi$ of the integer lattice in $t$. Thus $L(G_\Phi)x$ is isomorphic both to $H_2(O;\mathbb{Z})$ and to $\pi_1(G_\Phi)$, and classifies the possible reductions of structure group of $P \to \Sigma$ to the subgroup $G_\Phi$, in the case that $\Sigma$ is connected. If the reduction of structure group determined by $\varphi$ is classified by the lattice element $l \in L(G_\Phi)$, we refer to $l$ as the **monopole charge**. If $\Sigma$ has more than one component, we define the monopole charge by summing over the components of $\Sigma$.

We now wish to define a moduli space of anti-self-dual connections as
\[ M(X, \Sigma, P, \varphi) = \{ A \in \mathcal{C}^p \mid F_A^+ = 0 \} / \mathcal{G}^p. \]

As shown in [21], there is a Kuranishi model for the neighborhood of a connection $[A]$ in $M(X, \Sigma, P, \varphi)$ described by a Fredholm complex, as long as $p$ is chosen sufficiently close to 2. Specifically, if $x$ denotes the smallest
of all the real numbers $\alpha(\Phi)$ and $1 - \alpha(\Phi)$ as $\alpha$ runs through $\mathbb{R}^+\Phi$, then $p$ needs to be in the range

$$2 < p < 2 + \epsilon(x)$$

(17)

where $\epsilon$ is a continuous function which is positive for $x > 0$ but has $\epsilon(0) = 0$. We suppose henceforth that $p$ is in this range. The Kuranishi theory then tells us, in particular, that if $A$ is irreducible and the operator

$$d_A^+: L^p_{1,A}(X \setminus \Sigma; g_P \otimes \Lambda^1) \to L^p(X \setminus \Sigma; g_P \otimes \Lambda^+)$$

is surjective, then a neighborhood of $[A]$ in $M(X, \Sigma, P, \varphi)$ is a smooth manifold (or orbifold if the finite stabilizer is larger than $Z(G)$), and its dimension is equal to minus the index of the Fredholm complex

$$L^p_{2,A}(X \setminus \Sigma; g_P \otimes \Lambda^0) \xrightarrow{d_A} L^p_{1,A}(X \setminus \Sigma; g_P \otimes \Lambda^1) \xrightarrow{d_A^+} L^p(X \setminus \Sigma; g_P \otimes \Lambda^+).$$

No essential change is needed to carry over the proofs from [21]. The non-linear aspects come down to the multiplication theorems for the $W^p_k$ spaces, while the Fredholm theory for the linear operators reduces in the end to the case of a line bundle with the weighted norms.

We refer to minus the index of the above complex as the formal dimension of the moduli space $M(X, \Sigma, P, \varphi)$. We can write the formula for the formal dimension as

$$(4h^\nu)k + 2\langle c_1(\mathfrak{o} - \varphi), [\Sigma]\rangle + \frac{(\dim O)}{2} \chi(\Sigma) - (\dim G)(b^+ - b^+ + 1).$$

(18)

Here $\dim O$ denotes the dimension of $O$ as a real manifold: an even number, because $O$ is also a complex manifold. The proof of this formula can be given following [21] by using excision to reduce to the simple case that the reduction of structure group can extended to all of $X$. In this way, the calculation can eventually be reduced to calculating the index of a Fredholm complex of the same type as above, but with $g_P$ replaced by a complex line-bundle $\mathfrak{o}_\mu$ on $X$, equipped with a singular connection $d_\mu$ of the form

$$\nabla + i\beta(r)\mu \eta$$

with $\mu \in (0, 1)$ (cf. (13)). The index calculation in the case of such a line bundle is given in [21].

We can express the characteristic class $c_1(\mathfrak{o}_\varphi)$ that appears in the formula (18) in slightly different language. As was just mentioned, the manifold $O$
is naturally a complex manifold. To define the standard complex structure, we identify the tangent space to $O$ at $\Phi$ with $\mathfrak{o}$, and we give $\mathfrak{o}$ a complex structure $J$ by identifying it with $\mathfrak{o} \otimes \mathbb{C}$ using the linear projection. This gives a $G_{\Phi}$-invariant complex structure on $T_{\Phi}O$ which can be extended to an integrable complex structure on all of $O$ using the action of $G$. The bundle $O_P \to X$ is now a bundle of complex manifolds, and we use $c_1(O_P/X)$ to denote the first Chern class of its vertical tangent bundle. Then we can rewrite $c_1(\mathfrak{o}_\varphi)$ as
\[ c_1(\mathfrak{o}_\varphi) = \varphi^*(c_1(O_P/X)) \]
in $H^2(\Sigma)$. Using again the canonical trivialization of $P|_\Sigma$ up to homotopy, we can also think of $\varphi$ as simply a map $\Sigma \to O$ up to homotopy, and we can think of the characteristic class as $\varphi^*(c_1(O))$.

We can summarize the situation with a lemma:

**Lemma 2.3.** Let the reduction of structure group of $P \to \Sigma$ to the subgroup $G_{\Phi}$ have monopole charge $l$. Then the formula (18) for formal dimension of the moduli space can be rewritten as
\[ 4h^2k + 4\rho(l) + \frac{\dim O}{2} \chi(\Sigma) - (\dim G)(b^+ - b^1 + 1), \]
(19)
where $\rho$ is, as above, the Weyl vector.

**Proof.** The difference between this expression and the previous formula (18) is the replacement of the term involving $2c_1(\mathfrak{o}_\varphi)[\Sigma]$ by the term involving $4\rho(l)$. To see that these are equal, we may reduce the structure group of the $G_{\Phi}$ bundle over $\Sigma$ to the torus $T$, and again write $\xi$ for the vector in the integer lattice of $T$ that classifies this $T$-bundle. The bundle $\mathfrak{o}_\varphi$ now decomposes as a direct sum according to the positive roots in $R^+(\Phi)$:
\[ \mathfrak{o}_\varphi = \bigoplus_{\alpha \in R^+(\Phi)} \mathfrak{o}_\alpha. \]
We have
\[ c_1(\mathfrak{o}_\alpha)[\Sigma] = \alpha(\xi); \]
so,
\[ c_1(\mathfrak{o}_\varphi)[\Sigma] = \sum_{\alpha \in R^+(\Phi)} \alpha(\xi) = \sum_{\alpha \in R^+(\Phi)} \langle \alpha^\vee, \xi \rangle. \]
For each simple root $\beta$ in $S^0(\Phi)$, the Weyl group reflection $\sigma_\beta$ permutes the vectors
\[ \{ \alpha^\dagger \mid \alpha \in R^+(\Phi) \}. \]
It follows that when we write
\[ \sum_{\alpha \in R^+} \alpha^\dagger = \sum_{\alpha \in R^+(\Phi)} \alpha^\dagger + \sum_{\alpha \in R^0(\Phi) \cap R^+} \alpha^\dagger, \]
the first term on the right is in the kernel of $\beta$ for all $\beta$ in $S^0(\Phi)$; i.e. the first term belongs to $3(G_\Phi)$. The second term on the right belongs to the orthogonal complement of $3(G_\Phi)$, because it is in the span of the elements $\alpha^\vee$ as $\alpha$ runs through $S^0(\Phi)$. Recalling the definition of the Weyl vector $\rho$, we therefore deduce
\[ \sum_{\alpha \in R^+(\Phi)} \alpha^\dagger = 2\Pi \rho^\dagger. \]
Thus
\[ c_1(\omega^-)[\Sigma] = \sum_{\alpha \in R^+(\Phi)} \langle \alpha^\dagger, \xi \rangle \]
\[ = 2\langle \Pi \rho^\dagger, \xi \rangle \]
\[ = 2\langle \rho, \Pi \xi \rangle \]
\[ = 2\rho(l) \]
as desired, because $l = \Pi \xi$. 

2.3 Energy and monotonicity

Along with the formula (19) for the dimension of the moduli space, the other important quantity is the energy of a solution $A$ in $C^p(X, \Sigma, P, \varphi)$ to the equations $F_A^\pm = 0$, which we define as
\[ \mathcal{E} = 2 \int_{X \setminus \Sigma} |F_A|^2 \, d\text{vol} \]
\[ = 2 \int_{X \setminus \Sigma} -\text{tr}(\text{ad}(*F_A) \wedge \text{ad}(F_A)). \]
(Note that the norm on $g_P$ in the first line is again defined using the Killing form, $-\text{tr}(\text{ad}(a) \text{ad}(b))$.) The reason for the factor of two is to fit with our use of the path energy in the context of Floer homology later.

This quantity depends only on $P$ and $\varphi$, and can be calculated in terms of the instanton number $k$ and the monopole charges $l$. To do this, we again
suppose that the structure group of $P_{\varphi} \rightarrow \Sigma$ is reduced to the torus $T$, and we decompose the bundle $\mathfrak{g}_\varphi^+$ again as (11). The formula for $\mathcal{E}$ as a function of $P$ and $\varphi$ can then be written, following the argument of [21], as

$$\mathcal{E}(X, \Sigma, P, \varphi) = 32\pi^2 \left( h^\vee k + \sum_{\beta \in R^+} \beta(\Phi)c_1(\mathfrak{o}_\beta)[\Sigma] - \frac{1}{2} \sum_{\beta \in R^+} \beta(\Phi)^2(\Sigma \cdot \Sigma) \right)$$

where $\xi$ is again the integer vector in $t$ classifying the $T$-bundle to which $P_{\varphi}$ has been reduced. The two sums involving the positive roots in this formula each be interpreted as half the Killing form, which leads to the more compact formula

$$\mathcal{E} = 8\pi^2 \left( 4h^\vee k + 2\langle \Phi, \xi \rangle - \langle \Phi, \Phi \rangle (\Sigma \cdot \Sigma) \right).$$

Finally, using the fact that $\Phi$ belongs to $\mathfrak{z}(G_\Phi)$, we can replace $\xi$ by its projection $l$ and write

$$\mathcal{E} = 8\pi^2 \left( 4h^\vee k + 2\langle \Phi, l \rangle - \langle \Phi, \Phi \rangle (\Sigma \cdot \Sigma) \right). \tag{22}$$

An important comparison to be made is between the linear terms in $k$ and $l$ in the formula for $\mathcal{E}$ and the linear terms in the same variables in the formula (19) for the formal dimension of the moduli space. In the dimension formula, these linear terms are

$$4h^\vee k + 4\rho(l) \tag{23}$$

while in the formula for $\mathcal{E}$ they are

$$8\pi^2 \left( 4h^\vee k + 2\langle \Phi, l \rangle \right). \tag{24}$$

**Definition 2.4.** We shall say that the choice of $\Phi$ is monotone if the linear forms (24) and (23) in the variables $k$ and $l$ are proportional.

**Proposition 2.5.** Let $\Phi_0$ be any element in the fundamental Weyl chamber. Then there exists a unique $\Phi$ in the same Weyl chamber such that the stabilizers of $\Phi_0$ and $\Phi$ coincide, and such that the monotonicity condition holds. Furthermore, this $\Phi$ satisfies the constraint (9).

**Proof.** We are seeking a $\Phi$ with $G_\Phi = G_{\Phi_0}$. The Lie algebras of these groups therefore have the same center, in which both $\Phi$ and $\Phi_0$ lie, and we can write
Π for the projection of $t$ onto the center. From the formulae, we see that the monotonicity condition requires that $\langle \Phi, l \rangle = 2\rho(l)$ for all $l$ in the image of $\Pi$. This condition is satisfied only by the element

$$\Phi = 2\Pi(\rho^\dagger).$$

(25)

We can rewrite this as

$$\Phi = \sum_{\alpha \in \mathbb{R}^+} \alpha^\dagger.$$

(26)

It remains only to verify the bound $\theta(\Phi) < 1$. But we have

$$\theta(\Phi) < \sum_{\alpha \in \mathbb{R}^+} \theta(\alpha^\dagger)$$

$$= 2\langle \theta, \rho \rangle$$

$$= 1 - 1/h^\vee$$

$$< 1$$

by (8), as desired.

2.4 Geometric interpretation of the monotonicity condition

We can reinterpret these formulae in terms of cohomology classes on $O$. As a general reference for the following material, we cite [3, Chapter 8]. In the first line of (21), we see the characteristic class

$$\sum_{\beta \in \mathbb{R}^+} \beta(\Phi)c_1(\mathfrak{o}_\beta^-).$$

(27)

The decomposition of $\mathfrak{o}_\rho^-$ as the sum of $\mathfrak{o}_\beta^-$ reflects our reduction of the structure group to $T$. A more invariant way to decompose this bundle is as follows. We write $E^+ \subset \mathfrak{z}(G\Phi)^*$ for the set of non-zero linear forms on $\mathfrak{z}(G\Phi)$ arising as $\beta|_{\mathfrak{z}(G\Phi)}$ for $\beta$ in $R^+$. The elements of $E^+$ are weights for the action of $Z(G\Phi)$ on $\mathfrak{o}$, and we have a corresponding decomposition of the vector bundle $\mathfrak{o}_\rho^-$ into weight spaces,

$$\mathfrak{o}_\rho^- = \bigoplus_{\gamma \in E^+} \mathfrak{o}_\rho^-(\gamma).$$

The characteristic class (27) can be written more invariantly as

$$\sum_{\gamma \in E^+} \gamma(\Phi)c_1(\mathfrak{o}_\rho^-(\gamma)).$$

(28)
The tangent bundle $TO$ has a decomposition of the same form, as a complex vector bundle,

$$TO = \bigoplus_{\gamma \in E^+} TO(\gamma).$$

Using the canonical trivialization of $P \to \Sigma$ up to homotopy, interpret $\varphi$ again as a map

$$\varphi : \Sigma \to O,$$

and then rewrite the characteristic class (28) as $\varphi^*(\Omega_\Phi)$, where

$$\Omega_\Phi = \sum_{\gamma \in E^+} \gamma(\Phi)c_1(TO(\gamma)) \in H^2(O; \mathbb{R}). \quad (29)$$

In this way, we can rewrite the linear form (24) in $k$ and $l$ as

$$32\pi^2 \left( h^*k + \langle \varphi^*\Omega_\Phi, [\Sigma] \rangle \right) \quad (30)$$

Using this last expression, we see that the monotone condition simply requires that

$$c_1(\sigma^-) = 2\Omega_\Phi \quad (31)$$

in $H^2(O; \mathbb{R})$. This equality has a geometric interpretation in terms of the geometry of the orbit $O$ of $\Phi$ in $g$. Recall that we have equipped $O$ with a complex structure $J$, so that its complex tangent bundle is isomorphic to $\sigma^-$. There is also the Kirillov-Kostant-Souriau 2-form on $O$, which is the $G$-invariant form $\omega_\Phi$ characterized by the condition that at $\Phi \in O$ it is given by

$$\omega_\Phi([U, \Phi], [V, \Phi]) = \langle \Phi, [U, V] \rangle$$

where the angle brackets denote the Killing form. Together, $J$ and $\omega_\Phi$ make $O$ into a homogeneous Kähler manifold. The cohomology class $[\omega_\Phi]$ of the Kirillov-Kostant-Souriau form is $4\pi\Omega_\Phi$, so the monotonicity condition can be expressed as

$$[\omega_\Phi] = 2\pi c_1(O).$$

The fact that the Kähler class and the first Chern class are proportional means, in particular, that $(O, \omega_\Phi)$ is a monotone symplectic manifold, in the usual sense of symplectic topology. In the homogeneous case, this proportionality (with a specified constant) between the classes $[\omega_\Phi]$ and $c_1(O)$ on $O$ implies a corresponding relation between their natural geometric representatives, namely $\omega_\Phi$ itself and the Ricci form. That is to say, our monotonicity condition is equivalent to the equality

$$g_\Phi = \text{Ricci}(g_\Phi)$$
for the Kähler metric $g_\Phi$ corresponding to $\omega_\Phi$. Thus in the monotone case, $O$ is a Kähler-Einstein manifold with Einstein constant $1$.

### 2.5 The case of the special unitary group

We now look at the case of the special unitary group $SU(N)$. We continue to suppose that $X \supset \Sigma$ is a pair consisting of a $4$-manifold and an embedded surface, as in the previous subsections. Let $P \to X$ be a given principal $SU(N)$ bundle. An element $\Phi$ in the standard fundamental Weyl chamber in the Lie algebra $\mathfrak{su}(N)$ has the form the form

$$\Phi = i \text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_m, \ldots, \lambda_m)$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_m$. Let the multiplicities of the eigenspaces be $N_1, \ldots, N_m$, so that $N = \sum N_s$. We will suppose that

$$\lambda_1 - \lambda_N < 1,$$

this being the analog of the condition (9). Let $O \subset \mathfrak{su}(N)$ be the orbit of $\Phi$ under the adjoint action. Choose a section $\varphi$ of the associated bundle $O_P|\Sigma$, so defining a reduction of the structure group of $P|\Sigma$ to the subgroup

$$S(U(N_1) \times \cdots \times U(N_m)).$$

Let $P_s \to \Sigma$ be the principal $U(N+s)$-bundle arising from the $s$’th factor in this reduction and define

$$l_s = -c_1(P_s)[\Sigma].$$

Because we have a special unitary bundle, we have

$$\sum N_s \lambda_s = 0$$

$$\sum l_s = 0.$$

The integers $l_s$ are equivalent data to the monopole charge $l$ in $L(G_\Phi)$ as defined in Definition 2.2 for the general case: more precisely, the relationship is

$$l = i \text{diag}(l_1/N_1, \ldots, l_1/N_1, \ldots, l_m/N_m, \ldots, l_m/N_m).$$

The Killing form is $2N$ times the standard trace norm on $\mathfrak{su}(N)$, so we have for example

$$\langle \Phi, l \rangle = 2N \sum_{s=1}^m \lambda_s l_s$$
The dual Coxeter number is $N$ and the energy formula becomes:

$$\mathcal{E}(X, \Sigma, P, \varphi) = 32\pi^2 N \left( k + \sum_{s=1}^{m} \lambda_s l_s - \frac{1}{2} (\sum_{s=1}^{m} \lambda_s^2 N_s) \Sigma \cdot \Sigma \right).$$

If $E_s$ is the vector bundle associated to $P_s$ by the standard representation, then the bundle denoted $\mathcal{O}_\varphi$ in the previous sections (the pull-back by $\varphi$ of the vertical tangent bundle of $O_P$, equipped with its preferred complex structure) can be written as

$$\mathcal{O}_\varphi = \bigoplus_{s < t} E_s^* \otimes E_t,$$

and the first Chern class of this bundle evaluates on $\Sigma$ as

$$c_1(\mathcal{O}_\varphi)[\Sigma] = \sum_{s=1}^{m} \sum_{t=1}^{m} \text{sign}(t - s) N_t l_s.$$

The dimension formula becomes

$$4Nk + 2 \sum_{s,t} \text{sign}(t - s) N_t l_s + \left( \sum_{s < t} N_s N_t \right) \chi(\Sigma) - (N^2 - 1)(b^+ - b^1 + 1). \quad (33)$$

where $k$ is the second Chern number of $P$. Thus the monotone condition simply requires that

$$\lambda_s = \frac{1}{2N} \sum_{t=1}^{m} \text{sign}(t - s) N_t$$

for all $s$. Notice that the $\Phi$ whose eigenvalues are given by the formula (34) with multiplicities $N_s$ is already traceless, and satisfies the requirement (32), which we can take as confirming Proposition 2.5 in the case of $SU(N)$.

**Examples.** (i) The simplest example occurs when $P$ is an $SU(2)$ bundle and the reduction of structure group is to $U(1)$. In this case, we can write

$$\Phi = i \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

with $\lambda \in (0, 1/2)$, and the holonomy of $A^\varphi$ along small loops linking $\Sigma$ is asymptotically

$$\exp 2\pi i \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$
We can write \((l_1, l_2)\) as \((l, -l)\) and formula for the index becomes
\[
8k + 4l + \chi(\Sigma) - 3(b^+ - b^1 + 1).
\]

The action is given by
\[
64\pi^2(k + 2\lambda l - \lambda^2 \Sigma \cdot \Sigma).
\]

These are the formulae from [21]. The monotone condition requires that \(\lambda = 1/4\), and in this case the asymptotic holonomy on small loops is
\[
\begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix}.
\]

(ii) The next simplest case is that of \(SU(N)\) with two eigenspaces, of dimensions \(N_1 = 1\) and \(N_2 = N - 1\). In this case, \(O\) becomes \(\mathbb{CP}^{N-1}\). We can again write \((l_1, l_2)\) as \((l, -l)\) and we can write
\[
\begin{align*}
\lambda_1 &= \lambda \\
\lambda_2 &= -\lambda/(N - 1)
\end{align*}
\]
with \(\lambda\) in the interval \((0, 1)\). The formula for the dimension is
\[
4Nk + 2Nl + (N - 1)\chi(\Sigma) - (N^2 - 1)(b^+ - b^1 + 1)
\]
and the energy is given by
\[
32\pi^2N\left(k + \frac{N}{N - 1}\left(\lambda l - \frac{1}{2}\lambda^2 \Sigma \cdot \Sigma\right)\right).
\]

The monotone condition requires \(\lambda = (N - 1)/(2N)\), so that
\[
\begin{align*}
\lambda_1 &= (N - 1)/(2N) \\
\lambda_2 &= -1/(2N).
\end{align*}
\]

The asymptotic holonomy on small loops is
\[
\zeta \text{ diag}(-1, 1, \ldots, 1)
\]
where \(\zeta = e^{\pi i/N}\) is a \((2N)\)’th root of unity.
2.6 Avoiding reducible solutions

We return to the case of a general $G$ (still simple and simply connected). Recall that a connection is reducible if its stabilizer in $G^p(X,\Sigma, P, \varphi)$ has positive dimension. This is equivalent to saying that there is a non-zero section $\psi$ of the bundle $\mathfrak{g}^P$ on $X\backslash \Sigma$ that is parallel for the connection $A$.

Let us ask under what conditions this can occur for an $[A]$ belonging to the moduli space $M(X, \Sigma, P, \varphi)$. For simplicity, we suppose for the moment that $\Sigma$ is connected. If $\psi$ is parallel, then it determines a single orbit in $\mathfrak{g}$; we take $\Psi \in \mathfrak{g}$ to be a representative. The structure group of the bundle then reduces to the subgroup $G_\Psi$, the stabilizer of $\Psi$, which is a connected proper subgroup of $G$. Because $\Phi$ and $\Psi$ must commute, we may suppose they both belong to the Lie algebra $\mathfrak{t}$ of the maximal torus. As before, we shall suppose $\Phi$ belongs to the fundamental Weyl chamber.

The center of $G_\Psi$ contains a torus of dimension at least one, because $\Psi$ itself lies in the Lie algebra of the center. So there is a non-trivial character,

$$s : G_\Psi \to U(1).$$

Because $G_\Psi$ contains $T$, this character corresponds to a weight for this maximal torus: there is an element $w \in \mathfrak{t}^*$ in the lattice of weights such that

$$s(\exp(2\pi x)) = \exp(2\pi w(x))$$  \hspace{1cm} (35)

for $x$ in $\mathfrak{t}$. The fundamental group of $T$ maps onto that of $G_\Psi$, so we may assume that $w$ is a primitive weight (i.e. is not a non-trivial multiple of another integer weight).

In addition to taking $w$ to be primitive, we can further narrow down the possibilities as follows. Suppose first that $\Psi$ lies (like $\Phi$) in the fundamental Weyl chamber. In the complex group $G^c$, there are $r = \text{rank}(G)$ different maximal parabolic subgroups which contain the standard Borel subgroup corresponding to our choice of positive roots. These maximal parabolics are indexed by the set of simple roots $\Delta^+$; we let $G(\alpha)$, for $\alpha \in \Delta^+$, denote the intersection of these groups with the compact group $G$. Each group $G(\alpha) \subset G$ has 1-dimensional center, and the weight $w$ corresponding to a primitive character of $G(\alpha)$ is the fundamental weight $w_\alpha$. The group $G_\Psi$ lies inside one of the $G(\alpha)$, so the same fundamental weight $w_\alpha$ defines a character of $G_\Psi$. If $\Psi$ does not lie in the fundamental Weyl chamber, then we need to apply an element of the Weyl group $W$; so in general, we can always take $w$ to have the form

$$w = w_\alpha \circ \sigma$$
where $\sigma \in W$ and $w_\alpha$ is one of the fundamental weights.

Applying the character $w$ to the singular connection $A$, we obtain a singular $U(1)$ connection on $X \setminus \Sigma$, carried by the bundle obtained by applying $s$ to the principal $G_\Psi$-bundle $P_\psi \to X$. That is, we have a $U(1)$ connection differing by terms of regularity $L^1_{1,A}$ from the singular model connection
\[
\nabla + i \beta(r) w(\Phi) \eta.
\]
Here it is important that $\Sigma$ is connected: what we have really done, is picked a base-point near $\Sigma$, and used the values of $\phi$ and $\psi$ at that base-point to determine $T$ and $s$. When we apply the Chern-Weil to obtain an expression for $c_1$ of the line bundle $s(P_\psi)$, we obtain an additional contribution from the singularity, equal to the Poincaré dual of
\[
2\pi w(\Phi)[\Sigma].
\]
More precisely, if $F$ denotes the curvature of this $U(1)$ connection on $X \setminus \Sigma$ (as an $L^p$ form, extended by zero to all of $X$), then
\[
\frac{i}{2\pi} [F] = c_1(s(P_\psi)) - w(\Phi) \text{P.D.} [\Sigma].
\]
If $\Sigma$ has more than one component, say $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$, then the only change is that we will see a different element of the Weyl group for each component of $\Sigma$: so if we have a reducible solution then there will be a fundamental weight $w_\alpha$ and elements $\sigma_1, \ldots, \sigma_r$ in $W$ such that the curvature $F$ of the corresponding $U(1)$ connection satisfies
\[
\frac{i}{2\pi} [F] = c_1(s(P_\psi)) - \sum_{j=1}^r (w_\alpha \circ \sigma_j)(\Phi) \text{P.D.} [\Sigma_j].
\] (36)

Because the connection $A$ is anti-self-dual, the 2-form $F$ is also anti-self-dual, and is therefore $L^2$-orthogonal to every closed, self-dual form $h$ on $X$. By the usual argument [6], we deduce:

**Proposition 2.6.** Suppose that $b^+(X) \geq 1$, and let the components of $\Sigma$ be $\Sigma_1, \ldots, \Sigma_r$. Suppose that for every fundamental weight $w_\alpha$ and every choice of elements $\sigma_1, \ldots, \sigma_r$ in the Weyl group, the real cohomology class
\[
\sum_{j=1}^r (w_\alpha \circ \sigma_j)(\Phi) \text{P.D.} [\Sigma_j]
\] (37)
is not integral. Then for generic choice of Riemannian metric on $X$, there are no reducible solutions in the moduli space $M(X, \Sigma, P, \varphi)$. 

Examples. As an illustration, in the case $G = SU(N)$, if $\Sigma$ is connected and $[\Sigma]$ is primitive, then for (37) to be an integral class means that the sum of some proper subset of the eigenvalues of $\Phi$ (listed with repetitions) is equal to an integer multiple of $i$. If there are only two distinct eigenvalues $i\lambda_1$ and $i\lambda_2$ of multiplicities $N_1$ and $N_2$, and if we are in the monotone case, so that

$$\lambda_1 = N_2/(2N),$$
$$\lambda_2 = -N_1/(2N),$$

then this integrality means that

$$(2N) \mid (aN_2 - bN_1)$$

for some non-negative with $a \leq N_1$, $b \leq N_2$ and $0 < a + b < N$. This cannot happen if $N_1$ and $N_2$ are coprime.

Another family of examples satisfying the monotone condition occurs when $\Phi = 2\rho^\dagger$ (the case where the reduction of structure group is to the maximal torus, $T \subset G$) and the group $G$ is small. For example, if $\Phi = 2\rho^\dagger$ and $G$ is either the group $G_2$ or the simply-connected group of type $B_2$, $B_3$ or $B_4$, then (37) is never an integer. Thus we have:

**Corollary 2.7.** Suppose that $b^+ (X)$ is positive and that $\Sigma$ is connected and lies in a primitive homology class. Suppose $\Phi$ is chosen to satisfy the monotone condition and that we are in one of the following cases:

(i) $G$ is the group $SU(N)$, and $\Phi$ has two distinct eigenvalues whose multiplicities are coprime.

(ii) $G$ is the group $G_2$ and $\Phi$ is the regular element $2\rho^\dagger$.

(iii) $G$ is the group Spin(5), Spin(7) or Spin(9) and $\Phi$ is $2\rho^\dagger$.

Then for generic choice of Riemannian metric on $X$, there are no reducible solutions in the moduli space $M(X, \Sigma, P, \varphi)$.

**Proof.** We illustrate the calculation in the case of Spin(9). The $B_4$ root system is the following collection of integer vectors in $\mathbb{R}^4$: the vectors $\pm e_i \pm e_j$ for $i \neq j$, and the vectors $\pm e_i$. The simple roots are

$$\alpha_i = e_i - e_{i+1}, \quad (i = 1, 2, 3),$$
$$\alpha_4 = e_4$$
and the fundamental weights are

\[ w_1 = (1, 0, 0, 0) \]
\[ w_2 = (1, 1, 0, 0) \]
\[ w_3 = (1, 1, 1, 0) \]
\[ w_4 = (1/2, 1/2, 1/2, 1/2). \]

The orbit of these under the Weyl group consists of all vectors of the form

\[
\pm e_i \\
\pm e_i \pm e_j \\
\pm e_i \pm e_j \pm e_k \\
(1/2)(\pm e_1 \pm e_2 \pm e_3 \pm e_4)
\]

with \( i, j, k \) distinct. We identify \( t \) with its dual using the Euclidean inner product on \( \mathbb{R}^4 \), so that the coroots are \( \alpha_i^\vee = \alpha_i \) for \( i = 1, 2, 3 \) and \( \alpha_4^\vee = 2\alpha_4 \); and we note that the Killing form on \( t \) is 14 times the Euclidean inner product, because the dual Coxeter number is 7. Then we calculate the sum of the positive roots to obtain

\[ 2\rho = (7, 5, 3, 1) \]

and we deduce

\[ 2\rho^\dagger = (1/14)(7, 5, 3, 1). \]

It is straightforward to see that \( 2\rho^\dagger \) does not have integer pairing with any of the vectors in (38), with respect to the Euclidean inner product on \( \mathbb{R}^4 \). □

The examples in this corollary are not meant to be exhaustive: the authors have not attempted a complete classification of the monotone cases. Note that we cannot extend the above proof for Spin(9) to the case of Spin(11) (the \( B_5 \) Dynkin diagram) because the vector \( 2\rho^\dagger \) is then

\[ 2\rho^\dagger(1/18)(9, 7, 5, 3, 1) \]

which has inner product 0 in \( \mathbb{R}^5 \) with the vector \( (1, 0, -1, -1, -1) \), which belongs to the Weyl orbit of the fundamental weight \( (1, 1, 1, 1, 0) \).

Remark. We recall again from section 2.2 that in cases other than \( SU(N) \), it is possible for a connection to be irreducible and yet have finite stabilizer strictly larger than the center of the group \( G \). The examples of this phenomenon that were illustrated previously include the case that \( G \) is
Spin(2n + 1) as well as the case of $G_2$. Note that these examples include the examples mentioned in part (iii) of Corollary 2.7; so although we can avoid stabilizers of positive dimension in those cases, we will still be left with finite stabilizers larger than the center.

The examples in Corollary 2.7 are cases where $\Sigma$ is connected. The next corollary exhibits an interesting case where Proposition 2.6 can be applied to a disconnected $\Sigma$:

**Corollary 2.8.** Suppose $b^+(X)$ is positive. Let $G = SU(N)$ and suppose $\Sigma$ has $N + 1$ components, all belonging to the same primitive homology class. Let $\Phi$ be the element

$$\Phi = \left(\frac{i}{2N}\right) \text{diag}((N-1), -1, \ldots, -1)$$

so that the monotone condition holds. Then for a generic choice of Riemannian metric on $X$, the moduli space $M(X, \Sigma, P, \varphi)$ contains no reducible solutions.

**Proof.** From Proposition 2.6, we see that we must check that the rational number

$$\sum_{j=1}^{N-1} (w_\alpha \circ \sigma_j)(\Phi)$$

is never an integer. As a function on the maximal torus, the fundamental weight $\alpha$ can be taken to be the sum of the first $k$ eigenvalues for some $k$ with $1 \leq k \leq N - 1$, and so

$$(w_\alpha \circ \sigma_j)(\Phi) = \begin{cases} -k/(2N), & \text{or} \\ (N - k)/(2N), & \end{cases} \quad (39)$$

according to which Weyl group element $\sigma_j$ is involved. The above sum is therefore

$$(s/2) - k(N - 1)/(2N)$$

where $k$ depends on the choice of $\alpha$, and $s$ is the number of components for which the second case of (39) occurs. This quantity differs from an element of $(1/2)\mathbb{Z}$ by $k/(2N)$, so it cannot be an integer. \qed
2.7 Bubbles

Uhlenbeck's compactness theorem for instanton moduli spaces on a closed 4-manifold $X$ carries over to the case of instantons with codimension-2 singularities along a surface $\Sigma \subset X$. In the case of $SU(2)$, the proof can again be found in [21]. The proof carries over without substantial change to the case of a general group. We shall state here the version appropriate for a simply-connected simple Lie group $G$ and a moduli space $M(X, \Sigma, P, \varphi)$ of anti-self-dual connections with reduction along $\Sigma$.

**Proposition 2.9.** Let $[A_n]$ be a sequence of gauge-equivalence classes of connections in the moduli space $M(X, \Sigma, P, \varphi)$. Then, after replacing this sequence by a subsequence, we can find a bundle $P' \to X$, a section $\varphi'$ of the bundle $O_{P'} \to \Sigma$ defining a reduction of structure group to the same subgroup $G_\Phi \subset G$, an element $[A]$ in $M(X, \Sigma, P', \varphi')$ and a finite set of point $x \subset X$ with the following properties.

(i) There is a sequence of isomorphisms of bundles $g_n : P'|_{X\setminus x} \to P$ such that $g_n^*(\varphi) = \varphi'|_{\Sigma \setminus x}$ and such that

$$g_n^*(A_n) \to A|_{X\setminus x}$$

on compact subsets of $X\setminus x$.

(ii) In the sense of measures on $X$, the energy densities $2|F_{A_n}|^2$ converge to

$$2|F_A|^2 + \sum_{x \in X} \mu_x \delta_x$$

where $\delta_x$ is the delta-mass at $x$ and $\mu_x$ are positive real numbers.

(iii) For each $x \in X$, we can find an integer $k_x$ and an $l_x$ in the lattice $L(G_\Phi) \subset Z(G_\Phi)$ such that

$$\mu_x = 8\pi^2(\Phi, l_x)$$

If $x \notin \Sigma$, we can take $l_x = 0$ here. Furthermore, if $(k, l)$ and $(k', l')$ are the instanton numbers and monopole charges for $(P, \varphi)$ and $(P', \varphi')$ respectively, then we can arrange that

$$k = k' + \sum_{x \in X} k_x$$

$$l = l' + \sum_{x \in X} l_x$$
(iv) For each such pair \((k_x, l_x)\) with \(x \in \Sigma\) we can find an expression for these as finite sums,

\[
k_x = k_{x,1} + \cdots + k_{x,m}
\]

\[
l_x = l_{x,1} + \cdots + l_{x,m}
\]

and solutions \([A_{x,i}]\) in moduli spaces \(M(S^4, S^2, P_{x,i}, \varphi_{x,i})\) for the round metric on \((S^4, S^2)\), where \(P_{x,i}\) is the \(G\)-bundle on \(S^4\) with \(k(P_{x,i}) = k_{x,i}\) and \(\varphi_{x,i}\) is the reduction of structure group along \(S^2\) classifies by the element \(l_{x,i}\) in \(L(G_\Phi)\).

The content of the last three parts of the proposition is that the energy \(\mu_x\) that is “lost” at each of the point \(x\) in \(x\) is accounted for by the energy of a collection of solutions on \((S^4, S^2)\) that have bubbled off. (The expression

\[
8\pi^2(4h^\vee k + 2\langle \Phi, l \rangle)
\]

is the formula for the energy in the case of \((S^4, S^2)\).) In general, if no multiple of \(\Phi\) is an integer point, then the set of values realized by this function of \(k\) and \(l\) is dense in the real line; and while the proof of Uhlenbeck’s theorem does provide us with a constant \(\eta\) and a guarantee that \(\mu_x \geq \eta\) in all cases, we need better information than this to make use of the compactness theorem in applications. For example, the statement of the result as given does not guarantee that the formal dimension of \(M(X, \Sigma, P, \varphi)\) is not larger than that of \(M(X, \Sigma, P', \varphi')\).

The essential matter is to know which pairs \((k, l)\) in \(\mathbb{Z} \times L(G_\Phi)\) are realized by solutions on \((S^4, S^2)\).

**Proposition 2.10.** Let \(\Phi\) as usual lie in the fundamental Weyl chamber and satisfy the necessary constraint \(\theta(\Phi) < 1\). Then for any solution \([A]\) on \((S^4, S^2)\) with the round metric, the corresponding topological invariants \((k, l)\) in \(\mathbb{Z} \times L(G_\Phi)\) must satisfy the inequalities

\[
k \geq 0 \tag{41}
\]

and

\[
n_\alpha^\vee k + w_\alpha(l) \geq 0 \tag{42}
\]

for all simple roots \(\alpha\).

**Remark.** We will see in the course of the proof that the above inequalities are equivalent to a smaller set, namely the set consisting of the inequality (41) together with the inequalities (42) taken only for those \(\alpha\) belonging to the set of simple roots \(\alpha\) in \(S^+(\Phi)\) (the simple roots which are positive on \(\Phi\)).
Before proving the proposition, we note an important corollary for the formal dimensions of the non-empty moduli spaces on \((S^4, S^2)\). The dimension formula in this case can be written
\[
4h^\vee l + 4\langle \rho^\dagger, l \rangle - \dim G\Phi.
\]
We can interpret the first two terms
\[
4h^\vee l + 4\langle \rho^\dagger, l \rangle
\]
as the dimension of a framed moduli space, as follows. The gauge group \(G^p(X, \Sigma, P, \varphi)\) consists of continuous automorphisms of the bundle \(P \to S^4\) which preserve the section \(\varphi\) of the adjoint bundle; so if we pick a point \(s \in S^2 \subset S^4\) then there is a closed subgroup \(G_p^1\) consisting of elements with \(g(s) = 1\). The formula (43) can be interpreted as the formal dimension of a moduli space \(\tilde{\mathcal{M}}(S^4, S^2)\) where we divide the space of anti-self-dual singular connections by the smaller group \(G_p^1\) instead of the full gauge group. We then have:

**Corollary 2.11.** For any non-empty moduli space on \((S^4, S^2)\) with the round metric, the corresponding instanton number \(k\) and monopole charge are either both zero (in which case the moduli space contains only the flat connection) or satisfy
\[
4h^\vee k + 4\langle \rho^\dagger, l \rangle \geq 4.
\]

**Proof of Corollary 2.11.** We recall the relation (5) and the fact that \(\rho\) is the sum of the \(w_\alpha\) (taken over all simple roots \(\alpha\)). Using these, we see that the sum of all the inequalities in Proposition 2.10 gives us
\[
h^\vee k + \langle \rho^\dagger, l \rangle \geq 0.
\]
To refine this a little, let us break up the sum into two parts according to whether \(\alpha\) lies in \(S^+(\Phi)\) or \(S^-\Phi\): we obtain
\[
4h^\vee k + 4\langle \rho^\dagger, l \rangle = 4k + 4 \sum_{\alpha \in S^+(\Phi)} (h^\vee k + \langle w_\alpha^\dagger, l \rangle) + 4 \sum_{\beta \in S^0(\Phi)} (h^\vee k + \langle w_\beta^\dagger, l \rangle)
\]
\[
\geq 4k + 4 \sum_{\alpha \in S^+(\Phi)} (h^\vee k + \langle w_\alpha^\dagger, l \rangle).
\]
As well as being non-negative by the proposition, the terms under the final summation sign are all integers: this is because \(l\) is the projection in \(\frac{1}{2}(G\Phi)\) of an integer vector \(\xi \in t\) and \(\xi - l\) lies in the kernel of \(w_\alpha\) for \(\alpha\) in \(S^+(\Phi)\). (The similar terms involving the \(\beta\) in \(S^0(\Phi)\) need not be integers.)
This shows that $4h^\vee k + 4\langle \rho^\dagger, l \rangle$ is at least 4 unless $k$ is zero and $w_\alpha(l)$ is zero for all $\alpha$ in $S^+(\Phi)$. These are independent linear conditions which imply that $k$ and $l$ are both zero. \hfill \Box

**Corollary 2.12.** In the situation of Proposition 2.9, if the set of bubble-points $x$ is non-empty, then the formal dimension of the moduli space $M(\Sigma, P', \varphi')$ is smaller than the dimension of $M(\Sigma, P, \varphi)$, and the difference is at least 4.

**Proof of Corollary 2.12.** The difference in the dimensions is equal to a sum

$$\sum_x \sum_{i=1}^n \left(4h^\vee k_{x,i} + 4\langle \rho^\dagger, l_{x,i} \rangle \right)$$

and each term is at least four by the previous corollary and the condition in part (iv) of Proposition 2.9. \hfill \Box

**Proof of Proposition 2.10.** The proof rests on a theorem of Munari [28] which provides a correspondence between moduli spaces of singular instantons in $(S^4, S^2)$ and certain complex-analytic moduli spaces for holomorphic data on $\mathbb{CP}^2$. To state Munari’s theorem, fix $\Phi$ as usual, let $P \to S^4$ be a $G$-bundle and $\varphi$ a section of $O_P \to S^2$ defining a reduction of structure group. Pick a point $s \in S^2$ and let $\tilde{M}(S^4, S^2, P, \varphi)$ be the corresponding framed moduli space. Let $\pi : \mathbb{CP}^2 \to S^4$ be a map which collapses the line at infinity $\ell_{\infty} \subset \mathbb{CP}^2$ to the point $s$ and which maps another complex line $\Sigma$ to $S^2 \subset S^4$. Write $s_{\infty} \in \mathbb{CP}^2$ for the point where $\Sigma$ and $\ell_{\infty}$ meet. Then we have:

**Theorem 2.13 ([28]; see also [4]).** There is a bijection between the moduli space of singular anti-self-dual connections $\tilde{M}(S^4, S^2, P, \varphi)$ on the one hand, and on the other, the set of isomorphism classes of collections $(P, \psi, \tau)$ where

- $P \to \mathbb{CP}^2$ is a holomorphic principal $G^c$-bundle topologically isomorphic to $\pi^*(P)$,
- $\psi : \Sigma \to O_P$ is a holomorphic section of the associated bundle on $\Sigma = \mathbb{CP}^1$ with fiber $O$, homotopic to the section $\pi^*(\varphi)$,
- $\tau$ is a holomorphic trivialization of the restriction of $P$ to $\ell_{\infty}$, satisfying the constraint that the induced trivialization of the adjoint bundle carries $\psi(s_{\infty})$ to $\Phi$.
A special case of this theorem, which may help to understand the statement, is the case that $k = 0$ and the bundle $P$ on $S^4$ is trivial. In this case $P$ is topologically trivial; and the trivialization on the line at infinity forces $P$ to be analytically trivial also, so that $\tau$ extends uniquely to a holomorphic trivialization of $P \to \mathbb{CP}^2$. The data $\psi$ then becomes a based rational map: a holomorphic map from $\Sigma = \mathbb{CP}^1$ to $O$ sending $s_\infty$ to $\Phi$.

Staying with this special case, the inequalities of Proposition 2.10 have a straightforward interpretation. For a holomorphic map $\psi$ from $\mathbb{CP}^1$ to $O$, the pairing of $\psi(\mathbb{CP}^1)$ with any class in the closure of the Kähler cone of $O$ must be non-negative. The inequalities of the proposition when $k = 0$ can be seen as consequences of this statement. This is essentially the same argument that was used by Murray [29] to constrain the possible charges of monopoles on $\mathbb{R}^3$.

For the general case, the strategy is similar, but we use the energy $E$ of the anti-self-connection, rather than the energy of a holomorphic map. The essential point, which the following immediate corollary of Theorem 2.13 above:

**Corollary 2.14.** Suppose $\Phi$ and $\Phi_1$ are two elements of the fundamental Weyl chamber with the same stabilizer, so that $\mathfrak{z}(G_\Phi) = \mathfrak{z}(G_{\Phi_1})$. Suppose both satisfy the constraint (9). Then $M(S^4, S^2, P, \varphi)$ is homeomorphic to $M(S^4, S^2, P, \varphi_1)$ when $\varphi$ and $\varphi_1$ are sections of the bundles associated to the adjoint action of $G$ on the orbits of $\Phi_1$ and $\Phi_2$ respectively, with the same homotopy class. In particular, one of these moduli spaces is non-empty if and only if the other is.

To apply this corollary, suppose that a moduli space $M(S^4, S^2, P, \varphi)$ is non-empty. Let $k \in \mathbb{Z}$ and $l \in L(G_\Phi) \subset \mathfrak{z}(G_\Phi)$ be the topological invariants of $P$ and $\varphi$. Let $\mathfrak{A}$ be the alcove in $t$ defined as the intersection of the fundamental Weyl chamber with the half-space $\theta \leq 1$, where $\theta$ is the highest root. Thus $\mathfrak{A}$ is a closed simplex. The intersection $\mathfrak{A} \cap \mathfrak{z}(G_\Phi)$ is a simplex with possibly smaller dimension. In applying the corollary, the admissible values for $\Phi_1$ are precisely the interior points of the simplex $\mathfrak{A} \cap \mathfrak{z}(G_\Phi)$. A necessary condition for a moduli space to be non-empty is that the associated topological energy $E(S^4, S^2, P, \varphi_1)$ is non-negative, so the corollary tells us that

$$2k \cdot H + \langle \psi, l \rangle \geq 0$$

for all interior points of $\mathfrak{A} \cap \mathfrak{z}(G_\Phi)$, and hence for all points in the closed simplex $\mathfrak{A} \cap \mathfrak{z}(G_\Phi)$, by continuity. If $\Pi : t \to \mathfrak{z}(G_\Phi)$ again denotes the
orthogonal projection, then it is a fact about the geometry of $\mathfrak{A}$ that

$$\Pi(\mathfrak{A}) \subset \mathfrak{A} \cap \mathfrak{z}(G_{\Phi}).$$

As $l$ itself lies in $\mathfrak{z}(G_{\Phi})$, we deduce that the inequality (44) holds not just for $\psi$ in $\mathfrak{A} \cap \mathfrak{z}(G_{\Phi})$, but for all $\psi$ in $\mathfrak{A}$.

The vertices of the simplex $\mathfrak{A}$ are the point 0 and the points $\psi = w_{\alpha}^\dagger / \theta(w_{\alpha}^\dagger)$, as $\alpha$ runs through the simple roots. Applying (44) with $\psi$ at these vertices, we obtain $k \geq 0$ and

$$2k^\vee \theta(w_{\alpha}^\dagger)k + w_{\alpha}(l) \geq 0$$

for all simple roots $\alpha$. To complete the proof of the inequality (42), we calculate, using (6) and the definition of the coroots,

$$2k^\vee \theta(w_{\alpha}^\dagger) = 2(\theta, \theta)^{-1}w_{\alpha}(\theta^\dagger) = w_{\alpha}(\theta^\vee) = n_{\alpha}^\vee.$$

This completes the proof of the proposition.

Example. We illustrate the $SU(N)$ case. Arrange the eigenvalues of $\Phi$ as usual, as $i\lambda_1, \ldots i\lambda_m$ with $\lambda_1 > \cdots > \lambda_m$ and $\lambda_1 - \lambda_m < 1$. Let $N_s$ be the multiplicity of the eigenspace for $\lambda_s$, so that $\varphi$ defines a reduction of $P|S^2$ to the subgroup

$$S(U(N_1) \times \cdots \times U(N_m)).$$

Let $k$ be $c_2(P)[S^4]$, and let $l_1, \ldots, l_m$ be the first Chern numbers,

$$l_s = -c_1(E_s)[S^2]$$

where $E_s$ is the associated $U(N_s)$ bundle. We have $\sum l_s = 0$. Then the inequalities of Proposition 2.10, taken just for the extreme cases when $\alpha$ is in $S^+(\Phi)$, become

$$k \geq 0$$

$$k + l_1 \geq 0$$

$$\cdots$$

$$k + l_1 + l_2 + \cdots + l_{m-1} \geq 0.$$  

(45)

Note that the first inequality can also be written as the non-negativity of $k + l_1 + \cdots + l_m$, because the $l_s$ add up to zero. Let us write

$$K_s = k + \sum_{t<s} l_t,$$
so that the above inequalities assert $K_s \geq 0$. Then we observe that the formal dimension of the framed moduli space, given by the formula (33), can be written as

$$2(N_m + N_1)K_1 + 2(N_1 + N_2)K_2 + \cdots + 2(N_{m-1} + N_m)K_m.$$ 

This is bounded below by

$$4(K_1 + \cdots + K_m).$$

In particular the dimension of the moduli space is at least 4, unless $k$ and the $l_s$ are all zero. Slightly more precisely, we can state:

**Corollary 2.15.** For $\Phi$ as above, and $G = SU(N)$, the minimum possible formal dimension of any non-empty framed moduli space of positive formal dimension on $(S^4, S^2)$ is

$$\min \{ 2(N_{s-1} + N_s) \mid s = 1, \ldots, m \}$$

where we interpret $N_0$ as a synonym for $N_m$. In particular, no moduli space has dimension less than 4, except for the trivial zero-dimensional moduli space; and in the special case that there are only two distinct eigenvalues, the smallest positive-dimensional moduli space has dimension $2N$.

### 2.8 Orbifold metrics and connections

Up until this point, we have considered a moduli space $M(X, \Sigma, P, \varphi)$ of singular instantons defined using a space of connections $C^p(X, \Sigma, P, \varphi)$ modeled on an $L^p_1$ Sobolev space, with $p$ a little bigger than 2. There are disadvantages associated with having to use such a weak Sobolev norm: for example, these connections $A$ are not continuous, which creates difficulties if we want to use holonomy perturbations later. There is also a difficulty with proving the sort of vanishing theorems that are usually used to show that the moduli spaces of solutions on $S^4$, for example, are smooth.

Something that was exploited in $[21]$ is that we can use stronger Sobolev norms if we first make a slight change to the geometry of our picture. We will explain this here.

We shall equip $X$ with a singular metric $g^\nu$ which has an orbifold-type singularity along the surface $\Sigma$, with cone-angle $2\pi/\nu$ for some integer $\nu > 0$. This means that at each point of $\Sigma$ there is a neighborhood $U$ such that $(U \setminus \Sigma, g^\nu)$ is isometric to the quotient of a smooth Riemannian manifold by
a cyclic group of order $\nu$: the model for such a metric in the flat case is the metric
\[ du^2 + dv^2 + dr^2 + \left( \frac{r^2}{\nu^2} \right) d\theta^2. \]

As motivation, if $\Phi$ is an element of $t \subset g$ with the property that $\nu\Phi$ is integral, then our model singular connection $A^\nu$ from (13) can be constructed so that it becomes a smooth connection on passing to $\nu$-fold branched cover; so if we use the metric $g^\nu$, then we can regard $A^\nu$ as an orbifold connection, and we can reinterpret it as being a smooth connection in the orbifold sense.

We shall not use the orbifold language here, except in referring to the metric $g^\nu$ as having an “orbifold singularity”. Also, when using the metric $g^\nu$, we shall not require that $\nu\Phi$ be integral. What we will exploit is that, by making $\nu$ sufficiently large, the “Fredholm package” that is used in constructing the moduli spaces can be made to work in Sobolev spaces with any desired degree of regularity. More precisely, let $A^\nu$ be the model singular connection on $(X, \Sigma)$ equipped with the metric $g^\nu$, and let $d^+_A$ be the linearized anti-self-duality operator acting on $g_P$-valued 1-forms, defined using the metric $g^\nu$. On differential forms on $X \setminus \Sigma$, define the norms $\tilde{L}^p_{k,A^\nu}$ using the Levi-Civita derivative of $g^\nu$ and the covariant derivative of $A^\nu$ on $g_P$. Then let $\mathcal{D}_\varphi$ be the operator
\[ \mathcal{D}_\varphi = -d^*_A \oplus d^+_A \] (46)
acting on the spaces
\[ \tilde{L}^p_{k,A^\nu}(X \setminus \Sigma, g_P \otimes \Lambda^1) \to \tilde{L}^p_{k-1,A^\nu}(X \setminus \Sigma, g_P \otimes (\Lambda^0 \oplus \Lambda^+)) \] (47)

Then $\mathcal{D}_\varphi$ is Fredholm, as shown in [21, Proposition 4.17]:

**Proposition 2.16 ([21]).** Given any compact subinterval $I \subset (0,1)$ and any $p$ and $m$, there exists a $\nu_0 = \nu_0(I,p,m)$ such that for all $\nu \geq \nu_0$, all $k \leq m$ and all $\Phi$ in the fundamental Weyl chamber satisfying
\[ \alpha(\Phi) \in I, \forall \alpha \in R^+(\Phi), \]
the operator $\mathcal{D}_\varphi$ acting on the spaces (47) is Fredholm, as is its formal adjoint, and the Fredholm alternative holds.

This proposition gives us the linear part of the theory needed for the gauge theory; the non-linear aspects are the multiplication theorems and the Rellich lemma, which also go through in this setting; see [21] for details.
When using the orbifold-type metric, we will fix an integer \( m > 2 \) and define our space of connections as
\[
\mathcal{C}(X, \Sigma, P, \varphi) = \{ A \mid A - A^\varphi \in \tilde{L}^2_{m, A^\varphi} \}.
\]
We write \( \mathcal{G}(X, \Sigma, P, \varphi) \) for the corresponding gauge group, whose Lie algebra is \( L^2_{m+1, A^\varphi}(X \setminus \Sigma, g_P) \), and we let
\[
M(X, \Sigma, P, \varphi) \subset \mathcal{C}(X, \Sigma, P, \varphi) / \mathcal{G}
\]
be the moduli space of singular anti-self-dual connections for the metric \( g^\varphi \). The formula for the dimension of the moduli space at an irreducible regular point (i.e. the index of the operator \( \mathcal{D}_\varphi \) above) is given by the same formula (19) as before, as is the energy \( E \) of a solution.

2.9 Orienting moduli spaces
We next show that the moduli spaces of singular instantons are orientable, and discuss how to orient them. Again, for the case \( G = SU(2) \), the necessary material is in [21]. In the case that the \( K \) is absent, the orientability of the moduli spaces for a general simple Lie group \( G \) and simply-connected \( X \) is explained in [9]. For the case of \( SU(N) \) and arbitrary \( X \), a proof is given in [7]. In the following proposition, we treat a simple, simply-connected group \( G \). Recall that \( \Sigma \) is an oriented surface.

**Proposition 2.17.** In the moduli space \( M(X, \Sigma, P, \varphi) \), the set of regular points \( M^{\text{reg}} \) is an orientable manifold. If the dimension of \( G \) is even, then \( M^{\text{reg}} \) has a canonical orientation; while if \( G \) is odd-dimensional, the manifold \( M^{\text{reg}} \) can be canonically oriented once a homology orientation for \( X \) is given.

**Proof.** We first deal with the case that \( \Sigma \) is absent: we consider the orientability of the set of regular points in a moduli space \( M(X, P) \) of (non-singular) anti-self-dual connections. Following the usual argument, we consider the space of all connections modulo gauge, \( \mathcal{B}(X, P) \), and also the space of framed connections \( \tilde{\mathcal{B}}(X, P) \): the quotient of the space of connections by the based gauge group. Over \( \tilde{\mathcal{B}}(X, P) \) one has a real determinant line bundle \( \Omega(X, P) \), the determinant of the family of operators obtained by coupling \( -d^c \oplus d^c \) to the family of connections in the adjoint bundle. To show that \( M^{\text{reg}}(X, P) \) is orientable, we will show that \( \Omega(X, P) \) is trivial.

We will reduce the problem to the known case of an \( SU(2) \) bundle by applying (in the reverse direction) the same stabilization argument used in
[7]. Pick any long root for $G$, say the highest root $\theta$, and let $j : SU(2) \to G$ be the corresponding copy of $SU(2)$. The structure group of $P$ can be reduced to the subgroup $j(SU(2))$, giving us an $SU(2)$ bundle $Q \subset P$, and we have a map

$$j_* : \tilde{B}(X, Q) \to \tilde{B}(X, P)$$

whose domain is the space of based $SU(2)$ connections. From [7], we know that the corresponding line bundle $\Omega(X, Q)$ on $\tilde{B}(X, Q)$ is trivial.

The pair $(G, j(SU(2)))$ is 4-connected, so the inclusion of based gauge groups is surjective on $\pi_0$; and the map $j_*$ above is therefore surjective on $\pi_1$. To show that the determinant line $\Omega(X, P)$ on $\tilde{B}(X, P)$ is trivial, it is therefore enough to show that its pull back by $j_*$ is trivial. As stated in section 2.1, the adjoint representation of $G$ decomposes as a representation of $j(SU(2))$ as one copy of the adjoint representation of $SU(2)$, a number of copies of the 2-dimensional representation of $SU(2)$, and a number of copies of the trivial representation. Accordingly, the pull-back of $\Omega(X, P)$ by $j_*$ is a tensor product of a number of real line bundles: the one corresponding to the adjoint representation is a copy of $\Omega(X, Q)$; the determinant lines corresponding to the 2-dimensional representations are orientable using the complex orientations; and the remaining factors are trivial. Thus the triviality of $\Omega(X, P)$ is reduced to the known case of $\Omega(X, Q)$.

Once one knows that the moduli space is orientable, the next issue is to specify a standard orientation. We stay with the case that $\Sigma$ is absent. By “addition of instantons”, the matter is reduced to specifying a trivialization of $\Omega(X, P)$ in the case that $P$ is trivial. In this case we can look at the fiber of $\Omega(X, P)$ at the trivial connection, where operator is the standard operator $-d^* \oplus d^+$ coupled to the trivial bundle $g$. In the case that $g$ is even-dimensional, the determinant line can be canonically oriented; while in the case that $g$ is odd-dimensional, we need to specify and orientation the determinant of the operator $-d^* \oplus d^+$ with real coefficients, i.e. a homology orientation for $X$. Conventions for these choices can be set up so that the orientation of the moduli space agrees with its complex orientation when $X$ is Kähler: the arguments from [7] are adapted to the case of $SU(N)$ in [20], and the case of a general simple $G$ is little different.

We now consider how the determinant line changes when we introduce a codimension-2 singularity along $\Sigma \subset X$. There is again a determinant line bundle $\Omega(X, \Sigma, P, \varphi)$ over the space $\bar{B}(X, \Sigma, P, \varphi)$ of singular connections. We must show this line bundle is trivial. If we consider the restriction of the line bundle to a compact family $S \subset \bar{B}(X, \Sigma, P, \varphi)$ of singular connections, then the data is a family of $G$ bundles $P_s$ with a family of reductions of
structure group, $\varphi_s$. Let $A_s$ be a family of smooth connections in the bundles $P_s$. We may suppose that $A_s|_\Sigma$ respects the reduction $\varphi_s$ to the group $G_\Phi \subset G$. We may also suppose that $A_s$ can be identified with pull-back of $A_s|_\Sigma$ in a tubular neighborhood of $\Sigma$. Let $A_s^\varphi$ be constructed from $A_s$ by adding the singular term in the usual way, as at (13). Over $S$, we can consider two line bundles: first the line bundle $\Omega(X,P)$, which we have already seen is trivial; and second the line bundle $\Omega(X,\Sigma,P,\varphi)$, the determinant line of the deformation complex for the singular instantons. We must examine the ratio

$$\Omega(X,\Sigma,P,\varphi) \otimes \Omega(X,P)^{-1}$$

and show that it is trivial.

By excision, we can replace $X$ now by the sphere-bundle over $\Sigma$ which is obtained by doubling the tubular neighborhood, and we can replace the family of connections $A_s$ by the family of $G_\Phi$ connections obtained by pulling back $A_s|_\Sigma$. In this setting, the adjoint bundle $g_P$ over $S \times \Sigma$ decomposes as a sum of two sub-bundles, associated to the decomposition of $g$ as $g_\Phi \oplus \sigma$ in (10). There is a corresponding tensor product decomposition of each of the determinant lines in (48) above. On the summand $g_\Phi$, the two operators agree; so the ratio (48) is isomorphic to the ratio of determinant lines for the same operators coupled only to the subbundle coming from $\sigma$ instead of to all of $g$. Since $\sigma$ is complex, the ratio of determinant lines can be given its complex orientation, which completes the proof that the moduli space is orientable. This argument also shows that a choice of orientation for $\Omega(X,P)$ gives rise to a preferred orientation for $\Omega(X,\Sigma,P,\varphi)$. So the data needed to orient the moduli space is the same in the singular and non-singular cases.

2.10 The unitary and other non-simple groups

Up until this point, $G$ has always been a simple and simply-connected group. One very straightforward generalization is allow $G$ to be semi-simple and still simply-connected. In this case $G$ is a product of simple groups $G_1 \times \cdots \times G_m$, and our configuration spaces of connections are simply products. We can define the lattice $L(G_\Phi)$ and the the monopole charge $l$ just as before, with the understanding that we are now dealing with a root system that is reducible. The only new feature here is that the instanton charge $k$ is now an $m$-tuple, $k = (k_1, \ldots, k_m)$, and in the dimension and action formulae we see

$$\sum h^*_i k_i$$
with \( h' \) the dual Coxeter number of \( G_i \), where previously we had just \( h'k \). The monotone condition and Proposition 2.5 are unchanged.

There is a more interesting variation to consider when \( G \) has center of positive dimension, such as in the case of the unitary group. We will make the assumption that \( G \) is connected and that the commutator subgroup \([G, G]\) is simply connected. We shall write \( Z(G) \) for the center of \( G \), and we set

\[
\tilde{Z}(G) = \frac{G}{[G, G]} = \frac{Z(G)}{(Z(G) \cap [G, G])}.
\]

The quotient map will be written

\[
\delta: G \rightarrow \tilde{Z}(G),
\]
and we use the same notation also for the corresponding map on the Lie algebras. The abelian group \( Z(G) \) may not be connected, but \( \tilde{Z}(G) \) is a torus. We will run through some of the points to show how the theory adapts to this case.

**Instanton moduli spaces.** Let us temporarily omit the codimension-2 singularity along \( \Sigma \) from our discussion. An appropriate setting for gauge theory in a \( G \)-bundle \( P \rightarrow X \) when \( Z(G) \) has positive dimension is not to consider the space of all \( G \)-connections in \( P \), but instead to fix a connection \( \Theta \) in the associated \( \tilde{Z}(G) \)-bundle, \( \delta(P) \rightarrow X \), and to consider the space \( C(X, P) \) of connections \( A \) in \( P \) which induce the given connection \( \Theta \) in \( \delta(P) \):

\[
C(X, P) = \{ A \mid \delta(A) = \Theta \}
\]

(In the case of the unitary group \( U(N) \), this means looking at the unitary connections in a rank-\( N \) vector bundle \( E \) inducing a given connection \( \Theta \) in \( \Lambda^N E \).) Such a \( G \)-connection \( A \) in \( P \) is entirely determined by the induced connection \( \tilde{A} \) in the associated \( (G/Z(G)) \)-bundle \( \tilde{P} \). The appropriate gauge group in this context is not the group of all automorphisms of \( P \), but instead the group \( \mathcal{G}(X, P) \) consisting of automorphisms which take values in \([G, G]\) everywhere. That is, an element of \( \mathcal{G}(X, P) \) is a section of associated fiber bundle arising from the adjoint action of \( G \) on the subgroup \([G, G]\). In the case of a unitary vector bundle, this is the group of unitary automorphisms of a vector bundle \( E \rightarrow X \) having determinant 1 at every point. The moduli space \( M(X, P) \) is the subspace of the quotient \( \mathcal{B}(X, P) = C(X, P)/\mathcal{G}(X, P) \) consisting of all \([A]\) such that the curvature of \( \tilde{A} \) (not the curvature of \( A \)) is anti-self-dual:

\[
M(X, P) = \{ A \in C(X, P) \mid F^+_A = 0 \}/\mathcal{G}(X, P).
\]
Note that the chosen connection $\Theta$ really plays no role here, and we could equally well regard $\mathcal{C}(X, P)$ as parametrizing the connections $\tilde{A}$ in $\tilde{P}$. (Later, however, when we introduce holonomy perturbations in section 3.2, a choice of $\Theta$ will be important.) Indeed, $\mathcal{B}(X, P)$ and $M(X, P)$ really depend only on the adjoint group $\tilde{G}$ and the bundle $\tilde{P}$, because both the adjoint bundle with fiber $[g, g]$ and the bundle of groups with fiber $[G, G]$ (whose sections are the gauge transformations) are bundles associated to $\tilde{P}$. The choice of $G$ therefore only affects which bundles $\tilde{P}$ can arise.

Because of this last observation, we can if we wish start with the simply-connected semi-simple group $G_1$ with adjoint group $\tilde{G}_1$ and construct a group $G$ with $[G, G] \cong G_1$ as an extension. Such a $G$ can be described by taking a subgroup $H_1$ of the finite group $Z(G_1)$ together with a torus $S$ and an injective homomorphism $a : H_1 \to S$; one then defines

$$G = (G_1 \times S)/H$$

(50)

where $H \subset H_1 \times S$ is the graph of $a$. A given bundle $\tilde{P}$ with structure group $\tilde{G} = \tilde{G}_1$ lifts to a $G$-bundle $P$ if and only if its characteristic class

$$\tilde{c}(\tilde{P}) \in H^2(X; \pi_1(\tilde{G}_1))$$

lifts to a class

$$c \in H^2(X; \pi_1(G)) .$$

The image of the map $\pi_1(G) \to \pi_1(\tilde{G}_1) \cong Z(G_1)$ is our chosen subgroup $H_1$ and $\pi_1(G)$ is torsion-free; so the bundle $\tilde{P}$ has a lift to a $G$-bundle if and only if $\tilde{c}(\tilde{P})$ lies in the subgroup $H^2(X; H_1)$ and admits an “integer lift” to $\mathbb{Z}^k$ for one (and hence any) presentation of $H_1$ as

$$0 \to \mathbb{Z}^k \to \mathbb{Z}^k \to H_1 \to 0 .$$

This discussion shows us that, in order to allow the largest possible collection of $\tilde{G}_1$-bundles to lift, and to avoid redundancy, we may impose the following conditions.

**Condition 2.18.** In the construction of the non-semi-simple group $G$ from $G_1$ in (50), we may require

(i) the subgroup $H_1 \subset Z(G_1)$ is the whole of $Z(G_1)$;

(ii) the rank $k$ of the torus $S$ is chosen to be equal to the number of generators in a smallest possible generating set of $H_1$; or equivalently, the image of $H_1$ in $S$ is not contained in any proper sub-torus.
These conditions mean that if \( G_1 \) is simple, then \( G = G_1 \) in the case of type \( E_8, F_4 \) or \( G_2 \) (the simply connected cases); while \( S \) will be a circle group in all other cases except \( D_{2r} \), where \( S \) will be a 2-torus. These conditions do not determine \( G \) uniquely, in general. For example, in the case that \( G_1 = SU(N) \), the condition allows that \( G \) is \( U(N) \) but also allows \( G \) to be \( U(N)/C_m \), where \( C_m \) is a cyclic central subgroup of order prime to \( N \).

**Singular instantons.** We fix a maximal torus and a set of positive roots for \( G \). Because \( G \) is not semi-simple, the fundamental Weyl chamber in \( t = \text{Lie}(T) \) is the product of the fundamental Weyl chamber for \([g, g] \) with \( \mathfrak{z}(G) = \text{Lie}(Z(G)) \). The bundle \( P \) is no longer trivial on the 3-skeleton of \( X \), because \( \pi_1(G) \) is non-trivial. We can identify \( \pi_1(G) \) with the lattice

\[
L(G) \subset \mathfrak{z}(G)
\]

obtained as the projection of the integer lattice in \( t \). The bundle \( P \) has a 2-dimensional characteristic class which we write as

\[
c(P) \in H^2(X; L(G)). \tag{51}
\]

Now we introduce the codimension-2 singularity along a surface \( \Sigma \subset X \). Fix an element \( \Phi \) in the Lie algebra \( \mathfrak{g} \) belonging to the fundamental Weyl chamber and satisfying \( \theta(\Phi) < 1 \), where \( \theta \) is the highest root. Let \( O \subset \mathfrak{g} \) be the orbit of \( \Phi \) under the adjoint action. This lies in a translate of \([\mathfrak{g}, \mathfrak{g}] \) inside \( \mathfrak{g} \) consisting of all elements with the same trace as \( \Phi \). Choose a section \( \varphi \) of the associated bundle \( O_P|_\Sigma \), so defining a reduction of the structure group of \( P|_\Sigma \) to the subgroup \( G_\Phi \).

Let \( \Theta \) again be a fixed connection in the associated bundle \( \mathfrak{d}(P) \) on \( X \), and let \( A^0 \) be a \( G \)-connection with \( \mathfrak{d}(A^0) = \Theta \). Choose an extension of the section \( \varphi \) to the tubular neighborhood, and define a singular \( G \) connection on the restriction of \( P \) to \( X \setminus \Sigma \) by the same formula as in the previous case:

\[
A^\varphi = A^0 + \beta(r) \varphi \otimes \eta \tag{52}
\]

The induced connection on \( \mathfrak{d}(P) \) is

\[
\Theta^\varphi = \Theta + \beta(r) \mathfrak{d}(\varphi) \otimes \eta. \tag{53}
\]

We define a space of \( G \)-connections modeled on \( A^\varphi \) and having the same induced connection on \( \mathfrak{d}(P) \):

\[
C^p(X, \Sigma, P, \varphi) = \{ A^\varphi + a \mid a, \nabla_{A^\varphi} a \in L^p(X \setminus \Sigma; [\mathfrak{g}, \mathfrak{g}] \otimes \Lambda^1(X)) \}. 
\]
Our gauge group will consist of $[G, G]$-valued automorphisms of $P_{X \setminus \Sigma}$:

$$\mathcal{G}^p(X, \Sigma, P, \varphi) = \{ g \mid \nabla_{A^\varphi} g, \nabla_{A^\varphi}^2 g \in L^p(X \setminus \Sigma; [G, G]_P) \}.$$ 

For $p$ sufficiently close to 2, as in (17), the moduli space $M(X, \Sigma, P, \varphi)$ is defined as the quotient by $\mathcal{G}^p$ of the set of solutions to $F_A^+ = 0$ in $C^p(X, \Sigma, P, \varphi)$.

**Monopole charges.** We continue to write $Z(G_{\Phi})$ for the center of the commutant $G_{\Phi}$, and $L(G_{\Phi}) \subset \mathfrak{z}(G_{\Phi})$ for the image of the integer lattice in $\mathfrak{t}$ under the projection $\Pi : \mathfrak{t} \to \mathfrak{z}(G_{\Phi})$. The structure group of the bundle $P_{\varphi} \to \Sigma$ can still be reduced to the subgroup $T$, and the resulting $T$-bundle is classified by an element $\xi$ in the integer lattice. The element $l = \Pi(\xi)$ determines $P_{\varphi}$ up to isomorphism. The bundle $P$ itself may be non-trivial on $\Sigma$, and $l$ is constrained by the requirement that

$$\mathfrak{d}(l) = \langle c(P), [\Sigma] \rangle$$

in $L(G) \cong \pi_1(G)$.

**Dimension and energy.** The formula for the formal dimension of the moduli space can be written in the same way as before (see (19)), except for the term involving the 4-dimensional characteristic class:

$$-2p_1(g_P)[X] + 4\rho(l) + \frac{(\dim O)}{2} \chi(\Sigma) - (\dim G)(b^+ - b^1 + 1)$$

(54)

Note that the term $\rho(l)$ depends only on the projection of $l$ into $[\mathfrak{g}, \mathfrak{g}]$. The term involving $p_1(g_P)$ satisfies a congruence depending on the associated $Z(G)$-bundle $\mathfrak{d}(P)$, via the 2-dimensional characteristic class $c(P)$. In the case that the structure group of $P$ reduces to the maximal torus, we obtain a lift of $c(P)$ to a class $\hat{c} \in H^2(X; L(T))$, and we then have

$$-2p_1(g_P) = -\langle \hat{c}, \hat{c} \rangle$$

where the quadratic form on the right is defined using the semi-definite Killing form on the lattice $L(T)$ and the cup-square on $X$. Modulo $4\hat{h}^\vee$, the quantity on the right depends only on the image of $c(P)$ in the finite group $H^2(X; \pi(G))$. Furthermore, in the case that $[G, G]$ is simple, if $P$ is altered on a 4-cell in $X$, then $p_1(g_P)$ changes by a multiple of $2\hat{h}^\vee$ (as in the case of the 4-sphere); and since a general $P$ can be reduced to the maximal torus on the complement of a 4-cell, we have in general

$$-2p_1(g_P)[X] = -\langle \hat{c}, \hat{c} \rangle[X] \pmod{4\hat{h}^\vee}$$
where \( \hat{c} \) is any lift of \( c(P) \) to \( H^2(X; L(T)) \). In the case of the unitary group \( U(N) \), we identify \( c(P) \) as the first Chern class, and the formula becomes

\[
-2p_1(g_P)[X] = -2(N - 1)c_1^2(P)[X] \pmod{4N}.
\]

The appropriate definition of the energy \( E \) is still the formula (20), with the understanding that the norm defined by the Killing form is now only a semi-norm, so that the formula for the energy actually involves only \( \bar{A} \). In a similar manner, the formula (22) now becomes

\[
E = 8\pi^2 \left( -2p_1(g_P)[X] + 2\langle \Phi, l \rangle - \langle \Phi, \Phi \rangle (\Sigma \cdot \Sigma) \right),
\]

where the inner products are now only semi-definite.

In these two formulae, as elsewhere, the component of \( \Phi \) in the center of \( g \) is immaterial. The monotone condition (i.e. the condition that the terms in (54) and (56) which are linear in \( p_1(g) \) and \( l \) are proportional) is a constraint only on the component \( \bar{\Phi} \) which lies in \( [g, g] \). In formulae, the monotone condition still amounts to requiring that

\[
\langle \Phi, l \rangle = 2\rho(l)
\]

for all \( l \) in \( \mathfrak{z}(G_\Phi) \). Given any \( \Phi_0 \) in the fundamental Weyl chamber, there is a unique \( \Phi \) satisfying this condition with the additional constraints that (i) \( \Phi_0 \) and \( \Phi \) have the same centralizer and (ii) \( \Phi_0 \) and \( \Phi \) have the same central component in \( \mathfrak{z}(G) \).

**Isomorphic moduli spaces.** For the following discussion, we return temporarily to considering the moduli spaces \( M(X, P) \) of non-singular connections, in the absence of the embedded surface \( \Sigma \). If \( P \) and \( P' \) are isomorphic \( G \)-bundles, then the moduli spaces \( M(X, P) \) and \( M(X, P') \) are certainly homeomorphic also; but a particular identification \( M(X, P) \rightarrow M(X, P') \) depends on a choice of bundle isomorphism \( f : P \rightarrow P' \). Because we are dividing out by the action of the gauge group \( \mathcal{G}(X, P) \) consisting of all \( [G, G] \)-valued automorphisms, the map of moduli spaces \( M(X, P) \rightarrow M(X, P') \) depends on \( f \) only through the corresponding isomorphism of \( \check{Z}(G) \)-bundles, \( \bar{\sigma}(f) : \bar{\sigma}(P) \rightarrow \bar{\sigma}(P') \).

A convenient viewpoint on this is to fix a principal \( \check{Z}(G) \)-bundle \( \delta \) on \( X \), together with a connection \( \Theta \) on \( \delta \), and then regard \( \mathcal{B} \) as parametrizing isomorphism classes of triples consisting of:

(i) a principal \( G \)-bundle \( P \rightarrow X \) with specified instanton charges \( k = (k_1, \ldots, k_m) \);
(ii) an isomorphism of $G$-bundles $q : \mathfrak{d}(P) \to \delta$;

(iii) a connection $A$ in $P$ with $\mathfrak{d}(A) = q^*(\Theta)$, or equivalently just a connection $\bar{A}$ in $\bar{P}$.

Two such triples $(P, q, \bar{A})$ and $(P', q', \bar{A}')$ are isomorphic if there is an isomorphism of $G$-bundles, $f : P \to P'$, with $f^*(\bar{A}') = \bar{A}$ and $q = q' \circ \mathfrak{d}(f)$. From this point of view, it is natural to write the configuration space as $B_k^\delta$, indicating its dependence on $k$ and the $\bar{Z}(G)$-bundle. The corresponding moduli space can be written

$$M_k(X)^\delta \subset B_k(X)^\delta$$

It is clear that the automorphisms $g : \delta \to \delta$ act on $B_k(X)^\delta$, preserving the moduli space, by $(P, q, \bar{A}) \mapsto (P, g \circ q, \bar{A})$. Furthermore the action of $g$ on $B_k(X)^\delta$ is trivial if and only if $g = \mathfrak{d}(f)$ for some bundle isomorphism $f : P \to P$ with $f^*(\bar{A}) = \bar{A}$ for all $A$. This last condition on $f$ requires that $f$ take values in $Z(G)$. Thus the group which acts effectively is the quotient of $\text{Map}(X, \bar{Z}(G))$ by the image of $\text{Map}(X, Z(G))$ under the map $\mathfrak{d} : Z(G) \to \bar{Z}(G)$. This is a finite group. For example, in the case of $U(N)$, this is the quotient of $H^1(X; \mathbb{Z})$ by the image of multiplication by $N$; this is isomorphic to the subgroup of $H^1(X; \mathbb{Z}/N)$ consisting of elements with an integer lift.

There is another way in which isomorphisms arise between moduli spaces of this sort. The group operation provides a homomorphism of groups,

$$G \times Z(G) \to G.$$  

Given a $G$-bundle $P$ and a $Z(G)$-bundle $\epsilon$ on $X$, we can use this homomorphism to obtain a “product” $G$-bundle, which we will denote by $P \otimes \epsilon$. If we fix a connection $\omega$ in $\epsilon$, then to each connection $A$ in $\mathcal{C}(X, P)$ with $\mathfrak{d}(A) = \Theta$, we can associate a connection

$$A' = A + \omega$$

in $\mathcal{C}(X, P \otimes \epsilon)$ with $\mathfrak{d}(A') = \Theta + \mathfrak{d}(\omega)$. This operation descends to the quotient space $\mathcal{B}$ and preserves the locus of connections which satisfy the equations $F^+_A = 0$. It therefore gives an identification of moduli spaces

$$\mu_\epsilon : M(X, P) \to M(X, P \otimes \epsilon).$$

In terms of the data $k$ and $\delta$ which determine the moduli space up to isomorphism, this is a map

$$\mu_\epsilon : M_k(X)^\delta \to M_k(X)^{\delta \otimes \mathfrak{d}(\epsilon)},$$
where we have extended the use of $\otimes$ to denote also the product of two $\hat{Z}(G)$-bundles. (In the case of $U(N)$, the operation $\otimes$ in both cases becomes the tensor product by a line bundle.)

All of this discussion can be carried over to the case of connections with singularities along $\Sigma \subset X$. Once $\Phi$ is given, the moduli space $M(X, \Sigma, P, \varphi)$ is determined up to isomorphism by $\delta = \mathfrak{d}(P)$ together with the instanton charges $k$ and monopole charges $l$. (In the case that $\Sigma$ has more than one component, the monopole charges need to be specified for each component.) We can therefore write the moduli space as $M_{k,l}(X, \Phi)$. The automorphisms of $\delta$ then act on the moduli spaces, and it is again the quotient of $\text{Map}(X, \hat{Z}(G))$ by the image of $\text{Map}(X, Z(G))$ that acts effectively. If $\epsilon$ is a $Z(G)$-bundle, we also have a corresponding isomorphism

$$\mu_\epsilon : M_{k,l}(X, \Phi)_{\delta} \rightarrow M_{k,l}(X, \Phi)_{\delta \otimes \mathfrak{d}(\epsilon)}.$$ 

Reducibles. The main point at which the present discussion of non-semi-simple groups diverges from the previous case is in the discussion of reducible connections, because the characteristic class $c(P)$ now plays a role and because the integer lattice is no longer generated by the coroots. Let us write $L(T)$ for the integer lattice of the maximal torus; the lattice of weights is the dual lattice in $t^*$. The simple coroots $\alpha^\vee$ are a basis for $L(T) \cap [g, g]$, because $[G, G]$ is simply connected. Let $\bar{w}_\alpha$ denote the dual basis for $(t \cap [g, g])^*$. For each simple root $\alpha$, we can choose a weight $w_\alpha$ in $t^*$ such that the restriction of $w_\alpha$ to $t \cap [g, g]$ is $\bar{w}_\alpha$. The choice of $w_\alpha$ is uniquely determined by $\bar{w}_\alpha$ to within the addition of a weight that factors through $\mathfrak{d}$. Associated to $\alpha$, as before, is a subgroup $G(\alpha) \subset G$, the centralizer of $w_\alpha^\dagger$ (or equivalently of $\bar{w}_\alpha^\dagger$). Note that we will not always be able to choose $w_\alpha$ in such a way that its restriction to $\mathfrak{z}(G)$ is zero, and nor will its restriction to the lattice $L(G) \subset \mathfrak{z}(G)$ be integral: it will define a map

$$w_\alpha : L(G) \rightarrow \mathbb{Q}.$$ 

In the case of $U(2)$ for example, $w_\alpha$ will map the rank-1 lattice $L(G)$ onto $\frac{1}{2} \mathbb{Z}$.

To say that $[A]$ is reducible still means that its stabilizer in the gauge group $G^p(X, \Sigma, P, \varphi)$ has positive dimension, and this is equivalent to there being a non-zero covariant-constant section $\psi$ of the associated bundle $[g, g]_P \subset g_P$. As in subsection 2.6, we obtain a reduction of the structure group of $P$ to a subgroup $G_\psi$, and we write $P_\psi \subset P$ for this $G_\psi$-bundle. For some element $\sigma$ in the Weyl group, $G_{\sigma(\psi)}$ is contained in $G(\alpha)$ for some
simple root $\alpha$, and it follows that $w_\alpha \circ \sigma$ defines a non-central character of $G_\Psi$:

$$s : G_\Psi \to U(1).$$

Applying $s$ to the connection $A$ gives a $U(1)$ connection $s(A)$ whose curvature $F_{s(A)}$ is an $L^p$ form defines a de Rham class

$$\frac{i}{2\pi}[F_{s(A)}] = c_1(s(P_\psi)) - \sum_{j=1}^r (w_\alpha \circ \sigma_j)(\Phi)P.D.[\Sigma_j] \quad (58)$$

where the $\Sigma_j$ are the components of $\Sigma$, just as at (36). Unlike the previous case, the curvature form $F_{s(A)}$ is not anti-self-dual, because $F^+_A$ has a non-zero central component. The Chern-Weil formula for the central component is

$$\frac{i}{2\pi}[F_A] = c(P) - \sum_{j=1}^r \partial(\Phi)P.D.[\Sigma_j]$$

as an equality of $\mathfrak{g}(G)$-valued cohomology classes. The self-dual part of $F_A$ coincides with the self-dual part of $F_{\partial(A)}$. So, applying $w_\alpha \circ \sigma$ to this last formula and then subtracting it from (58), we learn that the class

$$c_1(s(P_\psi)) - w_\alpha(c(P)) - \sum_{j=1}^r (w_\alpha \circ \sigma_j)(\Phi - \partial(\Phi))P.D.[\Sigma_j]$$

is represented by an anti-self-dual form on $X$. The first term in this last formula is an integral class. The second term may not be integral: the class $c(P)$ takes values in $L(G)$, and $w_\alpha$ need not take integer values on this lattice. The terms in the last sum depend only on $\bar{w}_\alpha$, not on $w_\alpha$, because $\Phi - \partial(\Phi)$ lies in $[\mathfrak{g}, \mathfrak{g}]$. This leads to the following variant of Proposition 2.6.

**Proposition 2.19.** Suppose that $b^+(X) \geq 1$, and let the components of $\Sigma$ be $\Sigma_1, \ldots, \Sigma_r$. Write

$$\bar{\Phi} = \Phi - \partial(\Phi)$$

for the component of $\Phi$ in $[\mathfrak{g}, \mathfrak{g}]$. Suppose that for every fundamental weight $w_\alpha$ and every choice of elements $\sigma_1, \ldots, \sigma_r$ in the Weyl group, the real cohomology class

$$w_\alpha(c(P)) + \sum_{j=1}^r (\bar{w}_\alpha \circ \sigma_j)(\bar{\Phi})P.D.[\Sigma_j]$$

is not integral. Then for generic choice of Riemannian metric on $X$, there are no reducible solutions in the moduli space $M(X, \Sigma, P, \varphi)$. 
Note that, as a rational cohomology class, \( w_\alpha(c(P)) \) does depend on \( w_\alpha \), not just on \( \bar{w}_\alpha \). But different choices of how to extend \( \bar{w}_\alpha \) will be reflected in a change to \( w_\alpha(c(P)) \) by an integral class.

In the case that each component \( \Sigma_j \) is null-homologous, the criterion in the above proposition reduces to the requirement:

\[ w_\alpha(c(P)) \text{ be non-integral for each simple root } \alpha. \]  

(59)

To illustrate this, consider the familiar case of the unitary group \( U(N) \). There are \((N - 1)\) simple roots, \( \alpha_k, k = 1, \ldots, N - 1 \), and if we write a typical Lie algebra element in \( \mathfrak{u}(N) \) as

\[ x = i \text{ diag}(\epsilon_1, \ldots, \epsilon_N), \]

then we can choose \( w_{\alpha_k} \) so that

\[ w_{\alpha_k}(x) = \epsilon_1 + \cdots + \epsilon_k. \]

Then \( c(P) \) is related to the usual first Chern class \( c_1(P) \) in such a way that

\[ w_{\alpha_k}(c(P)) = (k/N)c_1(P). \]

So the criterion in the proposition is that the rational class \((k/N)c_1(P)\) should not be integral, for any \( k \) in the range \( 1 \leq k \leq N - 1 \). This is equivalent to requiring that the evaluation of \( c_1(P) \) on some integral homology class should be coprime to \( N \). We record this corollary, which is familiar from the case of non-singular instantons:

**Corollary 2.20.** Let \( G \) be the unitary group \( U(N) \). Suppose that \( b^+(X) \geq 1 \), and that all components of \( \Sigma \) are null-homologous. If there is an integral homology class in \( X \) whose pairing with \( c_1(P) \) is coprime to \( N \), then for generic choice of Riemannian metric on \( X \), there are no reducible solutions in the moduli space \( M(X, \Sigma, P, \varphi) \).

**Remark.** One might hope that there would be something analogous to this corollary in the case of other simply-connected groups with non-trivial center, but unfortunately, the case of the unitary group is rather special. If the commutator subgroup \( G_1 = [G, G] \) is a simply-connected simple group of any type other than \( A_r \), there will always be a fundamental weight \( w_\alpha \) which takes integer values on \( L(G) \); so the criterion (59) cannot be met. This phenomenon is noted in [35, Proposition 7.8].
**Orientations.** The discussion of orientations from section 2.9 adapts readily to the case of non-simple groups $G$ with $[G, G]$ simply-connected. Recall that Proposition 2.17 has two parts: the first is the assertion that the moduli spaces are orientable, and the second involves specifying a canonical orientation. Adapting the first part is routine. For the second task, we need to observe that the restriction of the homomorphism $(49)$ to the maximal torus $T ⊂ G$ admits a right-inverse,

$$\epsilon : \tilde{Z}(G) \to T$$

$$d \circ \epsilon = 1.$$  \hspace{1cm} (60)

Fix such an $\epsilon$ once and for all. From our chosen $\tilde{Z}(G)$-connection $\Theta$ in $\mathfrak{d}(P)$ we now obtain a $G$-connection $\epsilon(\Theta)$ on a bundle isomorphic to $P$, with $\mathfrak{d}(\epsilon(\Theta)) = \Theta$. This comes with a reduction of structure group (on the whole of $X$) to the maximal torus, and a fortiori to $G_{\Phi}$. Adding the singular term along $\Sigma$ in the usual way, we obtain a distinguished singular connection $A^\varphi$. Because $A^\varphi$ respects a reduction to the maximal torus, the adjoint bundle with fiber $[\mathfrak{g}, \mathfrak{g}]$ decomposes as a direct sum of a bundle with fiber $\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$ and a complex vector bundle, and there is a corresponding decomposition of the operator $(46)$ whose determinant line we wish to orient. The induced connection on the first summand is trivial. As in the previous argument, we can now proceed by making use of the complex orientation on the second summand and the homology orientation $o_W$ for the first summand. In this way, the moduli spaces $M(X, \Sigma, P, \varphi)$ become canonically oriented at all regular points.

Let us write $\delta$ again for the $\tilde{Z}(G)$-bundle $\mathfrak{d}(P)$, and so denote the moduli space by $M_{k,l}(X, \Phi)_{\delta}$ as above. Recall that in this setting, the automorphisms of $\delta$ act on the moduli space. The naturality of the construction of the orientation means that the automorphisms of this moduli space arising in this way are orientation-preserving diffeomorphisms on the regular part. A more interesting question arises if we ask whether the map

$$\mu_\epsilon : M_{k,l}(X, \Phi)_{\delta} \to M_{k,l}(X, \Phi)_{\delta \otimes \mathfrak{d}(\epsilon)}$$  \hspace{1cm} (61)

preserves orientation. The singularity along $\Sigma$ plays no role in the answer here, and we could equally well consider $M_k(X)_{\delta}$ instead. This is a question which was treated for $G = U(2)$ in [7], and the argument was adapted for $U(N)$ in [20]. The case of a general non-semisimple group is little different, and we shall summarize the results.

We shall write $G_1 = [G, G]$ again, and we shall suppose that $Z(G_1)$ is non-trivial (for otherwise $G$ is simply a product). We will also require that
$G_1$ is simple, and that $G$ is obtained from $G_1$ by the construction (50) and that Condition 2.18 holds. Then we have the following result:

**Proposition 2.21.** Under the above assumptions, the map $\mu_\epsilon$ in (61) is orientation preserving if the simple group $G_1$ is any group other than $E_6$ or $SU(N)$. In the case of $E_6$, the result depends on the choice of $\epsilon$; but $e : U(1) \to E_6$ may be chosen so that $\mu_\epsilon$ is orientation-preserving for all $\epsilon$ and $\delta$. In the case of $SU(N)$, if $G$ is chosen to the standard $U(N)$ and $\epsilon : U(1) \to U(N)$ is the inclusion in the first factor of the torus $U(1)^N$ in $U(N)$, then $\mu_\epsilon$ is orientation-preserving if $N$ is 0, 1 or 3 mod 4. If $N$ is 2 mod 4 then $\mu_\epsilon$ is orientation-preserving if and only if $c_1(\epsilon)^2[X]$ is even.

**Proof.** Both $e(\delta)$ and $e(\delta')$ are $T$-bundles on $X$. Let $a$ and $b$ be their respective characteristic classes in $H^2(X; L(T))$. The calculation for $U(N)$ from [20] adapts with little change to show that $\mu_\epsilon$ preserves or reverses orientation according to the parity of the quantity,

$$\left( \frac{1}{2} (a \sim b) + \frac{1}{4} (b \sim b) + \rho(b) \sim \rho(b) \right) [X]$$

in which $(a \sim b)$ denotes the pairing in $H^4(X; \mathbb{Z})$ obtained from the semi-definite Killing form on $L(T)$ and the cup product on $X$, and $\rho(b)$ is to be interpreted as an element of $H^2(X; \mathbb{Z})$. The quantity above plainly depends only on the images of $a$ and $b$ under the projection to $[g, g]$. Let us write $\bar{a}$, $\bar{b}$ for these projections (with the torsion parts of the cohomology dropped). We have

$$\bar{a} \in H^2(X; L(\bar{T}_1))/\text{torsion} \subset H^2(X; t_1)$$

$$\bar{b} \in H^2(X; L(\bar{T}_1))/\text{torsion} \subset H^2(X; t_1)$$

where $L(T_1)$ is the integer lattice for the maximal torus $T_1$ in the simply-connected group $G_1 = [G, G]$ and $L(\bar{T}_1)$ is the integer lattice for $T_1/Z(G_1)$ (the maximal torus of the adjoint form of $G_1$). Let $\langle -, - \rangle_2$ denote the inner product on $t_1 = \text{Lie}(T_1)$ normalized so that the coroots $\alpha^\vee$ corresponding to the long roots $\alpha$ for $G_1$ have length 2. We then have

$$\langle x, y \rangle = 2h^\vee(x, y)_2$$

where $h^\vee$ is the dual Coxeter number of $G_1$, so the formula (62) can be rephrased as

$$\left( h^\vee (\bar{a} \sim \bar{b})_2 + \frac{h^\vee}{2} (\bar{b} \sim \bar{b})_2 + \rho(\bar{b}) \sim \rho(\bar{b}) \right) [X]$$

(64)
The pairing \(\langle -, - \rangle_2\) gives a map
\[
L(\bar{T}_1) \times L(T_1) \to \mathbb{Z}
\]
and its restriction to the coarser lattice \(L(T_1)\) is an even form. The Weyl vector \(\rho\) takes integer values on \(L(T_1)\), so each of the three terms in (64) is an integer.

Let \(p\) be the least common multiple of the orders of the elements of \(Z(G_1)\): so \(p\) is 2 in the case of \(B_n, C_n, D_{2n}\) and \(E_7\), is 4 in the case of \(D_{2n+1}\), is 3 in the case of \(E_6\) and is \(n + 1\) in the case of \(A_n\). Condition 2.18 ensures that the map
\[
\vartheta : Z(G) \to \bar{Z}(G)
\]
has the property
\[
\vartheta_* (\pi_1(Z(G))) = p \cdot \pi_1(\bar{Z}(G)).
\]
This condition tells us that \(\bar{b}\) lies in
\[
p \cdot H^2(X; L(\bar{T}_1))/\ torsion \subset H^2(X; L(T_1))/\ torsion.
\]
We write
\[
\bar{b} = p\bar{e}
\]
\[
\bar{e} \in H^2(X; L(\bar{T}_1))/\ torsion.
\]
We also exclude the \(A_n\) case from our discussion, because the result of the Proposition for \(SU(N)\) is contained in \([20]\). We examine the parity of the three terms in (64), beginning with the first term, the quantity
\[
h^\vee \langle \bar{a} \sim \bar{b} \rangle_2 [X].
\]
This is even if \(h^\vee\) is even, and the remaining cases to look at (with the above exclusions in mind) are \(B_n\) for any \(n\) and \(C_n\) for \(n\) even. In both these cases, the pairing \(\langle -, - \rangle_2\) takes only even values on \(L(\bar{T}_1) \times L(T_1)\), so in all these cases this term is even. The second term in (64) is also even if \(h^\vee\) is even. If \(h^\vee\) is odd, then \(p\) is even and the term can be expressed as
\[
(h^\vee p/2) \langle \bar{e} \sim \bar{b} \rangle_2 [X].
\]
As with the first term, we are dealing with \(B_n\) or \(C_{2m}\), and the pairing is even by the same mechanism. The third term can be written
\[
p^2 \rho(\bar{e}) \sim \rho(\bar{e})[X].
\]
The case \(A_n\) has been excluded, and in all other cases \(\rho\) takes integer values on \(L(\bar{T}_1)\). So \(\rho(\bar{e})\) is an integral class. If \(p\) is even, then this term is therefore
even. The only case where \( p \) is odd is the case of \( E_6 \). In the \( E_6 \) case, one can check that there exists a coset representative \( \bar{e}_1 \) for a generator of \( L(\bar{T}_1)/L(T_1) \cong \mathbb{Z}/3 \) such that \( \rho(\bar{e}_1) \) is an even integer. We can choose \( \epsilon \) so that its image is spanned by this representative, and with such a choice this third term is again even.

\[ \square \]

3 Instanton Floer homology for knots

3.1 Configuration spaces and flat connections

Let \( Y \) be a closed, connected, oriented 3-manifold, and let \( K \subset Y \) be an oriented knot or link. Take a simple, simply-connected Lie group \( G \), and let \( P \to Y \) be a principal \( G \)-bundle (necessarily trivial). Fix a maximal torus and a set of positive roots as before, and choose \( \Phi \) in the fundamental Weyl chamber, satisfying the constraint \((9)\). Let \( O \subset \mathfrak{g} \) be its orbit, and let \( \varphi \) be a section of the associated bundle \( O_P \) along \( K \), defining a reduction of the structure group of \( P|_K \) to the subgroup \( G_\Phi \). Any two choices of section \( \varphi \) are homotopic: the only topological data is in the pair \((Y,K)\) and the choice of \( G \) and \( \Phi \).

We will equip \( Y \) with a Riemannian metric \( g^\nu \) that is singular along \( K \), as we did in dimension 4 in subsection 2.8 above: the cone-angle will be \( 2\pi/\nu \). To ensure sufficient regularity, we let \( I \) denote a compact interval in \((0,1)\) containing \( \alpha(\Phi) \) for all roots \( \alpha \) in \( R^+(\Phi) \), and take \( \nu \) to be at least as large as the integer \( \nu_0(I,2,m) \) supplied by Proposition 2.16, with \( m \) a chosen Sobolev exponent not less than 3. (We will impose further restrictions on \( \nu \) shortly.) We construct a model singular connection \( B^\varphi \) on the restriction of \( P \) to \( Y\setminus K \), just as in the 4-dimensional case (see \((13)\)), and we introduce the space of connections

\[
\mathcal{C}(Y,K,\Phi) = \{ B \mid B - B^\varphi \in \tilde{L}^2_{m,B^\varphi} \}.
\]

Here \( \tilde{L}^2_{k,B^\varphi} \) denote the 3-dimensional Sobolev spaces defined just as in subsection 2.8. Because they are trivial, we omit \( P \) and \( \varphi \) from our notation. There is a gauge group

\[
\mathcal{G}(Y,K,\Phi) = \{ g \mid g \in \tilde{L}^2_{m+1,B^\varphi} \}
\]

and a quotient space

\[
\mathcal{B}(Y,K,\Phi) = \mathcal{C}(Y,K,\Phi)/\mathcal{G}(Y,K,\Phi).
\]

Two connections \( B \) and \( B' \) belonging to \( \mathcal{C}(Y,K,\Phi) \) are gauge-equivalent as \( G \)-connections on \( Y\setminus K \) if and only if they differ by the action of an element
of $G(Y, K, \Phi)$. As in the 4-dimensional case, we call a connection $B$ reducible if its stabilizer has positive dimension.

The space of connections $\mathcal{C}(Y, K, \Phi)$ is an affine space, and on the tangent space $T_B \mathcal{C}$ we define an $L^2$ inner product (independent of $B$) by

$$
\langle b, b' \rangle_{L^2} = \int_Y -\text{tr}(\text{ad}(b) \wedge \text{ad}(b')),
$$

(65)

Thus we are using the Killing form to contract the Lie algebra indices, and the Hodge star on $Y$ and the wedge product to contract the form indices. The Hodge star is the one defined by the singular metric $g^\nu$. We define the Chern-Simons functional on $\mathcal{C}(Y, P, \Phi)$ to be the unique function $\text{CS}: \mathcal{C}(Y, K, \Phi) \to \mathbb{R}$ satisfying $\text{CS}(B^\varphi) = 0$ and having formal gradient (with respect to the above inner product)

$$(\text{grad} \text{CS})_B = \ast F_B.$$ 

From this characterization, one can derive as usual the formula

$$
\text{CS}(B^\varphi + b) = \langle \ast F_{B^\varphi}, b \rangle_{L^2} + \frac{1}{2} \langle \ast d_{B^\varphi} b, b \rangle_{L^2} + \frac{1}{3} \langle \ast [b \wedge b], b \rangle_{L^2}.
$$

(66)

The Chern-Simons functional is independent of the choice of Riemannian metric on $Y$, as can be seen by rewriting this formula using (65).

The homotopy type of $G(Y, K, \Phi)$ is that of the space of maps $g: Y \to G$ with $g(K) \subset G_\Phi$. It follows that the the space of components of the gauge group is

$$
\pi_3(G) \times [K, G_\Phi]
$$

which is isomorphic to

$$
\mathbb{Z} \oplus L(G_\Phi)^r
$$

(67)

where $r$ is the number of components of the link $K$ and $L(G_\Phi) \subset \mathfrak{z}(G_\Phi)$ is the lattice of Definition 2.2. In particular, there is a preferred homomorphism

$$
d: G(Y, K, \Phi) \to \mathbb{Z} \oplus L(G_\Phi),
$$

(68)

where the map to the second factor is obtained by taking the sum over all components of $K$. An alternative way to think of $d$ is to use a gauge transformation $g$ in $G(Y, K, \Phi)$ to form the bundle $S^1 \times g P$ over $S^1 \times Y$, together with its reduction $S^1 \times g \varphi$ over $S^1 \times K$, defined by $\varphi$. This data over $(S^1 \times Y, S^1 \times \Sigma)$ has an instanton number $k$ and monopole charge, as
in the previous section (see Definition 2.2 in particular). Then \(d(g)\) can be computed as \((k, l)\).

The Chern-Simons functional is invariant only under the identity component of the gauge group. To express this quantitatively, let \(B \in \mathcal{C}(Y, K, \Phi)\) be a connection, let \(g\) be a gauge transformation, and write \(d(g) = (k, l) \in \mathbb{Z} \times L(G_{\Phi})\). Then we have

\[
\text{CS}(B) - \text{CS}(g(B)) = 4\pi^2 (4h^*k + 2\langle \Phi, l \rangle).
\]

For a path \(\gamma : [0, 1] \to \mathcal{C}(Y, K, \Phi)\), we define the topological energy as twice the drop in the Chern-Simons functional; so we can reinterpret the last equation as saying that a path from \(B\) to \(g(B)\) has topological energy

\[
\mathcal{E} = 8\pi^2 (4h^*k + 2\langle \Phi, l \rangle).
\]

This is a formula which is familiar also for the energy of a solution on any closed-manifold pair \((X, \Sigma)\) with \(\Sigma \cdot \Sigma = 0\). For a path which formally solves the downward gradient-flow equation for the Chern-Simons functional on \(\mathcal{C}(Y, K, \Phi)\), the topological energy coincides with the modified path energy,

\[
\int_0^1 (\|\dot{\gamma}(t)\|^2 + \|\text{grad \text{CS}(\gamma(t))}\|^2) dt^2.
\]

From the definition of the Chern-Simons functional, it is apparent that critical points of CS are the flat connections in \(\mathcal{C}(Y, K, \Phi)\). The image of the critical points in the quotient space \(\mathcal{B}(Y, K, \Phi)\) can be identified with the quotient by the action of conjugation of the space of all homomorphisms \(\rho : \pi_1(Y \setminus K) \to G\) with the property that the holonomy around each positively-oriented meridian of \(K\) is conjugate to \(\exp(-2\pi \Phi)\). We shall write

\[
\mathcal{C} \subset \mathcal{B}(Y, K, \Phi)
\]

for this set of critical points.

Reducible critical points of the Chern-Simons functional can be ruled out on topological grounds, by the following criterion, whose proof follows the same line as the 4-dimensional version, Proposition 2.6.

**Proposition 3.1.** Let the components of \(K\) be \(K_1, \ldots, K_r\). Suppose that for every fundamental weight \(w_\alpha\) and every choice of elements \(\sigma_1, \ldots, \sigma_r\) in the Weyl group, the real cohomology class

\[
\sum_{j=1}^r (w_\alpha \circ \sigma_j)(\Phi) P.D.[K_j]
\]

(70)
is not integral. Then there are no reducible connections in the set of critical points $C \subset \mathcal{B}(Y, K, \Phi)$.

Because the criterion in this proposition is referred to a few times, we give it a name:

**Definition 3.2.** We will say that $(Y, K, \Phi)$ satisfies the *non-integral condition* if the expression (70) is a non-integral cohomology class for every choice of fundamental weight $w_\alpha$ and Weyl group elements $\sigma_1, \ldots, \sigma_r$.

### 3.2 Holonomy perturbations

We will perturb the Chern-Simons functional $CS$ by adding a term $f$: a real-valued function on $\mathcal{C}(Y)$ invariant under the action of the gauge group $\mathcal{G}(Y, K, \Phi)$. The type of perturbation that we use is essentially the same as that used in [13], though similar constructions appear in [41, 7] and elsewhere.

Let $q : S^1 \times D^2 \to Y \setminus K$ be a smooth immersion of a closed solid torus. Regard the circle $S^1$ as $\mathbb{R}/\mathbb{Z}$ and let $s$ be a corresponding periodic coordinate. Let $G_P \to Y$ be the bundle with fiber $G$ over $Y$ whose sections are the gauge transformations of $P$, and for each $z$ in $D^2$ let

$\text{Hol}_{q(-,z)}(B) \in (G_P)_{q(0,z)}$

be the holonomy of the connection $B$ around the corresponding loop based at $q(0,z)$. As $z$ varies, we obtain in this way a section $\text{Hol}_q(B)$ of the bundle $q^*(G_P)$ on the disk $D^2$.

Next suppose we have an $r$-tuple of maps,

$q = (q_1, \ldots, q_r),$

with $q_j : S^1 \times D^2 \to Y \setminus K$ an immersion. Suppose further that there is some interval $[-\eta, \eta]$ such that the restriction of $q_j$ to $[-\eta, \eta] \times D^2$ is independent of $j$:

$q_j(s, z) = q_j(s, z), \text{for all } s \text{ with } |s| \leq \eta. \quad (71)$

The pull-back bundles $q_j^*(G_P)$ are all canonically identified with each other on the subset $[-\eta, \eta] \times D^2$, and we can regard the holonomy maps as defining a section

$\text{Hol}_q(B) : D^2 \to q_1^*(G_P^r)$

of the $r$-fold fiber-product of the bundle $G_P$ pulled back to $D^2$. Pick any smooth function

$h : G_P^r \to \mathbb{R}$
that is invariant under the diagonal action of $G$ by the adjoint action on
the $r$ factors. Such an $h$ defines also a function on $q_1(G^r)$. Let $\mu$ be a
non-negative 2-form supported in the interior of $D^2$ and having integral 1,
and define
\[
f(B) = \int_{D^2} h(\text{Hol}_q(B))\mu.
\]
(72)
A function $f$ of this sort is invariant under the gauge group action.

**Definition 3.3.** A cylinder function on $\mathcal{C}(Y, K, \Phi)$ is a function
\[
f : \mathcal{C}(Y, K, \Phi) \rightarrow \mathbb{R}
\]
of the form (72), determined by an $r$-tuple of immersions as above and a
$G$-invariant function $h$ on $G^r$.

**Remark.** In the transversality arguments that arise later, the important
feature of the class of functions $f$ obtained in this way is that they separate
points in the quotient space $B(Y, K, \Phi)$, and also that they separate tangent
vectors at points where the gauge action is free. There is a slightly different
class of perturbations that one can use and which serves just as well in
the case that $G = SU(N)$: one can drop the requirement that $q_j = q_j'$ on
$[-\eta, \eta] \times D^2$, but instead put a more restrictive condition on $h$, namely that
it be invariant under the action of $G$ on each of the $r$ factors separately.
This alternative approach is laid out in detail in [8], where it is explained
that such functions do separate points of $B$ when $G = SU(N)$: the key point
is the following lemma.

**Lemma 3.4 ([8]).** Suppose $h_1, \ldots, h_m$ and $h_1', \ldots, h_m'$ are elements of
$SU(N)$ and suppose that for all words $W$, the elements $W(h_1, \ldots, h_m)$
and $W(h_1', \ldots, h_m')$ are conjugate. Then there is a $u \in SU(N)$ such that
$h_i' = uh_iu^{-1}$ for all $i$.

This lemma fails for other groups: this is essentially the observation of
Dynkin [11], that two homomorphisms $f_1, f_2 : H \rightarrow G$ between compact Lie
groups may be linearly equivalent without being equivalent, where linear
equivalence means that $\omega \circ f_1$ is equivalent to $\omega \circ f_2$ for all linear represen-
tations $\omega$ of $G$. The approach we have taken here is the one used in [13],
and works for any simple $G$.

We examine the formal gradient of such a cylinder function with respect
to our $L^2$ inner product on the tangent spaces of $\mathcal{C}(Y, K, \Phi)$. Let $\partial_j h$ be the
partial derivative of $h$ along the $j$'th factor: after trivializing the cotangent bundle of $G$ using left-translation, we may regard this as a map

$$\partial_j h : G^r \to g^*.$$ Using the Killing form, we can also construct the $g$-valued function $(\partial_j h)\dagger$. The $G$-invariance means that this also defines a map

$$(\partial_j h)\dagger : G^r_P \to g_P.$$ Let $H_j$ be the section of $q^*_j(g_P)$ on $D^2$ defined by

$$H_j = (\partial_j h)\dagger (\text{Hol}_q(B)).$$ We extend $H_j$ to a section of $q^*_j(g_P)$ on all of $S^1 \times D^2$ by using parallel transport along the curves $s \mapsto q(s, z)$: the resulting section $H_j$ has a discontinuity at $s = 0$ because the parallel transport around the closed loops may be non-trivial. The formal gradient of the cylinder function $f$, interpreted as a $g_P$-valued 1-form on $Y \setminus K$, is then given by

$$*\left(\sum_{j=1}^r (q_j)_*(H_j\mu)\right).$$ Note that on $[-\eta, \eta] \times D^2$ we can regard each $H_j$ as a section of the same bundle $q^*_1(g_P)$, and while each $H_j$ has a singularity at $s = 0$, the sum of the $H_j$'s does not, because of the $G$-invariance of $h$. The above 1-form is therefore continuous at $q_1(\{0\} \times D^2)$. Our connections are of class $L^2_m$ away from the link $K$, and as in [41, 20] a short calculation shows that the section defined by the holonomy is of the same class. So the $g_P$-valued 1-form (73) is indeed in $L^2_m$. It is also supported in a compact subset of $Y \setminus K$, so it defines a tangent vector to the space of connections $C(Y, K, \Phi)$. As an abbreviation, let us write $C_m$ for our space of connections modelled on $\tilde{L}^2_{m, B^r}$, and $T_m$ for its tangent bundle. For $k \leq m$ we have the bundle $T_k \to C_m$ obtained by completing the tangent bundle in the $L^2_k$ norm. We will write $V$ for the formal gradient of the cylinder function $f$. The following proposition details some of its properties.

**Proposition 3.5.** Let $f$ be a cylinder function and let $V$ be its formal gradient (73), regarded as a section of $T_m$ over $C(Y, K, \Phi) = C_m$. Then $V$ has the following properties:
(i) The formal gradient $V$ defines a smooth section, $V \in C^\infty(C_m, T_m)$.

(ii) For any $j \leq m$, the first derivative $DV \in C^\infty(C_m, \text{Hom}(T_m, T_m))$ extends to a smooth section

$$DV \in C^\infty(C_m, \text{Hom}(T_j, T_j)).$$

(iii) There is a constant $K$ such that $\|V(B)\|_{L^\infty} \leq K$ for all $B$.

(iv) For all $j$, there is a constant $K_j$ such that

$$\|DV(B)\|_{L^2_{j,B^\varphi}} \leq K_j (1 + \|B - B^\varphi\|_{L^2_{j,B^\varphi}})^j.$$

(v) There exists a constant $C$ such that for all $B$ and $B'$, and all $p$ with $1 \leq p \leq \infty$, we have

$$\|V(B) - V(B')\|_{L^p} \leq C\|B - B'\|_{L^p}.$$

In particular, $V$ is continuous in the $L^p$ topologies.

In order to work with a Banach space of perturbations $f$, we will consider functions $f : \mathcal{C}(Y, K, \Phi) \to \mathbb{R}$ which are obtained as the sum of a series

$$f = \sum_{i=1}^{\infty} a_if_i$$

where the $a_i$ are real and $\{f_i\}_{i \in \mathbb{N}}$ is some fixed countable collection of cylinder functions. We need to consider whether such a sum is convergent. More specifically, we would like the series of gradients $V_i = \text{grad } f_i$ to converge to a section

$$V = \sum_{i=1}^{\infty} a_iV_i$$

of the tangent bundle to $T_m$, and we would like the limit $V$ to share with $V_i$ the properties detailed in the above proposition. In the first part of this proposition, when we say that a section $V$ belongs to $C^\infty$, we do not imply that the norm of the $n$’th derivative $D^nV|_B$ is uniformly bounded independent of $B$. But the norm is bounded by a function of $\|B - B^\varphi\|_{L^2_{m,B^\varphi}}$; we have for each $n$ a continuous function $h_n(\cdot)$ such that

$$\|D^nV|_B(b_1, \ldots, b_n)\|_{L^2_{i,B^\varphi}} \leq h_n(\|B - B^\varphi\|_{L^2_{m,B^\varphi}}) \prod_{i=1}^{n} \|b_i\|_{L^2_{m,B^\varphi}}.$$
A similar remark applies to the derivatives of $DV$ in the norms that appear in the second part of the proposition. Because of this, given any countable collection of cylinder functions $\{f_i\}_{i \in \mathbb{N}}$, we can find constants $C_i$ such that the series (74) converges whenever the sum

$$\sum C_i |a_i|$$

converges, and such that the limit $V$ of the series is smooth.

Before proceeding further, we shall choose a suitable countable collection of cylinder functions $f_i$, sufficiently large to ensure that we can achieve transversality. For each integer $r > 0$, we choose a countable set of $r$-tuples of immersions $q : S^1 \times D^2 \to Y$,

$$(q^{r,j}_1, \ldots, q^{r,j}_r), \quad j \in \mathbb{N},$$

satisfying (71) which are dense in the $C^1$ topology on the space of such $r$-tuples of immersions. For each $r$, we also choose a collection of smooth $G$-invariant functions $\{h^r_k\}_{k \in \mathbb{N}}$ on $G^r$ which are dense in the $C^\infty$ topology. Finally we combine these to form a countable collection of cylinder functions

$$f_{j,k,r} = h^r_k(q^{r,j}_1, \ldots, q^{r,j}_r).$$

**Definition 3.6.** Fix a countable collection of cylinder functions $f_i$ and constants $C_i > 0$ as above. Let $\mathcal{P}$ denote the separable Banach space of all real sequences $\pi = \{\pi_i\}_{i \in \mathbb{N}}$ such that the series

$$\|\pi\|_{\mathcal{P}} \overset{\text{def}}{=} \sum_i C_i |\pi_i|$$

converges. For each $\pi \in \mathcal{P}$, let $f_\pi = \sum \pi_i f_i$ be the corresponding function on $C(Y,K,\Phi)$, and let

$$V_\pi = \sum_i \pi_i V_i$$

be the formal gradient of $f_\pi$ with respect to the $L^2$ inner product.

What our discussion has shown is that, for suitable choice of constants $C_i$, the series (74) will converge and the limit will also have the properties of cylinder functions that are given in Proposition 3.5. The next proposition records this, together with the fact that the dependence of the estimates on $\pi \in \mathcal{P}$ is as expected:
Proposition 3.7. If the constants $C_i$ in the definition of the Banach space $\mathcal{P}$ grow sufficiently fast, then the family of sections $V_\pi$ of $T_m$ satisfies the following conditions.

(i) The map 
$$(\pi, B) \mapsto V_\pi(B)$$
defines a smooth map $V_\bullet \in C^\infty(\mathcal{P} \times \mathcal{C}_m, T_m)$.

(ii) For any $j \leq m$, the first derivative in the $B$ variable, $DV_\bullet \in C^\infty(\mathcal{P} \times \mathcal{C}_m, \text{Hom}(T_m, T_m))$ extends to a smooth section 
$$DV_\bullet \in C^\infty(\mathcal{P} \times \mathcal{C}_m, \text{Hom}(T_j, T_j)).$$

(iii) There is a constant $K$ such that 
$$\|V_\pi(B)\|_{L^\infty} \leq K\|\pi\|_P$$ for all $\pi$ and $B$.

(iv) For all $j$, there is a constant $K_j$ such that 
$$\|V_\pi(B)\|_{L^2_{j,B\varphi}} \leq K_j\|\pi\|_P(1 + \|B - B^\varphi\|_{L^2_{j,B\varphi}})^j.$$ 

(v) There exists a constant $C$ such that for all $B$ and $B'$, and all $p$ with $1 \leq p \leq \infty$, we have 
$$\|V_\pi(B) - V_\pi(B')\|_{L^p} \leq C\|\pi\|_P\|B - B'\|_{L^p}.$$ 

(vi) The function $f_\pi$ whose gradient is $V_\pi$ is bounded on $\mathcal{C}_m(Y, K, \Phi)$.

We will refer to a perturbation $f = f_\pi$ of the Chern-Simons invariant which arises in this way as a holonomy perturbation.

3.3 Elliptic theory and transversality for critical points

We fix a Banach space $\mathcal{P}$ parametrizing holonomy perturbations as above, and we consider now the critical points of the perturbed Chern-Simons functional $CS + f_\pi$, for $\pi \in \mathcal{P}$. These critical points are the connections $B$ in $\mathcal{C}(Y, K, \Phi)$ satisfying 

$$*F_B + V_\pi(B) = 0.$$ (75) 

These equations are invariant under gauge transformation, and we denote by 
$$\mathcal{C}_\pi \subset \mathcal{B}(Y, K, \Phi)$$
the image of the set of critical points in the quotient space. The familiar
compactness properties of the space of flat connections modulo gauge trans-
formations extend to show that $\mathcal{C}_\pi$ is compact: we have, more generally, the
following lemma.

**Lemma 3.8.** Let $\mathcal{C}_\bullet \subset \mathcal{P} \times \mathcal{B}(Y,K,\Phi)$ be the parametrized union of the
critical sets $\mathcal{C}_\pi$ as $\pi$ runs through $\mathcal{P}$. Then the projection $\mathcal{C}_\bullet \to \mathcal{P}$ is proper.

**Proof.** Suppose that $[B_i]$ belongs to $\mathcal{C}_\pi$, and $\pi_i$ converges to $\pi$ in $\mathcal{P}$. The
terms $V_{\pi_i}(B_i)$ are bounded in $L^\infty$, so the curvatures of the connections $B_i$
are bounded in $L^\infty$ also. By Uhlenbeck’s theorem, we may assume (after
replacing the $B_i$ by suitable gauge transforms) that the connection forms
$B_i - B^\sharp$ are a bounded sequence in $\tilde{L}^p_1$, for all $p$. Uhlenbeck’s theorem also
allows us to find a finite covering of $Y$ by balls $U_\alpha$ (or orbifold balls cen-
tered at points of $K$) together with gauge transformations $g_{i,\alpha}$ such that the
connections $B_{i,\alpha} = g_{i,\alpha}(B_i)$ are in $g^\sharp$-Coulomb gauge with respect to some
trivialization of $P|U_\alpha$. The connection forms $B_{i,\alpha}$ on $U_\alpha$ are also bounded
in $\tilde{L}^p_2$ for all $p$, and the gauge transformations $g_{i,\alpha}$ are bounded in $\tilde{L}^p_2$.

The equations satisfied by $B_{i,\alpha}$ on $U_\alpha$ are

\[
\begin{align*}
* dB_{i,\alpha} &= -* [B_{i,\alpha} \wedge B_{i,\alpha}] - g_{i,\alpha}(V_{\pi_i}(B_i)) \\
d^* B_{i,\alpha} &= 0.
\end{align*}
\]

The terms $[B_{i,\alpha} \wedge B_{i,\alpha}]$ are bounded in $\tilde{L}^2_1$, because of the continuity of the
multiplication $\tilde{L}^p_1 \times \tilde{L}^p_1 \to \tilde{L}^2_1$ for $p > 3$. The term $V_{\pi_i}(B_i)$ is bounded in
$\tilde{L}^2_1$, as is $g_{i,\alpha}(V_{\pi_i}(B_i))$ therefore. On a smaller ball $U'_\alpha \subset U_\alpha$, these
equations therefore give us an $\tilde{L}^2_2$ bound on $B_{i,\alpha}$. The bootstrapping argument
now follows standard lines: on smaller balls $U''_\alpha$, the connections $B_{i,\alpha}$ are
bounded in all $\tilde{L}^2_j$ norms, and after passing to a subsequence, the gauge
transformations $g_{i,\alpha}$ can be patched together to form gauge transformations
gi such that $g_i(B_i)$ converges in the $\tilde{L}^2_m$ topology.

For fixed $\pi$, the left-hand side of (75) defines a smooth section,

\[ B \mapsto * F_B + V_{\pi}(B) \]

of the bundle $T_{m-1} \to \mathcal{C}_m$. Because of the gauge invariance of the functional,
this section is everywhere orthogonal to the orbits of the gauge group on
$\mathcal{C}(Y,K,\Phi)$, with respect to the $L^2$ inner product. To introduce some notation
to express this, let us first write $\mathcal{C}^*_m = \mathcal{C}^*(Y,K,\Phi)$ as usual for the subset of
$\mathcal{C}(Y,K,\Phi)$ consisting of irreducible connections, and let us decompose the
restriction of $T_j$ to $\mathcal{C}^*_m$ as

\[ T_j = J_j \oplus K_j, \]  

(76)
where \( \mathcal{J}_{j,B} \) is the \( L^2_j \) completion of the tangent space to the gauge-group orbit through \( B \) (i.e. the image of the map \( u \mapsto d_B u \)) and \( \mathcal{K}_{j,B} \) its \( L^2 \) orthogonal complement in \( L^2_j(Y; g_P) \). Thus \( \mathcal{K}_{j,B} \) is the space of \( b \) in \( L^2(Y; g_P) \) satisfying the Coulomb condition,

\[
d^*_B b = 0.
\]

On \( \mathcal{C}_m^\ast \), the decomposition (76) is a smooth decomposition of a Banach vector bundle. The gauge invariance of our perturbations means that \( V_\pi \) is a section the summand \( \mathcal{K}_m \subset T_m \) over \( \mathcal{C}_m^\ast \).

**Definition 3.9.** We say that a solution \( B_1 \in \mathcal{C}(Y, K, \Phi) \) to the equation (75) is **non-degenerate** if the section \( B \mapsto \ast F_B + V_\pi(B) \) of the bundle \( \mathcal{K}_{m-1} \to \mathcal{C}_m^\ast \) is transverse to the zero section at \( B = B_1 \).

The space of holonomy perturbations is sufficiently large to ensure that all critical points will be non-degenerate for a suitably chosen \( \pi \):

**Proposition 3.10.** There is a residual subset of the Banach space \( \mathcal{P} \) such that for all \( \pi \) in this subset, all the critical points of the perturbed functional \( CS + f_\pi \) in \( \mathcal{C}(Y, K, \Phi) \) are non-degenerate.

**Proof.** For critical points whose stabilizer coincides with the center \( Z(\mathcal{G}) \), this proposition follows from the fact that, given any compact (finite-dimensional) submanifold \( S \) of \( B^\ast(Y, K, \Phi) \), the functions \( f_\pi|_S \) are dense in \( C^\infty(S) \). This argument is just as in [13], [8] or [25]. For groups \( G \) other than \( SU(N) \), we must deal with the fact that irreducible connections may have finite stabilizer larger than \( Z(\mathcal{G}) \). Our holonomy perturbations are still dense in \( C^\infty(S) \) for \( S \) a compact sub-orbifold of \( B^\ast(Y, K, \Phi) \) however, so this extra complication can be dealt with as in [42].

This says nothing yet about the reducible critical points; but we will be working eventually with configurations \( (Y, K, \Phi) \) satisfying the non-integral condition of Definition 3.2, and we have the following version of Lemma 3.1 for small perturbations of the equations:

**Lemma 3.11.** Suppose \( (Y, K, \Phi) \) satisfies the non-integral condition. Then there exists \( \epsilon > 0 \) such that for all \( \pi \) with \( ||\pi||_P \leq \epsilon \), the critical points of \( CS + f_\pi \) in \( \mathcal{C}(Y, K, \Phi) \) are all irreducible.

**Proof.** This follows from Lemma 3.1 and the compactness result, Lemma 3.8.
When a critical point $B$ is non-degenerate, its gauge orbit $[B]$ in $\mathcal C_\pi$ is an isolated point of $\mathcal C_\pi$. It therefore follows that if $\pi$ has norm less than $\epsilon$ and belongs also to the residual set described in Proposition 3.10, then $\mathcal C_\pi$ is a finite subset of $\mathcal B(Y)$ and is contained in $\mathcal B^*(Y,K,\Phi) = \mathcal C^*(Y,K,\Phi)/\mathcal G(Y,K,\Phi)$. We record this as a proposition.

**Proposition 3.12.** If $(Y,K,\Phi)$ satisfies the non-integral condition of Definition 3.2, then there exists $\epsilon > 0$ and a residual subset of the $\epsilon$-ball in $\mathcal P$, such that for all $\pi$ in this subset the set of critical points $\mathcal C_\pi$ in the quotient space $\mathcal B(Y,K,\Phi)$ is a finite set and consists only of non-degenerate, irreducible critical points.

Another way to look at the condition of non-degeneracy is to look at the operator defined by the derivative of $\text{grad}(\text{CS} + f_\pi)$ on $\mathcal C(Y,K,\Phi)$, formally the Hessian of the functional. This Hessian is a section $\text{Hess} \in \mathcal C^\infty(\mathcal C_m, \text{Hom}(T_m, T_{m-1}))$ and is given by

$$\text{Hess}_B(b) = \ast d_B b + D\nu|_B(b).$$

At a critical point $B$, the Hessian annihilates $\mathcal J_B$ and maps $\mathcal K_B$ to itself; and as an operator $\mathcal K_{j,B} \to \mathcal K_{j-1,B}$ it is a compact perturbation of the Fredholm operator $\ast d_B$, because $D\nu_B$ maps $\check L^2_j$ to $\check L^2_{j-1}$. As an unbounded self-adjoint operator on $\mathcal K_{j,B}$ it has discrete spectrum: the spectrum consists of eigenvalues, the eigenspaces are finite-dimensional and the sum of the eigenspaces is dense. The non-degeneracy condition is the condition that $\text{Hess}_B$ is invertible, or equivalently the condition that $0$ is not an eigenvalue.

At a connection $B$ that is not a critical point of the perturbed functional, the operator $\text{Hess}_B$ does not leave invariant summands $\mathcal J_B$ and $\mathcal K_B$; and as an operator $T_m, B \to T_{m-1,B}$, it is not Fredholm. To correct this, one can introduce the extended Hessian, which is the operator

$$\tilde{\text{Hess}}_B = \begin{bmatrix} 0 & -d_B^* \\ -d_B & \text{Hess}_B \end{bmatrix}$$

acting on the spaces

$$\tilde{\text{Hess}}_B : \check L^2_j(Y; g_P) \oplus T_j \to \check L^2_{j-1}(Y; g_P) \oplus T_{j-1}.$$

This is a self-adjoint Fredholm operator which varies smoothly with $B \in \mathcal C(Y,K,\Phi)$; it is a compact perturbation of the family of elliptic operators

$$\begin{bmatrix} 0 & -d_B^* \\ -d_B & \ast d_B \end{bmatrix}.$$
The extended Hessian also has discrete spectrum consisting of (real) eigenvalues of finite multiplicity; the sum of the eigenspaces is again dense. At a critical point $B$, the extended Hessian can be decomposed into the direct sum of two operators, one of which is the restriction of $\text{Hess}_B$ as an operator $K_j \to K_{j-1}$. The other summand is invertible at irreducible critical points. It follows that, for a critical point $B$, the inevitability of $\tilde{\text{Hess}}_B$ is equivalent to the two conditions that $B$ be both irreducible and non-degenerate.

In the case that the perturbation is zero, the set of critical points $\mathcal{C} \subset \mathcal{B}(Y, K, \Phi)$ can be identified with a space of representations $\rho$ of the fundamental group of $Y \setminus K$, as explained at (69). In this situation, a representation $\rho$ determines a local coefficient system $\mathfrak{g}_\rho$ on $Y \setminus K$, with fiber $\mathfrak{g}$. This has cohomology groups $H^i(Y \setminus K; \mathfrak{g}_\rho)$. The following lemma (which is standard in the absence of the knot $K$) provides a criterion for the non-degeneracy of the corresponding connection $B$ as a critical point of CS.

**Lemma 3.13.** In the above situation, the kernel of $\text{Hess}_B$ on $K_j$ is isomorphic to

$$\ker : H^1(Y \setminus K; \mathfrak{g}_\rho) \to H^1(\mathfrak{m}; \mathfrak{g}_\rho)$$

where $\mathfrak{m}$ is any collection of loops representing the meridians of all the components of $K$. A critical point is therefore non-degenerate if and only if the above kernel is zero.

**Proof.** The kernel of the Hessian on $K_j$ is isomorphic to $\ker(d_B)/\text{im}(d_B)$ on our function spaces on $Y$ with the orbifold metric. We can decompose $Y$ as a union of two pieces, one of which is a tubular neighborhood of $K$ and the other of which is the complement of a smaller neighborhood. The isomorphism of the lemma then follows from a Mayer-Vietoris sequence, using this decomposition.

Suppose now that $B_0$ and $B_1$ are two irreducible, non-degenerate critical points. Let $B(t)$ be a path in $\mathcal{C}(Y, K, \Phi)$ from $B_0$ to $B_1$. We define

$$\text{gr}(B_0, B_1) \in \mathbb{Z}$$

to be the spectral flow of the one-parameter family of operators $\tilde{\text{Hess}}_{B(t)}$. Because $\mathcal{C}(Y, K, \Phi)$ is contractible, this number does not depend on the path, but only on the endpoints. Now let

$$\beta_0 = [B_0], \quad \beta_1 = [B_1]$$

be the corresponding critical points in the quotient space $\mathcal{B} = \mathcal{B}(Y, K, \Phi)$. The path $B(t)$ determines a path $\zeta$ from $\beta_0$ to $\beta_1$. Let $z \in \pi_1(\mathcal{B}, \beta_0, \beta_1)$ be
the relative homotopy class of $\zeta$. The homotopy class of $z$ again depends only on $B_0$ and $B_1$. To turn this around, if $\beta_0$ and $\beta_1$ belong to $\mathcal{C}_\pi \subset \mathcal{B}(Y, K, \Phi)$ and are both irreducible and non-degenerate, and if $z$ is a relative homotopy class of paths from $\beta_0$ to $\beta_1$, we define

$$\text{gr}_z(\beta_0, \beta_1) \in \mathbb{Z}$$

to be equal to $\text{gr}(B_0, B_1)$, where $B_0$ and $B_1$ are the endpoints of any path whose image in $\mathcal{B}(Y, K, \Phi)$ belongs to the homotopy class $z$.

The fundamental group of $\mathcal{B}(Y, K, \Phi)$ is equal to the group of components of $\mathcal{G}(Y, K, \Phi)$, which is described as (67) earlier. If $B$ is a non-degenerate, irreducible critical point, and if $B'$ is obtained from $B$ by applying a gauge transformation $g$ which is not in the identity component of $\mathcal{G}(Y, K, \Phi)$, then a path from $B'$ to $B$ gives rise to a homotopy class of closed loops $z$ based at the corresponding point $\beta$ in $\mathcal{B}(Y, K, \Phi)$. We can compute the spectral flow around this loop in terms of the data $(k, l) = d(g)$:

**Lemma 3.14.** Let $B' = g(B)$ in $\mathcal{C}(Y, K, \Phi)$, and write

$$d(g) = (k, l) \in \mathbb{Z} \times L(G_\Phi).$$

Then for the corresponding element $z$ of $\pi_1(\mathcal{B}(Y, K, \Phi), \beta)$ obtained from a path of connections from $B$ to $B'$, we have

$$\text{gr}_z(\beta, \beta) = 4h^c k + 4\rho(l)$$

where as usual $h^c$ is the dual Coxeter number of $G$ and $\rho$ is the Weyl vector.

The proof of this lemma follows the expected line, reinterpreting the spectral flow of the family of operators as the index of an operator associated to $S^1 \times Y$. The corresponding operator in this context is (a perturbation of) the linearized anti-self-duality equation with gauge fixing, so the index is the formal dimension of a moduli space of singular instantons on the pair $(S^1 \times Y, S^1 \times K)$. This relationship between the Chern-Simons functional on $\mathcal{C}(Y, K, \Phi)$ and singular instantons in dimension 4 is the subject of the next subsection.

### 3.4 The 4-dimensional equations and transversality for trajectories

Fix now some $\pi \in \mathcal{P}$, and write $V$ for $V_\pi$ and $f$ for $f_\pi$. Let $B(t)$ be a path in $\mathcal{C}(Y, K, \Phi)$, defined say on a bounded interval $I \subset \mathbb{R}$. The path
is a trajectory for the downward gradient flow of $\text{CS} + f$ if it satisfies the equation
\[
\frac{dB}{dt} = - \ast F_B - V(B).
\]
The path $B(t)$ defines a connection $A$ on the pull-back bundle over $I \times Y$. This connection $A$ is in temporal gauge (that is, it has no $dt$ component when expressed in local trivializations obtained by pull-back), and it has a singularity along $I \times K$ modelled on the singular connection $A^\varphi$ obtained by pulling back $B^\varphi$. In terms of $A$, the above equation can be written
\[
F_A^+ + (dt \wedge V(A))^+ = 0,
\]
where $V(A)$ denotes the 1-form in the $Y$ directions obtained by applying $V$ to each $B(t)$, regarded as giving a 1-form on $I \times Y$. In the form (77), the equation is fully gauge-invariant under the 4-dimensional gauge group.

As an abbreviation, let us write $\hat{V}$ for the perturbing term here,
\[
\hat{V}(A) = (dt \wedge V(A))^+,
\]
so that the equations are
\[
F_A^+ + \hat{V}(A) = 0. \tag{78}
\]

This perturbing term for the 4-dimensional equations shares the same basic properties as the perturbation $V(B)$ for the 3-dimensional equations. To state these, we suppose the interval $I$ is compact and write
\[
Z = I \times Y \quad \text{and} \quad L = I \times K,
\]
so that $L$ is an embedded 2-manifold with boundary in $Z$. We will continue to write $P \to Z$ for what is strictly the pull-back of $P$ from $Y$, and $\varphi$ for the translation-invariant section of $OP$ along $L$ obtained by pulling back the section $\varphi$ from $K$. We write $\mathcal{C}_m(Z, L, P, \varphi)$ for the space of connections singular connections $A = A^\varphi + a$ with $a$ of class $\mathcal{L}^2_m$. As an abbreviation, and to distinguish it from the similar space of 3-dimensional connections $\mathcal{C}_m = \mathcal{C}(Y, K, \Phi)$, we write
\[
\mathcal{C}^{(4)}_m = \mathcal{C}(Z, L, P, \varphi).
\]
In a similar way, we write $T^{(4)}_j$ for the $\mathcal{L}^2_j$ completion of the tangent bundle of $\mathcal{C}^{(4)}_m$, and we write $S_j \to \mathcal{C}^{(4)}_m$ for the (trivial) vector bundle with fiber $\mathcal{L}^2_j(Z; g_P \otimes \Lambda^+(Z))$. We assume from now on that $m$ is at least 3, so that our connections are again continuous on $Z \setminus L$. Then we have the following facts about $\hat{V}$, mirroring Proposition 3.7.
Proposition 3.15. Let $A \mapsto \hat{V}(A)$ be the perturbing term for the 4-dimensional equations, regarded as a section of the bundle $S_m$ over $C_m^{(4)}$. Then

(i) The section $\hat{V}$ is smooth:

$$\hat{V} \in C^\infty(C_m^{(4)}, S_m).$$

(ii) For any $j \leq m$, the first derivative

$$D\hat{V} \in C^\infty(C_m^{(4)}, \text{Hom}(T_m^{(4)}, S_m))$$

extends to a smooth section

$$D\hat{V} \in C^\infty(C_m^{(4)}, \text{Hom}(T_j^{(4)}, S_j)).$$

(iii) There is a constant $K$ such that $\|\hat{V}(A)\|_{L^\infty} \leq K$ for all $A$.

(iv) For all $j \leq m$, there is a constant $K_j$ such that

$$\|\hat{V}(A)\|_{L^2_j, A^\phi} \leq K_j (1 + \|A - A^\phi\|_{L^2_j, A^\phi})^j.$$

(v) There exists a constant $C$ such that for all $A$ and $A'$, and all $p$ with $1 \leq p \leq \infty$, we have

$$\|\hat{V}(A) - \hat{V}(A')\|_{L^p} \leq C\|A - A'\|_{L^p}.$$

In particular, $\hat{V}$ is continuous in the $L^p$ topologies.

In each of these cases, the dependence on $\pi \in \mathcal{P}$ can also be included, as in the statement of Proposition 3.7.

For a solution $A$ in $C_m^{(4)}$ on a compact cylinder $Z = [t_0, t_1] \times Y$, we define the (perturbed) topological energy as twice the change in the functional $\text{CS} + f_\pi$: that is,

$$E_\pi(A) = 2((\text{CS} + f_\pi)(B(t_0)) - (\text{CS} + f_\pi)(B(t_1))), \quad (79)$$

where $B(t)$ is the 3-dimensional connection obtained by restricting $A$ to $\{t\} \times Y$. Because of the last condition in Proposition 3.7, the perturbing term here only affects the energy by a bounded amount. The Chern-Simons functional is invariant only under the identity-component of the gauge group,
so $E_\pi(A)$ is not determined by knowing only the gauge-equivalence classes of the two endpoints, $\beta_0 = [B(t_0)]$ and $\beta_1 = [B(t_1)]$. The energy is determined by the endpoints $\beta_0, \beta_1$ in $B(Y, K, \Phi)$ and the homotopy class of the path $z$ between them given by $[B(t)]$. Accordingly, we may write the energy as

$$E_z(\beta_0, \beta_1).$$

We turn next to the Fredholm theory for solutions to the perturbed equations on the infinite cylinder. We write

$$Z = \mathbb{R} \times Y$$
$$L = \mathbb{R} \times K.$$

Let us suppose that the holonomy perturbation is chosen as in Proposition 3.12, so that the critical points are irreducible and non-degenerate. Let $\alpha$ and $\beta$ be two elements of $C_\pi$ and $z$ a homotopy class of paths between them. Let $B_\alpha$ and $B_\beta$ be corresponding elements of $C(Y, K, \Phi)$, chosen so that a path from $B_\alpha$ to $B_\beta$ projects to $B(Y, K, \Phi)$ to give a path belonging to the class $z$. Let $A_0$ be a singular connection on the pull-back of $P$ to the infinite cylinder $Z$, such that the restrictions of $A_0$ to $(-\infty, -T]$ and $[T, \infty)$ are equal to the pull-back of $B_\alpha$ and $B_\beta$ respectively, for some $T$. Define

$$C_z(\alpha, \beta) = \{ A \mid A - A_0 \in \tilde{L}^2_{m,A_0}(Z; g_P \otimes \Lambda^1(Z)) \}.$$  \hfill (80)

This space depends on the choice of $A_0$, not just on $\alpha$, $\beta$ and $z$. But any two choices are related by a gauge transformation. We define $\mathcal{G}(Z)$ to be the group of gauge transformations $g$ of $P$ on $Z$ satisfying

$$g - 1 \in \tilde{L}^2_{m+1,A_0}(Z; G_P),$$

and we have the quotient space

$$B_z(\alpha, \beta) = C_z(\alpha, \beta)/\mathcal{G}(Z).$$

It is an important consequence of the non-degeneracy of the critical points, that every solution $A$ to the perturbed equations on $\mathbb{R} \times Y$ which has finite total energy is gauge-equivalent to a connection in $C_z(\alpha, \beta)$, for some $\alpha$, $\beta$ and $z$.

**Definition 3.16.** The moduli space $M_z(\alpha, \beta) \subset B_z(\alpha, \beta)$ is the space of gauge-equivalence classes of solutions to the perturbed equations, $F^+_A + \tilde{V}(A) = 0.$
Because we have assumed that all critical points are irreducible, the configuration space $C_z(\alpha, \beta)$ consists also of irreducible connections. The action of the gauge group therefore has only finite stabilizers, and $B_z(\alpha, \beta)$ is a Banach orbifold (or a Banach manifold in the case that $G = SU(N)$): coordinate charts can be obtained in the usual way using the Coulomb condition. The local structure of $M_z(\alpha, \beta)$ is therefore governed by the linearization of the perturbed equations together with the Coulomb condition: at a solution $A$ in $C_z(\alpha, \beta)$, this is the operator

$$Q_A = -d_A^* \oplus (d_A^+ + D\hat{V}|A)$$

from $L^2_{m,A_0}(Z; \mathfrak{g}_P \otimes \Lambda^1(Z))$ to $L^2_{m-1,A_0}(Z; \mathfrak{g}_P \otimes (\Lambda^0 \oplus \Lambda^+)(Z))$.

This operator is Fredholm, and its index is equal to the spectral flow of the extended Hessian:

$$\text{index} Q_A = \text{gr}_z(\alpha, \beta).$$

**Definition 3.17.** A solution $A$ to the perturbed equations in $C_z(\alpha, \beta)$ is regular if $Q_A$ is surjective. We say that the moduli space $M_z(\alpha, \beta)$ is regular if $A$ is regular for all $[A]$ in the moduli space.

If the moduli space is regular, then it is a (possibly empty) smooth orbifold of dimension $\text{gr}_z(\alpha, \beta)$. At this point, one would like to argue that for a generic choice of perturbation $\pi$, all the moduli spaces $M_z(\alpha, \beta)$ are regular. However, although such a result is true for the case of $SU(2)$ (and is proved in [13] and [8]), the presence of non-trivial finite stabilizers is an obstruction to extending the transversality arguments to general simply-connected simple groups $G$. When the stabilizers are all equal to the center $Z(G)$, then the arguments from the $SU(2)$ case carry over without change. We therefore have:

**Proposition 3.18.** Suppose that $\pi_0$ is a perturbation such that all the critical points in $\mathcal{C}_{\pi_0}$ are non-degenerate and have stabilizer $Z(G)$. Then there exists $\pi \in \mathcal{P}$ such that:

(i) $f_\pi = f_{\pi_0}$ in a neighborhood of all the critical points of $CS + f_{\pi_0}$;

(ii) the set of critical points for these two perturbations are the same, so $\mathcal{C}_\pi = \mathcal{C}_{\pi_0}$;

(iii) for all critical points $\alpha$ and $\beta$ in $\mathcal{C}_\pi$ and all paths $z$, the moduli spaces $M_z(\alpha, \beta)$ for the perturbation $\pi$ are regular.
In order to proceed, we will require non-degeneracy for all critical points and regularity for all moduli spaces. We therefore impose the following conditions:

**Hypothesis 3.19.** We will assume henceforth that the triple \((Y, K, \Phi)\) satisfies the non-integral condition, and that a small perturbation \(\pi\) is chosen as in Proposition 3.12 so that the critical points are irreducible and non-degenerate. We assume furthermore that the stabilizer of each critical point is just \(Z(G)\), and that the moduli spaces \(M_z(\alpha, \beta)\) are all regular, as in the previous proposition.

In practice, we do not know how to ensure the condition in Hypothesis 3.19 that the stabilizers be \(Z(G)\) except by taking \(G = SU(N)\), in which case it is automatic, given the other conditions. Of course, for any given \((Y, K, \Phi)\), it is always possible that this condition is satisfied, as it were, “by accident”; but from this point on we really have \(SU(N)\) in mind. The notation we have set up for a general simply-connected simple group \(G\) is still appropriate, and we will continue to use it.

### 3.5 Compactness and bubbles

The basic compactness results for singular instantons on a compact pair \((X, \Sigma)\), which we summarized in Proposition 2.9, can be adapted to the case of solutions on a compact cylindrical pair

\[
Z = I \times Y  \\
L = I \times K.
\]

The main differences from the closed case are the following. First, in the case of a closed manifold, the energy \(E\) is entirely determined by the topology of \(P\) and \(\varphi\). In the case of a finite cylinder, the energy depends on the (perturbed) Chern-Simons invariants of the restriction of the connection to the two boundary components, and is therefore not constrained by the topology: in order to obtain a compactness results we need to impose a bound on the energy as part of the hypotheses. Second, since the proofs ultimately depend on interior estimates, the hypothesis of bounded energy for a sequence of solutions on \(Z\) will only ensure that we have a subsequence converging on some interior domain. Third, when bubbles occur, their effect is no longer local, because of our non-local holonomy perturbations. None of these issues are special to the case of instantons with singularities: they all occur in the standard construction of instanton Floer homology, and the
issues surrounding the non-local holonomy perturbations are treated in [8] and [20].

In the statement of the following proposition (which corresponds to the first parts of Proposition 2.9), the $\dot{L}_k^p$ topology refers to the topology on sections of $g_P \otimes \Lambda^i$ defined by using the covariant derivative of $A^x$ and the Levi-Civita derivative of the orbifold metric $g''$, just as $\dot{L}_k^2$ was defined earlier.

**Proposition 3.20.** Let $(Z, L)$ be the compact cylindrical pair defined above, and let $I'' \subset I$ be a compact sub-interval contained in the interior of $I$. Let $A_n$ be a sequence of solutions to the perturbed equations in $C(Z, L, P, \varphi)$, and suppose there is a uniform bound on the energy:

$$E_\pi(A_n) \leq C, \quad \text{for all } i.$$ 

Then after passing to a subsequence, we have the following situation. There is an interval $I'$ with $I'' \subset I' \subset I$, a finite set of points $x$ contained in the interior of the sub-cylinder $Z' = I' \times Y$, and a solution $A$ to the equations in $C(Z', L', P, \varphi)$, with the following properties.

(i) There is a sequence of isomorphisms of bundles $g_n : P|_{Z\setminus x} \rightarrow P$ of class $\dot{L}_{m+1}^2$ such that

$$g_n^*(A_n) \rightarrow A|_{Z\setminus x}$$

in the $\dot{L}_1^p$ topology on compact subsets of $Z\setminus x$ for all $p > 1$.

(ii) In the sense of measures on $Z'$, the energy densities $2|F_{A_n}|^2$ converge to

$$2|F_A|^2 + \sum_{x \in x} \mu_x \delta_x$$

where $\delta_x$ is the delta-mass at $x$ and $\mu_x$ are positive real numbers.

The reason for passing from a subinterval $I''$ to a larger one $I'$ in the statement above is to ensure that the set of bubble-points $x$ is contained entirely in the interior of $Z'$. This means in particular that the gauge-transformations $g_n$ in the statement of the proposition are defined on the two boundary components of $Z'$. Let us write $I'$ as $[t'_0, t'_1]$, so that the boundary components are $\{t'_0\} \times Y$ and $\{t'_1\} \times Y$. Using again the map $d$ defined at (68), we can consider the elements

$$d(g_n|_{t'_i}) \in Z \otimes L(G_{\Phi})$$
for $i = 0, 1$. If $x$ were empty, then these two would be equal, but in general
the difference is a topological quantity accounted for by the failure of $g_n$
to extend over the punctures. This is the same phenomenon that accounts
for the difference between $(k, l)$ and $(k', l')$ in item (iii) of Proposition 2.9.
Combining Proposition 2.9 with Proposition 2.10, we therefore obtain:

**Proposition 3.21.** In the situation of Proposition 3.20, we can choose the
subsequence so that the elements $d(g_n|_{t_i'})$ are independent of $n$ for $i = 0, 1$;
and for each $x \in x$, we can find $(k_x, l_x) \in \mathbb{Z} \oplus L(G_\Phi)$ such that

$$d(g_n|_{t_0'}) - d(g_n|_{t_1'}) = \left( \sum_{x \in x} k_x, \sum_{x \in x} l_x \right).$$

(If $x$ does not lie on the surface $L' = I' \times K$, then $l_x$ is zero.) The energy
$\mu_x$ that is lost at $x$ is then given by

$$\mu_x = 8\pi^2 \left( 4h'k_x + 2\langle \Phi, l_x \rangle \right).$$

Furthermore, the pairs $(k_x, l_x)$ are subject to the constraints of Proposi-
tion 2.10, namely

$$k_x \geq 0, \quad \text{and} \quad n_\alpha' k_x + w_\alpha(l_x) \geq 0$$

for all simple roots $\alpha$. 

As in the case of a closed manifold, the energy lost at the bubbles
is accounted for by solutions on the pair $(S^4, S^2)$ (equipped now with a
conformally-flat orbifold metric as in [21]).

The compactness results above, for solutions on a compact cylinder, lead
in a standard way to compactness results for solutions on the infinite cylinder
$\mathbb{R} \times Y$ when transversality hypotheses are assumed, as in Hypothesis 3.19.
To introduce notation for this, if $z$ is not the class of a constant path at
$\alpha = \beta$, we let $\bar{M}_z(\alpha, \beta)$ denote the quotient $M_z(\alpha, \beta)/\mathbb{R}$, where $\mathbb{R}$ acts by
translations. (For $\alpha = \beta$ and $z$ the constant path, we regard $\bar{M}_z(\alpha, \beta)$
as the empty set.) We call the elements of $\bar{M}_z(\alpha, \beta)$ the unparametrized
trajectories. By a broken (unparametrized) trajectory from $\alpha$ to $\beta$, we mean
a collection

$$[A_i] \in \bar{M}_{z_i}(\beta_{i-1}, \beta_i)$$

for $i = 1, \ldots, l$, with $\beta_0 = \alpha$ and $\beta_l = \beta$. The case $l = 0$ is allowed. We
write $\bar{M}_z^+(\alpha, \beta)$ for the space of all unparametrized broken trajectories from
$\alpha$ to $\beta$ with the additional property that the composite of the paths $z_i$ is in
the homotopy class $z$. 
In finite-dimensional Morse theory, the spaces of broken trajectories of this sort are compact. For the instanton theory, compactness holds only in situations where we can rule out the possibility that bubbles may occur. Given our transversality hypotheses, we can rule out bubbles on the grounds of the dimension of the moduli spaces involved. In particular, from Corollary 2.12, we deduce:

**Proposition 3.22.** If the dimension of $M_z(\alpha, \beta)$ is less than 4, then the space of unparametrized broken trajectories $\tilde{M}_z^+(\alpha, \beta)$ is compact. In particular, if $\text{gr}_z(\alpha, \beta) = 1$, then $\tilde{M}_z(\alpha, \beta)$ is a compact zero-dimensional manifold.

The bound of 4 in this proposition can be improved in particular cases, depending on the group $G$ and the choice of $\Phi$. The correct condition in general is that $\text{gr}_z(\alpha, \beta)$ is smaller than the smallest dimension of any positive-dimensional framed moduli space on $(S^4, S^2)$. See Corollary 2.15 for example.

There is a significant additional question that does not arise in the case that $K$ is absent. The compactness result that we have just stated concerns a single moduli space. There are only finitely many critical points, but for each pair $(\alpha, \beta)$ there are infinitely many possibilities for $z$. When $K$ is empty, $\pi_1(B(Y))$ is $\mathbb{Z}$ and $\text{gr}_z(\alpha, \beta)$ is a non-constant linear function of $z$: the moduli space will be empty when $\text{gr}_z(\alpha, \beta)$ is negative, and one should expect the moduli space to be non-empty (and of large dimension) once $\text{gr}_z(\alpha, \beta)$ becomes large. When $K$ is present, $\pi_1(B(Y, K, \Phi))$ is larger, and knowledge of $\text{gr}_z(\alpha, \beta)$ no longer determines $z$. There may be infinitely many non-empty moduli spaces, all of the same dimension. What we do have is a finiteness result when a bound on the energy is known. Let us again write

$$\mathcal{E}_z(\alpha, \beta) = 2\left( (CS + f_\pi)(B_\alpha) - (CS + f_\pi)(B_\beta) \right);$$

for the (perturbed) topological energy along a homotopy class of paths $z$. This is the energy for any solution in the moduli space $M_z(\alpha, \beta)$. For a proof of the following finiteness result, see [25, Proposition-something].

**Proposition 3.23.** Given any $C > 0$, there are only finitely many $\alpha, \beta$ and $z$ for which the moduli space $M_z(\alpha, \beta)$ is non-empty and has topological energy at most $C$.

For the construction of the Floer homology, the important comparison is between the topological energy $\mathcal{E}_z(\alpha, \beta)$ and the spectral flow, or relative grading, $\text{gr}_z(\alpha, \beta)$. We can look at the special case where $\alpha = \beta$ so that a
lift of a path in the class \( z \) gives a path of connections on \( Y \setminus K \) from \( B \) to \( B' \), where \( B' \) differs from \( B \) by a gauge transformation \( g \in G(Y, K, \Phi) \). We write \( B' = g(B) \). Let us again set
\[
d(g) = (k, l) \in \mathbb{Z} \times L(G_{\Phi})
\]
as in (68). Then for the corresponding homotopy class \( z \) of closed paths based at \( \beta = [B] \), we have
\[
gr_z(\beta, \beta) = 4\hbar' k + 4\rho(l)
\]
and
\[
E_z(\beta, \beta) = 8\pi^2 (4\hbar' k + 2\langle \Phi, l \rangle).
\]
These formulae can be computed, for example, by applying the dimension and energy formulae for the closed manifold \( S^1 \times Y \) containing the embedded surface \( S^1 \times K \). In the case that \( \Phi \) satisfies the monotone condition (Definition 2.4), these two linear forms in \( k \) and \( l \) are proportional. Since there are only finitely many critical points in all, we see:

**Lemma 3.24.** If \( \Phi \) satisfies the monotone condition, then there is a constant \( C_0 \) such that for all \( \alpha, \beta \) and \( z \), we have
\[
\left| E_z(\alpha, \beta) - 8\pi^2 gr_z(\alpha, \beta) \right| \leq C_0.
\]

From Proposition 3.23 we now deduce:

**Corollary 3.25.** If \( \Phi \) satisfies the monotone condition, then given any \( D > 0 \), there are only finitely many \( \alpha, \beta \) and \( z \) for which the moduli space \( M_z(\alpha, \beta) \) is non-empty and has formal dimension at most \( D \).

### 3.6 Orientations

If \( \alpha \) and \( \beta \) are not necessarily critical points, we can still construct the operator \( Q_A \) from an arbitrary \( A \) corresponding to a path \( \zeta \) joining \( \alpha \) to \( \beta \). The operator is Fredholm if the extended Hessian is invertible at both \( \alpha \) and \( \beta \). Under these circumstances, let us define
\[
\Lambda_\zeta(\alpha, \beta)
\]
to be the (two-element) set of orientations for the determinant line of the Fredholm operator \( Q_A \). As \( \zeta \) varies in the paths belonging to a particular
homotopy class of paths $z$, the family of determinant lines of the corresponding operators $Q_A$ forms an orientable real line bundle over the space of paths: this orientability can be deduced from the corresponding statement in the case of a closed manifold, Proposition 2.17. An orientation for any one determinant line in this connected family therefore determines an orientation for any other. Thus it makes sense to write

$$\Lambda_z(\alpha, \beta)$$

in place of $\Lambda_z(\alpha, \beta)$, for $z$ a homotopy class of paths from $\alpha$ to $\beta$. If $z'$ is a homotopy class of paths from $\beta$ to $\beta'$, then there is a natural composition law,

$$\Lambda_z(\alpha, \beta) \times \Lambda_{z'}(\beta, \beta') \to \Lambda_{z'\circ z}(\alpha, \beta').$$

(Note that our notation for a composite path puts the first path on the right.) Because of the requirement that the Hessian is invertible at the two end-points, the two-element set $\Lambda_z(\alpha, \beta)$ cannot be thought of as depending continuously on $\alpha$ and $\beta$ in $\mathcal{B}(Y, K, \Phi)$.

A priori, $\Lambda_z(\alpha, \beta)$ depends on $z$, not just on $\alpha$ and $\beta$; but we can specify a rule, compatible with the composition law, that identifies $\Lambda_z(\alpha, \beta)$ and $\Lambda_{z'}(\alpha, \beta)$ for different homotopy classes $z$ and $z'$. This can be done, for example, using excision to transfer the question to a closed pair $(X, \Sigma)$ and then using the constructions which were used to compare orientations in Proposition 2.17. This observation allows us to write $\Lambda(\alpha, \beta)$, omitting the $z$.

Because $G$ is simple, the bundle $P$ admits a product connection $B^0$ for which $\varphi$ is parallel. We add a singular term in the standard way, to obtain a connection $B^\varphi$ with a codimension-two singularity; the monodromy of this connection lies in the one-parameter subgroup generated by $\Phi$. We let $\theta^\varphi$ denote the corresponding point in $\mathcal{B}(Y, K, \Phi)$. This point is neither irreducible nor non-degenerate, so we cannot define $\Lambda(\theta^\varphi, \alpha)$ as above because the operator $Q_A$ will not be Fredholm as it stands. To remedy this, we we can regard $Q_A$ as acting weighted Sobolev space, on which this operator is Fredholm. That is, we choose a connection $A$ in $\mathcal{C}_{loc}$ from $B^\varphi$ to $B_\alpha$ and define $\Lambda(\theta^\varphi, \alpha)$ as the set of orientations of the determinant line of the operator $Q_A$ acting in the topologies

$$Q_A : e^{-\epsilon t} \tilde{L}^2_{m,A_0} \to e^{-\epsilon t} \tilde{L}^2_{m-1,A_0}$$

on the infinite cylinder. Here $\epsilon$ is a small positive constant, smaller than the smallest positive eigenvalue of the extended Hessians $\theta$ and $\alpha$. 

Now that we have a basepoint $\theta^\phi$, we can define a 2-element set

$$\Lambda(\alpha) = \Lambda(\theta^\phi, \alpha).$$

We could equally well define $\Lambda(\alpha)$ as $\Lambda(\alpha, \theta^\phi)$ (with the same weighted Sobolev spaces), because of the composition law and the fact that $\Lambda(\alpha, \alpha)$ is canonically trivial.

With this understood, the composition law for the orientation lines gives us a map

$$\Lambda(\alpha) \times \Lambda(\beta) \to \Lambda(\alpha, \beta).$$

If $\alpha$ and $\beta$ are now critical points and $[A]$ is a solution of the equations belonging to the (regular) moduli space $M_z(\alpha, \beta)$, then $\Lambda(\alpha, \beta)$ is isomorphic to the set of orientations of the moduli space at $[A]$. Using the above composition law, we can turn this round and say that an orientation of $M_z(\alpha, \beta)$ at $[A]$ determines an isomorphism $\Lambda(\alpha) \to \Lambda(\beta)$.

In particular, we can consider the case that $\text{gr}_z(\alpha, \beta) = 1$. In this case, the moduli space of unparametrized trajectories $M_z(\alpha, \beta)$ is a finite set of points, and $M_z(\alpha, \beta)$ is a finite set of copies of $\mathbb{R}$, acted on by the translations of the cylinder. Thus $M_z(\alpha, \beta)$ is canonically oriented. To be quite specific, if $\tau_t$ denotes the translation $(s, y) \mapsto (s + t, y)$ of $\mathbb{R} \times Y$, we make $\mathbb{R}$ act on $M_z(\alpha, \beta)$ by $[A] \mapsto \tau^*_t[A]$, and we use this to give each orbit of $\mathbb{R}$ an orientation. For each $[A]$ in $M_z(\alpha, \beta)$, we therefore obtain an isomorphism $\epsilon[A] : \Lambda(\alpha) \to \Lambda(\beta)$.

$$\epsilon[A] : \Lambda(\alpha) \to \Lambda(\beta) \quad (82)$$

### 3.7 Floer homology

We can now define the Floer homology groups. The situation is that we have a compact, connected, oriented 3-manifold $Y$ with an oriented knot or link $K \subset Y$, a choice of simple, simply-connected Lie group $G$ and a $\Phi$ in the fundamental Weyl chamber with $\theta(\Phi) < 1$. A Riemannian metric $g''$ with an orbifold singularity along $K$ is given. We continue to suppose that the non-integrality condition (Definition 3.2) holds and that a perturbation $\pi \in \mathcal{P}$ is chosen so as to satisfy Hypothesis 3.19. We also need to suppose that $\Phi$ satisfies the monotone condition, Definition 2.4.

For a 2-element set $\Lambda = \{\lambda, \lambda'\}$ we use $\mathbb{Z}\Lambda$ to mean the infinite cyclic group obtained from the rank-2 abelian group $\mathbb{Z}\lambda \oplus \mathbb{Z}\lambda'$ by imposing the condition $\lambda = -\lambda'$. Thus a choice of element of $\Lambda$ determines a generator for $\mathbb{Z}\Lambda$. We define $C_*(Y, K, \Phi)$ to be the free abelian group

$$C_*(Y, K, \Phi) = \mathbb{Z}\Lambda(\beta), \quad (83)$$
If \( \text{gr}_z(\alpha, \beta) = 1 \) and \([\tilde{A}]\) denotes the \( \mathbb{R} \)-orbit of some \([A]\) in \( M_z(\alpha, \beta) \), then from (82) above we obtain an isomorphism

\[
\epsilon[\tilde{A}] : \mathbb{Z}\Lambda(\alpha) \rightarrow \mathbb{Z}\Lambda(\beta).
\]

Combining all of these, we define

\[
\partial : C_*(Y, K, \Phi) \rightarrow C_*(Y, K, \Phi)
\]

by

\[
\partial = \sum_{(\alpha, \beta, z)} \sum_{[\tilde{A}]} \epsilon[\tilde{A}]
\]

where the first sum runs over all triples with \( \text{gr}_z(\alpha, \beta) = 1 \).

That the above sum is finite depends on the monotonicity condition. The point is that for any pair \((\alpha, \beta)\), there will be infinitely many homotopy classes of paths \( z \) with \( \text{gr}_z(\alpha, \beta) = 1 \) (as long as \( K \) is non-empty). Thus the first sum in the definition of \( \partial \) has an infinite range. The monotone condition, however, ensures that only finitely many of the 1-dimensional moduli spaces \( M_z(\alpha, \beta) \) will be non-empty: this is the statement of Corollary 3.25.

Based as usual on a gluing theorem and consideration of the compactification of moduli spaces \( \tilde{M}_z(\alpha, \gamma) \) with \( \text{gr}_z(\alpha, \gamma) = 2 \), one shows that \( \partial \circ \partial = 0 \). It is important here that the moduli spaces of broken trajectories \( \tilde{M}_z^+(\alpha, \gamma) \) are compact when \( \text{gr}_z(\alpha, \gamma) = 2 \), as follows from Proposition 3.22.

**Definition 3.26.** When the non-integrality and transversality assumptions of Hypothesis 3.19 holds and \( \Phi \) satisfies the monotone condition, we define \( \mathbb{L}_*(Y, K, \Phi) \) to be the homology of the complex \((C_*(Y, K, \Phi), \partial)\).

Since \( \text{gr}_z(\alpha, \beta) \) taken modulo 2 is independent of the path \( z \), we can regard \( \mathbb{L}_*(Y, K, \Phi) \) as having an affine grading by \( \mathbb{Z}/2 \). For particular choices of \( G \) and \( \Phi \), the greatest common divisor of \( \text{gr}_z(\beta, \beta) \), taken over all closed paths, may be a proper multiple of 2, in which case \( \mathbb{L}_*(Y, K, \Phi) \) has an affine \( \mathbb{Z}/(2d) \)-grading for \( d > 1 \). For example, if \( G = SU(N) \) and \( \Phi \) has just two distinct eigenvalues, then the homology is graded by \( \mathbb{Z}/(2N) \).

Rather than being left as a relative (i.e. affine) grading, the mod 2 grading can be made canonical. For a critical point \( \alpha \), the grading of \( \alpha \) mod 2 can be defined as the mod 2 reduction of \( \text{gr}_z(\theta^\varphi, \alpha) \), where \( \theta^\varphi \) is the reducible configuration constructed earlier and \( z \) is any homotopy class of paths. The result is independent of the choices made.
3.8 Cobordisms and invariance

The Floer group $I_*(Y, K, \Phi)$ depends only on $(Y, K)$ as a smooth oriented pair and on the choice of $\Phi$: it is independent of the remaining choices made. These choices include the choice of Riemannian metric and the perturbation $\pi$: changing either of these may change the set of critical points that form the generators of the complex. More subtly, the choice of cut-off function involved in the construction of the base connection $B^\varphi$ may effect the 2-element set $\Lambda(\alpha)$ used in fixing signs. As in Floer’s original approach, the independence of the Floer groups on these choices can be seen as a consequence of a more general property, namely the fact that a cobordism between pairs gives rise to a homomorphism on Floer homology.

To say this more precisely, let $(Y_0, K_0)$ and $(Y_1, K_1)$ be two pairs. By a cobordism between them we will mean a connected, oriented manifold-with-boundary, $W$, containing a properly embedded oriented surface-with-boundary, $S$, together with an orientation-preserving diffeomorphism of pairs

$$r : (\bar{Y}_0, K_0) \to (\bar{Y}_1, K_1) \to (\partial W, \partial S).$$

If $(W, S)$ and $(W', S')$ are two cobordisms between the same pairs, then an isomorphism between them means a diffeomorphism between the underlying manifolds commuting with $r$. Isomorphism classes of cobordisms can be composed in the obvious way, and in this manner we obtain a category, whose objects are the pairs $(Y, K)$ and whose morphisms are the isomorphism classes of cobordisms. If $(W_1, S_1)$ is a cobordism from $(Y_0, K_0)$ to $(Y_1, K_1)$ and $(W_2, S_2)$ is a cobordism from $(Y_1, K_1)$ to $(Y_2, K_2)$, we denote by

$$(W, S) = (W_2 \circ W_1, S_2 \circ S_1)$$

the composite cobordism from $(Y_0, K_0)$ to $(Y_2, K_2)$.

We adopt from [25] the appropriate definition of a homology orientation for a cobordism $W$ from $Y_0$ to $Y_1$: a homology orientation $o_W$ is a choice of orientation for the line

$$\Lambda^{\max} H^1(W; \mathbb{R}) \otimes \Lambda^{\max} I^+(W) \otimes \Lambda^{\max} H^1(Y_1; \mathbb{R})$$

where $I^+(W)$ is a maximal positive-definite subspace for the non-degenerate quadratic pairing on the image of $H^2(W, \partial W; \mathbb{R})$ in $H^2(W; \mathbb{R})$. (Note that the links $K_i$ and the 2-dimensional cobordism $S$ are not involved here, and are omitted from our notation.) This definition can be made to look less arbitrary by regarding this as an orientation for the determinant line of the operator $-d^* \oplus d^+$ on the cylindrical-end manifold obtained from $W$, acting
on weighted Sobolev spaces with a consistent choice of weights. There is a
composition law for homology orientations: if $W = W_2 \circ W_1$ and homology
orientations $o_{W_i}$ are given, we can construct a homology orientation $o_{W_2} \circ
o_{W_1}$ for $W$. This is most easily seen from the second description of what
a homology orientation is. We thus have a modified category in which
the morphisms are cobordisms of pairs, $(W, S)$, equipped with homology
orientations, up to isomorphism.

Let $(W, S)$ be a cobordism from $(Y_0, K_0)$ to $(Y_1, K_1)$. Suppose that
each $Y_i$ is equipped with a Riemannian metric and that perturbations $\pi_i$
are chosen satisfying Hypothesis 3.19. We continue to suppose also that $\Phi$
satisfies the monotone condition. Let base connections $B_{\pi_i}^\sigma$ be chosen for
each. In this case, we have Floer homology groups $\mathbb{F}_i(Y_i, K_i, \Phi)$, for $i = 0, 1$,
which depend a priori on the choices made. Let us temporarily denote this
collection of choices (of metric, perturbation and base connection) by $\sigma_i$,
and so write the groups as
\[ \mathbb{F}_i(Y_i, K_i, \Phi)_{\sigma_i}, \quad i = 0, 1. \]
The fact that cobordisms give rise to maps can be stated as follows.

**Proposition 3.27.** Suppose that $\Phi$ satisfies the monotone condition. For
$i = 0, 1$, let $(Y_i, K_i)$ be pairs as above, and suppose that Hypothesis 3.19
holds for both. Let $\sigma_i$ be choices of Riemannian metric, connection $B_{\pi_i}^\sigma$ and
perturbation as above. Let $(W, S)$ be a cobordism from $(Y_0, K_0)$ to $(Y_1, K_1)$,
and let a homology orientation $o_W$ for the cobordism $W$ be given. Then
$(W, S, o_W)$ gives rise to a homomorphism
\[ \mathbb{F}_i(W, S, \Phi, o_W) : \mathbb{F}_i(Y_0, K_0, \Phi)_{\sigma_0} \rightarrow \mathbb{F}_i(Y_1, K_1, \Phi)_{\sigma_1} \] (86)
which depends only on the isomorphism class of the cobordism with its ho-
mology orientation. Furthermore, composition of cobordisms becomes com-
position of maps and the trivial product cobordism gives the identity map.

**Remark.** The choice of homology orientation $o_W$ affects only the overall sign
of the map $\mathbb{F}_i(W, S, \Phi, o_W)$, and affects it non-trivially only if the dimension
of $G$ is odd: cf. Proposition 2.17.

In particular, by taking $W$ to be a cylinder and setting
\[ (Y_0, K_0) = (Y_1, K_1) = (Y, K), \]
we see that the Floer group $\mathbb{F}_i(Y, K)_{\sigma}$ is independent of the auxiliary choices
$\sigma$, up to canonical isomorphism. Usually, we omit mention of $o_W$ and $\sigma$ from
our notation (just as we have already silently omitted \(r\)), and we simply write

\[ \mathbb{I}_s(W, S, \Phi) : \mathbb{I}_s(Y_0, K_0, \Phi) \to \mathbb{I}_s(Y_1, K_1, \Phi) \]

with the unstated understanding that the identifications \(r\) (when needed) are implied and that \(o_W\) is needed to fix the overall sign of this map if the dimension of \(G\) is odd.

The proof of Proposition 3.27 follows standard lines, and can be modelled (for example) on the arguments from [25]. We content ourselves here with some remarks about the construction of the maps \(\mathbb{I}_s(W, S, \Phi)\).

For \(i = 0, 1\), let \(\beta_i\) be a critical point in \(\mathcal{B}(Y_i, K_i, \Phi)\). On the pair \((W, S)\), let us consider a \(G\)-bundle \(P\) equipped with a section \(\varphi\) of \(O_P\) along \(S\) and corresponding connection \(A\) with singularity along \(S\), subject to the constraint that the restrictions of \(A\) to the two ends should define singular connections belonging the gauge-equivalences classes of \(\beta_0\) and \(\beta_1\). There is an obvious notion of a continuous family of such data, \((P_t, \varphi_t, A_t)\) parametrized by any space \(T\), and we can therefore consider the set of deformation-classes of such data. We will refer to such an equivalence class as a path from \(\beta_0\) to \(\beta_1\) along the cobordism \((W, S)\). In the case of a cylindrical cobordism, such a path is the same as a homotopy class of paths from \(\beta_0\) to \(\beta_1\) in \(\mathcal{B}(Y, K, \Phi)\).

If \(S\) has any closed components, then different paths along \((W, S)\) may also be distinguished by having different monopole charges on the closed components. If \((W, S)\) is a composite cobordism, as in (85), and if \(z_1\) and \(z_2\) are paths along \((W_1, S_1)\) and \((W_2, S_2)\) from \(\beta_0\) to \(\beta_1\) and from \(\beta_1\) to \(\beta_2\) respectively, then there is a well-defined composite path along \((W, S)\), obtained by choosing any identification of the two bundles on \(Y_1\) respecting the sections \(\varphi_i\) and the connections.

**Remark.** There is a small point to take note of here. By assumption, the critical point \(\beta_1\), like all critical points, is irreducible and has stabilizer \(Z(G)\). When forming the composite path by identifying the two bundles along \(Y_1\), there is therefore a \(Z(G)\)'s worth of choice in how the identification is made. Despite this choice, the composite path is well-defined, because the automorphisms of the connection on \(Y_1\) extend to the 4-manifolds.

Let \(W^+\) be the manifold obtained by attaching cylindrical ends to the two boundary components of \(W\), and let this manifold be given a Riemannian metric \(g_W^+\) which is a product metric on each of the two cylindrical pieces. Let \(S^+ \subset W^+\) be obtained similarly from \(S\).

Let critical points \(\beta_i\) in \(\mathcal{B}(Y_i, K_i, \Phi)\) be given for \(i = 0, 1\), and let \(z\) be a path along \((W, S)\) from \(\beta_0\) to \(\beta_1\). Let \((P_W, \varphi_W, A_W)\) be a representative for \(z\), and extend this data to the cylindrical ends by pull-back. Imitating the
definition of $C_z(\alpha, \beta)$ from (80), we define a configuration space of singular connections $C_z(W, S, \Phi; \beta_0, \beta_1)$ as the space of all $A$ differing from $A_W$ by a term belonging to $\hat{L}_m^2$, and we write $B_z(W, S, \Phi; \beta_0, \beta_1)$ for the corresponding quotient space.

Let $\pi_0$ and $\pi_1$ be the chosen holonomy perturbations on $Y_0$ and $Y_1$ respectively. We perturb the 4-dimensional equations on $W^+$ by adding a term supported on the cylindrical ends: this term will be a $t$-dependent holonomy perturbation $\pi_W$ equal to $\pi_i$ on the two ends. In more detail, in a collar $[0, 1) \times Y_0$ of one of the boundary components $Y_0 \subset W$, the perturbed equations take the form

$$F^+_A + \beta(t)\hat{U}_0(A) + \beta_0(t)\hat{V}_0(A) = 0$$

where, as in (78), $\hat{V}_0$ is the perturbing term defined by $\pi_0 \in \mathcal{P}$ and $\hat{U}_0$ is defined by a choice of an auxiliary element of $\mathcal{P}$. The cut-off function $\beta$ is supported in the interior of the interval, while $\beta_0$ is equal to 1 near $t = 0$ and equal to 0 near $t = 1$. (This choice of perturbation follows [25, Section 24].) We write

$$M_z(W, S, \Phi; \beta_0, \beta_1) \subset B_z(W, S, \Phi; \beta_0, \beta_1)$$

for the moduli space of solutions to the perturbed equations on $W^+$. For generic choice of auxiliary perturbation $\hat{U}_i$ on the two collars, the moduli space is cut out transversely by the equations and (under our standing assumptions of Hypothesis 3.19) is a smooth manifold. A choice of homology orientation $o_W$ and an element of $\Lambda(\beta_0)$ and $\Lambda(\beta_1)$ determines an orientation of the moduli space. As in the closed case, if $G$ is even-dimensional, then $o_W$ is not needed. The map (86) is defined in the usual way by counting with sign the points of all zero-dimensional moduli spaces $M_z(W, S, \Phi; \beta_0, \beta_1)$.

As in the definition of the boundary map $\partial$, the monotonicity condition ensures that this is a finite sum, because for fixed $\beta_0$ and $\beta_1$, the dimension of the moduli space corresponding to a path $z$ along $(W, S)$ is an affine-linear function of the topological energy.

### 3.9 Local coefficients

There is a standard way in which the construction of Floer homology groups can be generalized, by introducing a local system of coefficients, $\Gamma$, on the configuration space (in this case, the configuration space $\mathcal{B}(Y, K, \Phi)$ of singular connections modulo gauge transformations on the 3-manifold). Thus we suppose that for each point $\beta$ in the configuration space we have an abelian group $\Gamma_\beta$ and for each homotopy class of paths $z$ from $\alpha$ to $\beta$ and isomorphism $\Gamma_z$ from $\Gamma_\alpha$ to $\Gamma_\beta$ satisfying the usual composition law. If we make
the same assumptions as before (the conditions of Hypothesis 3.19 and the monotonicity condition, Definition 2.4), then we can modify the definition of the chain group $C_*(Y,K,\Phi)$ by setting

$$C_*(Y,K,\Phi;\Gamma) = \bigoplus_{\beta \in \mathcal{E}_*} \mathbb{Z}\Lambda(\beta) \otimes \Gamma_{\beta}$$

and taking the boundary map to be

$$\partial = \sum_{(\alpha,\beta,z)} \sum_{[\tilde{A}]} \epsilon_{[\tilde{A}]} \otimes \Gamma_z. \quad (87)$$

The homology of this complex, $\mathbb{I}_*(Y,K,\Phi;\Gamma)$ is the Floer homology with coefficients $\Gamma$.

If we are given two pairs, $(Y_0,K_0)$ and $(Y_1,K_1)$ with local systems $\Gamma^0$ and $\Gamma^1$, and if $(W,S)$ is a cobordism between the pairs, then we have a natural notion of morphism, $\Delta$, of local systems along $(W,S)$: such a $\Delta$ assigns to each path $z$ from $\beta_0$ to $\beta_1$ along $(W,S)$ (in the sense of the previous subsection) a homomorphism

$$\Delta_z : \Gamma^0_{\beta_0} \rightarrow \Gamma^1_{\beta_1} \quad (88)$$

respecting the composition maps with paths in $B(Y_i,K_i,\Phi)$ on the two sides. (See [25], for example.) Using such a morphism $\Delta$, we can adapt the definition of the map $\mathbb{I}_*(W,S,\Phi)$ in an obvious way to obtain a homomorphism

$$\mathbb{I}_*(W,S,\Phi;\Delta) : \mathbb{I}_*(Y_0,K_0,\Phi;\Gamma^0) \rightarrow \mathbb{I}_*(Y_1,K_1,\Phi;\Gamma^1).$$

To give an example, we begin with a standard local system $\Gamma^{S^1}$ on the circle $S^1$, regarded as $\mathbb{R}/\mathbb{Z}$, defined as follows. We write $R$ for the ring of finite Laurent series with integer coefficients in a variable $t$. This is the group ring $\mathbb{Z}[\mathbb{R}]$, and we can regard it as lying inside the group ring $\mathbb{Z}[\mathbb{R}]$: the ring of formal finite series

$$\sum_{x \in \mathbb{R}} a_x t^x.$$

For each $\lambda$ in $\mathbb{R}$, we have an $R$-submodule $t^\lambda R \subset \mathbb{Z}[\mathbb{R}]$ generated by the element $t^\lambda$: this is the $R$-module of all finite series of the form

$$\sum_{x \in \lambda + \mathbb{Z}} a_x t^x.$$
As \( \lambda \) varies in \( \mathbb{R}/\mathbb{Z} \) these form a local system of \( \mathbb{R} \)-modules, \( \Gamma^{S^1} \) over \( S^1 \): the map \( \Gamma^\lambda_z \) corresponding to a path \( z \) is given by multiplication by \( t^{\lambda_1 - \lambda_0} \) if \( z \) lifts to a path in \( \mathbb{R} \) from \( \lambda_0 \) to \( \lambda_1 \). If we are now given a circle-valued function

\[
\mu : \mathcal{B}(Y, K, \Phi) \to S^1 = \mathbb{R}/\mathbb{Z}
\]

then we can pull back the standard local system \( \Gamma^{S^1} \) to obtain a local system

\[
\Gamma^\mu = \mu^*(\Gamma^{S^1})
\]
on \( \mathcal{B}(Y, K, \Phi) \).

This construction can be applied using a class of naturally-occurring circle-valued functions on the configuration space of singular connections. These functions can be defined, roughly speaking, by taking the holonomy of a connection \( B \) along a longitudinal curve close to a component of the link \( K \) and applying a character of \( G_{\Phi} \). To say this more precisely, we choose a framing of the link \( K \subset Y \), so has to have well-defined coordinates on the tubular neighborhood, up to isotopy, identifying the neighborhood with \( D^2 \times K \). Suppose first that \( K \) has just one component, and for each sufficiently small \( \epsilon > 0 \), let \( T_\epsilon \) be the torus obtained as the product of the circle of radius \( \epsilon \) in \( D^2 \) with knot \( K \). Use the coordinates to identify \( T_\epsilon \) with \( S^1 \times K \). If \( B \) is a connection in \( \mathcal{C}(Y, K, \Phi) \), then by restricting to \( T_\epsilon \) we obtain in this way a sequence of \( G \)-connections on \( S^1 \times K \); and the definition of the space \( \mathcal{L}^*_m \) in which we work guarantees that these have a well-defined limit, up to gauge transformation, which is a flat connection \( B_0 \) on \( S^1 \times K \). The holonomy of \( B_0 \) along a curve belonging to the \( S^1 \) factor is \( \exp(\varphi) \), and the holonomy along the longitudinal curve belonging to the \( K \) factor lies in the commutant. Choose a character

\[
s : G_{\Phi} \to U(1)
\]
and let

\[
w : \mathfrak{g}_{\Phi} \to \mathbb{R}
\]
be the corresponding weight, so that \( s(\exp(x)) = e^{2\pi i w(x)} \). We can apply \( s \) to the holonomy of \( B_0 \) along the longitudinal curve to obtain a well-defined element of \( U(1) \), depending only on the gauge-equivalence class of \( B \). Thus we obtain from \( s \) a function

\[
\mu_s : \mathcal{B}(Y, K, \Phi) \to U(1) = \mathbb{R}/\mathbb{Z}
\]
by applying \( s \) to the holonomy along the longitudinal curve. In this way we obtain a local system \( \Gamma^{\mu_s} \) by pull-back.
The choice of framing of $K$ is essentially immaterial. The set of framings is an affine copy of $\mathbb{Z}$; and if we change the chosen framing of $K$ by 1, then $\mu_s$ is changed by the addition of the constant $w(\Phi) \mod \mathbb{Z}$. The corresponding local systems are canonically isomorphic, via multiplication by $t^{w(\Phi)}$.

If $K$ has more than one component, we can apply this construction to each one, perhaps using different characters $s_i$ and form the product. Alternatively, one could define a local system over a ring of Laurent series in a number of variables $t_i$, one for each component of $K$.

In the above construction, the reason for taking such a specifically-defined function $\mu_s$, rather than a general circle-valued function belonging to the same homotopy class, is that the naturality inherent in the construction leads to a Floer homology group that is a topological invariant of the pair, rather than a group that is an invariant only up to isomorphism. The point is that if we have a cobordism of pairs, $(W,S)$, with a chosen framing of a tubular neighborhood of $S$ (or at least of the components of $S$ having non-empty boundary), then we obtain a natural morphism $\Delta$ between the corresponding local systems associated to the framed knots at the two ends. The map $\Delta_z$ corresponding to a path $z$ along $(W,S)$ can be defined as follows. Fix data $(P,\varphi,A)$ on $W$ corresponding to $z$. For each small positive $\epsilon$, we have a copy of $S^1 \times S$ in $W$, as the boundary of the $\epsilon$-neighborhood of $S$ in its framed tubular neighborhood $D^2 \times S$, and we therefore obtain connections $A_\epsilon$ on $S^1 \times S$. The limit of these connections is a connection $A_0$ on $S^1 \times S$ whose curvature 2-form has the $S^1$ direction in its kernel. Thus $A_0$ gives a $G_\Phi$-connection on $S^1 \times S$; and applying the character $s$ we obtain a $U(1)$-connection $s(A_0)$ on $S^1 \times S$. For any $p$ in $S^1$, we have a parallel copy of $S$ as $\{p\} \times S$, and the map $\Delta_z$ can then be defined as multiplication by $t^\nu$, where

$$\nu = \frac{i}{2\pi} \int_{\{p\} \times S} F_{s(A_0)}. \quad (90)$$

Because the curvature 2-form of $s(A_0)$ annihilates the circle directions, we see that we could have taken any section of $S^1 \times S$ instead of the constant section $\{p\} \times S$, and the above integral would be unchanged. So in the end, the map $\Delta_z$ is independent of the choice of framing of $S$.

Local systems can also be made use of to define Floer groups in the case that $\Phi$ does not satisfy the monotone condition. When $\Phi$ is not monotone, the sum (87) which defines the boundary operator may have infinitely many non-zero terms; but the sum can still be made sense of if each $\Gamma_\alpha$ is a topological group and the local system is such that the series converges. A typical instance of such a construction replaces the ring $R$ of finite Laurent
series which we used above by the ring of Laurent series that are infinite in one direction.

3.10 Non-simple groups

We have been considering instanton Floer homology in the case that $G$ is a simple group. When discussing instanton moduli spaces, we saw in section 2.10 how the definitions are readily adapted to the case which of a non-simple group such as the unitary group. We now carry this over to the Floer homology setting. We again suppose that $G$ has a simply-connected commutator subgroup. We write $Z(G)$ for its center and $\tilde{Z}(G)$ for $G/[G,G]$. Unlike the case in which $G$ itself is simply-connected, it is no longer the case that a $G$-bundle $P \to Y$ must be trivial: its isomorphism type is determined by the $\tilde{Z}(G)$-bundle $\vartheta(P)$, or equivalently by the characteristic class $c = c(P)$ in $H^2(Y; L(G))$ of (51).

To preserve the functoriality of the Floer homology groups, we need to adopt the alternative viewpoint for the configuration space and gauge group which we mentioned briefly in subsection 2.10. We fix $\tilde{Z}(G)$-bundle $\delta \to Y$ with an isomorphism $q : \vartheta(P) \to \delta$, and we fix a connection $\Theta$ in $\delta$. As before, we let $\Theta^\varphi$ denote the corresponding singular connection in $\delta$ (equation (53)), and we construct a space $\mathcal{C}(Y, K, \Phi)_\delta$ of singular connections, $B$, with the constraint that $\vartheta(B) = q^*(\Theta^\varphi)$. The gauge group $\mathcal{G}(Y, K, \Phi)$ consists of gauge transformations $g$ of class $L^2_{m+1}$ with $\vartheta(g) = 1$ and we have a quotient space $\mathcal{B}(Y, K, \Phi)_\delta$.

The construction of the Floer groups then proceeds as before, with straightforward modifications of the same type as we dealt with in section 2.10. We deal with some of these modifications in the next few paragraphs.

**The Chern-Simons functional.** The appropriate Chern-Simons functional on $\mathcal{C}(Y, K, \Phi)_\delta$ in the present setting is the one which ignores the central component of the connection: it can be defined by the same formula (66) as before, if we understand that the inner products in (66) are defined using the semi-definite Killing form. Critical points of the unperturbed Chern-Simons functional on $\mathcal{C}(Y, K, \Phi)_\delta$ are singular connections $B$ such that the induced connection $\bar{B}$ with structure group $G/Z(G)$ in the adjoint bundle is flat. The formal gradient flow lines of this functional correspond to connections $A$ in temporal gauge on the cylinder with the property that $A$ is anti-self-dual.
The non-integral condition. The most important change involves the non-integrality condition, Definition 3.2, which we used to rule out reducible critical points and which formed part of our standing Hypothesis 3.19. In the case that $G$ is not simple, the corresponding condition can be read off from the 4-dimensional version, Proposition 2.19:

**Definition 3.28.** Let the components of $K$ again be $K_1,\ldots,K_r$. For non-simple groups $G$, we will say that the bundle $P$ on $(Y,K,\Phi)$ satisfies the **non-integral condition** if, in the notation of section 2.10, the expression

$$w_\alpha(c(P)) + \sum_{j=1}^r (\bar{w}_\alpha \circ \sigma_j)(\bar{\Phi})P.D.[K_j]$$

is a non-integral cohomology class for every choice of fundamental weight $w_\alpha$ and Weyl group elements $\sigma_1,\ldots,\sigma_r$.

The simplest example in which this non-integrality holds is the case corresponding to Corollary 2.20, in which $G$ is the unitary group $U(N)$ and all the components of $K_i$ are null-homologous: in this case, the non-integral condition is equivalent to saying that the pairing of $c_1(P)$ with some integral homology class in $Y$ is coprime to $N$.

As previously, we need to suppose that this non-integrality condition holds and that further, as in Hypothesis 3.19, the stabilizer in $G(Y,K,\Phi)_\delta$ of every critical point is exactly $Z(G) \cap [G,G]$, rather than some larger finite group (a condition which is automatic in the non-integral case if $G = U(N)$). Under these conditions, and when $\Phi$ satisfies the monotone condition (57), we will arrive at a Floer homology group

$$\mathbb{I}_*(Y,K,\Phi)_\delta$$

depending on the choice of bundle $\bar{Z}(G)$-bundle $\delta$.

Holonomy perturbations. The definition of holonomy perturbations does not need any changes in the case of more general $G$. The basic ingredient is still a choice of function

$$h : G^r \to \mathbb{R}$$

invariant under the diagonal action of $G$, acting by the adjoint representation on each factor. Holonomy perturbations still separate points in the quotient space $\mathcal{B}(Y,K,\Phi)_\delta$. Note that the choice of connection $\Theta$ is involved in the construction, because we are taking the holonomy of a $G$-connection $B$ in the bundle $P$ which satisfies $\s(\delta(B)) = \Theta^p$. If we chose $h$ so that it was pulled
back from \((G/Z(G))^*\), then the choice of \(\Theta\) would again become irrelevant; but functions \(h\) of this sort are not a large enough class, as they do not allow our holonomy perturbations to separate points and tangent vectors in \(\mathcal{B}(Y, K, \Phi)_{\delta}\).

**Orientations.** In the case of a simple group \(G\), we defined a 2-element set \(\Lambda(\alpha, \beta)\) for a pair of configurations \(\alpha\) and \(\beta\); and we then defined \(\Lambda(\alpha)\) as being \(\Lambda(\theta^e, \alpha)\), where \(\theta^e\) was a specially chosen connection. The important features of our choice of \(\theta^e\) were first that \(\theta^e\) was reducible and second that, although the construction depended on details such as a choice of cut-off function, any two choices differed by a small isotopy, so that an essentially unique path connects any two choices.

When \(P\) is not simple and \(\vartheta(P)\) is non-trivial, we do not have a distinguished gauge-equivalence class of trivial connections in \(P\) from which to construct \(\theta^e\), but we can instead proceed as we did in section 2.10. We fix again a homomorphism \(\epsilon : \tilde{Z}(G) \to T\) which is right-inverse to \(\vartheta\) (see (60)). As in the 4-dimensional case, we obtain a \(G\)-connection \(\epsilon(\Theta)\) on a bundle isomorphic to \(P\), with \(\vartheta(\epsilon(\Theta)) = \Theta\). After adding the singular term along \(K\), we obtain a distinguished gauge-equivalence class of connections, \(\theta^e\), in \(\mathcal{B}(Y, K, \Phi)_{\delta}\). Once \(\epsilon\) is fixed, this gauge-equivalence class depends only on the details of how the singular term is constructed, through the choice of cut-off function for example. This puts us in a position to define \(\Lambda(\alpha)\) as we did before.

**Cobordisms.** Let \((W, S)\) now be a cobordism of oriented pairs, and write its two boundary components as \((Y_i, K_i)\) for \(i = 0, 1\), so that

\[
\partial(W, S) = (\bar{Y}_0, \bar{K}_0) \amalg (\bar{Y}_1, \bar{K}_1).
\]

(In the slightly more categorical language that we used earlier in section 3.8, we are supposing here that the identification map \(r\) is the identity.) Let \(\delta_W\) be a \(\tilde{Z}(G)\)-bundle on \(W\), and let write

\[
\delta_i = \delta_W|_{Y_i}, \quad i = 0, 1.
\]

Fix \(G\)-bundles \(P_0\) and \(P_1\) on \(Y_0\) and \(Y_1\) with isomorphisms \(q_i : \vartheta(P_i) \to \delta_i\). Let \(\Theta_0\) and \(\Theta_1\) be chosen connections in \(\delta_0\) and \(\delta_1\).

We wish to show how the data \((W, S, \delta_W)\) (together with a homology orientation of \(W\)) gives rise to a homomorphism from \(\mathbb{I}_*(Y_0, K_0, \Phi)_{\delta_0}\) to \(\mathbb{I}_*(Y_1, K_1, \Phi)_{\delta_1}\). The first step is to extend our previous notion of a “path along \((W, S)\)” between critical points \(\beta_0\) and \(\beta_1\) belonging the configuration...
spaces $B(Y_i, K_i, \Phi)_{\delta_i}$ for $i = 0, 1$. To do this, we let $\beta_i$ be represented by singular connections $B_i$ on $P_i \to Y_i$ and we define a path $z$ to be defined by data consisting of:

- a bundle $P \to W$;
- an isomorphism $q_W : \mathfrak{d}(P) \to \delta_W$;
- a reduction of structure group defined by a section $\varphi$ of $O_P$ along $S$; and
- an isomorphism $R$ from $(P_0 \amalg P_1)$ to $P|_{\partial W}$, respecting the reduction of structure group along $K_i$ and such that $\mathfrak{d}(R)$ fits into the obvious commutative diagram involving the other maps on the $\bZ(G)$-bundles – a condition which appears as $q_W \circ \mathfrak{d}(R) = (q_0 \amalg q_1)$.

For any path $z$ in this sense, we can construct a moduli space, generalizing our earlier $M_z(W, S, \Phi; \beta_0, \beta_1)$. To do this, we use the data $P, P_i, B_i$ and $R$ to construct a bundle $P^+$ on the cylindrical-end manifold $W^+$, together with a connection on the two cylindrical ends (obtained by pulling back the $B_i$). We extend this connection arbitrarily to a connection $A_W$ on the whole of $W^+$, with a singularity along $S^+$, and we write $\Theta_W^c$ for $\mathfrak{d}(A_W)$. We can then define a configuration space $C_z(W, S, \Phi; \beta_0, \beta_1)_{\delta_W}$ and quotient space $C_z(W, S, \Phi; \beta_0, \beta_1)_{\delta_W}$ using singular connections $A$ satisfying $\mathfrak{d}(A) = \Theta_W^c$ and with $A - A_W$ of class $L^2_m$. Introducing perturbations as before, we arrive at a moduli space $M_z(W, S, \Phi; \beta_0, \beta_1)_{\delta_W}$. The task of orienting this moduli space is the same as the case of a simple group $G$, with slight modifications drawn from section 2.10, so that when homology orientation of $W$ is given together with elements of $\Lambda(\beta_0)$ and $\Lambda(\beta_1)$, the moduli space is canonically oriented.

We summarize the situation with a proposition, generalizing Proposition 3.27 to the case of non-simple groups:

**Proposition 3.29.** Suppose that $\Phi$ satisfies the monotone condition (57). Let $(W, S)$ be a cobordism with boundary the two pairs $(Y_i, K_i)$ as above, let $\delta_W$ be a $\bZ(G)$-bundle and let $\delta_i$ be its restriction to $Y_i$. Suppose that the non-integrality condition Definition 3.28 holds at both ends and Hypothesis 3.19 holds, so that the Floer groups $I^*_*(Y_i, K_i, \Phi)_{\delta_i}$ are defined. Then, after choosing a homology orientation $\alpha_W$, there is a well-defined homomorphism

$$I(W, S, \Phi)_{\delta_W} : I^*_*(Y_0, K_0, \Phi)_{\delta_0} \to I^*_*(Y_1, K_1, \Phi)_{\delta_1}.$$ (91)
These homomorphisms satisfy the natural composition law when a cobordism \((W, S)\) is decomposed as the union of two pieces and \(\delta_W\) is restricted to each piece.

There is an important point about the naturality of this construction that is worth spelling out in more detail. Rather than taking \(\delta_i\) to be the restriction of \(\delta_W\) to the boundary component \(Y_i\), we could have taken the \(\bar{Z}(G)\)-bundles \(\delta_0\) and \(\delta_1\) to have been given in advance, in which case it is more natural to regard the necessary data on the cobordism \(W\) as consisting of

\begin{itemize}
  \item a \(\bar{Z}(G)\)-bundle \(\delta_W \to W\); and
  \item a pair of isomorphisms \(\tilde{r} = (\tilde{r}_0, \tilde{r}_1)\) from \(\delta_W|_{Y_i}\) to \(\delta_i\), \(i = 0, 1\).
\end{itemize}

In this setting, the induced map between the two instanton homology groups \(\mathbb{I}_*(Y, K, \Phi)\delta\) does depend on the choice of isomorphisms \(\tilde{r}_i\). The following corollary of Proposition 3.29 makes essentially the same point:

**Corollary 3.30.** Given \((Y, K)\) and a \(\bar{Z}(G)\)-bundle \(\delta\) satisfying as usual the conditions of Hypothesis 3.19, the group of components of the bundle automorphisms of \(\delta \to Y\) acts on the the instanton homology group \(\mathbb{I}_*(Y, K, \Phi)\delta\).

**Proof.** Given an automorphism \(g\) of \(\delta\), consider the cobordism \((W, S)\) that is the cylinder \([0, 1] \times (Y, K)\) and the bundle \(\delta_W\) which is the pull-back. We can identify \(\delta_W\) with \(\delta\) by using the identity map at the boundary component \(\{0\} \times Y\) and the map \(g\) at the other boundary component \(\{1\} \times Y\). From the data \((W, S)\) and \(\delta_W\) with these identifications, \((r_0, r_1) = (1, g)\), we obtain a homomorphism from \(\mathbb{I}_*(Y, K, \Phi)\delta\) to itself. \(\square\)

Some of the automorphisms of \(\delta\) act trivially on the Floer homology:

**Proposition 3.31.** Suppose that \(G\) is constructed from \(G_1 = [G, G]\) as in (50) and that Condition 2.18 holds. If \(G_1\) is \(SU(N)\), then suppose additionally that \(G = \tilde{U}(N)\) and that \(\epsilon\) is standard. (See Proposition 2.21.)

Then an automorphism \(g : \delta \to \delta\) of the \(\bar{Z}(G)\)-bundle \(\delta \to Y\) acts trivially on \(\mathbb{I}_*(Y, K, \Phi)\delta\) if \(g\) has the form \(\mathfrak{d}(f)\) for some \(Z(G)\)-valued automorphism \(f : P \to P\) of the corresponding bundle \(P \to Y\).

**Proof.** Take \(W\) to be the cylinder \([0, 1] \times Y\), and let \(\delta_W\) be as in the proof of the previous corollary. If \(g = \mathfrak{d}(f)\), then we can describe \(\delta_W\) as \(\delta_1 \otimes \mathfrak{d}(\epsilon)\), where \(\delta_1\) is the pull-back bundle \([0, 1] \times \delta\) and \(\epsilon \to W\) is a \(Z(G)\) bundle equipped with a trivialization at each boundary component of \(W\). That is,
we take \( \epsilon \) to be \( W \times Z(G) \), with the trivialization 1 at \( \{0\} \times Y \) and \( f \) at \( \{1\} \times Y \).

We are therefore left to compare two maps from \( \mathbb{I}_s(Y, K, \Phi)_\delta \) to itself: the first is the identity map, arising from the product cobordism \( W \) with \( \delta_1 \); and the second is the map obtained from \( W \) with \( \delta_1 \otimes \vartheta(\epsilon) \). This is essentially the same situation as the construction of the map \( \mu_\epsilon \) in (61). In particular, the moduli spaces that are involved in defining the two maps are identical, and the only question is whether the zero-dimensional moduli spaces arise with the same sign. Proposition 2.21 tells us that \( \mu_\epsilon \) preserves orientation in all cases except \( SU(N) \) and \( E_6 \). For the \( SU(N) \) case, \( \mu_\epsilon \) still preserves orientation because the cobordism \( W \) has even intersection form on its relative homology. For the case of \( E_6 \), an examination of the proof of Proposition 2.21 shows that orientation depends on a term \( \rho(\bar{e}) \sim \rho(\bar{e}) \); so again, the even intersection form ensures that \( \mu_\epsilon \) is orientation-preserving.

The bundle automorphisms of \( \delta \) are the maps \( Y \rightarrow \tilde{Z}(G) \) and the group of components is \( H^1(Y; \pi_1(\tilde{Z}(G))) \). The above Proposition tells us that the image of \( H^1(Y; \pi_1(Z(G))) \) acts trivially. Under the hypotheses of Condition 2.18, we have a short exact sequence

\[
\pi_1(Z(G)) \rightarrow \pi_1(\tilde{Z}(G)) \rightarrow Z(G_1)
\]

in which the first two groups are free abelian and the first map is multiplication by \( p \). From the corresponding long exact sequence in cohomology, we learn that the largest group that may act effectively on \( \mathbb{I}_s(Y, K, \Phi)_\delta \) via this construction is isomorphic to the subgroup

\[
\mathcal{H} \subset H^1(Y; Z(G_1))
\]

consisting of the elements with integer lifts:

\[
\mathcal{H} = \text{im}\left( H^1(Y; \pi_1(\tilde{Z}(G))) \rightarrow H^1(Y; Z(G_1)) \right).
\]

As explained in section 2.10 (where we treated the 4-dimensional case), we can also regard the automorphisms of \( \delta \) as defining, rather directly, automorphisms of the configuration space \( \mathcal{B}(Y, K, \Phi)_\delta \). Were it the case that the holonomy perturbations could be chosen to be invariant under the action of \( \mathcal{H} \) while still achieving the necessary transversality, then we would have a more direct way of understanding the action of \( \mathcal{H} \) on the instanton homology: the action on the set of critical points would give rise to an action of \( \mathcal{H} \).
on the chain complex $C_*(Y,K,\Phi)$ by chain maps. Although perturbations cannot always be chosen so as to realize the action in this way, the following situation does arise in some cases:

**Proposition 3.32.** Suppose that a subgroup $\mathcal{H}' \subset \mathcal{H}$ acts freely on the set of critical points for the unperturbed Chern-Simons functional $CS$ in $\mathcal{B}(Y,K,\Phi)_\delta$. Then a holonomy perturbation $\pi$ can be found, as in Proposition 3.12, that is invariant under $\mathcal{H}'$, such that the critical point set $\mathcal{C}_\pi$ is non-degenerate, the action of $\mathcal{H}'$ on $\mathcal{C}_\pi$ is still free, and the moduli spaces $M_z(\alpha,\beta)$ are all regular.

Proof. A cylinder function arising from a collection of loops and a map $h : G^r \to \mathbb{R}$ will be invariant under the action of $\mathcal{H}'$ on $\mathcal{B}(Y,K,\Phi)_\delta$ provided that $h$ is invariant under an associated action of $\mathcal{H}'$ on $G^r$ (given by multiplications by central elements). This gives us a means to construct invariant perturbations, and the statements about the critical point set are straightforward. For the moduli spaces $M_z(\alpha,\beta)$, we note that, by unique continuation, once the action on $\mathcal{C}_\pi$ is known to be free, it must also be free on the subset of $\mathcal{B}(Y,K,\Phi)$ consisting of all points lying on gradient trajectories between critical points. Once the action is known to be free here, the transversality arguments go through without change.

4 Classical knots and variants

4.1 Summing with a 3-torus

Take $G$ to be the group $U(N)$, let $Y$ be any closed, oriented 3-manifold and $K \subset Y$ an oriented knot or link. Let $y_0$ be a base-point in $Y \setminus K$, and let an oriented frame in $T_{y_0}Y$ be chosen. Using the base-point and frame, we can form the connected sum $Y \# T^3$ in a manner that makes the result unique to within a canonical isotopy class of diffeomorphisms. The knot or link $K$ now becomes a knot or link in the connected sum. Any topological invariant that we define for the pair $(Y \# T^3,K)$ becomes an invariant of the original pair $(Y,K)$ together with its framed basepoint.

Regard the 3-torus as a product, $S^1 \times T^2$, and let $\delta_1 \to T^3$ be a $U(1)$ bundle with $c_1(\delta_1)$ Poincaré dual to $S^1 \times \{\text{point}\}$. Extend $\delta_1$ trivially to the connected sum $Y \# T^3$, and call the resulting $U(1)$-bundle $\delta$. If $P$ is a $U(N)$ bundle on $Y \# T^3$ whose determinant is $\delta$, then $P$ satisfies the non-integral condition, Definition 3.28, for any choice of $\Phi$ in the positive Weyl chamber. (See the remarks immediately following that definition.) The remaining condition in Hypothesis 3.19 (that the stabilizer of each critical point is just
\( Z(\mathcal{G}) \) is automatically satisfied in the case of \( U(N) \) once the non-integrality condition holds. There is therefore a well-defined Floer homology group (in the notation of section 3.7),
\[
\mathbb{I}_*(Y \# T^3, K, \Phi)_\delta
\]
for any choice \( \Phi \) in the positive Weyl chamber which satisfies the monotone condition.

Let us consider a particular choice of \( \Phi \) in this context, namely
\[
\Phi = \text{diag}(i/2, 0, \ldots, 0).
\]
This \( \Phi \) satisfies the monotone condition, and its orbit in \( \mathfrak{g} = \mathfrak{u}(N) \) is a copy of \( \mathbb{C}P^{N-1} \). The group element \( \exp(2\pi \Phi) \) has order 2: it is
\[
\text{diag}(-1, 1, \ldots, 1).
\] (92)

We introduce a notation for the corresponding instanton homology group:

**Definition 4.1.** We write \( FI_*^N(Y, K) \) for the instanton homology group
\[
\mathbb{I}_*(Y \# T^3, K, \Phi)_\delta
\]
in the case that \( G = U(N) \), with \( \delta \) and \( \Phi \) as above. In the case that \( Y \) is \( S^3 \), we simply write \( FI_*^N(K) \); and in the case that \( N = 2 \), we write \( FI_*^N(Y, K) \) or \( FI_*^N(K) \). The group \( FI_*^N(Y, K) \) has an affine grading by \( \mathbb{Z}/(2N) \) (see the remark following Definition 3.26).

**Remark.** If \( \Phi \) is changed by the addition of a central element of \( \mathfrak{u}(N) \), then the resulting instanton homology is essentially unchanged. Thus, we could equally well have taken \( \Phi \) to be the element
\[
\Phi' = i\text{diag}(1/2, 0, \ldots, 0) - (i/(2N))\text{diag}(1, 1, \ldots, 1)
\] (93)
in \( \mathfrak{su}(N) \), so that
\[
\exp(2\pi \Phi') = e^{-\pi i/N} \text{diag}(-1, 1, \ldots, 1) \in SU(N).
\] (94)

To examine what comes of this definition, let us begin by looking at \( FI_*^N(\emptyset) \), i.e. the case of the the empty link in \( S^3 \). In other words, we are looking at \( \mathbb{I}_*(T^3)_\delta \). Let \( a, b \) and \( c \) be standard generators for the fundamental group of \( T^3 \), with \( a \) and \( b \) generating the fundamental group of the \( T^2 \) factor in \( T^3 = T^2 \times S^1 \). Let \( p \in T^2 \) be a point not lying on the \( a \) or \( b \) curves, and let \( D \) be a small disk around \( p \). We can take the line bundle \( \delta_1 \) to be
pulled back from $T^2$, with a trivialization on the complement of $D \times S^1$. Let $\Theta_1$ be a connection in the line bundle $\delta_1 \rightarrow T^3$ that is also pulled back from $T^2$. We can take the $\Theta_1$ to respect the trivialization of $\delta_1$ on the complement $D \times S^1$, so that the curvature of $\Theta_1$ is a 2-form supported in that neighborhood. If $A \in \mathcal{B}(T^3)_{\delta_1}$ is a critical point for the Chern-Simons functional, then the restriction of $A$ to $T^2 \setminus D$ is a flat $SU(N)$ connection whose holonomy around $\partial D$ is the central element $e^{2\pi i/N}$ in $SU(N)$. In this way, the critical points correspond to conjugacy classes of triples $\{h(a), h(b), h(c)\}$ in $SU(N)$ (the holonomies around the three generators) satisfying

$$[h(a), h(b)] = e^{2\pi i/N},$$

$$[h(a), h(c)] = 1,$$

$$[h(b), h(c)] = 1.$$

There are $N$ different solutions to these conditions (see also [20]): the element $h(c)$ can be any of the $N$ elements of the center of $SU(N)$, and up to the action of $SU(N)$, we must have

$$h(a) = \epsilon \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ 0 & 0 & \zeta^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \zeta^{N-1} \end{bmatrix},$$

$$h(b) = \epsilon \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$h(c) = \zeta^k \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad k \in \{0, 1, \ldots, N-1\}.$$  

(95)

Here, $\zeta = e^{2\pi i/N}$, and $\epsilon$ is 1 if $N$ is odd and an $N$th root of $-1$ if $N$ is even.

Thus the Chern-Simons functional has exactly $N$ distinct critical points in $\mathcal{B}(T^3)_{\delta_1}$. These critical points are irreducible (as they must be, on account of the coprime condition); and it is shown in [20] that these $N$ critical points
are non-degenerate. We can now describe the critical points in the case of a general \((Y, K)\).

**Proposition 4.2.** For any oriented pair \((Y, K)\), the set of critical points of the Chern-Simons functional on \(\mathcal{B}(Y \# T^3, K, \Phi) \delta\) consists of \(N\) disjoint copies of the space of representations

\[
\rho : \pi_1(Y \setminus K) \to SU(N)
\]  

satisfying the condition that, for each oriented meridian \(m\) of the knot or link \(K\), the element \(\rho(m)\) is conjugate to \((94)\).

**Proof.** Rather than consider \(\mathcal{B}(Y \# T^3, K, \Phi) \delta\) as in the statement of the proposition, we may alternatively consider the isomorphic space

\[
\mathcal{B}(Y \# T^3, K, \Phi') \delta
\]

with the alternative element \(\Phi' \in su(N)\) from \((93)\). As in the discussion of \(T^3\) above, the critical points in \(\mathcal{B}(Y \# T^3, K, \Phi') \delta\) correspond to flat \(SU(N)\) connections on the complement of \(K \amalg (D \times S^1)\) in \(Y \# T^3\), such that the holonomy around \(\partial D\) is \(e^{2\pi i/N}\) and \(\rho(m)\) is in the conjugacy class of \((94)\) for all oriented meridians \(m\). The fundamental group of the complement in this connected sum is a free product, so the result follows from the fact that there are \(N\) such flat connections on the \(T^3\) summand, all of which are irreducible: see the discussion surrounding \((4)\) in the introduction. \(\square\)

**Corollary 4.3.** In the case of the 3-sphere, and an oriented classical knot or link \(K \subset S^3\), the set of critical points of the Chern-Simons functional on \(\mathcal{B}(S^3 \# T^3, K, \Phi) \delta\) consists of \(N\) disjoint copies of the space of representations

\[
\rho : \pi_1(S^3 \setminus K) \to U(N)
\]  

satisfying the condition that, for each oriented meridian \(m\) of the knot or link \(K\), the element \(\rho(m)\) is conjugate to \((92)\).

**Proof.** In the case of \(S^3\), the meridians generate the first homology of the complement of the link, and the space of \(U(N)\) representations that appears here is therefore identical to the space of \(SU(N)\) representations in the proposition above. \(\square\)

As a special case, we have:
Corollary 4.4. For the unknot $K$ in $S^3$, the set of critical points

$$\mathcal{C} \subset \mathcal{B}(S^3 \# T^3, K, \Phi)$$

consists of $N$ disjoint copies $\mathbb{C}P^{N-1}$. Furthermore, the Chern-Simons function is Morse-Bott: at points of $\mathcal{C}$, the kernel of the Hessian is equal to the tangent space to $\mathcal{C}$.

Proof. Since the fundamental group of the knot complement is $\mathbb{Z}$, a homomorphism $\rho$ as in the previous corollary is determined by the image of a meridian $m$ in the conjugacy class (92). This conjugacy class in $U(N)$ can be identified with $\mathbb{C}P^{N-1}$ by sending an element to its $(-1)$-eigenspace. The kernel of the Hessian can be computed from Lemma 3.13, to establish the Morse-Bott property.

We introduce a notation for the space of representations that appears in the previous proposition:

Definition 4.5. We write

$$\mathcal{R}(Y, K, \Phi') \subset \text{Hom}(\pi_1(Y \setminus K), SU(N))$$

for the space of homomorphisms $\rho$ such that, for all meridians $m$, the element $\rho(m)$ is conjugate to (94).

Recall from section 3.10 that the set of critical points is acted on by the group $\mathcal{H}$, which in our present case is the subgroup

$$\mathcal{H} \subset H^1(Y \# T^3; \mathbb{Z}/N)$$

consisting of elements with integer lifts. Let

$$\mathcal{H}' \cong \mathbb{Z}/N$$

$$\mathcal{H}' \subset \mathcal{H}$$

be the subgroup of $H^1(T^3; \mathbb{Z}/N)$ consisting of elements which are non-zero only on the generator $c$ in $\pi_1(T^3)$ and are zero on the generators $a$ and $b$.

Lemma 4.6. The action of this copy, $\mathcal{H}'$ of $\mathbb{Z}/N$ on the set of critical points, $\mathcal{C}$, in $\mathcal{B}(Y \# T^3, K, \Phi)_\delta$ is to cyclically permute the $N$ copies of $\mathcal{R}(Y, K, \Phi')$ that comprise $\mathcal{C}$ according to the description in Proposition 4.2.

Proof. From our description of $\mathcal{C}$, it follows that it sufficient to check the lemma for the case of the $N$ critical points in $\mathcal{B}(T^3)_\delta$, (i.e. the case that $Y$ is $S^3$ and $K$ is empty). These critical points are described in (95), and the action of $\mathcal{H}' \cong \mathbb{Z}/N$ is to multiply $h(c)$ by the $N$th roots of unity. □
We can compare the values of the Chern-Simons functional on the $N$ copies of $\mathcal{R}(Y, K, \Phi')$ in $\mathfrak{c}$. For example, in the case of the unknot, because $\mathbb{C}P^{N-1}$ is connected, the functional is constant on each copy and we can compare the $N$ values. The general case is the next lemma.

**Lemma 4.7.** Let $s \in \mathcal{H}'$ be the generator that evaluates to 1 on the generator $c$ in $T^3$. Then for any $\alpha = [A]$ in $\mathcal{B}(Y \# T^3, K, \Phi)_{\delta}$ we have

$$CS(s(\alpha)) - CS(\alpha) = -16\pi^2(N - 1)$$

modulo the periods of the Chern-Simons functional.

**Proof.** The calculation reduces to a calculation for a connection $[A]$ on $T^3$ and its image under $s$. Pull back the $U(N)$ bundle $P_1$ (with $\vartheta(P_1) = \delta_1$) to the cylinder $[0, 1] \times T^3$. Identify the two ends to form $S^1 \times T^3$, gluing the bundle $P_1$ using an automorphism $f$ of $P_1$ with $\vartheta(f) = u$. Let $P \to S^1 \times T^3$ be the resulting $U(N)$ bundle. We have

$$c_1(P) = PD([S^1 \times c] + [a \times b])$$

modulo multiples of $N$. The change in the Chern-Simons functional is half the “topological energy” $E(P)$ on $S^1 \times T^3$, so

$$CS(s(\alpha)) - CS(\alpha) = -8\pi^2p_1(g_P)[S^1 \times T^3],$$

from (56). Using the relation (55), we obtain

$$CS(s(\alpha)) - CS(\alpha) = -8\pi^2c_1(P)^2[S^1 \times T^3]$$

modulo periods, and the final result follows from the above formula for $c_1(P)$, which gives

$$c_1(P)^2[S^1 \times T^3] = 2 \pmod{N}.$$ 

We can also consider the relative grading for the pair of points $\alpha$ and $s(\alpha)$ in $\mathcal{B}(Y \# T^3, K, \Phi)_{\delta}$ along a suitable chosen path $z$:

$$\text{gr}_z(s(\alpha), \alpha) \in \mathbb{Z}.$$

Note that this relative grading is easy to interpret unambiguously, even when the Hessian at $\alpha$ has kernel, essentially because the Hessians at $\alpha$ and $s(\alpha)$ are isomorphic operators.
Lemma 4.8. Let $z$ be a path in $\mathcal{B}(Y \# T^3, K, \Phi)_{\delta}$ along which a single-valued lift of the Chern-Simons functional satisfies (66). Then along this path we have

$$\text{gr}_z(s(\alpha), \alpha) = -4(N - 1).$$

Proof. Concatenating $z$ with its image under the maps $s$, $s^2$, $\ldots$, $s^{N-1}$, we obtain a closed loop, along which the total energy $E$ is $-32\pi^2(N - 1)N$. From the monotone relationship between the dimension and energy, we have that the spectral flow along the closed loop is $-4(N - 1)N$. The spectral flow along each part is therefore $-4(N - 1)$, because each part makes an equal contribution. \[\square\]

According to Proposition 3.32, we can choose a holonomy perturbation $\pi$ which is invariant under $\mathcal{H}'$ while still making the critical point set non-degenerate and the moduli spaces regular. If follows that we have an action of $\mathcal{H}'$ on $FI^N_*(Y,K)$ resulting from this geometric action on the configuration space. (Without using an invariant perturbation, the action can still be defined – using cobordisms – by the procedure described in subsection 3.10.) Since the grading in $FI^N_*(Y,K)$ is only defined modulo $2N$, we can interpret the last lemma above as saying that the action of $s$ gives an automorphism of $FI^N_*(Y,K)$ of degree 4:

$$s_* : FI^N_j(Y,K) \rightarrow FI^N_{j+4}(Y,K).$$

(The subscript is to be interpreted mod $2N$.)

Whereas the construction using cobordisms only gives us an action on the homology, the geometric action on the configuration space gives us an action on the chain complex. So, rather than consider the action of $\mathcal{H}' \cong \mathbb{Z}/N$ on the instanton homology group, we can instead consider dividing $\mathcal{B}(Y \# T^3, K, \Phi)_{\delta}$ by the action of $\mathcal{H}'$ and then taking the Morse homology. We can interpret Proposition 3.32 as telling us that $\pi$ can be chosen so that the Morse construction works appropriately on $\mathcal{B}(Y \# T^3, K, \Phi)_{\delta}/\mathcal{H}'$. The relative grading on the Morse complex is defined mod 4 if $N$ is even, and mod 2 if $N$ is odd.

Definition 4.9. We define $\tilde{FI}^N_*(Y,K)$ to be the homology of the quotient chain complex $\tilde{C}_*(Y \# T^3, K, \Phi)_{\delta}$: the quotient of $C_*(Y \# T^3, K, \Phi)_{\delta}$ by the action of $\mathbb{Z}/N$. We write $\tilde{FI}^N_*(K)$ in the case that $Y$ is the 3-sphere.

We can calculate this group in the case of the unknot.
Proposition 4.10. For the unknot $K$ in $S^3$, we have

$$\tilde{FI}^N_*(S^3, K) \cong H_*(\mathbb{C}P^{N-1}; \mathbb{Z})$$
$$\cong \mathbb{Z}^N,$$

$$FI^N_*(S^3, K) \cong H_*(\mathbb{C}P^{N-1} \sqcup \cdots \sqcup \mathbb{C}P^{N-1}; \mathbb{Z}), \quad (N \text{ copies})$$
$$\cong \mathbb{Z}^{N^2}.$$ 

Proof. The group $\tilde{FI}^N_*(S^3, K)$ is the homology of $\mathcal{C}_*(Y \# T^3, K, \Phi)_\delta$, and this chain complex has generators corresponding to the points of $\mathcal{C}_\pi/\mathcal{H}'$, for suitable holonomy perturbation $\pi$. Before perturbation, $\mathcal{C}_\pi$ consists of $N$ copies of $\mathbb{C}P^{N-1}$ and $\mathcal{H}'$ is a copy of $\mathbb{Z}/N$ which permutes the $N$ copies cyclically. So $\mathcal{C}_\pi/\mathcal{H}'$ is a single copy of $\mathbb{C}P^{N-1}$.

Choose a holonomy perturbation $\pi_1$ which is invariant under $\mathcal{H}'$ and is such that the corresponding function $f_1$ on $B(S^3 \# T^3, K)_\delta$ has the property that $f_1|_{\mathcal{C}}$ is a standard Morse function with even-index critical points on each copy of $\mathbb{C}P^{N-1}$. Then set $\pi_\epsilon = \epsilon \pi_1$ and take $\epsilon$ a small, positive quantity. Because the Chern-Simons functional is Morse-Bott, an application of the implicit function theorem and the compactness theorem for critical points shows that, for $\epsilon$ sufficiently small, the perturbed critical set $\mathcal{C}_{\pi_\epsilon}/\mathcal{H}'$ consists of exactly $N$ critical points. As $\epsilon$ goes to zero, these converge to the $N$ critical points of $f_1|_{\mathbb{C}P^{N-1}}$, and the relative grading of the points in $\mathcal{C}_{\pi_\epsilon}$ along paths in the neighborhood of $\mathcal{C}$ is equal to the relative Morse grading of the corresponding critical points of $f_1|_{\mathbb{C}P^{N-1}}$. It follows that, for this perturbation, the complex has $N$ generators, all of which are in the same grading mod 2. \[\square\]

In the case $N = 2$, the invariant $\tilde{FI}_*(K)$ of classical knots appears to resemble Khovanov homology in the simplest cases. As mentioned in the introduction, it is natural to ask whether we have

$$\tilde{FI}_*(K) \cong Kh(K)$$

for the $(2, p)$, $(3, 4)$ and $(3, 5)$ torus knots, for example.

4.2 Bridge number and representation varieties

For a knot $K$ in $S^3$, the knot group (i.e. the fundamental group of the knot complement) is generated by the conjugacy class of the meridian. If we choose meridional elements $m_1, \ldots, m_k$ which generate the knot group,
then a homomorphism $\rho$ as in (97) from the knot group to $U(N)$ is entirely determined by $k$ elements

$$A_i = \rho(m_i)$$

in the conjugacy class of the reflection (92); or equivalently, $k$ points in $\mathbb{CP}^{N-1}$. The $-1$-eigenspaces of the reflections $A_i$ will span at most a $k$-plane in $\mathbb{C}^N$. It follows that $\rho$ is conjugate in $U(N)$ to a representation whose image lies in $U(k) \subset U(N)$.

In this sense, the problem of describing the space of representations $\rho$ stabilizes at $N = k$. For larger $N$, the homomorphisms $\rho$ to $U(N)$ are obtained from the homomorphisms to $U(k)$ by conjugating by elements of the larger unitary group. An upper bound for the number $k$ of meridians that are needed to generate the knot group is the bridge number of the knot. So for a $k$-bridge knot, the problem of computing the set of critical points $\mathcal{C} \subset B(S^3 \# T^3, K, \Phi_\delta)$ for the group $U(N)$ can be reduced to the corresponding problem for $U(k)$ (though the critical point sets are not the same).

The simplest example after the unknot is the trefoil, a 2-bridge knot. The group is generated by a pair of meridians, and for $N = 2$ a representation $\rho$ is therefore determined by a pair of points in $\mathbb{CP}^1 \cong S^2$. The relation between the generators in the knot group implies that these two points in $S^2$ either coincide or make an angle $2\pi/3$. The set of all such representations for $N = 2$ is therefore parametrized by one copy of $S^2$ and one copy of $SO(3)$. When $N$ is larger, we have essentially the same classification: a representation is determined by a pair of points in $\mathbb{CP}^{N-1}$, and either these coincide, or they lie at angle $2\pi/3$ from each other along the unique $\mathbb{CP}^1$ that contains them both. These two components are a copy of $\mathbb{CP}^{N-1}$ itself and a copy of the unit sphere bundle in $T\mathbb{CP}^{N-1}$ respectively. The authors conjecture that for the trefoil, $FI^N_*(K)$ is isomorphic to the direct sum of the homologies of these two components of the critical set of the unperturbed functional.

For a general knot $K$, the critical set $\mathcal{C}$, after perturbation, determines the set of generators of the complex that computes $FI^N_*(K)$. It would be interesting to know whether there is any sort of stabilization that occurs for the differentials in the complex, as $N$ increases. The situation is reminiscent of the large-$N$ stabilization for Khovanov-Rozansky homology that is discussed in [10, 37].

### 4.3 A reduced variant

In the construction of $FI^N_*(Y, K)$, the important feature of the manifold $T^3$ with which we formed the connected sum was that, for a suitable choice of $U(N)$ bundle $P_1 \to T^3$, the corresponding set of critical points $\mathcal{C}(T^3, P_1)$ was
just a finite set of reducibles (a single orbit of the finite group $H' \cong \mathbb{Z}/N$). Rather than $T^3$, we can consider the pair $(S^1 \times S^2, L)$, where $L \subset S^1 \times S^2$ is the $(N+1)$-component link

$$L = S^1 \times \{p_0, \ldots, p_N\}.$$ 

Let $P_0 \to S^1 \times S^2$ be the trivial $SU(N)$ bundle and let $\Phi' \in \mathfrak{su}(N)$ be the element (93). In the space of singular connections $B(S^1 \times S^2, L, \Phi')$, consider again the set of critical points:

$$\mathcal{C}(S^1 \times S^2, L, \Phi') \subset B(S^1 \times S^2, L, \Phi').$$

The pair $(S^1 \times S^2, L)$ with this choice of $\Phi'$ fits the hypotheses of Corollary 2.8, and it follows that the critical set consists only of irreducible flat connections.

**Lemma 4.11.** The critical set $\mathcal{C}(S^1 \times S^2, L, \Phi')$ consists of exactly $N$ non-degenerate, irreducible points. These form a single orbit of the group $H = H^1(S^1 \times S^2; \mathbb{Z}/N)$.

**Proof.** The critical set parametrizes conjugacy classes of homomorphisms

$$\rho : \pi_1(S^1 \times S^2 \setminus L) \to SU(N)$$

such that $\rho$ of each meridian of $L$ is conjugate to $\exp(\Phi')$. The fundamental group is a product, with a $\mathbb{Z}$ factor coming from the $S^1$. The lemma will follow if we can show that there is only a single, irreducible, conjugacy class of homomorphisms

$$\sigma : \pi_1(S^2 \setminus \{p_1, \ldots, p_{N+1}\}) \to SU(N)$$

such that $\sigma$ sends the linking circle of each $p_j$ into the conjugacy class of $\exp(\Phi')$. The classification of such homomorphisms $\sigma$ can be most easily achieved by using the correspondence with stable parabolic bundles on the Riemann sphere with $(N+1)$ marked points. In this instance, the relevant parabolic bundles are holomorphic bundles $\mathcal{E} \to \mathbb{C}P^1$ of rank $N$ and degree 0, equipped with a distinguished line $\mathcal{L}_i$ in the fiber $\mathcal{E}_{p_i}$ for each $i$. The appropriate stability condition for such a parabolic bundle $(\mathcal{E}, \{\mathcal{L}_i\})$ is that, for every proper holomorphic subbundle $\mathcal{F} \subset \mathcal{E}$, we should have

$$\#\{i \mid \mathcal{L}_i \subset \mathcal{F}_{p_i}\} + 2\deg(\mathcal{F}) \leq \text{rank}(\mathcal{F}).$$

The only solution to these constraints is to take $\mathcal{E}$ to be the trivial bundle $\mathcal{O} \otimes \mathbb{C}^N$ and to take $\mathcal{L}_i$ to be $\mathcal{O}_{p_i} \otimes L_i$, where the lines $L_i$ define $N+1$ points in general position in $\mathbb{C}P^{N-1}$. There is therefore a single homomorphism $\sigma$ satisfying the given conditions. \qed
Now let \((Y, K)\) be an arbitrary pair, and let \(k_0\) be a basepoint on \(K\). Choose a framing of \(K\) at \(k_0\), and use this framing data to form the connected sum of pairs

\[(\hat{Y}, \hat{K}) = (Y, K) \# (S^1 \times S^2, L),\]

connecting the component of \(K\) containing \(k_0\) to the component \(S^1 \times \{p_0\}\) of \(L\). We define a reduced version of the framed instanton homology by setting

\[RI^N_\ast(Y, K) = \mathbb{I}_\ast(\hat{Y}, \hat{K}, \Phi').\]

Like \(FI^N_\ast(Y, K)\), this group has an affine grading by \(\mathbb{Z}/(2N)\).

The definition is such that, for the case that \(Y\) is \(S^3\) and \(K\) is the unknot, the pair \((\hat{Y}, \hat{K})\) is simply \((S^1 \times S^2, L)\). The lemma above thus tells us that, in this case, the complex that computes \(RI^N_\ast\) has \(N\) generators. The relative grading of these generators is even, and we therefore have

\[RI^N_\ast(S^3, \text{unknot}) = \mathbb{Z}^N.\]

The set of critical points in \(\mathcal{B}(\hat{Y}, \hat{K}, \Phi')\) for a general \((Y, K)\) can be described by a version of Proposition 4.2. Let us again write \(\mathcal{R}(Y, K, \Phi')\) for the space of homomorphisms described in Definition 4.5. As a basepoint for the fundamental group \(\pi_1(Y \setminus K)\) let us choose the push-off the the chosen point \(k_0 \in K\) using the framing. We then have a preferred meridian, \(m_0\), in \(\pi_1(Y \setminus K)\) linking \(K\) near this basepoint, and hence an evaluation map taking values in the conjugacy class \(C(\exp \Phi')\) of the element \(\exp(\Phi')\):

\[ev : \mathcal{R}(Y, K, \Phi') \to C(\exp \Phi')\]

\[\rho \mapsto \rho(m_0).\]  

(100)

The counterpart to Proposition 4.2 is then:

**Proposition 4.12.** For any oriented pair \((Y, K)\), the set of critical points of the Chern-Simons functional on \(\mathcal{B}(\hat{Y}, \hat{K}, \Phi')\) consists of \(N\) copies of the fiber of the evaluation map (100).

Thus, for example, in the case of the trefoil, the set of critical points consists of \(N\) points and \(N\) copies of the sphere \(S^{2N-3}\).

As in the case of \(FI^N_\ast\), it is possible to pass to the quotient of the configuration space by the action of the cyclic group \(\mathcal{H}'\). The result is a variant, \(RI^N_\ast(Y, K)\), which is isomorphic to \(\mathbb{Z}\) in the case of the unknot. In the quotient space \(\mathcal{B}(\hat{Y}, \hat{K}, \Phi')/\mathcal{H}'\), the set of critical points consists of just one copy of the fiber of the evaluation map \(ev\) above.
Rasmussen [39] has observed that the reduced Khovanov homology coincides with Heegaard-Floer knot homology group \( \widehat{HFK}(K) \), defined by Ozsváth and Szabó in [33] and by Rasmussen in [38], for many knots, but not for the \((4,5)\) torus knot. It would therefore be interesting to have some data comparing the reduced group \( \overline{RI}_*(K,k_0) \) with \( \widehat{HFK}(K) \).

### 4.4 Longitudinal surgery

Another variant briefly mentioned in the introduction is to begin with a null-homologous knot \( K \) in an arbitrary \( Y \), and form the pair \((Y_K, K_0)\), where \( Y_K \) is the 3-manifold obtained by 0-surgery (longitudinal surgery), and \( K_0 \) is the core of the solid torus used in the surgery. The knot \( K_0 \) represents a primitive element in the first homology of the manifold \( Y_K \). We can apply our basic construction to this pair (without taking a further connected sum), with \( G = SU(N) \) as usual. To avoid reducibles, we can take \( \Phi \) to have just two distinct eigenvalues with coprime multiplicity (see Corollary 2.7). In particular, we may again take the \( \Phi' \) given in equation (93). We make a definition to cover this case:

**Definition 4.13.** For a null-homologous knot \( K \) in a 3-manifold \( Y \), we define \( LI_*^N(Y,K) \) to be the result of applying the standard construction \( I_* \) to the oriented pair \((Y_K, K_0)\), with \( G = SU(N) \) and \( \Phi \) given by (93).

For \( N = 2 \), we just write \( LI_*(Y,K) \), and if \( Y = S^3 \) we drop the \( Y \) from our notation.

The complement of \( K_0 \) in \( Y_K \) is homeomorphic to the original complement of \( K \) in \( Y \), and the meridian of \( K_0 \) corresponds to the longitude of \( K \). Thus we see that the set of critical points of the Chern-Simons function in the configuration space for \((Y_K, K_0)\) can be identified with the space of conjugacy classes of homomorphisms

\[
\rho : \pi_1(Y \setminus K) \to SU(N)
\]

satisfying the constraint that \( \rho \) maps the longitude of \( K \) to an element conjugate to \( \exp(2\pi i \Phi) \). In the case of the unknot in \( S^3 \), the group is \( \mathbb{Z} \) and the longitude represents the identity element, so the set of critical points is empty. For the unknot therefore, the group \( LI_*^N(K) \) is zero. The same applies to an “unknot” in any \( Y \), i.e. a knot that bounds a disk.

In part because of the results of [24] and [23], it is natural to conjecture that \( LI_*(Y,K) \) is zero only if \( K \) is an unknot. An examination of the representation variety, and a comparison with Casson’s work [1], suggests
that the Euler characteristic of $LI_s(K)$ should be $2\Delta'_K(1)$, where $\Delta_K$ is the symmetrized Alexander polynomial. In the case of a torus knot $K$, the representation variety of homomorphisms $\rho$ satisfying the longitudinal constraint

$$\rho(\text{longitude}) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

consists of exactly $2\Delta''_K(1)$ points, and it would follow that

$$LI_s(K) = \mathbb{Z}^{2\Delta''_K(1)}$$

for all torus knots. (For the $(2,p)$ torus knot with $p > 3$, it would follow from this that $LI_s(K)$ is not isomorphic to Eftekhary’s longitude variant of Heegaard Floer homology [12]; because for the $(2,p)$ knot, the quantity $\Delta''_K(1)$ grows quadratically with $p$, while the rank of Eftekhary’s knot invariant grows linearly.)

5 Filtrations and genus bounds

In this section, we take up a theme from the introduction and explore how a lower bound for the slice genus of a knot can be obtained from (a variant of) instanton Floer homology. In doing so, we will see that the formal outline of the construction can be made to resemble closely the work of Rasmussen in [36], while the underlying mechanism of the proof draws on [19].

From this point on, we will work exclusively with the group $G = SU(2)$ and the element $\Phi = i \text{diag}(1/4, -1/4)$ in the fundamental alcove (this is the only balanced case for the group $SU(2)$). We will write $FI_s(K)$ for the framed instanton homology of a classical knot, as described in Definition 4.1, and $FI_s(Y, K)$ when the knot is in a 3-manifold other than $S^3$. To keep the notation to a minimum, we will treat only classical knots to begin with, though little changes when we generalize to knots in other 3-manifolds.

5.1 Laurent series and local coefficients

Let $K \subset S^3$ be an oriented knot, and let us write

$$\mathcal{B}(K) = \mathcal{B}(S^3 \# T^3, K, \Phi)_{\delta}.$$

for the configuration space that is used in defining the framed instanton Floer homology group, Definition 4.1. As in section 3.9, we will consider the Floer homology of $K$ coming from a system of local coefficients on the configuration space $\mathcal{B}(K)$. Specifically, we let $\mu : \mathcal{B} \to U(1) = \mathbb{R}/\mathbb{Z}$ be a
circle-valued function arising from the standard character $G \Phi \to U(1)$, using
the holonomy construction of (89); and we let $\Gamma = \Gamma^\mu$ be the corresponding
local system pulled back from $\mathbb{R}/\mathbb{Z}$, as described in section 3.9. Thus $\Gamma$ is a
local system of free modules of rank 1 over the ring

$$R = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{-1}, t].$$

Thus for each knot $K$, we have a finitely-generated $R$-module

$$FI_\ast(K; \Gamma).$$

We should recall at this point that the construction of the local system $\Gamma^\mu$
really depends on a choice of framing $n$ for the knot $K$, but that the local sys-
tems arising from different choices of framings are all canonically identified
(see section 3.9 again). So we should regard $\Gamma$ as the common identification
of a collection of local systems $\Gamma^\mu_n$ as $n$ runs through all framings.

A cobordism of pairs, $(W, S)$, from $(S^3, K_0)$ to $(S^3, K_1)$ gives rise to a
homomorphism of the corresponding Floer groups, by the recipe described
at (90). (The way we have set it up, a framing of the normal bundle to $S$ is
needed in order to define the map, but the resulting map is independent of
the choices made.) We will abbreviate our notation for the map and write

$$\psi_{W,S} : FI_\ast(K_0; \Gamma) \to FI_\ast(K_1; \Gamma).$$

A homology-orientation of $W$ is needed to fix the sign of the map $\psi_{W,S}$.

From the argument of Proposition 4.10 we obtain:

**Proposition 5.1.** For the unknot $U$ in $S^3$, the Floer group $FI_\ast(U; \Gamma)$ is a
free $R$-module of rank 4.

The definition of the local coefficient system $\Gamma$ and the maps induced
by a cobordism are related to the monopole number. To understand this
relationship, consider in general two different cobordisms of pairs, $(W, S)$ and
$(W', S')$ from $(S^3, K_0)$ to $(S^3, K_1)$, (with both $S$ and $S'$ being embedded
surfaces, not immersed). Let $\beta_0$ and $\beta_1$ be critical points in $B(K_0)$ and
$B(K_1)$ respectively, and let $z$ and $z'$ be paths from $\beta_0$ to $\beta_1$ along $(W, S)$
and $(W', S')$ respectively. Corresponding to $z$ and $z'$, the local system gives
us maps of the form (88), which in this case means we have

$$\Delta_z, \Delta_{z'} : t^{\mu(\beta_0)} R \to t^{\mu(\beta_1)} R.$$
Both of these maps are multiplication by a certain (real) power of $t$:

\[
\Delta_z = t^{\nu(z)} \\
\Delta_{z'} = t^{\nu(z')}.
\]

We can express the difference between $\nu(z)$ and $\nu(z')$ in topological terms. The surfaces $S$ and $S'$ are not closed, so there is not a well-defined monopole number $l$ for the classes $z$ and $z'$; but there is a well-defined relative monopole number: we can write

\[ d(z - z') = (k, l) \]

where $k$ is the relative instanton number and $l$ is the relative monopole number (both are integer-valued). There is also a “relative” self-intersection number $S^2 - (S')^2$ (the self-intersection number of the union of $S$ and $-S'$ in $W \cup (-W')$). In these terms, we have

\[ \nu(z) - \nu(z') = -l + (1/4)(S^2 - (S')^2). \]

This is essentially the formula (17) of [19], which expresses the curvature integral which defines $\nu$ in terms of the topological data. Our $\nu$ corresponds to $-\lambda$ in [19]. If we fix a reference cobordism $(W_*, S_*)$ and a path $z_*$ along it from $\beta_0$ to $\beta_1$, then the contribution involving $\beta_0$ and $\beta_1$ to the map $\psi_{W,S}$ can be written (in the style of definition (84)) as

\[
\sum_z \sum_{[\tilde{A}]} \epsilon[\tilde{A}] \Delta_z = \sum_z \sum_{[\tilde{A}]} \epsilon[\tilde{A}] t^{-l+1/4}(S^2 - (S')^2)+\nu_*. \tag{102}
\]

### 5.2 Immersed surfaces and canonical isomorphisms

A cobordism of pairs from $(Y_0, K_0)$ to $(Y_1, K_1)$, as considered so far, consists of a 4-dimensional cobordism $W$ and an embedded surface $S$ with $\partial S = K_1 - K_0$. It is convenient to follow [19] and consider also immersed surfaces $S$. We will always consider only smoothly immersed surfaces $f : S \leftrightarrow W$ with normal crossings (transverse double-points), all of which should be in the interior of $W$. As is common, we often omit mention of $f$ and confuse $S$ with its image in $W$. By blowing up $W$ at each of the double points (forming a connected sum with copies of $\mathbb{CP}^2$) and taking the proper transform of $S$ (cf.[19]), we have a canonical procedure for replacing any such immersed cobordism $(W, S)$ with an embedded version, $(\tilde{W}, \tilde{S})$. We then define a map

\[ \psi_{W,S} : FI_*(K_0; \Gamma) \rightarrow FI_*(K_1; \Gamma) \]
corresponding to the immersed cobordism by declaring it to be equal to the map obtained from its resolution:

$$\psi_{W,S} := \psi_{\tilde{W},\tilde{S}}.$$  

Now suppose that $f_0 : S \to W$ and $f_1 : S \to W$ are two immersions in $W$ with normal crossings, and suppose that they are homotopic as maps relative to the boundary. Then the image $S_1$ of $f_1$ can be obtained from $S_0 = f(S_0)$ by a sequence of standard moves, each of which is either

(0) an ambient isotopy of the image of the immersed surface in $W$,

(1) a twist move introducing a positive double point,

(2) a twist move introducing a negative double point,

(3) a finger move introducing two double points of opposite sign,

or the inverse of one of these [14]. To analyze the relation between the maps $\psi_{W,S_0}$ and $\psi_{W,S_1}$, we must therefore analyze the effect of each of these types of elementary changes to an immersed surface. This was carried out in [19] for the case of closed surfaces in a closed 4-manifold, and the same arguments work as well in the relative case, leading to the following result.

**Proposition 5.2.** Let $S$ be obtained from $S'$ by one of the three basic moves (1)–(3). Then the maps

$$\psi_{W,S'} : FI_*(K_0, \Gamma) \to FI_*(K_1, \Gamma)$$

$$\psi_{W,S} : FI_*(K_0, \Gamma) \to FI_*(K_1, \Gamma)$$

are related by, respectively,

(1) $\psi_{W,S} = (t^{-1} - t)\psi_{W,S'},$

(2) $\psi_{W,S} = \psi_{W,S'}$ (no change), and

(3) $\psi_{W,S} = (t^{-1} - t)\psi_{W,S'}$ (the same as case (1)).

**Remark.** The three cases of this proposition can be summarized by saying that the map $\psi_{W,S}$ acquires a factor of $(t^{-1} - t)$ for every positive double point that is introduced.
Proof. As indicated above, this is essentially Proposition 3.1 of [19]. In that proposition, the monopole number $l$ contributes to the power of $t$ in the coefficients of the map $\psi_{W,S}$, according to the formula (102). The other contribution to the exponent is the self-intersection number of the proper transform of the immersed surface $S$, which changes by $-4$ when $S$ acquires a positive double point and is unaffected by negative double-points.

Let us now return to situation where we have two homotopic maps $f_i : S \to W$ with images $S_i$, $i = 0, 1$. If we form the ring $R'$ by inverting the element $(t^{-1} - t)$ in $R$, so

$$R' = R[(t^{-1} - t)^{-1}],$$

and if we write

$$\psi'_{W,S} = \psi_{W,S} \otimes 1 : FI_*(K_0; \Gamma) \otimes_R R' \to FI_*(K_1; \Gamma) \otimes_R R'$$

then the proposition tells us:

**Corollary 5.3.** If $S_0$ and $S_1$ are the images of homotopic immersions as above, then the maps

$$\psi'_{W,S_1} : FI_*(K_0; \Gamma) \otimes_R R' \to FI_*(K_1; \Gamma) \otimes_R R'$$

differ by multiplication by a unit. More specifically, if $\tau(S_i)$ is the number of positive double-points in $S_i$, then we have

$$\psi'_{W,S_1} = (t^{-1} - t)^{\tau(S_1) - \tau(S_0)} \psi'_{W,S_0}.$$  

**Corollary 5.4.** For any two knots $K_0$ and $K_1$, we have

$$FI_*(K_0; \Gamma) \otimes_R R' \cong FI_*(K_1; \Gamma) \otimes_R R'$$

Proof. In the cylindrical cobordism $W = [0, 1] \times S^3$, let $S$ be any immersed annulus from $K_0$ to $K_1$, and let $\bar{S}$ be any annulus from $K_1$ to $K_0$. The concatenation of these immersed annular cobordisms, in either order, give annular immersed cobordisms from $K_0$ to $K_0$ and from $K_1$ to $K_1$. These composite annuli are each homotopic to a trivial product annulus; so the composite maps

$$\psi'_{W,S} \circ \psi'_{W,\bar{S}}$$

and

$$\psi'_{W,S} \circ \psi'_{W,\bar{S}}$$

are both the identity map, and it follows that $\psi'_{W,S}$ and $\psi'_{W,\bar{S}}$ are isomorphisms.  

\[ \square \]
Extending this line of thought a little further, we see that the group $\text{FI}_*(K; \Gamma) \otimes_R R'$ is not just independent of $K$ up to isomorphism, but up to canonical isomorphism. That is, if $K_0$ and $K_1$ are any two knots, we may choose any immersed annulus $S$ from $K_0$ to $K_1$ in the 4-dimensional product cobordisms $W = [0,1] \times S^3$ and construct the isomorphism

$$(t^{-1} - t)^{-r(S)} \psi_{W,S}' : \text{FI}_*(K_0; \Gamma) \otimes_R R' \to \text{FI}_*(K_1; \Gamma) \otimes_R R'.$$

This isomorphism is independent of the choice of annulus $S$. In particular, for any knot $K$, the $R'$-module $\text{FI}_*(K; \Gamma) \otimes_R R'$ is canonically isomorphic to the $R'$-module arising from the unknot. From Proposition 5.1 we therefore obtain:

**Corollary 5.5.** For any knot $K$ in $S^3$, there is a canonical isomorphism

$$\Psi : (R')^4 \to \text{FI}_*(K; \Gamma) \otimes_R R'.$$

### 5.3 Filtrations and double-point bounds

The inclusion $R \to R'$ gives us a canonical copy of $R^4$ in $(R')^4$, the image of $\text{FI}_*(U; \Gamma)$ in $\text{FI}_*(U; \Gamma) \otimes_R R'$ for the unknot $U$. We define an increasing filtration of $\text{FI}_*(K; \Gamma) \otimes_R R'$,

$$\cdots \subset \mathcal{F}^{-1}(K) \subset \mathcal{F}^0(K) \subset \mathcal{F}^1(K) \subset \cdots$$

by first setting

$$\mathcal{F}^0(K) = \Psi(R^4) \subset \text{FI}_*(K; \Gamma) \otimes_R R',$$

where $\Psi$ is the canonical isomorphism of Corollary 5.5, and then defining

$$\mathcal{F}^i(K) = (t^{-1} - t)^{-i} \mathcal{F}^0(K).$$

Although $\mathcal{F}^0(U)$ is the image of $\text{FI}_*(U; \Gamma)$ in the tensor product for the case of the unknot, this does not hold for a general knot. We make the following definition, modelled on similar constructions in [36, 32, 27].

**Definition 5.6.** For any knot $K$ in $S^3$ we define $\varrho(K)$ to be the smallest integer $i$ such that $\mathcal{F}^{-i}(K)$ is contained in the image of $\text{FI}_*(K; \Gamma)$ in $\text{FI}_*(K; \Gamma) \otimes_R R'$.

To see that the definition makes sense, choose an immersed annular cobordism from the unknot $U$ to $K$, and let $\tau$ be the number of positive
double points in this annulus. As $R$-submodules of $FI_* (K; \Gamma) \otimes_R R'$, we have, from the definitions,

$$F^{\tau} (K) = \Psi ((t^{-1} - t)^{\tau} R^4)$$

$$= \psi'_{W,S} (FI_* (U; \Gamma))$$

$$\subset \text{im} (FI_* (K; \Gamma) \to FI_* (K; \Gamma) \otimes_R R')$$

where the last inclusion holds because passing to the ring $R'$ commutes with the maps $\psi_{W,S}$ and $\psi'_{W,S}$ induced by the cobordism. From this observation and the definition of $\varrho (K)$, we obtain

$$\varrho (K) \leq \tau.$$ 

Since the annulus $S$ was arbitrary, we have:

**Theorem 5.7.** Let $K$ be a knot in $S^3$ and let $D$ be an immersed disk with normal crossings in the 4-ball, with boundary $K$. Then the number of positive double points in $D$ is at least $\varrho (K)$. $\square$

### 5.4 Algebraic knots

The lower bound for the number of double points in an immersed disk, given by Theorem 5.7, is sharp for the case of an algebraic knot (a knot arising as the link of a singularity in a complex plane curve, such as a torus knot). The reason this is so comes down to the same mechanisms that were involved in [21, 22] and [19], where singular instantons were used to obtain bounds on unknotted numbers and slice genus.

To explain this, we recall some background from [21, 19]. Let $(X, \Sigma)$ be a closed pair, with $\Sigma$ connected for simplicity, and let $P \to X$ be a $U(2)$ bundle. Suppose that $c_1 (P)$ satisfies the non-integrality condition, that

$$\frac{1}{2} c_1 (P) \pm \frac{1}{2} [\Sigma]$$

is not an integer class, for either choice of sign. Denoting by $k$ the instanton number of $P$, we have for any choice of monopole number $l$ a moduli space $M_{k,l} (X, \Sigma)_{\delta}$, which we label by $k l$ and $\delta$, where $\delta$ is the line bundle det $P$. If the formal dimension of this moduli space is zero, then there an integer invariant

$$\varrho_{k,l}^\delta (X, \Sigma) \in \mathbb{Z}.$$ 

(A homology orientation is needed as usual to fix the sign.) In [19], these integer invariants are combined into a Laurent series (with only finitely many
non-zero terms):

\[ R^\delta(X, \Sigma)(t) = 2^{-g(\Sigma)} \sum_{(k,l) : \dim M_{k,l} = 0} t^{-l} q_{k,l}^\delta(X, \Sigma). \]

The normalizing factor \(2^{-g}\) was convenient in [19] but is not significant here. The definition of \(q_{k,l}^\delta(X, \Sigma)\) and \(R^\delta(t)\) is extended to the case of immersed surfaces with normal crossings by blowing up.

Suppose now that \((X, \Sigma)\) decomposes along a 3-manifold \(Y\), meeting \(\Sigma\) transversely in a knot \(K\). Suppose also that the restriction of \(P\) to \((Y, \Sigma)\) satisfies the non-integrality condition. Then there is a gluing formula which expresses the Laurent series \(R^\delta(X, \Sigma)(t)\) as a pairing between a cohomology class and a homology class in the Floer group

\[ \mathbb{L}_s(Y, K, P; \Gamma). \]

The coefficient system \(\Gamma\) is the one we have been using, and keeps track of the power of \(t\).

Suppose now that we have a cobordism \((W, S)\) (with \(S\) immersed perhaps) from \((Y_0, K_0)\) to \((Y_1, K_1)\), giving us a map

\[ \psi : \mathbb{L}_s(Y_0, K_0, P; \Gamma) \to \mathbb{L}_s(Y_1, K_1, P; \Gamma). \]

Suppose we wish to show that the image of \(\psi\) is not contained in the image of multiplication by \((t^{-1} - t)\). From the functorial properties and the gluing formulae, it will be sufficient if we can find a closed pair \((X, \Sigma)\) (together with a \(U(2)\) bundle \(P\)) such that:

- \((X, \Sigma, P)\) contains \((W, S, P)\) as a separating subset, as indicated in the figure; and
- the Laurent series \(R^\delta(X, \Sigma)(t)\) does not vanish at \(t = 1\).

Summarizing this discussion, we therefore have:

**Proposition 5.8.** Suppose \((X, \Sigma)\) is a pair (with \(\Sigma\) perhaps immersed) such that, for some \(\delta\), the finite Laurent series \(R^\delta(X, \Sigma)(t)\) is non-vanishing at \(t = 1\). Suppose that \((X, \Sigma)\) has a decomposition as shown, and suppose:

- \(Y_0 \cong Y_1 \cong S^3 \# T^3\);
- \(W\) the 4-dimensional product cobordism;
- \(c_1(\delta)\) is dual to a standard circle in \(T^3\);
Figure 1: A closed pair \((X, \Sigma)\) separated by \((W, S)\), with \(S\) immersed.

- \(K_0, K_1\) arise from classical knots in \(S^3\), with \(K_0\) the unknot;
- the surface \(S\) arises from an immersed annulus in \([0,1] \times S^3\) with \(\tau\) double positive points.

Then \(\varrho(K_1) = \tau\) and the bound of Theorem 5.7 is sharp for \(K_1\).

Consider now the 4-torus \(T^4\) as a complex surface, containing an algebraic curve \(C\) with a unibranch singularity at a point \(p\). Let \(B_1\) be a small ball around \(p\) so that the curve \(C\) meets \(\partial B_1\) in a knot \(K_1\). In a \(C^\infty\) manner, we can alter \(C\) in the interior of \(B_1\) to obtain an immersed surface \(\tilde{C}\), so that the part of \(\tilde{C}\) that is in the interior of \(B_1\) is isotopic to a complex-analytic immersed disk with \(\tau\) positive double-points. Let \(B_0 \subset B_1\) be a smaller 4-ball, meeting \(\tilde{C}\) in a standard embedded disk, so that the part of \(\tilde{C}\) that lies between \(B_0\) and \(B_1\) is an immersed annulus with \(\tau\) double points. Let \(T\) be a real 2-torus in \(T^4\) disjoint from \(C \cup B\), and let \(\delta\) be a line bundle with \(c_1(\delta)[T] = 1\). Let \(Y_1 \cong S^3 \# T^3\) be obtained as an internal connected sum of \(\partial B_1\) with the boundary of a tubular neighborhood of \(T\). Similarly, let \(Y_0\) be obtained as the internal connected sum of \(\partial B_0\) with the boundary of a smaller tubular neighborhood of \(T\). Then the pair \((T^4, \tilde{C})\) has a decomposition as shown in the diagram, satisfying the itemized conditions of the theorem above.

To show that \(\varrho(K_1) = \tau\) for this algebraic knot \(K_1\), we therefore need only show that the corresponding Laurent series \(R^\delta(T^4, \tilde{C})(t)\) is non-zero at \(t = 1\). Because of the results of [19], this is equivalent to showing that \(R^\delta(T^4, \Sigma)(1)\) is non-zero, where \(\Sigma\) is a smooth algebraic curve (embedded in
the abelian surface). Using the results of [21] however, we can calculate this Laurent series. We are free to arrange that $\Sigma$ has odd genus, that $c_1(\delta)[\Sigma]$ is zero, and that $c_1(\delta)^2 = 0$. The terms in the series come from the invariants $q^\delta_{k,l}$ with

$$2k + l - \frac{1}{2}(g - 1) = 0,$$

and from [21] we learn that

$$q^\delta_{k,l} = \begin{cases} 2^g q^0_0(T^4), & k = 0 \text{ and } l = (g - 1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

where $q^0_0(T^4)$ is the Donaldson invariant of $T^4$, which is 2. The Laurent series is therefore a non-zero multiple of a certain power of $t$, and in particular is non-zero at $t = 1$, as required.

5.5 The involution on the configuration space

As we have defined it, the group $Fl_*(K; \Gamma)$ is a free $\mathbb{R}$-module of rank 4 in the case that $K$ is the unknot. The four generators come from the two 2-spheres that make up the set of critical points of the unperturbed Chern-Simons functional on $B(K)$. However, there is an involution on $B(K)$, interchanging these two copies: this is the action of the cyclic group $\mathcal{H}'$ of order 2. Recall that, with $\mathbb{Z}$ coefficients, we defined $\tilde{Fl}_*(K)$ (in Definition 4.9, where we dealt with $SU(N)$ for arbitrary $N$) by passing to the quotient $B(K)/\mathcal{H}'$ and taking the Morse theory in this quotient.

Because the local coefficient system $\Gamma$ on $B(K)$ is actually pulled back from the quotient $B(K)/\mathcal{H}'$, we can adapt this construction to define an $R$-module

$$\tilde{Fl}_*(K; \Gamma) \quad (104)$$

for any knot $K$. For a suitable choice of perturbation, the complex that computes $\tilde{Fl}_*(K; \Gamma)$ is the quotient of the complex that computes $Fl_*(K; \Gamma)$ by an involution that acts freely on the generators. In the case of the unknot, this Floer group would be $\mathbb{R}^2$ instead of $\mathbb{R}^4$. Little else in our discussion would need to be changed.

5.6 Genus bounds

Let

$$f : S \to W$$

$$f' : S' \to W$$
be two immersions with transverse self-intersections, having as boundary the same knots $K_0 \subset Y_0$ and $K_1 \subset Y_1$. We have seen that if $S = S'$ and $f \simeq f'$ relative to the boundary, then the two resulting maps $FI_*(K_0; \Gamma) \to FI_*(K_1; \Gamma)$ differ only by factors of $(t^{-1} - t)$ (Proposition 5.2 and Corollary 5.3). Another situation to consider is the case that $S'$ is obtained from $S$ by adding a handle: forming an internal connected sum with a 2-torus contained in a ball in $W$.

The effect of adding a handle in this way was examined for the case of closed pairs $(X, \Sigma)$ in [19]. In our present context, the relevant calculation is the following. Let $W$ be the 4-dimensional product cobordism $[0, 1] \times S^3$, and let $S_1 \subset W$ be a cobordism from the unknot to the unknot and having genus 1. This gives rise to a homomorphism

$$\psi_1 : FI_*(U; \Gamma) \to FI_*(U; \Gamma).$$

If we pass to the “bar” version of the Floer groups (104), then we have also a map

$$\bar{\psi}_1 : \bar{FI}_*(U; \Gamma) \to \bar{FI}_*(U; \Gamma).$$

The group $\bar{FI}_*(U; \Gamma)$ is a free $R$-module of rank 2, with generators in different degrees mod 4. We can therefore identify it with $R \oplus R$ with an ambiguity consisting of multiplication by units on each summand. The map $\bar{\psi}_1$ must be off-diagonal in this basis, because its degree is 2 mod 4; so we have a map

$$\bar{\psi}_1 : R \oplus R \to R \oplus R$$

of the form

$$\bar{\psi}_1 = \begin{pmatrix} 0 & p(t) \\ q(t) & 0 \end{pmatrix}$$

(105)

for certain Laurent polynomials $p(t)$ and $q(t)$, well-defined up to units. From [19] we know the effect of adding two handles to a surface, and from that we deduce the relation

$$p(t)q(t) = 4(t - 2 + t^{-1}),$$

or in other words

$$\bar{\psi}_1^2 = 4(t - 2 + t^{-1}) = 4t^{-1}(t - 1)^2.$$  

(106)

as an endomorphism of $\bar{FI}_*(U; \Gamma)$.

At this point, because of the factor of 4 in the above formula, we shall pass to rational coefficients rather than integer coefficients: without change...
of notation, let us redefine $R$ as $\mathbb{Q}[t^{-1}, t]$. We again define $R'$ by inverting $(1 - t)$ and $(1 + t)$ in $R$.

Suppose now that we have an embedded cobordism $S$ of genus $g$ from the unknot $U$ to $K$, inside $[0, 1] \times S^3$. This gives rise to a map

$$\tilde{\psi}_S : \bar{FI}_*(U; \Gamma) \to \bar{FI}_*(K; \Gamma)$$

and similarly

$$\tilde{\psi}'_S : \bar{FI}_*(U; \Gamma) \boxtimes R R' \to \bar{FI}_*(K; \Gamma) \boxtimes R R'.$$

The surface $S$ is homotopic to an immersed cobordism $S^+$ which is a composite of two parts: the first part is $g$ copies of the standard genus-1 cobordism $S_1$ from $U$ to $U$; and the second part is an immersed annulus. Let $\tau$ be the number of positive double points in the immersed annulus. From Corollary 5.3 and the definition of the canonical isomorphism $\Psi$, we obtain

$$\tilde{\psi}'_S = \Psi \circ (\tilde{\psi}'_1)^g$$

where $\tilde{\psi}'_1$ is the map defined by (105). This provides a constraint on the genus $g$: it must be that $\Psi \circ (\tilde{\psi}'_1)^g$ carries $R \oplus R$ into the image of $\bar{FI}_*(U; \Gamma)$ in $\bar{FI}_*(U; \Gamma) \boxtimes R R'$. This constraint gives us a lower bound for $g$, just as we obtained a lower bound $\rho(K)$ for the number of double points previously. For algebraic knots again, the bound will be sharp.

It is not inconceivable that, by working over $\mathbb{Z}$ and paying attention to the factor 4 above instead of passing to $\mathbb{Q}$, one could obtain a stronger bound for $g$ in some cases, but the authors have no evidence one way or the other.

References


