

FINITE β_e UNIVERSAL MODE TURBULENCE AND ALCATOR SCALING

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ABSTRACT

An outline for a self-consistent theory of finite β_e universal mode turbulence is given. Saturation results from resonance broadening of the electron response due to magnetic shear. Electron diffusion, for $\beta_e > m_e/m_i$, is due to the magnetic part of the fluctuations. The diffusion coefficient, $D = .1 (T_e/(T_i + T_e))^4 (m_e/m_i \beta_e) (L_s/L_n)^2 \times v_i \rho_i^2 / L_n$, scales inversely with density, is independent of magnetic field, and is in excellent quantitative agreement with observations on the Alcator tokamak.

One of the principal theoretical goals in tokamak research is the development of a self-consistent turbulence theory for the short wavelength fluctuations thought to be responsible for anomalous transport. This paper presents a resonance broadening turbulence theory for the finite β_e universal instability. An approximate analytic solution of the coupled, non-linear, eigenmode equations is obtained. The accuracy of this solution is verified by numerical computations. The resulting formula for the anomalous electron thermal conductivity, eq. (14), has many similarities with experimental observations, including absolute magnitude, and scaling with density, temperature and ion mass. For typical tokamak regimes where $\beta_e > m_e/m_i$, the calculation constitutes an example of a self-consistent theory of stochastic¹ magnetic fluctuations.

Until recently,² most turbulence theories ignored shear in the equilibrium magnetic field. Without shear, turbulence mainly effects the ions. The basic saturation picture, as developed by Dupree,³ balanced linear electron growth, γ_e^L , against nonlinear (turbulent) ion damping, γ_i^{NL} . Taking $\gamma_i^{NL} = k_\perp^2 D$ is the basis for the γ_e^L/k_\perp^2 estimates of the anomalous diffusion coefficient. However, recent theory⁴ has found that $\gamma_i^{NL} \ll k_\perp^2 D$, because the ion-wave interaction is weak for low frequency modes, and consequently, for tokamak parameters, that ion non-linearity is not a viable saturation mechanism.

With shear and because of their rapid mobility along the field lines, electrons are the species the most strongly effected by turbulence.² Here, the ions cause damping linearly, due to the shear, γ_\perp^S . Electron growth is modified by shear induced resonance broadening, to γ_e^{NL} . Saturation, $\gamma_e^{NL} = \gamma_\perp^S$, can occur at low turbulence levels, consistent with observations.

In a sense, the main point of this paper is to show that shear induced resonance broadening is applicable to the electromagnetic problem and provides an effective mechanism for the saturation of such instabilities.

Of course, as before,² the shear also produces the instability due to the same effect for finite radial diffusion, $D \sim \nu_{ei} \rho_e^2$.

The specific example considered here is basically a drift wave, although finite β_e does modify the shear damping in an important way, and when $\beta_e > m_e/m_i$, most of the transport is due to the magnetic part of the fluctuation.⁵

We consider a cylindrical tokamak, with equilibrium field components $\underline{B} = B_0 \underline{e}_z + B_\theta(r) \underline{e}_\theta$. Since $\beta_e \equiv 4\pi n T_e / B^2 \ll 1$, the compressional mode of fluctuations may be neglected, and it suffices to consider the parallel component of the vector potential \tilde{A}_\parallel . Expanding the modes in a Fourier series in poloidal and toroidal angle $\sim \exp[i m \theta - i n \phi - i \omega t]$, the parallel wave vector is given by $k_\parallel \equiv (m - nq) / Rq$, where $q = r B_\theta / R B_0$ is the safety factor. For each mode, the rational surface is at r_{mn} , such that $q(r_{mn}) = m/n$, and we let $x = r - r_{mn}$ be the distance from the rational surface. Then $k_\parallel = k'_\parallel x \equiv k_\theta x / L_s$. As in linear theory the modes have a definite parity with respect to x . The drift wave parity considered here is $\tilde{\Phi}$ even, \tilde{A}_\parallel odd, so that $\hat{E}_\parallel = -i k_\parallel (\tilde{\Phi} - \tilde{\Psi})$ is odd, where $\tilde{\Psi}$, defined as $\tilde{A}_\parallel \omega / k_\parallel c$, is even in x .

The non-linear electron response to $\tilde{\Phi}$ and $\tilde{\Psi}$ may be computed in the manner indicated previously.² Writing the electron fluctuation as

$$\tilde{f}_e = \frac{e}{T_e} F_M \left[\tilde{\Phi} - \left(1 - \frac{\omega_* e}{\omega} \right) \tilde{\Psi} \right] + \tilde{h}_e, \quad (1)$$

where the first term is the adiabatic response (or the $k_\parallel \nu_\parallel \rightarrow \infty$ limit of the drift kinetic equation), the result of this "renormalization" is equivalent to computing \tilde{h}_e from the diffusion equation.⁶

$$[-i(\omega - k_{||} v_{||}) - D d^2/dx^2] \tilde{h}_e = \frac{e}{T_e} F_M i(\omega - \omega_{*e})(\tilde{\phi} - \tilde{\psi}) . \quad (2)$$

Now D contains magnetic as well as electrostatic contributions,

$$D \approx \sum_{m,n} \frac{c^2 k_{\perp}^2}{B^2} \left| \tilde{\phi}_{mn} - \frac{v_{||}}{c} \tilde{A}_{||mn} \right|^2 R(\omega - k_{||} v_{||}) , \quad (3)$$

where $R = \int_0^{\infty} dt \exp[i(\omega - k_{||} v_{||})t - \frac{1}{3} (k_{||} v_{||})^2 Dt^3]$, is a turbulently broadened resonance function. Note, since $\tilde{A}_{||} \approx \frac{c}{v_A} \tilde{\phi}$ (see eq. (10)), where $v_A = (B^2/4\pi n m_i)^{1/2}$, that when $v_e^2/v_A^2 > 1$ or $\beta_e > m_e/m_i$, diffusion due to the magnetic part of the fluctuations is dominant. The physical properties of shear induced resonance broadening can be inferred directly from eq. (2)⁷. It is derived by detailed renormalization techniques, but represents also a plausible physical model.

Equation (2) is appropriate when the particle orbits in the presence of the fluctuations exhibit the stochasticity property, thus justifying diffusive behavior on the spatial scale of a wavelength. To verify the stochasticity property we examine the orbit equations representing the deviation from the linear or unperturbed trajectory

$$\frac{d\delta r}{dt} = \tilde{v}_r = \sum_{mn} \frac{ck_{\theta}}{B} \left(\tilde{\phi}_{mn} - \frac{v_{||}}{c} \tilde{A}_{||mn} \right) \sin(m\theta - n\phi - \omega t) \quad (4)$$

$$\frac{d\delta\theta}{dt} = \frac{v_{||}}{Rq} \frac{d \ln q}{dr} \delta r + \frac{\tilde{v}_{\theta}}{r} ,$$

where $\theta = \frac{v_{||}}{Rq} t + \delta\theta$, $\phi = \frac{v_{||}}{R} t$, and $n\delta\phi$, of order $\frac{r}{Rq^2} m\delta\theta$, is neglected.

The shearless contribution to the θ fluctuation is small and can be discarded⁸.

For small enough amplitude these orbits generate chains of resonances, or

islands, in the r, θ phase space, each chain centered at the radius given by $\omega = k_{\parallel} v_{\parallel}$. In the electrostatic limit, $\beta_e < m_e/m_i$, the radial island widths are

$$\Delta_I^{ES} = \rho_e \left(\frac{L_s}{\rho_e} \frac{e\phi_{mn}}{T_e} \right)^{1/2}, \quad (5)$$

while for $\beta_e > m_e/m_i$ when the perturbation of the orbits is mostly magnetic, the islands, for the drift wave parity, have width⁵

$$\Delta_I^B = \left(\frac{x_\beta L_s}{k_{\perp}} \frac{\tilde{B}_{rnm}}{B} \right)^{1/3}, \quad (6)$$

where x_β , to be defined shortly, is essentially the Alfvén layer thickness. The spacing between island chains is basically a geometric condition giving the distance between zeroes of $m - nq(r)$ (ignoring the frequency in $\omega - k_{\parallel} v_{\parallel} = 0$). For modes (m, n) and $(m + \mu, n + \eta)$, the distance is given by $dq/dr \delta r = (\mu n - \eta m)/(n(n + \eta))$. For mode numbers in the 10^2 range which, as we find, dominate the spectrum, the numerator $\mu n - \eta m$ can be made of order one, so that approximately,

$$\Delta_{RES} \approx r/(m^2 d \ln q / d \ln r), \quad (7)$$

as noted previously¹. The Chirikov condition for stochasticity $\Delta_I^{ES} / \Delta_{RES} > 1$ or $\Delta_I^B / \Delta_{RES} > 1$, will be satisfied for electrostatic fluctuation levels on the order of $e\phi_{mn}/T_e \sim 10^{-4}$, or magnetic fluctuations of $\tilde{B}_{rnm}/B \sim 10^{-7}$. Thus, for these high mode number fluctuations the rational surfaces are packed very densely together and the stochasticity condition is, for all practical purposes, always satisfied.

Returning to the eigenmode problem, the electron density and current fluctuations can be computed from eq. (2). Finite gyroradius ion fluctuations are computed from linear theory. Combining these, using quasineutrality $\tilde{n}_e = \tilde{n}_i$ and Ampere's law $-\nabla_{\perp}^2 \tilde{A}_{\parallel} = \frac{4\pi}{c} (\tilde{J}_{\parallel e} + \tilde{J}_{\parallel i})$, leads to

$$\frac{d^2 \phi}{dx^2} - \xi \phi = (\phi - \psi) \left[\Lambda - \xi - \mu^2 x^2 + \frac{\sigma_e(|x|)}{|x|} \right], \quad (8)$$

$$\left(\frac{d^2}{dx^2} - b \right) \frac{k_{\parallel} c}{\omega} \psi = (\phi - \psi) \frac{c^2}{v_A^2} \frac{\omega d}{k_{\parallel} c} \left[\Lambda - \xi - \mu^2 x^2 + \frac{\sigma_e(|x|)}{|x|} (1 + i \frac{\omega_c}{\omega}) \right]. \quad (9)$$

We have retained the standard notation wherever possible, defining

$$b = k_{\perp}^2 \rho_i^2, \quad \Gamma_n(b) = e^{-b} I_n(b), \quad \tau = T_e/T_i, \quad x_e = \omega/k'_{\parallel} v_e, \quad x_c = \frac{\omega_c}{k'_{\parallel} v_e}$$

$$d = (\Gamma_0 - \Gamma_1)(\tau + \omega_{*e}/\omega), \quad \xi = (1 - \Gamma_0)/(\Gamma_0 - \Gamma_1), \quad \Lambda = d^{-1} [1 + \tau(1 - \Gamma_0) - \Gamma_0 \omega_{*e}/\omega],$$

$$\mu^2 = d^{-1} (\tau + \omega_{*e}/\omega) \Gamma_0 (\omega_{*e}/\omega \tau)^2 (L_n/L_s)^2, \quad k'_{\parallel} \equiv k_{\perp}/L_s,$$

$$\sigma_e(|x|) / |x| = (\omega - \omega_{*e}) d^{-1} (x_e/|x|) Z((x_e + ix_c)/|x|).$$

Note that the underlying electromagnetic modes persist in the presence of stochasticity, at least for the odd parity of A_{\parallel} .

To simplify the solution to Eqs. (8) and (9) we use an approximate algebraic relation between ϕ and ψ (obtained by interpolating between the asymptotic relations) in Eq. (8) to give a self-adjoint equation for ϕ . The approximation is based on the following argument.⁹ We are concerned with what is basically a drift wave, and the principal dynamics are

described by eq. (8). The primary finite beta effect (as is known from previous work¹⁰) is the reduction of the parallel electric field, or $\phi - \psi$, within the Alfvén layer, $x_A = \omega/k_{\parallel} v_A$, due to inductive effects. To represent this algebraically, note that $\psi \rightarrow \phi$ as $x \rightarrow 0$ (this follows from eq. (9), by requiring finiteness of the right hand side). The large x behavior is found by combining (8) and (9) to give $d^2\phi/dx^2 - \xi\phi = (k_{\parallel} c/\omega d)(v_A^2/c^2)(d^2/dx^2 - b) \times k_{\parallel} c\psi/\omega$. As $x \rightarrow \infty$, the derivative terms are dominant, implying $\psi \sim \phi x_A^2/x^2$, so by a simple interpolation

$$\psi \approx \phi x_{\beta}^2 / (x^2 + x_{\beta}^2), \quad (10)$$

where $x_{\beta}^2 = x_A^2 d$. The approximate eigenmode equation is then obtained from eqs. (10) and (8).

$$\frac{d^2\phi}{dx^2} - \phi \frac{x^2(\Lambda - \mu^2 x^2 + \alpha_e) + \xi x_{\beta}^2}{x^2 + x_{\beta}^2} = 0 \quad (11)$$

Equation (11) passes to the usual electrostatic equation² as $x_{\beta} \rightarrow 0$.

A quadratic form may be constructed by multiplying eq. (11) by ϕ and integrating over x to give

$$0 = S = \int dx \left[\left(\frac{d\phi}{dx} \right)^2 + \phi^2 \times \frac{x^2(\Lambda - \mu^2 x^2 + \alpha_e) + \xi x_{\beta}^2}{x^2 + x_{\beta}^2} \right]. \quad (12)$$

Requiring the first variation of S to be zero gives eq. (11). Since we are interested in the dispersion relation near saturation, we pass to the limit

$\omega\tau_c \equiv \omega \left[\frac{1}{3} (k_{\parallel} v_e)^2 D \right]^{-1/3} < 1$ of the electron response², or $\alpha_e = i \frac{1}{3} \Gamma(\frac{1}{3}) d^{-1} \times (\omega - \omega_{*e})\tau_c$. Taking the normalized trial function to be $\phi = (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2)$, with α the variational parameter, and doing the integrals for $\sqrt{\alpha} x_{\beta} < 1$, yields $S = \alpha/2 - \mu^2/2\alpha + \sqrt{\pi} \xi x_{\beta} \sqrt{\alpha} + \Lambda + \alpha_e$, which again reproduces the electrostatic

form when $x_\beta \rightarrow 0$. The parameter α is determined by $\delta S/\delta x = 0 = 1/2 + \mu^2/2\alpha^2 + \sqrt{\pi} x_\beta \xi \alpha^{-1/2}/2$. Dominance of the first two terms gives the Weber trial function. However, when $\pi x_\beta^2 \xi^2 > \mu$, or $\beta_e > (L_n/L_s)^3 (1 + \tau)^2 \tau^{-1} \pi^{-1} (8b)^{-1/2}$, which is typically satisfied for tokamaks, the last two terms dominate and we obtain $\alpha^{1/2} = e^{i\pi/3} (\mu^2/\sqrt{\pi} \xi x_\beta)^{1/3}$ as the variational parameter. Here $e^{i\pi/3}$ is chosen uniquely as the cube root of -1 , by applying the outgoing wave boundary condition. The dispersion relation now follows as,

$$S = 0 = \frac{3}{4} (\sqrt{\pi} \xi x_\beta \mu)^{2/3} + \Lambda + i \left[\frac{3^{3/2}}{4} (\sqrt{\pi} \xi x_\beta \mu)^{2/3} + \frac{1}{3} \Gamma\left(\frac{1}{3}\right) d^{-1} (\omega - \omega_{*e}) \tau_c \right] \quad (13)$$

Note that the primary finite β_e effect is to modify the shear damping. Finite x_β pushes out the eigenmode, enhancing the outward convection of wave energy; the resultant damping involves a geometric mean of x_β and μ .

At marginal stability, the frequencies are real, and the real and imaginary parts of eq. (13) give the frequency, $\omega = \omega(k_\perp)$, and diffusion coefficient, $D = D(k_\perp)$, respectively. Here $D(k_\perp)$, which is interpreted as the amount of diffusion necessary to stabilize mode k_\perp , is given by

$$D = .07 \left(\frac{T_e}{T_e + T_i} \right)^4 \frac{m_e}{m_i \beta_e} \left(\frac{L_s}{L_n} \right)^2 \frac{v_i \rho_i^2}{L_n} \sqrt{b} \quad (14)$$

This is not yet the transport coefficient since it depends on the wave parameter, k_\perp . Actually ion collisions, as well as higher order spatial turbulent broadening effects and ion non-linearities¹¹ act to give additional damping for $b > 1$, with the consequence that $D(b)$ will have a maximum near $b_{\max} = 2$. The dependence of b_{\max} on plasma parameters will effect the scaling of D very weakly. For convenience, in practical applications, we simply put $b = 2$ in eq. (14) which gives the formula quoted in the abstract.

The D of eq. (14) is an electron test particle diffusion coefficient. When inserted in the electron kinetic equation, appropriately using the ambipolar potential¹², one can obtain transport equations in the usual way. The transport equations in Ref. (12) apply to the present case when $\beta_e > m_e/m_i$. For crude estimates D equals the electron thermal conductivity, χ_e , and particle diffusion is not anomalous (ion diffusion being slow for $k_i \rho_i > 1$).

Rewriting (14) to display the scaling (with lengths in cm. and temperatures in eV, μ the ion mass in units of the proton mass), gives

$$D = 2.65 \times 10^{16} \frac{1}{n\mu^{1/2}} \left(\frac{T_e}{T_i + T_i} \right)^4 \frac{T_i^{3/2}}{T_e} \left(\frac{L_s}{L_n} \right)^2 \frac{1}{L_n}, \quad (15)$$

which is independent of magnetic field. For $T_e = T_i$, eq. (15) scales unfavorably with temperature, but if the dependence on $\tau = T_e/T_i$ is considered, (15) is consistent with recent observations on PLT¹³ where confinement improved with increased T_i . The behavior with ion mass, $D \propto \mu^{-1/2}$ is in agreement with ISX-A measurements.¹⁴ Further, eq. (15) shows a tendency for improved confinement with decreased aspect ratio, (note $L_s/L_n = (Rq/r) \ln n/\ln q$ is approximately R/a) a feature which, qualitatively, has been seen in the data.¹⁵

Note that equation (14) can be written $D = \frac{v_e}{L_s} \frac{c^2}{2\omega_{pe}} [6 \times 10^{-3} \sqrt{m_e/m_i} (L_s/L_n)^3]$. This, to the extent that the factor in brackets is one (it is of this order for tokamaks), is the Ohkawa¹⁶ formula.

Of course, the physics underlying this diffusion is much different, being due to low frequency modes, as is the scaling with ion mass, the T_e/T_i ratio, q , and several other parameters.

Finally taking Alcator¹⁷ profiles at $r = 5$ cm, $L_s \approx 100$ cm., $L_n \approx 5$ cm., and $T = 1$ KeV, and defining $\tau_E \equiv a_L^2/D$, we find $\tau_E = 2.4 \times 10^{-19} n a_L^2$, which is within 50% of the Alcator empirical scaling law,¹⁸ $\tau_E = 3.77 \times 10^{-19} n a_L^2$. We might note that this transport mechanism is also consistent with the interpretation of the soft X-ray anomaly given in Ref. (12), and that the absolute value of D from eq. (14) agrees very well with that found empirically from the X-ray data.

Obviously, the theory as described in this paper is not accurate to 50%, and many omitted effects and questions about the finite β_e eigenvalue problem remain to be considered. However, the main attributes of Eq. (14), namely absolute magnitude and density scaling, have been verified by the numerical solution of Eqs. (8) and (9). The significant point is that, when saturated by shear induced resonance broadening, the simplest, almost archetypal, drift instability, the universal mode, gives inverse density scaling of the anomalous thermal conductivity. One then feels that this feature will be retained when complicating effects are added, and that on this basis, the established empirical scaling, $\tau_E \propto n$, can be understood.

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