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Formulation of the Arbitrary n Stability Problem in an Axisymmetric Torus with a Finite Resistivity Vacuum Chamber and PF System

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Contents

Chapter 1

General Theory

1.1 Introduction

This work is within the framework of MHD. We study a confined plasma surrounded by a vacuum region and bounded by a resistive wall. Confined to axisymmetric systems, Chapter 1 deals with a fairly general theory for such systems. To be more specific, we are interested here in the case where the configuration is stable in the presence of an infinitely conducting wall, but unstable without the wall. For physical reasons an infinitely conducting wall cannot be made. It is therefore of interest to study the effect of a resistive wall. This is done in Chapter 1.

The results in Chapter 1 depend upon knowledge of the solution for the stability problem in the two limiting cases.

1. The wall and the conductors in the vacuum region are not taken into account (wall and conductors at infinity).

2. The wall and the conductors in the vacumm region are taken into account as infinitely conducting elements in the proximity of the plasma.

It appears to be the case that even the last of these problem areas has yet to be comprehensively studied, probably because it is rather complex. This is the subject of Chapter 2.

1.2 The Lagrangian

We start from a Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = \delta W_F + \frac{1}{2\mu_0}\int_{S_p} dS\mathbf{n}\cdot\boldsymbol{\xi}_\perp\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1\,, \tag{1.1}$$

where the notation is the usual one associated with the MHD-energy Principle, see J. P. Freidberg [1].

We consider a plasma region V_p, where S_p is the boundary of this region, and a vacuum region, which contains a resistive wall, with V_i the region between the plasma and the wall and V_o the region outside the wall. Let the perturbation in the vacuum magnetic field inside the resistive wall be given by

$$\hat{\mathbf{B}}_1 = \nabla\phi_i\,. \tag{1.2}$$

This representation of the magnetic field has an obvious weakness, which is the problem of multivaluedness in a doubly connected region, as we have it in a typical toroidal configuration.[1] In spite of this, the simplicity of this analysis as compared to a description in terms of a vector potential, hopefully justify this shortcoming.

The boundary condition (the continuity of the normal component of $\mathbf{B}$ across the plasma vacuum interface) is given by

$$\begin{aligned}\mathbf{n}\cdot\nabla\phi_i|_{S_p} &= \mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}\times\hat{\mathbf{B}})|_{S_p} = -\nabla_s\cdot\{\mathbf{n}\times(\boldsymbol{\xi}\times\hat{\mathbf{B}})\}|_{S_p}\\ &= -\nabla_s\cdot\{\boldsymbol{\xi}\mathbf{n}\cdot\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{n}\cdot\boldsymbol{\xi}\}|_{S_p} = \nabla_s\cdot\{\mathbf{n}\cdot\boldsymbol{\xi}\hat{\mathbf{B}}\}|_{S_p}\,,\end{aligned} \tag{1.3}$$

since $\mathbf{n}\cdot\hat{\mathbf{B}} = 0$ at the surface. We obtain the last representation by noting that $\nabla_s \stackrel{\text{def}}{=} \nabla - \mathbf{n}\mathbf{n}\cdot\nabla$ (the surface gradient at a surface having the unit normal vector $\mathbf{n}$). The following identities exist

$$\nabla_s\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla_s\times\mathbf{u} - \mathbf{u}\cdot\nabla_s\times\mathbf{v}\,, \tag{1.4}$$

$$\nabla_s\times\mathbf{n}\equiv 0\,,\ \text{(whenever } \mathbf{n} \text{ is the surface unitnormal)}\,. \tag{1.5}$$

[1] This is of special concern for the $n = 0$ and $m = 0$ modes.

Take $\mathbf{u} = \mathbf{n}$ and $\mathbf{v} = \boldsymbol{\xi} \times \hat{\mathbf{B}}$ in eq.(1.4) to prove the second last step in eq.(1.3). We now consider the integral

$$\delta W_V^{(i)} \stackrel{def}{=} \frac{1}{2\mu_0} \int_{V_i} |\nabla \phi_i|^2 dS = \frac{1}{2\mu_0} \int_{V_i} \nabla \cdot (\phi_i \nabla \phi_i) dS\,. \tag{1.6}$$

Then we may write

$$\delta W_V^{(i)} = -\frac{1}{2\mu_0} \int_{S_p} \phi_i \mathbf{n} \cdot \nabla \phi_i dS + \frac{1}{2\mu_0} \int_{S_w} \phi \mathbf{n} \cdot \nabla \phi_i dS\,, \tag{1.7}$$

where $\mathbf{n}$ is a unit normal pointing outward from the plasma surface. Also notice that we have used the fact that $\nabla^2 \phi_i = 0$. Here S_p and S_w are the plasma and wall surfaces, respectively. At the wall surface we adopt the convention that the unit normal vector is pointing towards the wall on the inside, and away from the wall on the outside.

Using the boundary condition eq.(1.3), we may write

$$\delta W_V^{(i)} = \frac{1}{2\mu_0} \int_{S_p} \phi_i \nabla_s \cdot \{\mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}}\} dS + \frac{1}{2\mu_0} \int_{S_w} \phi_i \mathbf{n} \cdot \nabla \phi_i dS\,, \tag{1.8}$$

and

$$\begin{aligned} \frac{1}{2\mu_0} \int_{S_p} \phi_i \nabla_s \cdot \{\mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}}\} dS &= \\ \frac{1}{2\mu_0} \int_{S_p} \nabla_s \cdot \{\phi_i \mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}})\} dS &- \frac{1}{2\mu_0} \int_{S_p} \mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}}) \cdot \nabla_s \phi_i dS \end{aligned}$$

to show that the first of the integrals on the right-hand side above is zero, we use the formula

$$\int \{\nabla_s \cdot \mathbf{v} + J \mathbf{n} \cdot \mathbf{v}\} dS = \oint \mathbf{m} \cdot \mathbf{v} ds\,, \tag{1.9}$$

where the integrals on the left-hand side are over a surface bounded by a closed curve over which the integral on the right-hand side is performed. Also $\mathbf{m} = \mathbf{T} \times \mathbf{n}$, where $\mathbf{T}$ is the unit tangent vector to the curve, $\mathbf{n}$ is the unit normal vector to the surface and $J = \nabla_s \cdot \mathbf{n}$. (See Brand: Vector and Tensor analysis p. 222 [2]).

In our case the surface is closed and the left-hand side vanish, also

$$\mathbf{v} = \phi_i \mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}}), \tag{1.10}$$

thus $\mathbf{v} \cdot \mathbf{n} = 0$, and the first integral on the right-hand side of eq.(1.9) vanish and we have the result

$$\frac{1}{2\mu_0} \int_{S_p} \phi_i \nabla_s \cdot \{\mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}}\} dS = \frac{1}{2\mu_0} \int_{S_{wi}} \mathbf{n} \cdot \boldsymbol{\xi}_\perp \hat{\mathbf{B}} \cdot \nabla \phi_i dS \,. \tag{1.11}$$

Notice that $\hat{\mathbf{B}} \cdot \nabla \phi = \hat{\mathbf{B}} \cdot \nabla_s \phi = \hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1$ where $\hat{\mathbf{B}}_1$ is the perturbation in the vacuum magnetic field. We may then write

$$\delta W_V^{(i)} = \frac{1}{2\mu_0} \int_{S_p} \mathbf{n} \cdot \boldsymbol{\xi}_\perp \hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1 dS + \frac{1}{2\mu_0} \int_{S_{wi}} \phi_i \mathbf{n} \cdot \nabla \phi_i dS \,. \tag{1.12}$$

For the outer region we have

$$\delta W_V^{(o)} = \frac{1}{2\mu_0} \int_{V_o} |\nabla \phi_o|^2 dS = -\frac{1}{2\mu_0} \int_{S_{wo}} \phi_o \mathbf{n} \cdot \nabla \phi_o dS \,, \tag{1.13}$$

with $\nabla^2 \phi_o = 0$.

The solutions ϕ_i and ϕ_o are connected over the resistive wall region by the solution within the wall. For the solution within the wall we shall use the thin wall approximation and adopt the solution given by Haney and Freidberg [3] eqs.(30) - (40). For this problem the solution for the magnetic field is derived from a vector potential, and by expansion in the thin wall parameter it is given by

$$\mathbf{B}_{1w} = \nabla \times \mathbf{A}_{w0} + \nabla \times \mathbf{A}_{w1} + \ldots \tag{1.14}$$

The first or leading order term is a surface quantity not changing across the wall. Continuity of the normal component of the magnetic field requires to leading order that

$$\mathbf{n} \cdot \nabla \phi_i = \mathbf{n} \cdot \nabla \times \mathbf{A}_{w0} = -\nabla_s \cdot (\mathbf{n} \times \mathbf{A}_{w0}), \tag{1.15}$$

where we again have used the identity given by eq.(1.4). In a similar way we obtain

$$\mathbf{n}\cdot\nabla\phi_o = \mathbf{n}\cdot\nabla\times\mathbf{A}_{w0} = -\nabla_s\cdot(\mathbf{n}\times\mathbf{A}_{w0})\,. \tag{1.16}$$

From this we conclude that the jump in $\mathbf{n}\cdot\nabla\phi$ across the wall is zero to leading order (in the thin wall approximation). Since we also assume that there is no surface current present on the wall surface, the parallel component of the magnetic field must also be continuous across these surfaces (inner and outer), that is

$$\mathbf{n}\times\nabla\phi_i|_{S_{wi}} = \mathbf{n}\times(\nabla\times\mathbf{A}_{w0})|_{S_{wi}} + \mathbf{n}\times(\nabla\times\mathbf{A}_{w1})|_{S_{wi}} + \ldots \tag{1.17}$$

$$\mathbf{n}\times\nabla\phi_o|_{S_{wo}} = \mathbf{n}\times(\nabla\times\mathbf{A}_{w0})|_{S_{wo}} + \mathbf{n}\times(\nabla\times\mathbf{A}_{w1})|_{S_{wo}} + \ldots \tag{1.18}$$

From Ref. 3, eq.(40) we have

$$\mathbf{n}\times\mathbf{A}_{w1} = \mathbf{a}_1(s) + \mathbf{C}_1(s)u + \mu_0\sigma\gamma d^2\mathbf{n}\times\mathbf{A}_{w0}(s)\frac{u^2}{2}\,, \tag{1.19}$$

where u is a parameter varying across the wall, being zero on the inside and one on the outside and s is a surface parameter for the poloidal direction. Since we are considering axisymmetric systems, no reference to the toroidal direction is necessary. We find

$$\begin{aligned}
\nabla\times\mathbf{A}_{w1} &= (\nabla_s + \mathbf{nn}\cdot\nabla)\times\mathbf{A}_{w1} \\
&= \mathbf{nn}\cdot\nabla\times\mathbf{A}_{w1} + \text{higher order terms} \\
&= \mathbf{n}\cdot\nabla(\mathbf{n}\times\mathbf{A}_{w1}) + \ldots \\
&= \frac{1}{d}\frac{\partial}{\partial u}(\mathbf{n}\times\mathbf{A}_{w1}) + \ldots \\
&= \frac{\mathbf{C}_1(s)}{d} + \mu_0\sigma\gamma d\,\mathbf{n}\times\mathbf{A}_{w0}u + \ldots
\end{aligned}$$

By using this result in eqs.(1.17) and (1.18) we obtain

$$\mathbf{n}\times\nabla\phi_i|_{S_i} = \mathbf{n}\times(\nabla\times\mathbf{A}_{w0}) + \frac{1}{d}\mathbf{n}\times\mathbf{C}_1(s)\,, \tag{1.20}$$

$$\begin{aligned}
\mathbf{n}\times\nabla\phi_o|_{S_o} &= \mathbf{n}\times(\nabla\times\mathbf{A}_{w0}) + \frac{1}{d}\mathbf{n}\times\mathbf{C}_1(s) \\
&+ \mu_0\sigma\gamma d\mathbf{n}\times(\mathbf{n}\times\mathbf{A}_{w0}(s))\,,
\end{aligned} \tag{1.21}$$

or written as a jump condition

$$[\![\mathbf{n}\times\nabla\phi]\!] = -\mu_0\sigma\gamma d\mathbf{n}\times(\mathbf{n}\times\mathbf{A}_{w0}(s))\,, \tag{1.22}$$

where $[\![Q]\!]$ means $Q_i - Q_o$. By adding eqs.(1.12) and (1.13) we obtain

$$\begin{aligned}\delta W_V^{(i)} + \delta W_V^{(o)} &= \frac{1}{2\mu_0}\int_{S_p}\mathbf{n}\cdot\boldsymbol{\xi}_\perp\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 dS \\ &+ \frac{1}{2\mu_0}\int_{S_{wi}}\phi_i\mathbf{n}\cdot\nabla\phi_i dS - \frac{1}{2\mu_0}\int_{S_{wo}}\phi_o\mathbf{n}\cdot\nabla\phi_o dS\end{aligned} \tag{1.23}$$

Using the boundary conditions eqs.(1.15) and (1.16) we obtain

$$\begin{aligned}\delta W_V^{(i)} + \delta W_V^{(o)} &= \frac{1}{2\mu_0}\int_{S_p}\mathbf{n}\cdot\boldsymbol{\xi}_\perp\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 dS \\ &\quad -\frac{1}{2\mu_0}\int_{S_{wi}}(\phi_i-\phi_o)\nabla_s\cdot(\mathbf{n}\times\mathbf{A}_{wo})dS \\ &= \frac{1}{2\mu_0}\int_{S_p}\mathbf{n}\cdot\boldsymbol{\xi}_\perp\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 dS \\ &\quad +\frac{1}{2\mu_0}\int_{S_{wi}}(\mathbf{n}\times\mathbf{A}_{wo})\cdot\nabla_s(\phi_i-\phi_o)dS \\ &\quad -\frac{1}{2\mu_0}\int_{S_{wi}}\nabla_s\cdot\{(\mathbf{n}\times\mathbf{A}_{wo})(\phi_i-\phi_o)\}dS \\ &= \frac{1}{2\mu_0}\int_{S_p}\mathbf{n}\cdot\boldsymbol{\xi}_\perp\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 dS \\ &\quad +\frac{1}{2\mu_0}\int_{S_{wi}}(\mathbf{n}\times\mathbf{A}_{wo})\cdot\nabla_s(\phi_i-\phi_o)dS\,,\end{aligned} \tag{1.24}$$

where we have again used eq.(1.9). From eq.(1.22) we obtain

$$\mathbf{n}\times[\mathbf{n}\times\nabla(\phi_i-\phi_o)] = -\mu_0\sigma\gamma d\mathbf{n}\times[\mathbf{n}\times(\mathbf{n}\times\mathbf{A}_{w0}(s))]\,, \tag{1.25}$$

$$\nabla_s(\phi_i-\phi_o) = -\mu_0\sigma\gamma d\mathbf{n}\times\mathbf{A}_{w0}(s)\,, \tag{1.26}$$

where eq.(1.26) is a simplified form of eq.(1.25). From eqs.(1.24) and (1.26) we obtain

$$\delta W_V^{(i)} + \delta W_V^{(o)} = \frac{1}{2\mu_0}\int_{S_p} \mathbf{n}\cdot\boldsymbol{\xi}_\perp\,\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 dS - \frac{\sigma\gamma d}{2}\int_{S_w} |\mathbf{n}\times\mathbf{A}_{w0}|^2 dS\,, \quad (1.27)$$

and

$$\frac{1}{2\mu_0}\int_{S_p} \mathbf{n}\cdot\boldsymbol{\xi}_\perp\,\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 dS = \delta W_V^{(i)} + \delta W_V^{(o)} + \frac{\sigma\gamma d}{2}\int_{S_w} |\mathbf{n}\times\mathbf{A}_{w0}|^2 dS\,. \quad (1.28)$$

Finally we may write our Lagrangian in the desired form

$$\mathcal{L} = \delta W_F + \delta W_V^{(i)} + \delta W_V^{(o)} + \frac{\sigma\gamma d}{2}\int_{S_w} |\mathbf{n}\times\mathbf{A}_{w0}|^2 dS\,. \quad (1.29)$$

Concerning the next section we shall simplify notation slightly and write eq. (1.29) as

$$\mathcal{L} = \delta W_F + \delta W_i + \delta W_o + \delta W_W\,, \quad (1.30)$$

where

$$\delta W_W \stackrel{def}{=} \frac{\sigma\gamma d}{2}\int_{S_w} |\mathbf{n}\times\mathbf{A}_{w0}|^2 dS\,.$$

Notice that in the last integration over S_w, no distinction needs to be made between inner and outer wall surfaces, since only surface quantities are involved.

The next step is to prove that the Lagrangian above is a variational form that has the pressure jump condition at the plasma vacuum surface and condition (1.26) as natural boundary conditions.

1.3 A Variational Principle

First we shall assume the following boundary conditions to apply, and that these conditions also apply to the variation. These conditions reflect the fact that the normal component of the magnetic field $\mathbf{B}$, is always continuous across any boundary, due to $\nabla\cdot\mathbf{B} = 0$.

$$\mathbf{n}\cdot\nabla\phi_i|_{S_p} \;=\; \mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}\times\hat{\mathbf{B}})|_{S_p}$$

$$
\begin{aligned}
&= -\nabla_s \cdot \{\mathbf{n} \times (\boldsymbol{\xi} \times \hat{\mathbf{B}})\}|_{S_p} \\
&= \nabla_s \cdot \{\mathbf{n} \cdot \boldsymbol{\xi}_\perp \hat{\mathbf{B}}\}|_{S_p}\,, \qquad (1.31)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{n} \cdot \nabla\phi_i|_{S_w} &= \mathbf{n} \cdot \nabla \times \mathbf{A}_{w0} = -\nabla_s \cdot \{\mathbf{n} \times \mathbf{A}_{w0}\}\,, \qquad (1.32) \\
\mathbf{n} \cdot \nabla\phi_o|_{S_w} &= \mathbf{n} \cdot \nabla \times \mathbf{A}_{w0} = -\nabla_s \cdot \{\mathbf{n} \times \mathbf{A}_{w0}\} \qquad (1.33) \\
&= \mathbf{n} \cdot \nabla\phi_i|_{S_w}\,, \\
\phi_o(\infty) &= 0\,. \qquad (1.34)
\end{aligned}
$$

We now take the variation of eq.(1.30). After a considerable amount of algebra (see Appendix A for details) this can be written in the following way:

$$
\delta(\delta W_F) = -\int_{V_p} \delta\boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) dV - \frac{1}{\mu_0}\int_{S_p} \mathbf{n} \cdot \delta\boldsymbol{\xi}(\mathbf{B}\cdot\mathbf{Q} - \gamma p \nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p)\,, \quad (1.35)
$$

where

$$
\mathbf{F}(\boldsymbol{\xi}) \stackrel{def}{=} -\frac{1}{\mu_0}\mathbf{Q} \times (\nabla \times \mathbf{B}) - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{Q}) + \nabla\{\boldsymbol{\xi}\cdot\nabla p + \gamma p \nabla\cdot\boldsymbol{\xi}\}\,,
$$

and

$$
\mathbf{Q} \stackrel{def}{=} \nabla \times (\boldsymbol{\xi} \times \mathbf{B})\,.
$$

For $\delta(\delta W_i)$ we find

$$
\begin{aligned}
\delta(\delta W_i) &= \frac{1}{\mu_0}\int_{S_p} \mathbf{n} \cdot \delta\boldsymbol{\xi}\hat{\mathbf{B}} \cdot \nabla\phi_i dS + \frac{1}{\mu_0}\int_{S_w} \phi_i \mathbf{n} \cdot \nabla\delta\phi_i dS \\
&\quad - \frac{1}{\mu_0}\int_{Vi} \phi_i \nabla^2 \delta\phi_i dS\,, \qquad (1.36)
\end{aligned}
$$

and

$$
\delta(\delta W_o) = -\frac{1}{\mu_0}\int_{S_w} \phi_o \mathbf{n} \cdot \nabla\delta\phi_o dS - \frac{1}{\mu_0}\int_{Vo} \phi_o \nabla^2 \delta\phi_o dS\,. \qquad (1.37)
$$

Notice that

$$
\begin{aligned}
\int_{S_p} \phi_i \mathbf{n} \cdot \nabla\delta\phi_i dS &= -\int_{S_p} \phi_i \nabla_s \cdot \{\mathbf{n} \times (\delta\boldsymbol{\xi} \times \hat{\mathbf{B}})\} dS \\
&= -\int_{S_p} [\nabla_s \cdot \{\phi_i \mathbf{n} \times (\delta\boldsymbol{\xi} \times \hat{\mathbf{B}})\} - \mathbf{n} \times (\delta\boldsymbol{\xi} \times \hat{\mathbf{B}}) \cdot \nabla_s \phi_i] dS \\
&= -\int_{S_p} \mathbf{n} \cdot \delta\boldsymbol{\xi}\hat{\mathbf{B}} \cdot \nabla_s \phi_i dS\,,
\end{aligned}
$$

which has been used when obtaining eq.(1.36). When integrating the inner vacuum region we need the following expressions obtained from eqs.(1.32) and (1.33)

$$\begin{aligned}\frac{1}{\mu_0}\int_{S_w}\phi_i\mathbf{n}\cdot\nabla\delta\phi_i dS &= -\frac{1}{\mu_0}\int_{S_w}\phi_i\nabla_s\cdot(\mathbf{n}\times\delta\mathbf{A}_{wo})dS \\ &= -\frac{1}{\mu_0}\int_{S_w}\nabla_s\cdot(\phi_i\mathbf{n}\times\delta\mathbf{A}_{wo})dS \\ &+ \frac{1}{\mu_0}\int_{S_w}\mathbf{n}\times\delta\mathbf{A}_{wo}\cdot\nabla_s\phi_i dS\,, \end{aligned}\tag{1.38}$$

and also when integrating the outer vacuum region we need

$$\begin{aligned}-\frac{1}{\mu_0}\int_{S_w}\phi_o\mathbf{n}\cdot\delta\phi_o dS &= \frac{1}{\mu_0}\int_{S_w}\phi_o\nabla_s\cdot(\mathbf{n}\times\delta\mathbf{A}_{wo}dS \\ &= \frac{1}{\mu_0}\int_{S_w}\nabla_s\cdot(\phi_o\mathbf{n}\times\delta\mathbf{A}_{wo})dS \\ &- \frac{1}{\mu_0}\int_{S_w}\mathbf{n}\times\delta\mathbf{A}_{wo}\cdot\nabla_s\phi_o dS\,. \end{aligned}\tag{1.39}$$

We summarize the results:

$$\delta(\delta W_F) = -\int_{V_p}\delta\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})dV - \frac{1}{\mu_0}\int_{S_p}\mathbf{n}\cdot\delta\boldsymbol{\xi}(\mathbf{B}\cdot\mathbf{Q}-\gamma p\nabla\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla p)dS\,, \tag{1.40}$$

$$\begin{aligned}\delta(\delta W_i) &= \frac{1}{\mu_0}\int_{S_p}\mathbf{n}\cdot\delta\boldsymbol{\xi}\hat{\mathbf{B}}\cdot\nabla_s\phi_i dS - \frac{1}{\mu_0}\int_{Vi}\phi_i\nabla^2\delta\phi_i dS \\ &-\frac{1}{\mu_0}\int_{S_w}\nabla_s\cdot\{\phi_i\mathbf{n}\times\delta\mathbf{A}_{wo}\}dS \\ &+\frac{1}{\mu_0}\int_{S_w}\mathbf{n}\times\delta\mathbf{A}_{wo}\cdot\nabla_s\phi_i dS\,, \end{aligned}\tag{1.41}$$

$$\delta(\delta W_o) = \frac{1}{\mu_0}\int_{S_w}\nabla_s\cdot\{\phi_o\mathbf{n}\times\delta\mathbf{A}_{wo}dS\}$$

$$-\frac{1}{\mu_0}\int_{S_w} \mathbf{n}\times\delta\mathbf{A}_{wo}\cdot\nabla_s\phi_o dS - \frac{1}{\mu_0}\int_{V_o}\phi_o\nabla^2\delta\phi_o dS\,, \quad (1.42)$$

$$\delta(\delta W_w) \;=\; \sigma\gamma d\int_{S_w}(\mathbf{n}\times\delta\mathbf{A}_{wo})\cdot\mathbf{n}\times\mathbf{A}_{wo}dS\,. \quad (1.43)$$

Adding these results and making another integration by parts, we obtain

$$\begin{aligned}
\delta\mathcal{L} = \; & - \int_{V_p}\delta\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})dV \\
& - \frac{1}{\mu_0}\int_{S_p}\mathbf{n}\cdot\delta\boldsymbol{\xi}(\mathbf{B}\cdot\mathbf{Q}-\gamma p\nabla\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla p)dS \\
& + \frac{1}{\mu_0}\int_{S_p}\mathbf{n}\cdot\delta\boldsymbol{\xi}\hat{\mathbf{B}}\cdot\nabla\phi_i dS \\
& + \frac{1}{\mu_0}\int_{S_w}\mathbf{n}\times\delta\mathbf{A}_{wo}\cdot\{\nabla_s(\phi_i-\phi_o)+\mu_0\sigma\gamma d\mathbf{n}\times\mathbf{A}_{wo}\}dS \\
& - \frac{1}{\mu_0}\int_{Vi}\phi_i\nabla^2\delta\phi_i dS - \frac{1}{\mu_0}\int_{V_o}\phi_o\nabla^2\delta\phi_o dS \\
& - \frac{1}{\mu_0}\int_{S_w}\nabla_s\cdot\{\mathbf{n}\times\delta\mathbf{A}_{wo}(\phi_i-\phi_o)\}dS\,. \qquad (1.44)
\end{aligned}$$

From eq.(1.44) we see that we have a variational principle that produces the equations of motion $\mathbf{F}(\boldsymbol{\xi})=\mathbf{0}$ in the plasma region, and with the proper boundary conditions, i.e., the pressure jump condition at the plasma vacuum surface S_p, and the corresponding condition given by eq.(1.22) or (1.26) at the conductor surface, as natural boundary conditions. There is in addition one restriction to be observed. *The solutions for the scalar potential have to satisfy Laplace's equation also for the variations.* This is still, not a severe restriction since for our problems it will be natural to choose the trial functions as solutions of Laplace's equation.

Finally, notice that the last term in eq.(1.44) integrates to zero when S_w is a closed surface, otherwise this term may contribute.

1.4 Trial Functions

We shall now discuss functions that we shall use as a base for our trial functions. We consider two associated problems:

Problem 1: There is no conducting wall present. The corresponding solutions are given by $\boldsymbol{\xi}_\infty$ and ϕ_∞.

Problem 2: The wall is infinitely conducting and makes a closed surface in the vacuum region. The corresponding solutions are given by $\boldsymbol{\xi}_b$ and ϕ_b.

Assumption: We shall make one assumption about these solutions, namely $\boldsymbol{\xi}_\infty = \boldsymbol{\xi}_b$ or the perturbation in the plasma is the same for the modes considered in the two situations.

We then make the following expansion, let

$$\phi_i = c_1\phi_\infty + c_2\phi_b\,, \qquad (1.45)$$

$$\phi_o = c_3\phi_\infty\,, \qquad (1.46)$$

with the boundary conditions

$$\phi_\infty(\infty) = 0 \qquad \text{and} \qquad \mathbf{n}\cdot\nabla\phi_b|_{S_b} = 0\,, \qquad (1.47)$$

and let both solutions satisfy the same boundary condition at the plasma vacuum interface ($\boldsymbol{\xi}_\infty = \boldsymbol{\xi}_b$), according to the assumption above. We shall use our Lagrangian as given in eq.(1.30)

$$\mathcal{L} = \delta W_F + \delta W_i + \delta W_o + \delta W_W\,. \qquad (1.48)$$

From the boundary conditions at the conducting wall and at the plasma - vacuum interface, we obtain

$$\mathbf{n}\cdot\nabla\phi_i|_{S_b} = \mathbf{n}\cdot\nabla\phi_o|_{S_b} \text{ and } \mathbf{n}\cdot\nabla\phi_b|_{S_b} = 0 \quad \Rightarrow \quad c_1 = c_3\,, \qquad (1.49)$$

$$\mathbf{n}\cdot\nabla\phi_i|_{S_p} = \mathbf{n}\cdot\nabla\phi_\infty|_{S_p} = \mathbf{n}\cdot\nabla\phi_b|_{S_p} \quad \Rightarrow \quad c_1 + c_2 = 1\,, \qquad (1.50)$$

$$
\begin{aligned}
\delta W_i &= \frac{1}{2\mu_0}\int_{V_i}|\nabla\phi_i|^2 dV \\
&= c_1^2\frac{1}{2\mu_0}\int_{V_i}|\nabla\phi_\infty|^2 dV + c_2^2\frac{1}{2\mu_0}\int_{V_i}|\nabla\phi_b|^2 dV \\
&\quad +2c_1c_2\frac{1}{2\mu_0}\int_{V_i}\nabla\phi_\infty\cdot\nabla\phi_b dV\,. \qquad (1.51)
\end{aligned}
$$

Now we have

$$
\int_{V_i}\nabla\phi_\infty\cdot\nabla\phi_b dV = \int_{V_i}\nabla\cdot(\phi_\infty\nabla\phi_b)dV = -\int_{S_p}\phi_\infty\mathbf{n}\cdot\nabla\phi_b dS = \delta W_V^\infty\,. \quad (1.52)
$$

Notice that in eq.(1.52) there is no contribution from the other surfaces, i.e., wall and conductors, as will subsequently be considered, because $\mathbf{n}\cdot\nabla\phi_b|_{S_b} = 0$ on these surfaces, which has been used in order to obtain eq.(1.52). Here δW_V^∞ refers to the vacuum part of the energy for the first problem, i.e., wall at ∞, and δW_V^b to be used shortly, refers similarly to the problem with an infinitely conducting wall present. Notice that $\mathbf{n}\cdot\nabla\phi_\infty|_{S_p} = \mathbf{n}\cdot\nabla\phi_b|_{S_p}$ due to assumptions made. Furthermore

$$
\delta W_o = \frac{1}{2\mu_0}\int_{V_o}|\nabla\phi_o|^2 dV = c_3^2\frac{1}{2\mu_0}\int_{V_o}|\nabla\phi_\infty|^2 dV\,. \qquad (1.53)
$$

Now since $c_3 = c_1$ and

$$
\frac{1}{2\mu_0}\int_{V_i}|\nabla\phi_\infty|^2 dV + \frac{1}{2\mu_0}\int_{V_o}|\nabla\phi_\infty|^2 dV = \delta W_V^\infty\,, \qquad (1.54)
$$

we may write

$$
\begin{aligned}
\mathcal{L} &= \delta W_F + \delta W_i + \delta W_o + \delta W_W \\
&= \delta W_F + c_1^2\delta W_V^\infty + c_2^2\delta W_V^b + 2c_1c_2\delta W_V^\infty + \delta W_W \\
&= \delta W_F + (1-c_2^2)\delta W_V^\infty + c_2^2\delta W_V^b + \delta W_W \\
&= \delta W_\infty + c_2^2(\delta W_V^b - \delta W_V^\infty) + \delta W_W\,.
\end{aligned}
$$

Thus

$$\mathcal{L} = \delta W_\infty + c_2^2(\delta W_V^b - \delta W_V^\infty) + \delta W_W \,, \tag{1.55}$$

where

$$\delta W_W = \frac{\sigma\gamma d}{2}\int_{S_b} |\mathbf{n} \times \mathbf{A}_{wo}|^2 dS \,, \tag{1.56}$$

and

$$\mathbf{n} \cdot \nabla\phi_o|_{S_b} = \mathbf{n} \cdot \nabla \times \mathbf{A}_{w0} = -\nabla_s \cdot (\mathbf{n} \times \mathbf{A}_{w0}) \,, \tag{1.57}$$

is the boundary condition that connects $\mathbf{A}_{w0}$ to ϕ_o.

Since $\phi_o = c_3\phi_\infty$ it is natural to scale $\mathbf{A}_{w0}$ in a similar way, thus let

$$\mathbf{A}_{w0} = c_3\hat{\mathbf{A}}_{w0} \,, \tag{1.58}$$

and we have

$$\mathbf{n} \cdot \nabla\phi_\infty|_{S_b} = -\nabla_s \cdot (\mathbf{n} \times \hat{\mathbf{A}}_{w0}) \,, \tag{1.59}$$

and

$$c_3^2\frac{\sigma\gamma d}{2}\int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_{w0}| dS \stackrel{def}{=} c_3^2\delta\hat{W}_W \,. \tag{1.60}$$

The Lagrangian can now be written as

$$\mathcal{L} = \delta W_\infty + c_2^2(\delta W_b - \delta W_\infty) + (1 - c_2)^2\delta\hat{W}_W \,. \tag{1.61}$$

We find the stationary value of $\mathcal{L}$ with respect to the variational parameter c_2 by taking the derivative of $\mathcal{L}$ with respect c_2 and set the result equal zero, this way we obtain

$$c_2 = \frac{\delta\hat{W}_W}{\delta W_b - \delta W_\infty + \delta\hat{W}_W} \,. \tag{1.62}$$

Let the stationary value of the Lagrangian be given as $\hat{\mathcal{L}}$, then we obtain

$$\hat{\mathcal{L}} = \delta W_\infty + \frac{\delta\hat{W}_W(\delta W_b - \delta W_\infty)}{\delta W_b - \delta W_\infty + \delta\hat{W}_W} \,. \tag{1.63}$$

In order to make an estimate of the growth rate we solve the equation

$$\hat{\mathcal{L}} = 0\,, \tag{1.64}$$

which yields

$$\frac{\delta \hat{W}_W}{\delta W_b - \delta W_\infty} = -\frac{\delta W_\infty}{\delta W_b}\,, \tag{1.65}$$

with

$$\begin{aligned} \delta W_\infty &= \delta W_F + \int_V |\nabla \phi_\infty|^2 dV\,, \quad V = V_i + V_o\,, & (1.66) \\ \delta W_b &= \delta W_F + \int_V |\nabla \phi_b|^2 dV\,, \quad V = V_i\,, & (1.67) \\ \delta \hat{W}_W &= \frac{\sigma \gamma d}{2} \int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_{wo}|^2 dS\,, & (1.68) \end{aligned}$$

and $\hat{\mathbf{A}}_{w0}$ satisfies the boundary condition

$$\mathbf{n} \cdot \nabla \phi_\infty|_{S_b} = \mathbf{n} \cdot \nabla \times \hat{\mathbf{A}}_{w0} = -\nabla_s \cdot (\mathbf{n} \times \hat{\mathbf{A}}_{w0})\,. \tag{1.69}$$

Finally we may write in correspondence with the notations used by Haney and Freidberg[3],

$$\gamma \tau_D = -\frac{\delta W_\infty}{\delta W_b}\,, \tag{1.70}$$

with

$$\tau_D = \mu_0 \sigma d \bar{b}\,, \tag{1.71}$$

and

$$\bar{b} = \frac{\frac{1}{2\mu_0} \int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_{w0}|^2 dS}{\delta W_b - \delta W_\infty}\,. \tag{1.72}$$

The boundary condition eq.(1.22) is a natural boundary condition, and therefore it does not have to be satisfied by the variation of the solution. This boundary condition normally is satisfied only by the exact solution, which in general is unknown. However, the error made by assuming this condition satisfied also by the variation must be small of first order. By making the assumption that the variations satisfy this boundary condition we can derive a less accurate formula that may be simpler in practical use for the purpose of evaluating $\bar{b}$,

$$\bar{b} = \frac{\frac{1}{2\mu_0}\int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_{w0}|^2 dS}{\delta W_b - \delta W_\infty} \cong \frac{\delta W_b - \delta W_\infty}{\frac{1}{2\mu_0}\int_{S_b} |\nabla_s \phi_b|^2 dS} \,. \tag{1.73}$$

where the last step in the formula above rests on the assumptions made above. The last expression for $\bar{b}$ in eq.(1.73) has the virtue of being independent of $\mathbf{A}_{w0}$, it depends only on the solutions of Problem 1 and Problem 2 and the geometry of the conducting surface. A derivation of this alternative formula is presented in Appendix B.

1.5 Resistive Wall and Resistive Conductors

In order to include resistive conductors inside a resistive wall we need to modify the Lagrangian by adding two terms

$$\delta W_C = \frac{1}{2\mu_0}\int_{V_c} |\nabla \times \mathbf{A}_c|^2 dV + \frac{\sigma\gamma}{2}\int_{V_c} \mathbf{A}_c^2 dV\,. \tag{1.74}$$

In order to prove this we take the variation of eq.(1.74)

$$\begin{aligned}
\delta(\delta W_C) &= \frac{1}{\mu_0}\int_{V_c}(\nabla \times \mathbf{A}_c)\cdot(\nabla \times \delta\mathbf{A}_c)dV + \sigma\gamma\int_{V_c}\mathbf{A}_c \cdot \delta\mathbf{A}_c dV \\
&= \frac{1}{\mu_0}\int_{V_c}\nabla \cdot \{\delta\mathbf{A}_c \times (\nabla \times \mathbf{A}_c)\}dV \\
&\quad + \frac{1}{\mu_0}\int_{V_c}\delta\mathbf{A}_c \cdot \{\nabla \times (\nabla \times \mathbf{A}_c) + \sigma\gamma\mathbf{A}_c\}dV \\
&= -\frac{1}{\mu_0}\int_{S_c}\mathbf{n} \times \delta\mathbf{A}_c \cdot \nabla \times \mathbf{A}_c dS \\
&\quad + \frac{1}{\mu_0}\int_{V_c}\delta\mathbf{A}_c \cdot \{\nabla \times (\nabla \times \mathbf{A}_c) + \mu_0\sigma\gamma\mathbf{A}_c\}dV\,.
\end{aligned}$$

Notice that the surface unit normal vector **n** is pointing into the conductor.

1.5.1 Conductor Inside the Wall

We shall first study the case where the conductor is positioned inside the wall. From the vacuum region outside the conductor there is a contribution from the vacuum energy

$$\delta W_i = \frac{1}{2\mu}\int_{V_i}|\nabla\phi_i|^2 dV \tag{1.75}$$

$$\begin{aligned}
\delta(\delta W_i) &= \frac{1}{\mu}\int_{V_i}\nabla\phi_i \cdot \nabla\delta\phi_i dV \\
&= \frac{1}{\mu}\int_{V_i}\nabla \cdot \{\phi_i \cdot \nabla\delta\phi_i\}dV - \frac{1}{\mu_0}\int_{V_i}\phi_i\nabla^2\delta\phi_i dV \\
&= -\frac{1}{\mu}\int_{S_p}\phi_i\mathbf{n}\cdot\nabla\delta\phi_i\, dS + \frac{1}{\mu_0}\int_{S_w}\phi_i\mathbf{n}\cdot\nabla\delta\phi dS \\
&\quad + \frac{1}{\mu_0}\int_{S_c}\phi_i\mathbf{n}\cdot\nabla\delta\phi_i dS\,,
\end{aligned} \tag{1.76}$$

where the last integral is over the conductor surface. Equation (1.76) now replaces eq.(1.36). Let

$$\delta(\delta\hat{W}_i) \stackrel{def}{=} \frac{1}{\mu_0}\int_{S_c} \phi_i \mathbf{n}\cdot\nabla\delta\phi_i dV\,. \tag{1.77}$$

The same kind of boundary condition that applies to a resistive wall also applies to a resistive conductor. Here we make use of the condition corresponding to eq.(1.33), and obtain

$$\begin{aligned}\delta(\delta\hat{W}_i) &= -\frac{1}{\mu_0}\int_{S_c} \phi_i \nabla_s\cdot(\mathbf{n}\times\delta\mathbf{A}_c)dS \\ &= -\frac{1}{\mu_0}\int_{S_c} \nabla_s\cdot\{\phi_i\mathbf{n}\times\mathbf{A}_c\}dS \\ &\quad +\frac{1}{\mu_0}\int_{S_c} \mathbf{n}\times\delta\mathbf{A}_c\cdot\nabla_s\phi_i dS\,.\end{aligned}$$

The integral of the surface divergence integrates to zero and we obtain

$$\begin{aligned}\delta(\delta\hat{W}_i)+\delta(\delta W_C) &= \frac{1}{\mu_0}\int_{S_c} \mathbf{n}\times\delta\mathbf{A}_c\cdot\{\nabla_s\phi_i-\nabla\times\mathbf{A}_c\}dS \\ &\quad +\frac{1}{\mu_0}\int_{V_c} \delta\mathbf{A}_c\cdot\{\nabla\times(\nabla\times\mathbf{A}_c)+\mu\sigma\gamma\mathbf{A}_c\}dS\,.\end{aligned}$$

Since $\delta\mathbf{A}_c$ is arbitrary and $\mathbf{n}\times\delta\mathbf{A}_c$ is an arbitrary vector in the tangent plane of the conductor surface S_c, we obtain

$$\nabla\times(\nabla\times\mathbf{A}_c)+\mu_0\sigma\gamma\mathbf{A}_c=0 \tag{1.78}$$

in the conductor and on the conductor surface we have

$$\mathbf{n}\times\{\mathbf{n}\times[\nabla_s\phi-\nabla\times\mathbf{A}_c]\}|_{S_c}=0\,. \tag{1.79}$$

The last condition means that the parallel component of the magnetic field is continuous across the conductor surface. This way the condition becomes a natural boundary condition for the variational problem.

1.5.2 Conductor Outside the Wall

As the next case we consider the situation where the conductor is positioned outside the wall. For this case the equation corresponding to eq.(1.37) now becomes

$$\begin{aligned}
\delta(\delta W_o) &= \frac{1}{\mu}\int_{V_o} \nabla\phi_o \cdot \nabla\delta\phi_o dV \\
&= \frac{1}{\mu}\int_{V_o} \nabla\cdot\{\phi_o \cdot \nabla\delta\phi_o\} dV - \frac{1}{\mu_0}\int_{V_o} \phi_o \nabla^2\delta\phi_o dV \\
&= -\frac{1}{\mu}\int_{S_w} \phi_o \mathbf{n} \cdot \nabla\delta\phi_o \, dS + \frac{1}{\mu_0}\int_{S_c} \phi_o \mathbf{n} \cdot \nabla\delta\phi_o dS . \qquad (1.80)
\end{aligned}$$

Let

$$\delta(\delta\hat{W}_o) \overset{def}{=} \frac{1}{\mu_0}\int_{S_c} \phi_o \mathbf{n} \cdot \nabla\delta\phi_o dS . \qquad (1.81)$$

We again use the boundary condition corresponding to eq.(1.33), and obtain results similar to the case with the conductor inside the wall.

1.5.3 Lagrangian Containing Wall and Resistive Conductors

The preceeding analysis is strictly speaking restricted to considering just one conductor, but the generalization to an arbitrary number of conductors is obvious.

The full Lagrangian takes the following form

$$\mathcal{L} = \delta W_F + \delta W_i + \delta W_o + \delta W_W + \delta W_C , \qquad (1.82)$$

where $\delta\hat{W}_i$ is now a part of δW_i and δW_C is the sum over all conductors taken into account. Again we consider the fields to be given by eqs.(1.45) and (1.46)

$$\delta W_V^{(i)} = c_1^2 \frac{1}{2\mu_0}\int_{V_i} |\nabla\phi_\infty|^2 dV + c_2^2 \frac{1}{2\mu_0}\int_{V_i} |\nabla\phi_b|^2 dV + 2c_1 c_2 \delta W_V^\infty ,$$

$$\delta W_V^{(o)} = c_3^2 \frac{1}{2\mu_0}\int_{V_0} |\nabla\phi_\infty|^2 dV ,$$

$$\delta W_C = \sum_i \left[\frac{1}{2\mu_0}\int_{V_{c_i}} |\nabla\times\mathbf{A}_{c_i}|^2 dV + \frac{\sigma\gamma}{2}\int_{V_{c_i}} \mathbf{A}_{c_i}^2 \, dV\right] \overset{def}{=} \delta W_C^\infty + \delta W_C^R ,$$

and the sum is extended over all conductors. To leading order we obtain [2]

$$\nabla \times \mathbf{A}_c = \nabla \times \mathbf{A}_{c0} = c_1 \nabla \phi_\infty \,, \tag{1.83}$$

and we obtain

$$\delta W_C^\infty = \frac{1}{2\mu_0} \int_{V_c} |\nabla \times \mathbf{A}_c|^2 \, dV = \frac{c_1^2}{2\mu_0} \int_{V_c} |\nabla \phi_\infty|^2 \, dV \,.$$

Moreover

$$\frac{1}{2\mu_0} \int_{V_i} |\nabla \phi_\infty|^2 dV + \frac{1}{2\mu_0} \int_{V_o} |\nabla \phi_\infty|^2 dV + \frac{1}{2\mu_0} \int_{V_c} |\nabla \phi_\infty|^2 dV = \delta W_V^\infty \,.$$

Thus

$$\begin{aligned} \mathcal{L} &= \delta W_F + c_1^2 \delta W_V^\infty + c_2^2 \delta W_V^b \\ &\quad + 2 c_1 c_2 \delta W_V^\infty + \delta W_W + \delta W_C^R \\ &= \delta W_\infty + c_2^2 (\delta W_V^b - \delta W_V^\infty) + \delta W_W + \delta W_C^R \,, \end{aligned} \tag{1.84}$$

$$\delta W_W = c_3^2 \delta \hat{W}_W \,, \tag{1.85}$$

$$\delta W_C^R = c_3^2 \delta \hat{W}_C^R \,, \quad c_3 \mathbf{n} \cdot \nabla \phi_\infty |_{S_c} = c_3 \mathbf{n} \cdot \nabla \times \hat{\mathbf{A}}_c |_{S_c} \,, \tag{1.86}$$

$$\delta \hat{W}_W = \frac{\sigma \gamma d}{2} \int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_{w0}|^2 dS \,, \tag{1.87}$$

$$\delta \hat{W}_C^R = \frac{\sigma \gamma}{2} \int_{V_c} \hat{\mathbf{A}}_c^2 dV \,, \tag{1.88}$$

$$\mathbf{n} \cdot \nabla \times \hat{\mathbf{A}}_{w0} |_{S_w} = \mathbf{n} \cdot \nabla \phi_\infty |_{S_w} \,, \tag{1.89}$$

$$\mathbf{n} \cdot \nabla \times \hat{\mathbf{A}}_c |_{S_c} = \mathbf{n} \cdot \nabla \phi_\infty |_{S_c} \,. \tag{1.90}$$

Since $\delta W_b = \delta W_F + \delta W_V^b$ and $\delta W_\infty = \delta W_F + \delta W_V^\infty$ we obtain

$$\mathcal{L} = \delta W_\infty + c_2^2 (\delta W_b - \delta W_\infty) + c_1^2 (\delta \hat{W}_W + \delta \hat{W}_C^R) \,, \tag{1.91}$$

[2]The error here depends on the resistivity of the conductor, and the error obviously depends on the volume of the conductor as compared to the total volume of the vacuum region. Remember that for the limiting case of a conductor with infinite conductivity, the normal component of the magnetic field remains zero at all times.

and

$$c_1^2 = (1 - c_2)^2 .$$

Notice that for the case with the conductor in the outer region, the only formal change is that c_1 in eq.(1.83) is replaced by c_3, but since eq.(1.49) requires $c_1 = c_3$, this does not make any formal changes in the formulas. This does not mean that there are no real changes, because the final result will depend on the actual location of the conductor. However, the computational procedure is the same whether it is located outside or inside the wall. Now we make $\mathcal{L}$ stationary by determining c_2 such that

$$\frac{d\mathcal{L}}{dc_2} = 0 ,$$

and we obtain from $\mathcal{L} = 0$ that

$$\frac{\delta \hat{W}_V + \delta \hat{W}_C^R}{\delta W_b - \delta W_\infty} = -\frac{\delta W_\infty}{\delta W_b} . \tag{1.92}$$

Thus

$$\gamma \tau_D = -\frac{\delta W_\infty}{\delta W_b} , \qquad \tau_D = \mu_0 \sigma d \bar{b} , \tag{1.93}$$

and

$$\bar{b} = \frac{\frac{1}{2\mu_0} \int_{S_w} |\mathbf{n} \times \hat{\mathbf{A}}_{w0}|^2 dS + \frac{1}{2\mu_0 d} \int_{V_c} |\hat{\mathbf{A}}_c|^2 dV}{\delta W_b - \delta W_\infty} . \tag{1.94}$$

We are then left with a problem of determining the integral

$$I_c = \frac{1}{2\mu_0 d} \int_{V_c} |\hat{\mathbf{A}}_c|^2 dV .$$

This is the subject of the next section.

1.5.4 Magnetic Field Perturbation Inside a Resistive Conductor

We shall assume axisymmetric conductors. This assumption is not a severe restriction for tokamak systems. In fact axisymmetry is a virtue of a properly designed tokamak system. We shall furthermore make the assumption of circular cross section of the conductors and use the long thin approximation, permitting us to approximate the toroidal conductors with a straight cylinder to leading order.

Assuming the perturbed magnetic field inside the conductor is described by a vector potential $\mathbf{A}$, the governing equation in the low frequency limit is given by

$$\nabla \times (\nabla \times \mathbf{A}) = -\alpha \mathbf{A}\,, \qquad \alpha = \mu_0 \sigma \gamma\,, \tag{1.95}$$

where the time dependency is of the form $\exp(\gamma t)$. Written in components, we have

$$\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} = -\alpha A_r\,, \tag{1.96}$$

$$\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} = -\alpha A_\theta\,, \tag{1.97}$$

$$\nabla^2 A_z = -\alpha A_z\,. \tag{1.98}$$

Boundary Conditions

The solution has to satisfy the boundary condition

$$\mathbf{n} \cdot \nabla \phi_\infty |_{S_c} = \mathbf{n} \cdot \nabla \times \mathbf{A} |_{S_c}\,. \tag{1.99}$$

In the local cylindrical coordinate system the centerline of the circular cross section conductor is the $z-$ axis. In this system we have the coordinates r' and θ' being related to the r and θ in the 'plasma'- coordinate system as follows

$$\begin{aligned}
r^2 &= r_0^2 + r'^2 + 2r_0 r' \cos(\theta_0 - \theta') \\
\tan\theta &= \frac{r_0 \sin\theta_0 + r' \sin\theta'}{r_0 \cos\theta_0 + r' \cos\theta'} \\
\frac{\partial r}{\partial r'} &= \frac{r'}{r} + \frac{r_0}{r}\cos(\theta - \theta') \\
\frac{\partial \theta}{\partial r'} &= \cos^2\theta \frac{r' r_0 \cos(\theta + \theta') + r'^2 \cos 2\theta'}{(r_0 \cos\theta_0 + r' \cos\theta')^2}
\end{aligned}$$

$$\cos^2\theta = \frac{1}{1+s^2}\,, \qquad s = \frac{r_0 \sin\theta_0 + r' \sin\theta'}{r_0 \cos\theta_0 + r' \cos\theta'}\,.$$

We need to know $\mathbf{n} \cdot \nabla\phi_\infty|_{S_c}$, and there are several ways of obtaining this quantity, a few of which are outlined in Appendix C. At this point we shall merely assume that this quantity is known and given as

$$\mathbf{n} \cdot \nabla\phi_\infty|_{S_c} = \sum_k \frac{1}{r_c} f_k e^{2\pi i k v} \,,$$

where v is the parameter representing the poloidal direction of the surface, and r_c is the radius of the conductor. Thus for the rest of this problem we consider $\mathbf{n} \cdot \nabla\phi_\infty|_{S_c}$ to be a given quantity on the conductor surface. We recall that

$$\mathbf{n} \cdot \nabla \times \mathbf{A} = \frac{\partial A_z}{r'\partial\theta'} - \frac{\partial A_\theta}{\partial z} = \frac{im'}{r'} A_z - ikA_\theta \,. \qquad (1.100)$$

This expression represents a real physical quantity and must therefore be real, thus we arrive at the conclusion: The following quantities must be real

$$\hat{A}_z \stackrel{def}{=} iA_z \,, \qquad \text{and} \qquad \hat{A}_\theta \stackrel{def}{=} iA_\theta \,.$$

Resistive Conductor, Solutions

For simplicity we shall from now on omit the primes as reference to the local coordinate system, but remember that in the present context r, θ and m refer to the local coordinate system. Turning back to eqs.(1.96) - (1.98), we may write these equations as

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})A_r - \frac{(m^2+1)}{r^2}A_r - (k^2-\alpha)A_r = \frac{2m}{r^2}iA_\theta \,, \qquad (1.101)$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})A_\theta - \frac{(m^2+1)}{r^2}A_\theta - (k^2-\alpha)A_\theta = -\frac{2m}{r^2}iA_r \,, \qquad (1.102)$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})A_z - \frac{m^2}{r^2}A_z - (k^2-\alpha)A_z = 0 \,. \qquad (1.103)$$

Multiplying eqs.(1.102) and (1.103) by i, we may rewrite the equations as

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})A_r - \frac{(m^2+1)}{r^2}A_r - (k^2-\alpha)A_r = \frac{2m}{r^2}\hat{A}_\theta \,, \qquad (1.104)$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})\hat{A}_\theta - \frac{(m^2+1)}{r^2}\hat{A}_\theta - (k^2-\alpha)\hat{A}_\theta = \frac{2m}{r^2}A_r\,, \tag{1.105}$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})\hat{A}_z - \frac{m^2}{r^2}\hat{A}_z - (k^2-\alpha)\hat{A}_z = 0\,. \tag{1.106}$$

By adding and subtracting eqs.(1.104) and (1.105) we obtain

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})(A_r+\hat{A}_\theta) - \frac{(m^2+2m+1)}{r^2}(A_r+\hat{A}_\theta) - (k^2-\alpha)(A_r+\hat{A}_\theta) = 0\,,$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})(A_r-\hat{A}_\theta) - \frac{(m^2-2m+1)}{r^2}(A_r-\hat{A}_\theta) - (k^2-\alpha)(A_r-\hat{A}_\theta) = 0\,.$$

By inspection we see that

$$A_r + \hat{A}_\theta = c_1 I_{m+1}(k_0 r)\,, \tag{1.107}$$

$$A_r - \hat{A}_\theta = c_2 I_{m-1}(k_0 r)\,, \tag{1.108}$$

$$A_z = c_3 I_m(k_0 r)\,, \tag{1.109}$$

where $I_m(k_0 r)$ is the modified Bessel function and $k_0 = \sqrt{k^2-\alpha}$. Since the solution has to be finite at $r=0$, there are no K_m- functions present. Also notice that our notation is a bit sloppy since our r and m actually are r' and m', i.e., it is the coordinates in the local system.

We have already chosen $\nabla \cdot \mathbf{A} = 0$ as our gauge condition. Thus we have

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} = 0\,,$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{m}{r}\hat{A}_\theta + k\hat{A}_z = 0\,.$$

Here $k = n/R_c$, where n is the toroidal modenumber and R_c is the R-coordinate for the center of the conductor. From eqs.(1.107) - (1.109) we obtain

$$A_r = \frac{1}{2}\left(c_1 I_{m+1} + c_2 I_{m-1}\right)\,, \tag{1.110}$$

$$\hat{A}_\theta = \frac{1}{2}\left(c_1 I_{m+1} - c_2 I_{m-1}\right)\,, \tag{1.111}$$

$$\hat{A}_z = c_3 I_m\,. \tag{1.112}$$

The gauge condition can be written as

$$\frac{d}{dx}\{x(c_1 I_{m+1} + c_2 I_{m-1})\} + m\{c_1 I_{m+1} - c_2 I_{m-1}\} + krc_3 I_m \equiv 0\,. \quad (1.113)$$

We list some properties of Bessel functions:

$$\begin{aligned}
\frac{d}{dx}(xI_m) &= I_m + xI'_m = \\
&= (1-m)I_m + xI_{m-1} \\
&= (m+1)I_m + xI_{m+1}\,, \\
\frac{d}{dx}(xI_{m+1}) &= -mI_{m+1} + xI_m\,, \\
\frac{d}{dx}(xI_{m-1}) &= mI_{m-1} + xI_m\,.
\end{aligned}$$

Using these relations we may rewrite eq.(1.113) as

$$c_1(-mI_{m+1} + xI_m) + c_2(-mI_{m-1} + xI_m) + c_1 mI_{m+1} + c_2 mI_{m-1} + krc_3 I_m \equiv 0$$

$$\Longrightarrow c_1 = -c_2\,, \qquad c_3 = 0\,.$$

The solutions are therefore given by

$$A_r = c_m(I_{m+1} - I_{m-1}) = -c_m \frac{2m}{k_0 r} I_m(k_0 r)\,, \quad (1.114)$$

$$\hat{A}_\theta = c_m(I_{m+1} + I_{m-1}) = c_m 2I'_m(k_0 r)\,. \quad (1.115)$$

From this result we conclude that $A_z = 0$, and that the boundary condition eqs.(1.99) and (1.100) simpifies to

$$\sum \frac{1}{r_c} f_m e^{2\pi i m v} = -k \sum_m c_m (I_{m+1} + I_{m-1})|_{r=r_c} e^{2\pi i m v}\,. \quad (1.116)$$

It then follows that

$$\frac{1}{r_c} f_m = -kc_m(I_{m+1} + I_{m-1})|_{r=r_c}\,,$$

and

$$2c_m = -\frac{2f_m}{kr_c(I_{m+1} + I_{m-1})|_{r=r_c}} = -\frac{f_m}{kr_c I'_m(k_0 r_c)}\,. \quad (1.117)$$

Technical details for a Copper Conductor

The solutions we just obtained contain a critical parameter k_0 where

$$k_0 \stackrel{def}{=} \sqrt{k^2 - \alpha} = \frac{1}{R_c^2}\sqrt{n^2 - \mu_0 \sigma R_c^2 \gamma}\,,$$

thus k_0 depends on conductivity and the characteristic time for the resistive wall instability. The dependency on γ makes the problem complicated because this is the primary unknown to be determined in this work. Therefore by considering a conductor of finite thickness we no longer obtain an explicit formula for γ as given in eq.(1.93), since the integral over the conductors are implicit functions of γ. However, γ is likely to be weakly dependent on k_0, in which case we can solve the problem by iteration. The first step could be to solve the problem with $\gamma = 0$ or some predetermined value from experience. Numerical trial and error is necessary to determine the validity of this procedure. We shall now discuss actual numerical values for k_0. Using the conductivity for copper we find

$$\begin{aligned}
\sigma &= 5.88 \times 10^7 \Omega^{-1}\mathrm{m}^{-1} = 5.88 \times 10^7 \mathrm{AV}^{-1}\mathrm{m}^{-1} \\
\mu_0 &= 1.26 \times 10^{-6}\mathrm{Hm}^{-1} = 1.26 \times 10^{-6}\mathrm{VsA}^{-1}\mathrm{m}^{-1}
\end{aligned}$$

thus

$$\mu_0 \sigma = 1.26 \times 5.88 \times 10 \mathrm{AV}^{-1}\mathrm{m}^{-1}\mathrm{VsA}^{-1}\mathrm{m}^{-1} = 74.1 \mathrm{sm}^{-2}\,.$$

If we take $R_c = 1\,\mathrm{m}$ we find $\mu_0 \sigma R_c^2 = 74\mathrm{s}$. At this point we restrict ourselves to situations where the resistive wall instability occurs on a timescale of the order of 100s or longer, i.e., $\gamma < 10^{-2}\mathrm{s}^{-1}$. We then have

$$\mu_0 \sigma R_c^2 \gamma < 1\,,$$

and for $n \neq 0$, it then follows that

$$k_0^2 = \frac{1}{R_c^2}(n^2 - \alpha R_c^2) > 0\,. \tag{1.118}$$

Therefore, as long as the characteristic timescale for the resistive wall instability is sufficiently long for the inequality in eq.(1.118) to be satisfied, the solutions can be written the way they are presented in eqs.(1.114) and (1.115). If the inequality in eq.(1.118) is not satisfied, then the solutions have to be rewritten in terms of ordinary Bessel functions. We are not investigating this possibility here. However, we notice that for the case considered

$$x_0 = k_0 r_c = \frac{r_c}{R_c}\sqrt{n^2 - \alpha R_c^2}\,,$$

would normally be a small number, as long as the square root is not too large. When this is the case we can use the small argument expansion for the Bessel functions. We shall discuss this limit subsequently, but first we consider the energy integral.

Energy Integral

The purpose of this exercise is to evaluate the energy integral over the conductor, see eq.(1.93) and the following equation.

$$\begin{aligned}\int_{V_c} |\hat{\mathbf{A}}_c|^2 dV &= \int_{V_c} \left(|A_r|^2 + |A_\theta|^2\right) r dr d\theta R d\phi \\ &= \frac{4\pi^2 R_c}{k_0^2}\int_{V_c} \left(\sum_m A_{rm}\right)^2 + \left(\sum_m A_{\theta m}\right)^2 x dx\,,\end{aligned}$$

where $x = k_0 r$, $x_0 = k_0 r_c$ and

$$\begin{aligned}A_{rm} &= c_m(I_{m+1}) - I_{m-1}) = -c_m \frac{2m}{x} I_m(x)\,, \\ A_{\theta m} &= c_m(I_{m+1}) + I_{m-1}) = c_m I'_m(x)\,,\end{aligned}$$

$$\begin{aligned}\left(\sum_m A_{rm}\right)^2 + \left(\sum_m A_{\theta m}\right)^2 &= \sum_{m,n} c_m c_n \{(I_{m+1} - I_{m-1})(I_{n+1} - I_{n-1}) \\ &\quad +(I_{m+1} + I_{m-1})(I_{n+1} + I_{n-1})\} \\ &= \sum_{m,n} 2c_m c_n \{I_{m+1}I_{n+1} + I_{m-1}I_{n-1}\}\,.\end{aligned}$$

We define the following matrices and vectors

$$\hat{C}_{lm} \stackrel{def}{=} \int_0^{x_0} I_l(x) I_m(x) x dx \,, \tag{1.119}$$

$$C_{lm} \stackrel{def}{=} \hat{C}_{l+1\,m+1} + \hat{C}_{l-1\,m-1} \,, \tag{1.120}$$

$$\mathbf{c} \stackrel{def}{=} \{c_1 \cdots c_n\} \,. \tag{1.121}$$

Then we may write

$$\int_{V_c} \left(|A_r|^2 + |A_\theta|^2\right) Rr d\theta d\phi dr = \frac{8\pi^2 R_c}{k_0^2} \mathbf{c} \cdot \mathbf{C} \cdot \mathbf{c}^t \,. \tag{1.122}$$

We have not yet succeeded in solving the integrals given by eq.(1.119) analytically. Therefore, in the next section we shall evaluate these intgrals approximately.

The Small Argument Limit

An approach to evaluate the integrals given by eq.(1.119) is to expand in the small argument of the Bessel functions

$$I_m(x) \sim (\frac{1}{2}x)^m \frac{1}{\Gamma(m+1)}\,, \qquad I'_m(x) \sim \frac{m}{2}(\frac{1}{2}x)^{m-1}\frac{1}{\Gamma(m+1)}\,,$$

$$\begin{aligned}
2c_m &= -\frac{f_m}{kr_c}\frac{1}{I'_m(k_0 r_c)}\,, \\
\hat{A}_r &= \frac{f_m}{kr_c}\frac{mI_m(x)}{xI'_m(k_0 r_c)} \sim \frac{f_m}{kr_c}(\frac{r}{r_c})^{|m|-1} \\
\hat{A}_\theta &= -\frac{f_m}{kr_c}\frac{I'_m(x)}{I'_m(k_0 r_c)} \sim -\frac{f_m}{kr_c}(\frac{r}{r_c})^{|m|-1}\,.
\end{aligned}$$

From which it follows that $\int A_r^2 dV = \int \hat{A}_\theta^2 dV$, and we find

$$\begin{aligned}
\int_0^{r_c} A_r^2 r dr + \int_0^{r_c} \hat{A}_\theta^2 r dr &\sim 2\int_0^{r_c}\left(\sum_m \frac{f_m}{kr_c}\left(\frac{r}{r_c}\right)^{|m|-1}\right)^2 r dr \\
&= \sum_{m,n}\frac{2f_m f_n}{k^2}\frac{1}{|m|+|n|}\,.
\end{aligned}$$

For $k = 0$, i.e., axisymmetric modes one has to re-do the problem. The proper solution now is $A_r = A_\theta = 0$, $A_z \neq 0$, or

$$\hat{A}_z = c_3 I_m(k_0 r)\,.$$

With a similar expression for c_3

$$c_3 = \frac{1}{|m|}\frac{f_m}{I_m(k_0 r_c)}\,.$$

Thus

$$\begin{aligned}\int_0^{r_c} \hat{A}_z^2 r dr &= \int_0^{r_c}\left(\sum_m \frac{f_m}{|m|}\frac{I_m(x)}{I_m(x_0)}\right)^2 \\ &= \sum_{m,n}\frac{r_c^2 f_m f_n}{|m|\,|n|}\int_0^1 y^{|m|+|n|+1}dy \\ &= \sum_{m,n}\frac{r_c^2 f_m f_n}{|m|\,|n|}\frac{1}{|m|+|n|+2}\,,\end{aligned}$$

where the dummy variable $y = \frac{r}{r_c}$. A full toroidal calculation has not yet been done. This approximation may, however, be sufficiently accurate to be of practical value.

1.6 The Screw Pinch with Resistive Wall

According to J.P. Freidberg: Ideal Magnetohydrodynamics, Ref. 1, eq.(9.105), we have

$$\gamma\tau_d = \frac{k^2b^2 + m^2}{k^2b^2K_b'I_b'\{1-(I_a'K_b'/I_b'K_a')\}}\frac{\delta W_\infty}{\delta W_b}\,, \tag{1.123}$$

where $\tau_d = \mu_0\sigma d b$.

1.6.1 The Basic Formula

We shall first test our basic formula eq.(1.93), which we write as

$$\gamma\tau_D = -\frac{\delta W_\infty}{\delta W_b}\,, \qquad \tau_D = \mu_0\sigma d\bar{b}\,, \tag{1.124}$$

with τ_D given by eq.(1.70) and $\bar{b}$ by eq.(1.72) from which we obtain

$$\frac{\gamma b}{\bar{\gamma}\bar{b}} = -\frac{k^2b^2+m^2}{k^2b^2K_b'I_b'\{1-(I_a'K_b'/I_b'K_a')\}}\,, \tag{1.125}$$

or

$$\frac{\gamma}{\bar{\gamma}} = -\frac{\bar{b}}{b}\frac{k^2b^2+m^2}{k^2b^2K_b'I_b'\{1-(I_a'K_b'/I_b'K_a')\}}\,. \tag{1.126}$$

We then proceed by computing $\bar{b}$. According to the formula given in eq.(1.72) we have

$$\bar{b} = \frac{\frac{1}{2\mu_0}\int_{S_b}|\mathbf{n}\times\hat{\mathbf{A}}_{w0}|^2dS}{\delta W_b - \delta W_\infty}\,. \tag{1.127}$$

First we consider the numerator in the expression for $\bar{b}$. Thus let

$$I_{\bar{b}} = \frac{1}{2\mu_0}\int_{S_b}|\mathbf{n}\times\hat{\mathbf{A}}_{w0}|^2dS\,, \tag{1.128}$$

where the vector potential $\mathbf{A}_{w0}$ satisfies the boundary condition

$$\mathbf{n}\cdot\nabla\phi_\infty|_{S_b} = \nabla_s\cdot\{\mathbf{n}\times\mathbf{A}_{w0}\}|_{S_b}\,. \tag{1.129}$$

Furthermore

$$\begin{aligned} \mathbf{n}\times\mathbf{A}_{w0} &= \mathbf{a}\,, & (1.130)\\ \nabla_s &= i\mathbf{K} = i\{\frac{m}{b}\mathbf{e}_\theta + k\mathbf{e}_z\}\,, & (1.131)\\ \nabla_s\cdot\mathbf{a} &= i\mathbf{K}\cdot\mathbf{a} = i\mathbf{K}\cdot\mathbf{n}\times\mathbf{A}_{w0}\,, & (1.132)\end{aligned}$$

and assume

$$\mathbf{n}\cdot\mathbf{A}_{w0} = 0 \quad \& \quad \nabla_s\cdot\mathbf{A}_{w0} = 0\,,$$

$$\Longrightarrow \mathbf{K}\cdot\mathbf{A}_{w0} = 0 \Longrightarrow \mathbf{A}||\mathbf{n}\times\mathbf{K}\,,$$

or

$$i\mathbf{K}\cdot\mathbf{n}\times\mathbf{A}_{w0} = i|\mathbf{K}|\,|\mathbf{n}\times\mathbf{A}_{w0}|\,,$$

from which we obtain

$$\mathbf{n}\cdot\nabla\phi_\infty|_{S_b} = \nabla_s\cdot\{\mathbf{n}\times\mathbf{A}_{w0}\}|_{S_b} = i|\mathbf{K}|\,|\mathbf{n}\times\mathbf{A}_{w0}|\,, \tag{1.133}$$

and

$$\begin{aligned}|\mathbf{n}\times\mathbf{A}_{w0}|^2 &= \frac{|\mathbf{n}\cdot\nabla\phi_\infty|_{S_b}|^2}{K^2} = \frac{\left|\frac{\partial\phi_\infty}{\partial r}|_{r=b}\right|^2}{\frac{m^2}{b^2}+k^2} \\ &= \frac{b^2\left|\frac{\partial\phi_\infty}{\partial r}|_{r=b}\right|^2}{m^2+k^2b^2}\,. \end{aligned}\tag{1.134}$$

From Ref. 1 we also have

$$\phi_\infty = AK_r e^{i(m\theta+kz)} = \frac{iF(a)\xi_a}{kK_a'}K_m(kr)e^{i(m\theta+kz)}\,, \tag{1.135}$$

thus

$$\left|\frac{\partial\phi_\infty}{\partial r}|_{r=b}\right|^2 = \xi_a^2F^2(a)(\frac{K_b'}{K_a'})^2\,, \tag{1.136}$$

and

$$\frac{1}{2\mu_0}\int_{S_b}|\mathbf{n}\times\hat{\mathbf{A}}_{w0}|^2dS = 4\pi^2R_cb^3\frac{1}{2\mu_0}\frac{\xi_a^2F^2(a)}{m^2+k^2b^2}(\frac{K_b'}{K_a'})^2\,. \tag{1.137}$$

From Ref. 1, eq.(9.79) we have

$$\delta W_b - \delta W_\infty = \frac{2\pi^2R_c}{\mu_0}\left.\frac{r^2F^2}{|m|}\right|_{r=a}(\Lambda_b-\Lambda_\infty)\xi_a^2\,, \tag{1.138}$$

where

$$\Lambda_b - \Lambda_\infty = -\frac{|m|K_a}{kaK_a'}\frac{(I_a'/K_a')-(I_a/K_a)}{(I_b'/K_b')-(I_a'/K_a')}\,, \tag{1.139}$$

or

$$\Lambda_b - \Lambda_\infty = -\frac{|m|K_b'}{kaI_b'K_a'^2}\,\frac{I_a'K_a - I_aK_a'}{1-\frac{I_a'K_b'}{I_b'K_a'}}\,. \tag{1.140}$$

We now use the fact that

$$\begin{aligned} I'_a K_a - I_a K'_a &= I'_m(ka)K_m(ka) - I_m(ka)K'_m(ka) \\ &= W(I_m, K_m) = (I_{m-1} - \frac{m}{ka}I_m)K_m - I_m(-K_{m-1} - \frac{m}{ka}K_m) \\ &= I_{m-1}K_m + I_m K_{m-1} \\ &= I_\nu K_{\nu+1} + I_{\nu+1}K_\nu = \frac{1}{ka}, \end{aligned}$$

where $W(I_m, K_m)$ is the Wronskian and $\nu = m - 1$. (Fore more details see Abramowitz & Stegun: Handbook of Mathematical Functions, p. 375, eq.(9.6.15).) Thus we obtain

$$\Lambda_b - \Lambda_\infty = -\frac{|m|K'_b}{k^2a^2 I'_b K'^2_a}\frac{1}{1 - \frac{I'_a K'_b}{I'_b K'_a}} > 0\,, \tag{1.141}$$

since

$$-K'_b > 0\,, \quad 0 < \frac{I'_a}{I'_b} < 1\,, \quad 0 < \frac{K'_b}{K'_a} < 1\,.$$

We may now write eq.(1.138) as

$$\delta W_b - \delta W_\infty = -\frac{2\pi^2 R_c}{\mu_0}\frac{a^2F^2(a)}{k^2a^2}\frac{K'_b}{I'_b K'^2_a}\frac{\xi_a^2}{1 - \frac{I'_a K'_b}{I'_b K'_a}}\,. \tag{1.142}$$

We are then able to compute $\bar{b}$ given by eq.(1.127) and obtain

$$\frac{\bar{b}}{b} = \frac{1}{b}\frac{\frac{1}{2\mu_0}\int_{S_b}|\mathbf{n}\times\hat{\mathbf{A}}_{w0}|^2 dS}{\delta W_b - \delta W_\infty}\,, \tag{1.143}$$

or

$$\frac{\bar{b}}{b} = -\frac{k^2b^2}{m^2 + k^2b^2} I'_b K'_b\{1 - \frac{I'_a K'_b}{I'_b K'_a}\}\,. \tag{1.144}$$

From eq.(1.126) we then obtain the desired result

$$\bar{\gamma} = \gamma\,, \tag{1.145}$$

which proves that our variational formulation gives exactly the same result as obtained by standard methods.

1.6.2 Alternative Formula

We shall now derive the growth rate using the alternative formula given by eq.(1.73), from which we have

$$\frac{\bar{b}}{b} = \frac{\delta W_b - \delta W_\infty}{b\frac{1}{2\mu_0}\int_{S_b} |\nabla_s \phi_b|^2 dS} \; .$$

Using eq.(9.37) in Ref. 1[3] we find

$$\begin{aligned} \nabla_s \phi &= i\left(\mathbf{e}_\theta \frac{m}{r} + \mathbf{e}_z k\right) \phi_b|_{r=b} \\ &= -\left(\mathbf{e}_\theta \frac{m}{b} + \mathbf{e}_z k\right) \frac{F(a)\xi_a}{kK'_a} \frac{K_b - \frac{K'_b}{I'_b} I_b}{1 - \frac{I'_a K'_b}{I'_b K'_a}} e^{i(m\theta + kz)} \, , \end{aligned}$$

from which we obtain

$$\begin{aligned} |\nabla_s \phi(b)|^2 &= \left(\frac{m^2}{b^2} + k^2\right) \frac{F^2(a)\xi_a^2}{k^2 K'^2_a} \frac{\left\{K_b - \frac{K'_b}{I'_b} I_b\right\}^2}{\left\{1 - \frac{I'_a K'_b}{I'_b K'_a}\right\}^2} \\ &= \left(\frac{m^2}{b^2} + k^2\right) \frac{F^2(a)\xi_a^2}{k^2 K'^2_a} \frac{\frac{1}{k^2 b^2 I'^2_b}}{\left\{1 - \frac{I'_a K'_b}{I'_b K'_a}\right\}^2} \, , \end{aligned}$$

and finally a short calculation gives

$$\frac{\bar{b}}{b} = -\frac{k^2 b^2}{m^2 + k^2 b^2} I'_b K'_b \{1 - \frac{I'_a K'_b}{I'_b K'_a}\} \, , \qquad (1.146)$$

which is exactly the same result as obtained in eq.(1.144). We then conclude that the two formulas given in eqs.(1.72) and (1.73) give exactly the same result for the case considered.

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[3] Notice that there are some misprints in eq.(9.39) of Ref. 1

Chapter 2

Vacuum Region

2.1 Preliminary Considerations

According to the general theory we have presented in Chapter 1, we have to consider two cases.

1. No wall or conductors are present or taken into account. This case provides information about δW_{∞}.

2. The problem is solved by considering the wall and conductors present to be of infinite conductivity. This provides information about δW_{w}.

Regarding the first class of problems, there exists vast literature on the subject, and numeric codes routinely solve such problems. When it comes to the second class of problems, it appears that not many problems have been solved in this catogory, although special cases have been studied like the screw pinch considered in Chapter 1, section 1.7. Here we present an analytic study of a system with a wall and an arbitrary number of conductors present in the vacuum region, and the wall and the conductors considered have infinite conductivity.[1] By using Green's functions techniques and fast Fourier transforms, we reduce the problems to problems in linear algebra. The final solution have, however, to be obtaind by numerical methods. The aim here is to provide the analytical basis for such an approach; that means,

[1]According to our formula for the computed growth rate γ given by eq.(1.93), this is the solution requierd in addition to the solution with boundary and conductors absent.

numerically the problem is reduced to finding fast Fourier transforms and subsequent matrix manipulations.

2.2 Green's 'Theorem'

Green's second identity can be written as

$$\begin{aligned}\int_V \{V\nabla^2 U - U\nabla^2 V\} dV &= -\int_{S_p} \{V\mathbf{n}\cdot\nabla U - U\mathbf{n}\cdot\nabla V\} dS_p \\ &+ \int_{S_w} \{V\mathbf{n}\cdot\nabla U - U\mathbf{n}\cdot\nabla V\} dS_w \\ &+ \sum_i \int_{S_{c_i}} \{V\mathbf{n}\cdot\nabla U - U\mathbf{n}\cdot\nabla V\} dS_{c_i} \,. \quad (2.1)\end{aligned}$$

We shall take V to be the vacuum region bounded by the plasma-vacuum interface S_p, the infinitely conducting wall S_w, and all the conductor surfaces S_{c_i} . Here $\mathbf{n}$ refers to the surface unit normal vector for the surface under consideration. We adopt the following conventions:

At S_p, $\mathbf{n}$ points outward into the vacuum region,

at S_c, $\mathbf{n}$ points away from the vacuum region, (into the conductor),

at S_w, $\mathbf{n}$ points outward (away from the plasma).

Let the perturbation in the vacuum magnetic field be given as ∇V (notice that V may not be single valued in the general case). We take U to be given by

$$U = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \stackrel{def}{=} G(\mathbf{r},\mathbf{r}') \,. \quad (2.2)$$

Here $G(\mathbf{r},\mathbf{r}')$ is the Green's function for Laplace's equation, and it satisfies the equation

$$\nabla^2 G = \delta(\mathbf{r}-\mathbf{r}') \,. \quad (2.3)$$

Since $\nabla\cdot\mathbf{B} = 0 \rightarrow \nabla^2 V = 0$, we can integrate the left-hand side of eq.(2.1) over the vacuum region to obtain

$$
\begin{aligned}
\sigma V(\mathbf{r}) &= -\int_{S_p} \{V(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla V(\mathbf{r}')\} dS_p \\
&\quad + \int_{S_w} \{V(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}')\} dS_w \\
&\quad + \int_{S_{c_i}} \{V(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}')\} dS_{c_i} \,. \qquad (2.4)
\end{aligned}
$$

This is basically Green's third identity. As the observation point moves on to a regular point[2] on one of the bounding surfaces $\sigma = \frac{1}{2}$, otherwise it has the value 1. Notice that the solid angle over which integration is to be performed, in the case when the observation point moves on to the surface, is reduced from 4π to 2π. Notice also that $\mathbf{n}\cdot\nabla V(\mathbf{r}) = 0$ over a surface with infinite conductivity. From eq.(2.4) we notice that there are basically two kinds of integrals to be considered, that is

$$
I_1 = \int_S V(\mathbf{r}')\mathbf{n}'\cdot\nabla' G(\mathbf{r},\mathbf{r}') dS \,, \qquad (2.5)
$$

and

$$
I_2 = \int_{S_p} G(\mathbf{r},\mathbf{r}')\mathbf{n}'\cdot\nabla' V(\mathbf{r}') dS \,. \qquad (2.6)
$$

The seconed integral eq.(2.6) is nonzero only over S_p, the first integral eq.(2.5) contribute from all surfaces under consideration.

2.3 Coordinates

We shall use cylindrical coordinates R, ϕ, Z , to describe the axisymmetric toroidal configuration.

[2]Regular here means that the surface has a welldefined normal vector $\mathbf{n}$ at the point in question, otherwise appropriate adjustments must be made.

The following relation to Cartesian coordinates exist

$$\mathbf{r} = \{R\cos\phi, R\sin\phi, Z\}, \tag{2.7}$$

$$\mathbf{r}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = \{-R\sin\phi, R\cos\phi, 0\} = R\mathbf{e}_\phi. \tag{2.8}$$

We also have

$$\nabla = \mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} + \mathbf{e}_Z\frac{\partial}{\partial Z}, \tag{2.9}$$

where $\mathbf{e}_R, \mathbf{e}_\phi, \mathbf{e}_Z$ are unit vectors in the respective directions.

Let the plasma vacuum interface be given as

$$R = R_0\{1 + \epsilon x(v)\}, \tag{2.10}$$

$$Z = R_0\epsilon y(v). \tag{2.11}$$

The surface unit normal vector pointing outward from the plasma region is given by

$$\begin{aligned}\mathbf{n} &= \frac{\mathbf{r}_\phi \times \mathbf{r}_v}{|\mathbf{r}_\phi \times \mathbf{r}_v|} \\ &= \frac{R\mathbf{e}_\phi \times \left(\mathbf{e}_R\frac{\partial R}{\partial v} + \mathbf{e}_Z\frac{\partial Z}{\partial v}\right)}{|\mathbf{r}_\phi \times \mathbf{r}_v|} \\ &= \frac{1}{Q}\{\dot{y}\mathbf{e}_R - \dot{x}\mathbf{e}_Z\},\end{aligned}$$

where

$$\dot{y} = \frac{d}{dv}y(v),\ \dot{x} = \frac{d}{dv}x(v),\ \text{and } Q = Q(v) = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{2.12}$$

We shall also at times use the shorthand notation for any quantity $q = q(v)$ and write $q' = q(v')$. We compute $\mathbf{n}\cdot\nabla$ and find

$$\mathbf{n}\cdot\nabla = \frac{1}{Q}\left\{\dot{y}\frac{\partial}{\partial R} - \dot{x}\frac{\partial}{\partial Z}\right\}. \tag{2.13}$$

The surface element is given by

$$d\mathbf{S} = \mathbf{r}_\phi \times \mathbf{r}_v d\phi dv = \mathbf{n} R R_0 \epsilon Q d\phi dv \tag{2.14}$$

or

$$dS = R d\phi dl_p, \tag{2.15}$$

where dl_p is the arclength element in the poloidal direction given by

$$dl_p = \sqrt{dR^2 + dZ^2} = \epsilon R_0 \sqrt{\dot{x}^2 + \dot{y}^2} dv = \epsilon R_0 Q dv\,. \tag{2.16}$$

Thus we have

$$\frac{dl_p}{dv} = \epsilon R_0 Q \qquad \text{and} \qquad \epsilon R_0 \int_0^1 Q dv = C_p\,,$$

C_p being the circumference in the poloidal direction at ϕ =constant. We have chosen the range of v to be $[0, 1]$.

2.3.1 Boundary conditions

The boundary condition at the plasma vacuum interface given by eq.(1.3) is

$$\mathbf{n} \cdot \nabla V|_{S_p} = \nabla_\mathbf{s} \cdot \{\mathbf{n} \cdot \boldsymbol{\xi} \hat{\mathbf{B}}\}|_{S_p}\,.$$

Notice that the perturbation in the magnetic field is $\mathbf{B_1} = \nabla \mathbf{V}$.

We now introduce a poloidal unit tangent vector to the surface by

$$\mathbf{t} \stackrel{def}{=} \frac{\dot{x}\mathbf{e}_R + \dot{y}\mathbf{e}_\phi}{Q}\,, \qquad \text{where again} \qquad Q = \sqrt{\dot{x}^2 + \dot{y}^2}\,, \tag{2.17}$$

with $\dot{x} = \frac{dx}{dv}$ and $\dot{y} = \frac{dy}{dv}$. We may then write the magnetic field as

$$\mathbf{B} = B_p \mathbf{t} + B_\phi \mathbf{e}_\phi\,, \tag{2.18}$$

and we have the surface gradient given as

$$\nabla_s = \mathbf{t}\mathbf{t} \cdot \nabla + \frac{\mathbf{e}_\phi}{R} \frac{\partial}{\partial \phi}\,. \tag{2.19}$$

From eqs.(2.18) and (2.19) we obtain

$$
\begin{aligned}
\nabla_s \cdot \mathbf{B} &= \{\mathbf{tt}\cdot\nabla + \frac{\mathbf{e}_\phi}{R}\frac{\partial}{\partial\phi}\}\cdot\{B_p\mathbf{t} + B_\phi\mathbf{e}_\phi\} \\
&= \mathbf{t}\cdot\nabla B_p + \frac{B_p}{R}\mathbf{e}_\phi\cdot\frac{\partial}{\partial\phi}\mathbf{t}\,.
\end{aligned}
$$

Notice that $\mathbf{tt}:\nabla\mathbf{t} = 0$, $\mathbf{t}\cdot\mathbf{e}_\phi = 0$, $\mathbf{e}_\phi\cdot\frac{\partial\mathbf{e}_\phi}{\partial\phi} = 0$ and $\frac{\partial B_\phi}{\partial\phi} = 0$, where the last relation is due to axisymmetry. For $\frac{\partial\mathbf{t}}{\partial\phi}$ we find

$$
\frac{\partial\mathbf{t}}{\partial\phi} = \frac{\partial}{\partial\phi}\frac{\dot{x}\mathbf{e}_R + \dot{y}\mathbf{e}_Z}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\dot{x}}{Q}\frac{\partial\mathbf{e}_R}{\partial\phi} = \frac{\dot{x}}{Q}\mathbf{e}_\phi\,,
$$

and we have the following result

$$
\nabla_s\cdot\mathbf{B} = \mathbf{t}\cdot\nabla B_p + \frac{B_p}{R}\frac{\dot{x}}{Q}\,. \tag{2.20}
$$

Since we have

$$
\mathbf{t}\cdot\nabla R = \left(\frac{\dot{x}}{Q}\frac{\partial}{\partial R} + \frac{\dot{y}}{Q}\frac{\partial}{\partial Z}\right)R = \frac{\dot{x}}{Q}\,,
$$

we may write $\nabla_s\cdot\mathbf{B}$ as

$$
\nabla_s\cdot\mathbf{B} = \mathbf{t}\cdot\nabla B_p + \frac{B_p}{R}\mathbf{t}\cdot\nabla R = \frac{1}{R}\mathbf{t}\cdot\nabla(RB_p)\,.
$$

Thus we obtain

$$
\begin{aligned}
\nabla_s\cdot\{\hat{\mathbf{B}}\,\xi_\perp\} &= \nabla_s\cdot\hat{\mathbf{B}}\,\xi_\perp + \hat{\mathbf{B}}\cdot\nabla\xi_\perp \\
&= \frac{\xi_\perp}{R}\mathbf{t}\cdot\nabla(R\hat{B}_p) + \hat{B}_p\mathbf{t}\cdot\nabla\xi_\perp + \hat{B}_\phi\mathbf{e}_\phi\cdot\nabla\xi_\perp \\
&= \frac{1}{R}\mathbf{t}\cdot\nabla(R\hat{B}_p\xi_\perp) + \hat{B}_\phi\frac{1}{R}\frac{\partial\xi_\perp}{\partial\phi}\,,
\end{aligned}
$$

where $\xi_\perp = \mathbf{n}\cdot\boldsymbol{\xi}$ is the normal component of $\boldsymbol{\xi}$ at the boundary. We finally obtain the result we want

$$
\mathbf{n}\cdot\nabla V = \frac{1}{R}\mathbf{t}\cdot\nabla(R\hat{B}_p\xi_\perp) + \hat{B}_\phi\frac{1}{R}\frac{\partial\xi_\perp}{\partial\phi}\,, \tag{2.21}
$$

where we remember that $R = R_0\{1+\epsilon x(v)\}$ in our notation.

2.4 Green's Functions

From Sec.**2.2** we are left with the problem of evaluating the following two integrals

$$I_1 = \int_S V(\mathbf{r}')\mathbf{n}' \cdot \nabla' G(\mathbf{r}, \mathbf{r}')dS\,, \tag{2.22}$$

and

$$I_2 = \int_{S_p} G(\mathbf{r}, \mathbf{r}')\mathbf{n}' \cdot \nabla' V(\mathbf{r}')dS\,. \tag{2.23}$$

In the first integral, integration is to be performed over the plasma vacuum interface as well as any conducting wall surface and conductor surfaces. The last integral is to be performed only over the plasma vacuum interface, since contribution to this integral from any infinitely conducting surface is zero ($\mathbf{n} \cdot \mathbf{B} = 0$ on such surfaces).

Using a parameterization of the surfaces S_i and S_j by the variables v and v', and representing $V(\mathbf{r}')$ as $V(\mathbf{r}') = V(R', Z')\exp(in\phi')$, we may write

$$I_1^{(S_i,S_j)} = \int_0^1 dv' \int_0^{2\pi} d\phi' R' V_n(R', Z') e^{in\phi'} \epsilon R_0 \left\{ \dot{y}(v') \frac{\partial}{\partial R'} - \dot{x}(v') \frac{\partial}{\partial Z'} \right\} G(\mathbf{r}, \mathbf{r}') \Bigg|_{\substack{\mathbf{r} \in S_i \\ \mathbf{r}' \in S_j}} \tag{2.24}$$

where

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') &= -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\frac{1}{4\pi} \frac{1}{\{R^2 + R'^2 + (Z' - Z)^2 - 2RR' \cos(\phi - \phi')\}^{\frac{1}{2}}} \end{aligned} \tag{2.25}$$

and

$$\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial R'} = \frac{1}{4\pi} \frac{R' - R\cos(\phi - \phi')}{\{R^2 + R'^2 + (Z' - Z)^2 - 2RR' \cos(\phi - \phi')\}^{\frac{3}{2}}}\,, \tag{2.26}$$

$$\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial Z'} = \frac{1}{4\pi} \frac{Z' - Z}{\{R^2 + R'^2 + (Z' - Z)^2 - 2RR' \cos(\phi - \phi')\}^{\frac{3}{2}}}\,. \tag{2.27}$$

We then use the following identity

$$
\begin{aligned}
R' - R\cos(\phi-\phi') &= R' + \frac{1}{2R'}\{R^2 + R'^2 + (Z'-Z)^2 - 2RR'\cos(\phi-\phi')\} \\
&\quad -\frac{1}{2R'}\{R^2 + R'^2 + (Z'-Z)^2\} \\
&= \frac{R'^2 - R^2 - (Z'-Z)^2}{2R'} \\
&\quad +\frac{1}{2R'}\{R^2 + R'^2 + (Z'-Z)^2 - 2RR'\cos(\phi-\phi')\},
\end{aligned}
$$

and obtain

$$
\begin{aligned}
I_1^{(S_i,S_j)} &= \int_0^1 dv' \int_0^{2\pi} d\phi' R' V_n(R',Z') e^{in\phi'} \epsilon R_0 \left\{ \dot{y}(v')\frac{\partial}{\partial R'} - \dot{x}(v')\frac{\partial}{\partial Z'} \right\} G(\mathbf{r},\mathbf{r}')\Big|_{\substack{\mathbf{r}\in S_i \\ \mathbf{r}'\in S_j}} \\
&= \frac{1}{4\pi}\int_0^1 dv' \int_0^{2\pi} V_n(R',Z')\frac{\epsilon R_0}{2}\frac{\dot{y}(v')e^{in\phi'}d\phi'}{\{R^2+R'^2+(Z'-Z)^2-2RR'\cos(\phi-\phi')\}^{\frac{1}{2}}} \\
&\quad \frac{1}{4\pi}\int_0^1 dv' \int_0^{2\pi} V_n(R',Z')\epsilon R_0 R' \frac{\{\dot{y}(v')\frac{R'^2-R^2-(Z'-Z)^2}{2R'} - \dot{x}(v')(Z'-Z)\}e^{in\phi'}d\phi'}{\{R^2+R'^2+(Z'-Z)^2-2RR'\cos(\phi-\phi')\}^{\frac{3}{2}}} \\
&= \frac{e^{in\phi}}{2\pi}\int_0^1 V_n^{(S_j)}(R',Z')\left\{\epsilon \dot{y}^{(S_j)}(v') I_n^{(S_i,S_j)}(v,v') \right. \\
&\quad \left. + F^{(S_i,S_j)}(v,v')\hat{I}_n^{(S_i,S_j)}(v,v')\right\} dv', \qquad (2.28)
\end{aligned}
$$

where

$$
\begin{aligned}
F^{(S_i,S_j)}(v,v') &= \frac{\epsilon}{R_0^2}\left\{\dot{y}(v')[R'^2 - R^2 - (Z-Z')^2] - 2R'\dot{x}(v')(Z'-Z)\right\}\Big|_{\substack{R,Z\in S_i \\ R',Z'\in S_j}} \\
&= a(v)(v'-v)^2 + O(\{v'-v\}^3)|_{v,v'\in S_i} \quad \text{for} \quad i=j, \qquad (2.29)
\end{aligned}
$$

where the last step in this formula is valid only for the case $i = j$ and then

$$a(v) = \{[1 + \epsilon x(v)]\kappa Q - \epsilon^3 \dot{y}(v)\} Q^2 \, , \tag{2.30}$$

here κ is the curvature of the surface S_i in the poloidal direction given as

$$\kappa = \epsilon^2 \frac{\dot{x}(v)\ddot{y}(v) - \dot{y}(v)\ddot{x}(v)}{\{\dot{x}^2(v) + \dot{y}^2(v)\}^{\frac{2}{3}}} \, . \tag{2.31}$$

See Appendix D, for details. The last representation of $F(v, v')$ is useful for studying the limit $v' \to v$, in the case when v and v' refer to the same surface.

$$I_n^{(S_i,S_j)}(v, v') = \frac{1}{4} \int_0^{2\pi} \left. \frac{R_0 e^{in(\phi'-\phi)} d\phi'}{\{R^2 + R'^2 + (Z' - Z)^2 - 2RR' \cos(\phi - \phi')\}^{\frac{1}{2}}} \right|_{\substack{R,Z \in S_i \\ R',Z' \in S_j}} , \tag{2.32}$$

$$\hat{I}_n^{(S_i,S_j)}(v, v') = \frac{1}{4} \int_0^{2\pi} \left. \frac{R_0^3 e^{in(\phi'-\phi)} d\phi'}{\{R^2 + R'^2 + (Z' - Z)^2 - 2RR' \cos(\phi - \phi')\}^{\frac{3}{2}}} \right|_{\substack{R,Z \in S_i \\ R',Z' \in S_j}} . \tag{2.33}$$

It is assumed that all the surfaces considerd are determined by some parametric representation cast in the following form

$$\begin{aligned} R &= R_0\{1 + \epsilon x(v)\} \, , \\ Z &= R_0 \epsilon y(v) \, . \end{aligned}$$

We then consider the integral I_2 and obtain

$$\begin{aligned} I_2^{(S_i,S_p)} &= \int_{S_p} G(\mathbf{r}, \mathbf{r}')\mathbf{n}' \cdot \nabla' V(\mathbf{r}') dS \\ &= -\frac{1}{4\pi} \int_0^1 dv' \frac{A_n(v') e^{in\phi}}{\epsilon R' Q'} \int_0^{2\pi} \frac{\epsilon R_0 R' Q' e^{in(\phi'-\phi)} d\phi'}{\{R^2 + R'^2 + (Z' - Z)^2 - 2RR' \cos(\phi - \phi')\}^{\frac{1}{2}}} \\ &= -\frac{1}{\pi} \int_0^1 A_n(v') e^{in\phi} I_n^{(S_i,S_p)}(v, v') dv' \, . \end{aligned} \tag{2.34}$$

Notice that the integration over v' is restricted to the surface S_p, since this integral vanishes over all other surfaces. A_n is defined by (see eq.(2.21))

$$\begin{aligned} \mathbf{n}' \cdot \nabla' V(\mathbf{r}') &= \frac{1}{R'} \left\{ \mathbf{t}' \cdot \nabla' (R' \hat{B}_p \xi_\perp) + \hat{B}_\phi i n \xi_\perp \right\} \\ &\stackrel{def}{=} \frac{A_n(v')}{\epsilon Q' R'} e^{in\phi'} , \end{aligned} \tag{2.35}$$

or

$$A_n(v') = \epsilon Q' \{ \mathbf{t}' \cdot \nabla' (R' \hat{B}_p \xi_\perp) + \hat{B}_\phi i n \xi_\perp \} |_{v' \in S_p} .$$

Notice that

$$Q' \mathbf{t}' \cdot \nabla' = \dot{x}(v') \frac{\partial}{\partial R'} + \dot{y}(v') \frac{\partial}{\partial Z'} = \frac{1}{\epsilon R_0} \frac{d}{dv'} . \tag{2.36}$$

In order to show that eq.(2.36) is correct, let $F(R(v'), Z(v')) = f(v')$ be an arbitrary differentiable function of its arguments. Then we have

$$\frac{df}{dv'} = \frac{dR'}{dv'} \frac{\partial F}{\partial R'} + \frac{dZ'}{dv'} \frac{\partial F}{\partial Z'} ,$$

and with $R' = R_0 \{1 + \epsilon x(v')\}$ and $Z' = \epsilon R_0 y(v')$ we easily obtain eq.(2.36). The boundary condition, eq.(2.35) is worked out in detail in Sec.**2.3.1** (eq.(2.21)). Notice also that we may consider only one toroidal modenumber n ($\xi_\perp = \xi_\perp e^{in\phi}$), at a time, and that $\mathbf{t}$ is the unit tangent vector in the poloidal direction given by

$$\mathbf{t} = \frac{1}{Q} \{ \dot{x}(v) \mathbf{e}_R + \dot{y}(v) \mathbf{e}_Z \} .$$

Making use of these results it is convenient to write

$$A_n(v') = \frac{d}{dv'} \left\{ [1 + \epsilon x(v')] \hat{B}_p(v') \xi_\perp(v') \right\} + i n \epsilon Q(v') \hat{B}_\phi(v') \xi_\perp(v') |_{v' \in S_p} . \tag{2.37}$$

In order to proceed we have to evaluate the elliptic integrals $I_n^{(S_i,S_j)}(v, v')$ and $\hat{I}_n^{(S_i,S_j)}(v, v')$, as given by eqs.(2.32) and (2.33).

2.5 Elliptic Integrals

2.5.1 The I_n and $\hat{I}_n$ Integrals

We shall simplify notation by writing I_n and $\hat{I}_n$ instead of the more precise notation $I_n^{(S_i,S_j)}$ and $\hat{I}_n^{(S_i,S_j)}$ used in eq.(2.32) and (2.33). At this point specification of the surfaces over which integration is performed is not essential. In the integrals I_n and $\hat{I}_n$ we change the variable of integration to $\psi = \phi' - \phi$ and also use the fact that $e^{in\psi} = \cos n\psi + i \sin n\psi$ as well as some new parameters defined by the following equations, (we are using notations similar to Hakkarainen[4]).

$$R'^2 + R^2 + (Z' - Z)^2 - 2R'R\cos(\phi' - \phi) = h^2\{1 + k^2 - 2k\cos\psi\}$$

where

$$R'^2 + R^2 + (Z' - Z)^2 \stackrel{def}{=} h^2(1 + k^2),$$

$$2R'R \stackrel{def}{=} 2h^2 k,$$

$$\frac{1 + k^2}{k} = 2\alpha \stackrel{def}{=} \frac{R'^2 + R^2 + (Z' - Z)^2}{R'R}.$$

Thus

$$k^2 - 2\alpha k + 1 = 0,$$

and

$$k = \alpha \pm \sqrt{\alpha^2 - 1} = \alpha - \sqrt{\alpha^2 - 1} < 1,$$

where we have chosen the lower sign to make $k < 1$. By use of these substitutions we can cast the integrals I_n and $\hat{I}_n$ eqs.(2.32) and (2.33) in the following form

$$I_n(v, v') = \frac{R_0}{4h} \int_{-\pi}^{\pi} \frac{\cos n\psi}{\{1 + k^2 - 2k\cos\psi\}^{\frac{1}{2}}} d\psi, \tag{2.38}$$

$$\hat{I}_n(v, v') = \frac{R_0^3}{4h^3} \int_{-\pi}^{\pi} \frac{\cos n\psi}{\{1 + k^2 - 2k\cos\psi\}^{\frac{3}{2}}} d\psi, \tag{2.39}$$

where

$$h = \sqrt{\frac{R'R}{k}}. \tag{2.40}$$

Since the integrands in both integrals are periodic functions with period 2π, we may choose the range of integration to be any interval of length 2π. Also notice that the integral containing $\sin\psi$ integrates to zero, being an odd function. In order to arrive at the standard form of these integrals we make a new change of integration variable by the substitution

$$\psi = 2\theta - \pi ,$$

thus $\cos\psi = -1 + 2\sin^2\theta$ and $d\psi = 2d\theta$. This way we obtain

$$\begin{aligned} I_n(v,v') &= 2\frac{R_0}{4h}\int_0^{\pi}(-1)^n\frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{1}{2}}}d\theta \\ &= (-1)^n\frac{R_0}{h}\int_0^{\frac{\pi}{2}}\frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{1}{2}}}d\theta . \end{aligned} \tag{2.41}$$

The last step is easily accomplished by looking at the integral from $\frac{\pi}{2}$ to π, changing variable in this integral by the substitution $\theta = \theta' + \pi$, and then noticing that the sum of these integrals amounts to integrating an even function from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Performing the same steps on the other integral $\hat{I}_n$ we obtain

$$\hat{I}_n(v,v') = (-1)^n\frac{R_0^3}{h^3}\int_0^{\frac{\pi}{2}}\frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{3}{2}}}d\theta . \tag{2.42}$$

For simplicity we now write

$$L_{2n}(k) = \frac{1}{1+k}\int_0^{\frac{\pi}{2}}\frac{\cos 2n\theta}{\{1 - \hat{k}^2\sin^2\theta\}^{\frac{1}{2}}}d\theta , \tag{2.43}$$

$$\hat{L}_{2n}(k) = \frac{1}{(1+k)^3}\int_0^{\frac{\pi}{2}}\frac{\cos 2n\theta}{\{1 - \hat{k}^2\sin^2\theta\}^{\frac{3}{2}}}d\theta , \tag{2.44}$$

where

$$\hat{k} \stackrel{def}{=} \frac{2\sqrt{k}}{1+k}. \tag{2.45}$$

Thus L_0 is an elliptic integral and

$$I_n = (-1)^n \frac{R_0}{h} L_{2n}(k), \tag{2.46}$$

$$\hat{I}_n = (-1)^n \frac{R_0^3}{h^3} \hat{L}_{2n}(k). \tag{2.47}$$

In order to evaluate $\hat{L}_{2n}(k)$ we take the derivative of $L_{2n}(k)$ with respect to k and obtain

$$\frac{dL_{2n}(k)}{dk} = -\int_0^{\frac{\pi}{2}} \frac{[1+k-2\sin^2\theta]\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{3}{2}}} d\theta. \tag{2.48}$$

Then we use the identity

$$\begin{aligned} 1+k-2\sin^2\theta &= \frac{1}{2k}\{(1+k)^2 - 4k\sin^2\theta\} + 1 + k - \frac{(1+k)^2}{2k} \\ &= \frac{1}{2k}\{(1+k)^2 - 4k\sin^2\theta\} - \frac{1-k^2}{2k}, \end{aligned}$$

and obtain

$$\begin{aligned} \frac{dL_{2n}(k)}{dk} &= -\frac{1}{2k}\int_0^{\frac{\pi}{2}} \frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{1}{2}}} d\theta \\ &+ \frac{1-k^2}{2k}\int_0^{\frac{\pi}{2}} \frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{3}{2}}} d\theta. \end{aligned}$$

This yields

$$\hat{L}_{2n}(k) \stackrel{def}{=} \int_0^{\frac{\pi}{2}} \frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{3}{2}}} d\theta = \frac{2k}{1-k^2}\frac{dL_{2n}}{dk} + \frac{1}{1-k^2} L_{2n}. \tag{2.49}$$

Thus, as long as we know $L_{2n}(k)$ and $\frac{d}{dk}L_{2n}(k)$ we can determine $\hat{L}_{2n}(k)$. Before we pursue the evaluation of the integrals $L_{2n}(k)$ and $\frac{d}{dk}L_{2n}(k)$, we shall list some basic properties of elliptic integrals that will be useful.

2.5.2 Basic Properties of Elliptic Integrals

The complete elliptic integrals of first and second kind, $K(k)$ and $E(k)$ are definied as

$$K(k) \stackrel{def}{=} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{2.50}$$

and

$$E(k) \stackrel{def}{=} \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2 x}\, dx\,. \tag{2.51}$$

In eq.(2.45) we have $\hat{k} \stackrel{def}{=} 2\sqrt{k}/(1+k)$ for obvious reasons. From standard tables[5] we find the following useful relations,

$$\begin{aligned} K(\hat{k}) &= (1+k)K(k)\,, && (2.52)\\ E(\hat{k}) &= \frac{1}{1+k}\{2E(k)-(1-k^2)K(k)\}\,, && (2.53) \end{aligned}$$

and

$$\begin{aligned} K'(k) = \frac{d}{dk}K(k) &= \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}\,, && (2.54)\\ E'(k) = \frac{d}{dk}E(k) &= \frac{1}{k}\{E(k)-K(k)\}\,. && (2.55) \end{aligned}$$

Thus we have from eq.(2.43)

$$L_0 = \frac{K(\hat{k})}{1+k} = K(k)\,, \qquad L_0' = K'(k)\,. \tag{2.56}$$

2.5.3 Useful Recurrence Relations

For the purpose of evaluating $L_{2n}(k)$ for $n > 0$ the following recurrence relation that can be obtained from Gradshteyn & Ryzhik: Integral Tables[5] p.157 (2.581) with $m = 0$ and $r = -1$, is useful

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2n} x\Delta^{-1}dx &= \frac{1}{(2n-1)k^2}\left\{\left[(2n-2)k^2-(2n-2)(1-k^2)\right]\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\Delta^{-1}dx\right.\\ &\left.+(2n-3)(1-k^2)\int_0^{\frac{\pi}{2}}\cos^{2n-4}x\Delta^{-1}dx\right\}\\ &= \frac{2n-2}{2n-1}\,\frac{2k^2-1}{k^2}\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\Delta^{-1}dx\\ &+\frac{2n-3}{2n-1}\,\frac{1-k^2}{k^2}\int_0^{\frac{\pi}{2}}\cos^{2n-4}x\Delta^{-1}dx\,,\end{aligned}$$

where

$$\Delta = \sqrt{1-k^2\sin^2 x} \qquad \text{and} \qquad n \geq 2\,.$$

Let

$$\hat{M}_{2n}(\hat{k}) = \int_0^{\frac{\pi}{2}} \frac{\cos^{2n} x}{\sqrt{1-\hat{k}^2\sin^2 x}}dx\,. \tag{2.57}$$

It then follows that $\hat{M}_{2n}(\hat{k})$ is given by the following recurrence relation

$$\hat{M}_{2n}(\hat{k}) = \frac{2n-2}{2n-1}\,\frac{2\hat{k}^2-1}{\hat{k}^2}\hat{M}_{2n-2}(\hat{k}) + \frac{2n-3}{2n-1}\,\frac{1-\hat{k}^2}{\hat{k}^2}\hat{M}_{2n-4}(\hat{k})\,. \tag{2.58}$$

In order to start the iteration we need to know $\hat{M}_{2n}(\hat{k})$ for $n = 0$ and $n = 1$, which can easily be determined as

$$\hat{M}_0(\hat{k}) = K(\hat{k})\,, \tag{2.59}$$

and

$$\begin{aligned}\hat{M}_2(\hat{k}) &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sqrt{1-\hat{k}^2\sin^2 x}}dx\\ &= \int_0^{\frac{\pi}{2}} \frac{\frac{\hat{k}^2-1}{\hat{k}^2}+\frac{1}{\hat{k}^2}(1-\hat{k}^2\sin^2 x)}{\sqrt{1-\hat{k}^2\sin^2 x}}dx\\ &= \frac{1}{\hat{k}^2}\left\{E(\hat{k})-(1-\hat{k}^2)K(\hat{k})\right\}\,.\end{aligned}$$

We shall find it more convenient to express all quantities in terms of k rather than $\hat{k}$. Therefore we define

$$M_{2n}(k) \stackrel{def}{=} \frac{1}{1+k}\hat{M}_{2n}(\hat{k})\,. \tag{2.60}$$

We then compute these quantities and the derivatives for $n = 0$ and $n = 1$, and obtain

$$M_0(k) = K(k)\,, \tag{2.61}$$

$$M_2(k) = \frac{1}{2k}\left\{E(k) - (1-k)K(k)\right\}\,, \tag{2.62}$$

$$M_0'(k) = \frac{1}{k}\left\{\frac{E(k)}{(1-k^2)} - K(k)\right\}\,, \tag{2.63}$$

$$M_2'(k) = -\frac{1}{2k^2}\left\{\frac{E(k)}{1+k} - (1-k)K(k)\right\}\,. \tag{2.64}$$

By expressing $\cos 2nx$ in terms of powers in $\cos x$ we may write

$$\begin{aligned}
(1+k)L_{2n}(k) &= \int_0^{\frac{\pi}{2}} \frac{\cos 2nx}{\sqrt{1-\hat{k}^2\sin^2 x}}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{2^{2n-1}\cos^{2n}x - \frac{2n}{1}2^{2n-3}\cos^{2n-2}x + \cdots}{\sqrt{1-\hat{k}^2\sin^2 x}}dx \\
&= 2^{2n-1}\hat{M}_{2n}(\hat{k}) - \frac{2n}{1}2^{2n-3}\hat{M}_{2n-2}(\hat{k}) \\
&\quad + \frac{2n}{2}\binom{2n-3}{1}2^{2n-5}\hat{M}_{2n-4}(\hat{k}) - \frac{2n}{3}\binom{2n-4}{2}2^{2n-7}\hat{M}_{2n-6}(\hat{k}) + \cdots \\
&= 2^{2n-1}\hat{M}_{2n}(\hat{k}) + \sum_{m=1}^{n}(-1)^m\frac{2n}{m}\binom{2n-(m+1)}{m-1}2^{2n-(2m+1)}\hat{M}_{2n-2m}(\hat{k})\,,
\end{aligned}$$

where $\hat{M}_{2n}$ is determined by the recurrence relation eq.(2.58). Changing to k representation and M instead of $\hat{M}$ and $\hat{k}$ we obtain for $n > 0$ the relation (notice that the $1+k$ - factor is now absorbed in the definition of M_{2n})

$$L_{2n}(k) = 2^{2n-1}M_{2n}(k) + \sum_{m=1}^{n}(-1)^m\frac{n}{m}\binom{2n-(m+1)}{m-1}2^{2n-2m}M_{2n-2m}(k)\,. \tag{2.65}$$

For $\frac{d}{dk}L_{2n}(k) = L'_{2n}(k)$ we thus have the relation

$$L'_{2n}(k) = 2^{2n-1}M'_{2n}(k) + \sum_{m=1}^{n}(-1)^m \frac{n}{m}\binom{2n-(m+1)}{m-1}2^{2n-2m}M'_{2n-2m}(k) . \quad (2.66)$$

The recurrence relations for M_{2n} and M'_{2n} becomes

$$M_{2n}(k) = \frac{2n-2}{2n-1}f(k)M_{2n-2}(k) + \frac{2n-3}{2n-1}g(k)M_{2n-4}\,, \quad (2.67)$$

$$\begin{aligned} M'_{2n}(k) &= \frac{2n-2}{2n-1}f(k)M'_{2n-2}(k) + \frac{2n-3}{2n-1}g(k)M'_{2n-4} \\ &+ \frac{2n-2}{2n-1}f'(k)M_{2n-2}(k) + \frac{2n-3}{2n-1}g'(k)M_{2n-4}\,, \end{aligned} \quad (2.68)$$

where

$$f(k) = \frac{-1+6k-k^2}{4k}, \qquad f'(k) = \frac{1-k^2}{4k^2}, \quad (2.69)$$

$$g(k) = \frac{(1-k)^2}{4k}, \qquad g'(k) = -\frac{1-k^2}{4k^2}. \quad (2.70)$$

We then find

$$\begin{aligned} M_4(k) &= \frac{2}{3}f(k)M_2(k) + \frac{1}{3}g(k)M_0(k) \\ &= \frac{2}{3}\left[\frac{-1+6k-k^2}{4k}\right]\left[\frac{1}{2k}\{E(k)-(1-k)K(k)\}\right] \\ &\quad + \frac{1}{3}\left[\frac{(1-k)^2}{4k}\right][K(k)] \\ &= \frac{1}{3k^2}\{f_4(k)E(k) + g_4(k)(1-k)K(k)\}\,, \end{aligned} \quad (2.71)$$

where

$$f_4(k) = \frac{1}{4}\{-1+6k-k^2\}\,, \qquad g_4(k) = \frac{1}{4}\{1-5k\}\,,$$

$$
\begin{aligned}
M_4'(k) &= \frac{2}{3}f'(k)M_2(k) + \frac{1}{3}g'(k)M_0(k) \\
&+ \frac{2}{3}f(k)M_2'(k) + \frac{1}{3}g(k)M_0'(k) \\
&= \frac{2}{3}\left[\frac{1-k^2}{4k^2}\right]\left[\frac{1}{2k}\left\{E(k)-(1-k)K(k)\right\}\right] \\
&\quad +\frac{1}{3}\left[-\frac{1-k^2}{4k^2}\right][K(k)] \\
&\quad +\frac{2}{3}\left[\frac{-1+6k-k^2}{4k}\right]\left[-\frac{1}{2k^2}\left\{\frac{E(k)}{1+k}-(1-k)K(k)\right\}\right] \\
&\quad +\frac{1}{3}\frac{(1-k)^2}{4k}\left[\frac{1}{k}\left\{\frac{E(k)}{(1-k^2)}-K(k)\right\}\right] \\
&= \frac{1}{3k^3}\left\{\hat{f}_4(k)E(k)+\hat{g}_4(k)(1-k)K(k)\right\}, \qquad (2.72)
\end{aligned}
$$

where

$$
\hat{f}_4(k) = \frac{2-4k-k^2-k^3}{4(1+k)}, \qquad \hat{g}_4(k) = k - \frac{1}{2},
$$

$$
\begin{aligned}
M_6(k) &= \frac{4}{5}f(k)M_4(k) + \frac{3}{5}g(k)M_2(k), \\
&\;\vdots
\end{aligned}
$$

$$
\begin{aligned}
M_6'(k) &= \frac{4}{5}f(k)M_4'(k) + \frac{3}{5}g(k)M_2'(k) \\
&\quad +\frac{4}{5}f'(k)M_4(k) + \frac{3}{5}g'(k)M_2(k), \\
&\;\vdots
\end{aligned}
$$

this stage one ortant point should be observed. That is: M_2 and well as their derivatives are all regular functions of k in the limit $k \to 1$, which corresponds to $|v' - v| \to 0$. From this fact we can conclude that in our representation of the integrals I_n we have:

The singular behavior when $k \to 1$ is associated with the terms containing M_0 and M_0' only.

We are now in a position to determine L_{2n} and $\hat{L}_{2n}$ and thereby I_n and $\hat{I}_n$ for $n \geq 1$. We have already determined L_0 and L_0' by eq.(2.56). From eq.(2.65) we obtain with $n = 1$

$$\begin{aligned} L_2(k) &= 2M_2 - M_0 \\ &= 2\left(\frac{1}{2k}\left\{E(k) - (1-k)K(k)\right\}\right) - K(k) \\ &= \frac{1}{k}\{E(k) - (1-k)K(k)\} - K(k) \\ &= \frac{1}{k}\{E(k) - K(k)\}, \end{aligned}$$

$$\begin{aligned} L_2'(k) &= 2M_2' - M_0' \\ &= 2\left(-\frac{1}{2k^2}\left\{\frac{E(k)}{1+k} - (1-k)K(k)\right\}\right) - \frac{1}{k}\left\{\frac{E(k)}{(1-k^2)} - K(k)\right\} \\ &= \frac{1}{k^2}\{\frac{-1}{(1-k^2)}E(k) + K(k)\}, \end{aligned}$$

which also can easily be obtained by taking the derivative of the expression for $L_2(k)$ directly. From eq.(2.65) we obtain with $n = 2$

$$\begin{aligned} L_4(k) &= 8M_4 - 8M_2 + M_0 \\ &= 8\left(\frac{1}{3k^2}\left\{f_4(k)E(k) + g_4(k)(1-k)K(k)\right\}\right) \\ &\quad -8\left(\frac{1}{2k}\left\{E(k) - (1-k)K(k)\right\}\right) + K(k) \\ &= \frac{2}{3k^2}\{-(1+k^2)E(k) + (1-k^2)K(k)\} + K(k)\}, \end{aligned}$$

$$
\begin{aligned}
L_4'(k) &= 8M_4' - 8M_2' + M_0' \\
&= 8\left(\frac{1}{3k^3}\left\{\hat{f}_4(k)E(k) + \hat{g}_4(k)(1-k)K(k)\right\}\right) \\
&\quad -8\left(-\frac{1}{2k^2}\left\{\frac{E(k)}{1+k} - (1-k)K(k)\right\}\right) + \frac{1}{k}\left\{\frac{E(k)}{(1-k^2)} - K(k)\right\} \\
&= \frac{1}{3k^3}\left\{\frac{4-3k^2+2k^4}{1-k^2}E(k) + (k^2-4)K(k)\right\}.
\end{aligned}
$$

For $n > 2$ it becomes increasingly difficult to work out the analytic solutions in detail. The algebra probably becomes "prohibitive" for practical reasons with regard to determining M_{2n} and M_{2n}' for $n > 3$ in detail, as an explicit function of k. If this is the case there exist the possibility to do it numerically. The recurrence relations eqs.(2.67), (2.68) and eqs.(2.65) and (2.66) can be used to determine $L_{2n}(k)$ and $L_{2n}'(k)$ and this can be implemented as a numerical scheme.

In principle the integrals I_n and $\hat{I}_n$ are now determined for arbitrary values of n.

In the numerical evaluation of these integrals over v' we have to pay special attention to the singularities of M_0 and M_0'. In the next section we do the basic analysis for proper treatment of these singularities. For the purpose of reference we list some basic results here,

$$
I_n = (-1)^n \frac{R_0}{h} L_{2n}(k)\,, \tag{2.73}
$$

$$
\begin{aligned}
\hat{I}_n &= (-1)^n \frac{R_0^3}{h^3} \hat{L}_{2n}(k) \\
&= (-1)^n \frac{R_0^3}{h^3(1-k^2)}\{2kL_{2n}'(k) + L_{2n}(k)\}\,,
\end{aligned} \tag{2.74}
$$

and

$$
h = \sqrt{\frac{RR'}{k}}\,, \quad k = \alpha - \sqrt{\alpha^2 - 1}\,, \quad \alpha = \frac{R'^2 + R^2 + (Z'-Z)^2}{2R'R}\,. \tag{2.75}
$$

The specialization to $I_n^{(S_i,S_j)}$ and $\hat{I}_n^{(S_i,S_j)}$ is obtained from the above formulas by letting $(R,Z) \in S_i$ and $(R',Z') \in S_j$. In addition to the singularity for $k = 1$ that appears explicitly in eq.(2.74), there are also singularities associated with M_0 and M_0'. This will be the subject of the next section.

2.5.4 Singularity of the $K(k)$

When the observation point is located on the surface over which the integration is performed, we encounter a situation where $|v' - v| \to 0$ or $k \to 1$. For this value of k, $K(k)$ is singular. From Abramowitz and Stegun: Mathematical Tables[6] p. 591 (17.3.26) we obtain

$$\lim_{k\to 1} K(k) = 2\ln 2 - \frac{1}{2}\lim_{k\to 1}\ln(1-k^2)\,.$$

Notice also that $k \to 1 \quad \sim \quad |v'-v| \to 0$, thus

$$\lim_{k\to 1} K(k) = \lim_{|v'-v|\to 0} K(k)\,.$$

We shall now evaluate this singularity in terms of the physical variables v and v'. We have from eq.(2.75)

$$k = \alpha - \sqrt{\alpha^2 - 1}\,,$$

$$\begin{aligned} \alpha &= \alpha(v,v') = \frac{R'^2 + R^2 + (Z'-Z)^2}{2R'R} \\ &= 1 + \frac{\epsilon^2}{2}\frac{(x(v') - x(v))^2 + (y(v') - y(v))^2}{(1+\epsilon x(v'))(1+\epsilon x(v))}\,. \end{aligned} \tag{2.76}$$

Thus

$$k = 1 \quad \Rightarrow (1-\alpha)^2 = \alpha^2 - 1 \quad \text{or} \quad \alpha = 1\,,$$

$$\alpha = 1 \quad \Rightarrow \quad (R'-R)^2 + (Z'-Z)^2 = 0 \quad \Rightarrow \quad R' = R,\ Z' = Z\,.$$

And we also have

$$\frac{h}{R_0} = \sqrt{\frac{RR'}{R_0^2 k}},$$

where

$$\begin{aligned} R &= R_0\{1+\epsilon x(v)\}, \\ Z &= R_0\epsilon y(v), \\ R' &= R_0\{1+\epsilon x(v')\}, \\ Z' &= R_0\epsilon y(v'). \end{aligned}$$

All the surfaces considered, i.e., the plasma-vacuum interface, the wall and the conductor surfaces are assumed to have a parametric representation of the form presented here. Thus $x(v)$ and $y(v)$ will be different for the different surfaces. However, the following analysis is general in the sense that it applies to all these surfaces for the case where the "observation point" is located on the surface over which the integration is performed. And when the integration variable v' approches the "observation point" v, we make a Taylor expansion and obtain

$$\begin{aligned} \alpha(v,v') &= \alpha(v,v) + \frac{\epsilon^2}{2}\frac{Q^2(v'-v)^2}{(1+\epsilon x(v))^2} + \mathcal{O}((v'-v)^3) \\ &= 1 + \frac{\epsilon^2}{2}\frac{Q^2(v'-v)^2}{(1+\epsilon x(v))^2} + \mathcal{O}((v'-v)^3), \\ \alpha^2(v,v') &= 1 + \epsilon^2\frac{Q^2(v'-v)^2}{(1+\epsilon x(v))^2} + \mathcal{O}((v'-v)^3), \end{aligned} \tag{2.77}$$

$$\begin{aligned} k &= \alpha - \frac{\epsilon Q|v'-v|}{1+\epsilon x(v)} + \mathcal{O}((v'-v)^2) \\ &= 1 - \frac{\epsilon Q|v'-v|}{1+\epsilon x(v)} + \mathcal{O}((v'-v)^2), \end{aligned} \tag{2.78}$$

$$1-k^2 = 1 - \left(1 - \frac{2\epsilon Q|v'-v|}{1+\epsilon x(v)}\right) + \mathcal{O}((v'-v)^2))$$

$$= \frac{2\epsilon Q|v'-v|}{1+\epsilon x(v)} + \mathcal{O}((v'-v)^2), \tag{2.79}$$

$$\ln(1-k^2) = \ln|v'-v| + \ln\frac{2\epsilon Q}{1+\epsilon x(v)}, \tag{2.80}$$

$$\begin{aligned} K(k) &= -\frac{1}{2}\ln(1-k^2) + 2\ln 2 + \mathcal{O}(1-k^2) \\ &= -\frac{1}{2}\ln\frac{|v'-v|}{2} - \frac{1}{2}\ln\frac{\epsilon Q}{1+\epsilon x(v)} + \ln 2 + \mathcal{O}(|v'-v|) \\ &= -\frac{1}{2}\ln|\sin\pi(v'-v)| - \frac{1}{2}\ln\frac{\epsilon Q}{8\pi[1+\epsilon x(v)]} + \mathcal{O}(|v'-v|), \end{aligned} \tag{2.81}$$

where we have used that asymptotically for small $|x|$, $\frac{|x|}{|\sin x|} = 1 + \mathcal{O}(x^2)$ and $\ln[1+\mathcal{O}(x^2)] = \mathcal{O}(x^2) \Rightarrow \ln|x| = \ln|\sin x| + \mathcal{O}(x^2)$. The last representation for $K(k)$ is a conveninent form for further elaboration, based on some useful integrals that can be found in Gradshteyn & Ryzhik: Integral Tables[5] p. 584 (4.384 (3) & (7), which we list here for convenience

$$I_p = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dx\,\ln(\sin x)\cos 2px = -\frac{1}{2p}, \qquad p > 0, \tag{2.82}$$

$$I_p = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dx\,\ln(\sin x)\cos 2px = -\ln 2, \qquad p = 0, \tag{2.83}$$

$$J_p = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dx\,\ln(\sin x)\sin 2px = 0, \qquad \forall p. \tag{2.84}$$

2.5.5 Singularities of L_{2n} and $\hat{L}_{2n}$

In eq.(2.28) we first consider the last part of the integral given by

$$\frac{e^{in\phi}}{2\pi}\int_0^1 V_n^{(S_j)}(R',Z')F^{(S_i,S_j)}(v,v')\hat{I}_n^{(S_i,S_j)}(v,v')dv'. \tag{2.85}$$

Again we shall omit the references to the specific surfaces other than observing that we consider the situation where the integration in v' is performed over the same surface as v is evaluated on, i.e., we have $i = j$ or $S_i = S_j$. These cases are the only cases we need to consider with regard to singular behavior in the limit $k \to 1$ or $|v' - v| \to 0$. We shall evaluate the integral given by (2.85) in this limit. From eqs.(2.33) and (2.29) we obtain

$$\begin{aligned}\hat{I}_n^{(S_i,S_j)}(v,v') &= \frac{1}{4}\int_0^{2\pi}\frac{R_0^3 e^{in(\phi'-\phi)}d\phi'}{\{R^2+R'^2+(Z'-Z)^2-2RR'\cos(\phi-\phi')\}^{\frac{3}{2}}}\Bigg|_{\substack{R,Z\in S_i\\ R',Z'\in S_j}} \\ &= (-1)^n\frac{R_0^3}{h^3}\hat{L}_{2n}(v,v'). \end{aligned}\tag{2.86}$$

$$\hat{L}_{2n} = \frac{1}{1-k^2}\{2kL'_{2n} + L_{2n}\} \tag{2.87}$$

$$F(v,v') = (v-v')^2 a(v,v') + \mathcal{O}((v-v')^3). \tag{2.88}$$

It is convenient to write L_{2n} and L'_{2n} in terms of a regular and singular part, thus we define

$$\tilde{L}_{2n}(k) \stackrel{def}{=} L_{2n}(k) - (-1)^n M_0(k) = L_{2n}(k) - (-1)^n K(k), \tag{2.89}$$
$$\tilde{L}'_{2n}(k) \stackrel{def}{=} L'_{2n}(k) - (-1)^n M'_0(k) = L'_{2n}(k) - (-1)^n K'(k), \tag{2.90}$$

or

$$L_{2n}(k) = \tilde{L}_{2n}(k) + (-1)^n K(k), \tag{2.91}$$
$$L'_{2n}(k) = \tilde{L}'_{2n}(k) + (-1)^n K'(k). \tag{2.92}$$

Evaluation of the integral (2.85) makes it necessary to calculate the limit

$$\begin{aligned}
\lim_{|v'-v|\to 0}(v'-v)^2\hat{L}_{2n}(k(v,v')) &= \lim_{|v'-v|\to 0}\frac{(v'-v)^2}{1-k^2}\left\{2kL'_{2n}+L_{2n}\right\} \\
&= \lim_{|v'-v|\to 0}\frac{(v'-v)^2}{1-k^2}(-1)^n\left\{2kK'(k)+K(k)\right\} \\
&= \lim_{|v'-v|\to 0}(-1)^n\frac{|v'-v|^2}{(1-k^2)^2}2E(k) \\
&= \frac{(-1)^n[1+\epsilon x(v)]^2}{2\epsilon^2Q^2}, \qquad (2.93)
\end{aligned}$$

where we have used that

$$kK'(k) = \left\{\frac{E(k)}{(1-k^2)} - K(k)\right\},$$

$$1-k^2 = \frac{2\epsilon Q|v'-v|}{1+\epsilon x(v)} + \mathcal{O}((v'-v)^2),$$

which are easily obtained from eqs.(2.54) and (2.79). L_{2n} has at most a logarithmic singularity which behave as $\ln|v'-v|$ when $|v'-v|\to 0$, associated with $K(k)$ through M_0. It is then clear that

$$\lim_{|v'-v|\to 0}|v'-v|L_{2n}(k(v,v')) = 0.$$

Finally we obtain

$$\begin{aligned}
&\lim_{|v'-v|\to 0}\frac{1}{2\pi}e^{in\phi}V_n(R',Z')F(v,v')\hat{I}_n(v,v') \\
&= \frac{e^{in\phi}}{2\pi}\lim_{|v'-v|\to 0}V_n(R',Z')a(v,v')(-1)^n\frac{R_0^3}{h^3}(v-v')^2\hat{L}_{2n}(v,v') \\
&= \frac{e^{in\phi}}{4\pi}V_n(R,Z)\frac{a(v,v)}{\epsilon^2Q^2[1+\epsilon x(v)]} \\
&= \frac{e^{in\phi}}{4\pi}V_n(R,Z)\left\{\frac{\kappa}{\epsilon^2}Q - \frac{\epsilon\dot{y}(v)}{1+\epsilon x(v)}\right\} < \infty, \qquad (2.94)
\end{aligned}$$

where κ is given by eq.(2.31) and $Q = Q(v, v') > 0$. Equations (2.30), (2.44) and (2.47) as well as the limit given in eq.(2.93) has been employed. Also note that $\lim_{|v'-v|\to 0} h = R$, and that $R_0/R = [1 + \epsilon x(v)]^{-1}$.

This result shows that the integrands of the integrals containing $\hat{L}_{2n}$ is finite in the singular limit, and therefore numeric integration can be performed across the singularity without any problems. The formula given by eq.(2.94) may be useful as a test for the numerical scheme to be used.

These results reflect the fact that L_{2n} has only a logarithmic singularity associated with M_0, and L'_{2n} has a singularity of the form $(v' - v)^{-1}$ as well as a logarithmic singularity. It is the singularity $(v' - v)^{-1}$ that gives rise to the finite contribution in the limit considered in eq.(2.94). We conclude that the integrand of $I^{(S_i,S_j)}$ in eq.(2.28) is nonsingular concerning the part dependent on $\hat{I}_n$. The only singularity left to give special consideration is associated with L_{2n}, where again it derives from the logarithmic singularity of $M_0 = K(k)$.

We then turn to the first part of the I_1 integral eq.(2.28) and I_2, eq.(2.34), both of these integrals depends on L_{2n} through I_n.

Singularity of L_{2n}

We first consider a few asymptotic limits that make it easier to evaluate $\lim_{|v'-v|\to 0} L_{2n}$. From eqs.(2.81) and (2.79) we have the asymptotic relations

$$K(k) = -\frac{1}{2}\ln(1 - k^2) + 2\ln 2 + \mathcal{O}(1 - k^2)\,, \tag{2.95}$$

and

$$1 - k^2 = \frac{2\epsilon Q(v)}{\pi[1 + \epsilon x(v)]}\left|\frac{2\pi(v' - v)}{2}\right| + \mathcal{O}((v' - v)^2)\,. \tag{2.96}$$

By combining eqs.(2.95) and (2.96) we may write

$$K(k) = -\frac{1}{2}\ln\left|\frac{2\pi(v' - v)}{2}\right| - \frac{1}{2}\ln\frac{\epsilon Q(v)}{8\pi[1 + \epsilon x(v)]} + \mathcal{O}(v' - v)\,. \tag{2.97}$$

It is convenient to define a function $\bar{K}(v, v')$ which is regular at $v = v'$ by

$$\bar{K}(v, v') \stackrel{def}{=} K(k) + \frac{1}{2}\ln|\sin\pi(v' - v)|\,. \tag{2.98}$$

Since

$$\lim_{|v'-v|\to 0} \ln \left| \frac{\sin \pi(v'-v)}{\pi(v'-v)} \right| = 0 \,,$$

we obtain

$$\begin{aligned} \lim_{|v'-v|\to 0} \bar{K}(v,v') &= \lim_{|v'-v|\to 0} \ln \left| \frac{\sin \pi(v'-v)}{\pi(v'-v)} \right| - \frac{1}{2} \ln \frac{\epsilon Q(v)}{8\pi(1+\epsilon x(v))} \\ &= -\frac{1}{2} \ln \frac{\epsilon Q(v)}{8\pi(1+\epsilon x(v))} \,. \end{aligned}$$

By eq.(2.98) we also have

$$K(k) = \bar{K}(k) - \frac{1}{2} \ln |\sin \pi(v'-v)| \,. \tag{2.99}$$

The reason for the particular choice of function to add in eq.(2.98) is of course the ease of integration in the next step to be considered.

The I_1 Integral

Returning to eq.(2.28) we now consider the part of the integral containing $I_n^{(S_i,S_j)}$. We restrict ourselves to the cases where $i = j$ or $S_i = S_j$, because only in these cases do we have to integrate through the logarithmic singularity. Again we omit the reference to a particular surface, in order to ease notation.

We first consider

$$I_n = (-1)^n \frac{R_0}{h} L_{2n}(k) \,,$$

obtained from eqs.(2.41) and (2.46). By using eqs.(2.91) and (2.99) we represent $L_{2n}(k)$

as

$$L_{2n}(k) = \bar{L}_{2n}(k) - (-1)^n \frac{1}{2} \ln |\sin \pi(v'-v)| \,. \tag{2.100}$$

Notice that $\bar{L}_{2n}(k)$ is just L_{2n} with $M_0 = K(k)$ replaced by $\bar{K}(v,v')$. $\bar{L}_{2n}$ is now a regular function of v and v' also when $v' = v$. This way I_n splits naturally into parts

$$I_n = \bar{I}_n + \tilde{I}_n\,, \tag{2.101}$$

where

$$\bar{I}_n \stackrel{def}{=} (-1)^n \frac{R_0}{h} \bar{L}_{2n}(v, v')\,, \qquad \tilde{I}_n \stackrel{def}{=} -\frac{R_0}{2h} \ln|\sin\pi(v'-v)|\,, \tag{2.102}$$

and I_1 as given by eq. (2.28) splits naturally into two parts

$$I_1 = \bar{I}_1 + \tilde{I}_1\,. \tag{2.103}$$

A Useful Integral Formula

In order to numerically evaluate the double fast Fourier transforms of the fields, it is convenient to evaluate the integrals over the singular part of $K(k)$ analytically. Thus, we must evaluate integrals of the form

$$\begin{aligned}
I_s &= \int_0^1 e^{2\pi i(m'v'-mv)} \ln|\sin\pi(v'-v)|dv' \\
&= e^{2\pi i(m'-m)v} \int_0^1 e^{2\pi i(m'(v'-v)} \ln|\sin\pi(v'-v)|dv' \\
&= e^{2\pi i(m'-m)v} \int_{-v}^{1-v} \{\cos 2\pi m'x + i\sin 2\pi m'x\} \ln|\sin\pi x|dx \\
&= e^{2\pi i(m'-m)v} \int_0^1 \{\cos 2\pi m'x + i\sin 2\pi m'x\} \ln|\sin\pi x|dx \\
&= e^{2\pi i(m'-m)v} \int_0^1 \cos 2\pi m'x \ln|\sin\pi x|dx \\
&= e^{2\pi i(m'-m)v} \times \begin{cases} -\frac{1}{2m'} & (m' \neq 0) \\ -\ln 2 & (m' = 0) \end{cases} \\
&= e^{2\pi i(m'-m)v} f(m')\,,
\end{aligned} \tag{2.104}$$

where

$$f(q) \stackrel{def}{=} \begin{cases} -\frac{1}{2q} & (q \neq 0) \\ -\ln 2 & (q = 0) \end{cases}\,. \tag{2.105}$$

This result is easily obtained by applying the formulas given by eqs.(2.82), (2.83) and (2.84).

The Singular Part of the Integral I_1

We proceed to consider that part of the integral (eq.(2.28)) that contains $\tilde{I}_n$, and which we call $\tilde{I}_1$, and is defined by

$$\tilde{I}_1 \stackrel{def}{=} \frac{e^{in\phi}}{2\pi} \int_0^1 V_n(v')\epsilon\dot{y}(v')\tilde{I}_n(v,v')dv' . \tag{2.106}$$

Let

$$V_n(v') = iR_0B_0 \sum_{m=-\infty}^{\infty} \bar{V}_m e^{2\pi i m v'} . \tag{2.107}$$

From now on we shall omit the reference to the toroidal mode number n, since it appears only as a parameter that has to be set initially.

Looking at eqs.(2.102) and (2.106), we find it convenient to Fourier transform the following quantity

$$-\frac{R_0}{4\pi h(v,v')}\epsilon\dot{y}(v') = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} Y_{kl} e^{2\pi i(kv'+lv)} . \tag{2.108}$$

We can now compute the integral $\tilde{I}_1$ in terms of given Fourier transforms.

$$\begin{aligned}
\tilde{I}_1(v) &= \frac{e^{in\phi}}{2\pi}\int_0^1 V_n(v')\epsilon\dot{y}(v')\{-\frac{R_0}{2h}\ln|\sin\pi(v'-v)|\}dv' \\
&= iR_0B_0e^{in\phi}\int_0^1\{\sum_{m=-\infty}^{\infty}\bar{V}_m e^{2\pi i m v'}\}\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} Y_{kl}e^{2\pi i(kv'+lv)}\}\ln|\sin\pi(v'-v)|\,dv' \\
&= iR_0B_0e^{in\phi}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\bar{V}_m Y_{kl}e^{2\pi i(k+l+m)v}\int_0^1 e^{2\pi i(m+k)(v'-v)}\ln|\sin\pi(v'-v)|\,dv' \\
&= iR_0B_0e^{in\phi}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\bar{V}_m Y_{kl}e^{2\pi i(k+l+m)v}\times\begin{cases} -\frac{1}{2(m+k)} & m+k\neq 0 \\ -\ln 2 & m+k=0 \end{cases} ,
\end{aligned}$$

or

$$\tilde{I}_1 = iR_0B_0e^{in\phi}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\bar{V}_m Y_{kl}e^{2\pi i(k+l+m)v}f(m+k) . \tag{2.109}$$

where eq.(2.104) has been used and $f(m+k)$ is defined by eq.(2.105). In the subsequent analysis we shall sometimes simplify notation by writing

$$\sum_m \stackrel{def}{=} \sum_{m=-\infty}^{\infty} \quad \text{and} \quad \sum_{m,k} \stackrel{def}{=} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} .$$

We have made no special reference to any surface here, since the procedure is the same for all surfaces. However, when evaluating these integrals for different surfaces one has to pick the right functions $\dot{y}(v')$, $h(v,v')$ and so forth, for that particular surface. When we later on may want to be specific in referring to a particular surface, we shall do so by superscripts α, β, where α, β can be S_p, S_w and S_{c_i}, as previously introduced as reference parameters for the plasma vacuum interface, the wall and conductor surface number i, respectively. Thus in specific notation it will be like

$$\bar{V}_m^{(\alpha)} \quad \text{and} \quad Y_{kl}^{(\alpha,\beta)} . \tag{2.110}$$

The Singular Part of the Integral I_2

We then consider the singular part of the integral I_2. From eq.(2.34) we have

$$I_2 = -\frac{e^{in\phi}}{\pi} \int_0^1 A_n(v') I_n(v,v') dv' , \tag{2.111}$$

where $A_n(v')$ is given by eq.(2.37) and I_n is given by eq.(2.32) or in elaborated form by eqs.(2.41) and (2.46). Thus we have

$$\begin{aligned} I_n(v,v') &= \frac{1}{4} \int_0^{2\pi} \frac{R_0 e^{in(\phi'-\phi)} d\phi'}{\{R^2 + R'^2 + (Z'-Z)^2 - 2RR' \cos(\phi-\phi')\}^{\frac{1}{2}}} \Big|_{\substack{R,Z \in S_i \\ R',Z' \in S_j}} \\ &= (-1)^n \frac{R_0}{h(v,v')} L_{2n}(k(v,v')) \\ &= (-1)^n \frac{R_0}{h(v,v')} \{\bar{L}_{2n}(k) - (-1)^n \frac{1}{2} \ln|\sin \pi(v'-v)|\} , \end{aligned}$$

where eq.(2.100) has been used. Then we write I_2 the obvious way as

$$I_2 = \bar{I}_2 + \tilde{I}_2 \tag{2.112}$$

where $\bar{I}_2$ is that part of the integral which contains the logarithmic singularity. By definition we introduce

$$\bar{I}_2 \stackrel{def}{=} \frac{e^{in\phi}}{2\pi} \int_0^1 A_n(v') \frac{R_0}{h(v,v')} \ln|\sin \pi(v'-v)|\, dv' \,, \tag{2.113}$$

and from eq.(2.37) we have

$$A_n(v') = \frac{d}{dv'} \left\{ [1+\epsilon x(v')] \hat{B}_p(v') \xi_n(v') \right\} + in\epsilon Q(v') \hat{B}_\phi(v') \xi_n(v')|_{v' \in S_p} \,. \tag{2.114}$$

Let

$$\begin{aligned} \xi_n &= R_0 \sum_k \bar{\xi}_k e^{2\pi i k v'} \,, \\ [1+\epsilon x(v')] \hat{B}_p(v') &= B_0 \sum_l \tilde{B}_l e^{2\pi i l v'} \,, \\ \epsilon Q(v') \hat{B}_\phi(v') &= B_0 \sum_m \bar{B}_m e^{2\pi i m v'} \,, \end{aligned}$$

thus we obtain

$$[1+\epsilon x(v')] \hat{B}_p(v') \xi_n = B_0 R_0 \sum_{k\,l} \bar{\xi}_k \tilde{B}_l e^{2\pi i (k+l) v'} \,,$$

from which we substitute in eq.(2.114) to obtain

$$\begin{aligned} A_n(v') &= \frac{d}{dv'} \{ [1+\epsilon x(v')] \hat{B}_p(v') \xi_n \} + in\epsilon Q(v') \hat{B}_\phi(v') \xi_n(v') \\ &= i R_0 B_0 \sum_k \sum_l \left(2\pi (k+l) \tilde{B}_l + n \bar{B}_l \right) \bar{\xi}_k e^{2\pi i (k+l) v'} \,. \end{aligned} \tag{2.115}$$

Furthermore let

$$\frac{1}{2\pi} \frac{R_0}{h(v,v')} = \sum_{pq} h_{pq} e^{2\pi i (p v' + q v)} \,. \tag{2.116}$$

Using the results in eqs.(2.115) and (2.116) we obtain

$$\begin{aligned}
\bar{I}_2 &= iR_0B_0e^{in\phi}\sum_{pq}\sum_{kl}(2\pi(k+l)\tilde{B}_l + n\bar{B}_l)\bar{\xi}_k h_{pq}e^{2\pi i(k+l+p+q)v} \\
&\quad \times \int_0^1 e^{2\pi i(k+l+p)(v'-v)} \ln|\sin\pi(v'-v)|\, dv' \\
&= iR_0B_0e^{in\phi}\sum_{pq}\sum_{kl}(2\pi(k+l)\tilde{B}_l + n\bar{B}_l)\bar{\xi}_k h_{pq}e^{2\pi i(k+l+p+q)v} \\
&\quad \times \begin{cases} -\frac{1}{2(k+l+p)} & k+l+p \neq 0 \\ -\ln 2 & k+l+p = 0 \end{cases} .
\end{aligned} \tag{2.117}$$

The Regular Parts of the Integrals I_1 and I_2

We then proceed to evaluate the regular parts of the integrals I_1 and I_2. From eq.(2.28) we find that the regular part of the integral I_1, which we called $\bar{I}_1$ is defined by

$$\bar{I}_1 \stackrel{def}{=} \frac{e^{in\phi}}{2\pi}\int_0^1 V_n(v')\left\{\epsilon\dot{y}(v')\bar{I}_n(v,v') + F(v,v')\hat{I}_n(v,v')\right\} dv' , \tag{2.118}$$

where $\bar{I}_n(v,v')$ is given by eq.(2.102) as the regular part of I_n, also notice that we have already checked that $F(v,v')\hat{I}_n(v,v')$ does not have any singular limits. We let

$$\frac{\epsilon}{2\pi}\dot{y}(v')\bar{I}_n(v,v') = \sum_{k,m} H_{km}e^{2\pi i(kv'+mv)} , \tag{2.119}$$

$$\frac{1}{2\pi}F(v,v')\hat{I}_n(v,v') = \sum_{k,m} F_{km}e^{2\pi i(kv'+mv)} , \tag{2.120}$$

and use the representation of V_n,

$$V_n(v') = iR_0B_0\sum_l \bar{V}_l e^{2\pi i l v'} . \tag{2.121}$$

For $\bar{I}_1$ we then find

$$\begin{aligned}
\bar{I}_1 &= iR_0B_0e^{in\phi}\sum_{k,l,m}\bar{V}_l(H_{km} + F_{km})e^{2\pi i m v}\delta_{(k+l)0} \\
&= iR_0B_0e^{in\phi}\sum_{l,m}\bar{V}_l(H_{-lm} + F_{-lm})e^{2\pi i m v} .
\end{aligned} \tag{2.122}$$

Then adding the results in eqs.(2.109) and (2.122) we obtain

$$I_1 = iR_0B_0e^{in\phi}\left\{\sum_{k,l,m} f(m+k)\bar{V}_m Y_{kl} e^{2\pi i(k+l+m)v} + \sum_{l,m}\bar{V}_m(H_{-m\,l}+F_{-m\,l})e^{2\pi ilv}\right\},$$

where again $f(m+k)$ is defined by eq.(2.105). We may then write

$$I_1 = iR_0B_0e^{in\phi}\sum_{l,m}\bar{V}_m\left[(H_{-m\,l}+F_{-m\,l})+\sum_k f(m+k)Y_{k\,(l-k-m)}\right]e^{2\pi ilv}. \tag{2.123}$$

We define a matrix $\mathbf{\Gamma}$ by its elements

$$\Gamma_{m\,l} = 2\{H_{-m\,l}+F_{-m\,l}+\sum_k f(m+k)Y_{k\,(l-k-m)}\}, \tag{2.124}$$

and we represent the Fourier transforms as vectors, so that we may write

$$I_1 = \frac{i}{2}R_0B_0e^{in\phi}\sum_{l,m}\bar{V}_m\Gamma_{m\,l}e^{2\pi ilv} = \frac{i}{2}R_0B_0e^{in\phi}\sum_l(\mathbf{V}\cdot\mathbf{\Gamma})_l\,e^{2\pi ilv}, \tag{2.125}$$

where

$$\mathbf{V} \stackrel{def}{=} \{\bar{V}_1,\bar{V}_2,\cdots\bar{V}_n\}. \tag{2.126}$$

Referring to I_2 given by eqs.(2.111) and (2.112) we write

$$\bar{I}_2 \stackrel{def}{=} -\frac{e^{in\phi}}{\pi}\int_0^1 A_n(v')\bar{I}_n(v'v')dv'.$$

By using the representation for A_n given in eq.(2.115), we have

$$A_n(v,v') = iR_0B_0\sum_k\sum_l\left(2\pi(k+l)\bar{B}_l+n\bar{B}_l\right)\bar{\xi}_k e^{2\pi i(k+l)v'}.$$

Then we introduce the Fourier transform representation

$$-\frac{1}{\pi}\bar{I}_n(v,v') = \sum_{pq}\bar{H}_{p\,q}e^{2\pi i(pv'+qv)}, \tag{2.127}$$

from which we obtain

$$
\begin{aligned}
\bar{I}_2 &= iR_0B_0e^{in\phi}\sum_{pq}\sum_{kl}\{2\pi(k+l)\tilde{B}_l + n\bar{B}_l\}\bar{\xi}_k\bar{H}_{pq}e^{2\pi iqv} \\
&\quad\times\int_0^1 e^{2\pi i(k+l+p)v'}dv' \\
&= iR_0B_0e^{in\phi}\sum_{q}\sum_{kl}\{2\pi(k+l)\tilde{B}_l + n\bar{B}_l\}\bar{\xi}_k\bar{H}_{-(k+l)\,q}e^{2\pi iqv}\,. \quad (2.128)
\end{aligned}
$$

From eqs.(2.117) and (2.128) we find $I_2 = \tilde{I}_2 + \bar{I}_2$

$$
\begin{aligned}
I_2 &= iR_0B_0e^{in\phi}\sum_{pq}\sum_{kl}(2\pi(k+l)\tilde{B}_l + n\bar{B}_l)\bar{\xi}_k h_{pq}e^{2\pi i(k+l+p+q)v} \\
&\quad\times\begin{cases} -\frac{1}{2(k+l+p)} & k+l+p\neq 0 \\ -\ln 2 & k+l+p=0 \end{cases} \\
&\quad + iR_0B_0e^{in\phi}\sum_{q}\sum_{kl}\{2\pi(k+l)\tilde{B}_l + n\bar{B}_l\}\bar{\xi}_k\bar{H}_{-(k+l)\,q}e^{2\pi iqv} \\
&= \sum_{q}\sum_{km}\{2\pi m\tilde{B}_{m-k} + n\bar{B}_{m-k}\} \\
&\qquad\times\{\bar{H}_{-m\,q} + f(m+p)\sum_p h_{p\,(q-m-p)}\}\bar{\xi}_k e^{2\pi iqv}\,. \quad (2.129)
\end{aligned}
$$

We define a matrix $\mathbf{\Lambda}$ by

$$
\Lambda_{k\,q} = 2\sum_{pm}\{2\pi m\tilde{B}_{m-k} + \bar{B}_{m-k}\}\{\bar{H}_{-m\,q} + f(m+p)\sum_p h_{p\,(q-m-p)}\}\,, \quad (2.130)
$$

where again $f(m+p)$ is defined by eq.(2.105), and we obtain

$$
I_2 = \frac{i}{2}R_0B_0e^{in\phi}\sum_q(\boldsymbol{\xi}\cdot\Lambda)_q e^{2\pi iqv}\,. \quad (2.131)
$$

2.6 Matrix Representation

In the previous section we derived expressions for I_1 and I_2 given by eqs.(2.125) and (2.131)

$$I_1 = \frac{ie^{in\phi}}{2} R_0 B_0 \sum_{l,m} \bar{V}_m \Gamma_{m\,l} e^{2\pi i l v} = \frac{ie^{in\phi}}{2} R_0 B_0 \sum_{l} (\mathbf{V} \cdot \mathbf{\Gamma})_l \, e^{2\pi i l v} . \tag{2.132}$$

$$I_2 = \frac{1}{2} i R_0 B_0 e^{in\phi} \sum_{q} (\boldsymbol{\xi} \cdot \mathbf{\Lambda})_q e^{2\pi i q v} . \tag{2.133}$$

Going back to eq.(2.4) we obtain the following generic equation in terms of the vector representation of the Fourier harmonics

$$\frac{ie^{in\phi}}{2} R_0 B_0 \sum_{m} \bar{V}_m e^{2\pi i m v} = -I_1^{(p)} + I_2 + I_1^{w} + \sum_{i=1}^{l} I_1^{(c_i)} , \tag{2.134}$$

where we have adopted the convention that the unit normal vector is pointing outward on the plasma-vacuum interface, i.e., into the vacuum region. For more details see eqs.(2.125), (2.107) and (2.126). On the other surfaces $\mathbf{n}$ is oriented out of the vacuum region considered. With this convention we rewrite eq.(2.134) as

$$\mathbf{V}^{(\alpha)} + \mathbf{V}^{(p)} \cdot \mathbf{\Gamma}^{(p,\alpha)} - \mathbf{V}^{(w)} \cdot \mathbf{\Gamma}^{(w,\alpha)} - \sum_{i=1}^{l} \mathbf{V}^{(c_i)} \cdot \mathbf{\Gamma}^{(c_i,\alpha)} = \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(\alpha)} . \tag{2.135}$$

This is a linear system of equations where the unknowns $\mathbf{V}^{(\alpha)}$ are vectors. The label α indicates the surface where the 'observation point', i.e., on which surface the integration variable v is located. Thus α takes on the values p, w, c_i $(i = 1, 2 \cdots l)$ with

p referring to plasma - vacuum interface,

w to the infinitely conducting wall,

c_i to be conductor number i, of which there are l all together.

This is a rather complicated system of $l+2$ equations with $l+2$ unknowns. Remember the unknowns themselves are vectors with of the order 100 components, and all capital Greek quantities are matrices of the order 100×100, with $l = 4$ this becomes a 6×6 system. There is still one simplifying fact, and that is: We only need to determine one of the unknowns, namely $\mathbf{V}^{(p)}$. If we consider a triangularization procedure for solving this system, this means that we have to carry this procedure only to the stage where the system matrix has zeroes above the diagonal in the first row. Remember, however, that the elements in our system matrix are themselves matrices.

Wall but no Conductors

As a first simple step we consider a wall with no conductors taken into account, resulting in the following system.

$$\mathbf{V}^{(p)} \cdot \mathbf{\Psi}^{(p)} - \mathbf{V}^{(w)} \cdot \mathbf{\Gamma}^{(w,p)} = \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(p)} \tag{2.136}$$

$$\mathbf{V}^{(p)} \cdot \mathbf{\Gamma}^{(p,w)} + \mathbf{V}^{(w)} \cdot \mathbf{\Omega}^{(w,w)} = \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(w)} \tag{2.137}$$

where we have introduced

$$\mathbf{\Psi}^{(p)} \stackrel{def}{=} \mathbf{I} + \mathbf{\Gamma}^{(p,p)} \quad \text{and} \quad \mathbf{\Omega}^{(w)} \stackrel{def}{=} \mathbf{I} - \mathbf{\Gamma}^{(w,w)} . \tag{2.138}$$

Multiplying the first equation by $(\mathbf{\Gamma}^{(w,p)})^{-1}$ and the second equation by $(\mathbf{\Omega}^{(w,w)})^{-1}$ and adding the resulting equations we obtain

$$\mathbf{V}^{(p)} \cdot \mathbf{A} = \boldsymbol{\xi} \cdot \mathbf{B} , \tag{2.139}$$

or

$$\mathbf{V}^{(p)} = \boldsymbol{\xi} \cdot \mathbf{B}\mathbf{A}^{-1} , \tag{2.140}$$

where

$$\mathbf{A} = \left\{ \mathbf{\Psi}^{(p)} \cdot (\mathbf{\Gamma}^{(w,p)})^{-1} + \mathbf{\Gamma}^{(p,w)} \cdot (\mathbf{\Omega}^{(w,w)})^{-1} \right\} , \tag{2.141}$$

and

$$\mathbf{B} = \left\{ \mathbf{\Lambda}^{(p)} \cdot (\mathbf{\Gamma}^{(w,p)})^{-1} + \mathbf{\Lambda}^{(w)} \cdot (\mathbf{\Omega}^{(w,w)})^{-1} \right\} . \tag{2.142}$$

Thus the infinitely conducting wall without any conductors has a somewhat simple solution. Now we return to the general case.

General Case

In general we can write the system of equations, eq.(2.135) as

$$
\begin{aligned}
\mathbf{V}^{(p)} \cdot \mathbf{\Psi}^{(p)} - \mathbf{V}^{(w)} \cdot \mathbf{\Gamma}^{(w,p)} - \mathbf{V}^{(c_1)} \cdot \mathbf{\Gamma}^{(c_1,p)} - \mathbf{V}^{(c_2)} \cdot \mathbf{\Gamma}^{(c_2,p)} + \cdots &= \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(p)} \\
\mathbf{V}^{(p)} \cdot \mathbf{\Gamma}^{(p,w)} + \mathbf{V}^{(w)} \cdot \mathbf{\Omega}^{(w)} - \mathbf{V}^{(c_1)} \cdot \mathbf{\Gamma}^{(c_1,w)} - \mathbf{V}^{(c_2)} \cdot \mathbf{\Gamma}^{(c_2,w)} + \cdots &= \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(w)} \\
\mathbf{V}^{(p)} \cdot \mathbf{\Gamma}^{(p,c_1)} - \mathbf{V}^{(w)} \cdot \mathbf{\Gamma}^{(w,c_1)} + \mathbf{V}^{(c_1)} \cdot \mathbf{C}_1 - \mathbf{V}^{(c_2)} \cdot \mathbf{\Gamma}^{(c_2,c_1)} - \cdots &= \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(c_1)} \\
&\vdots \\
\mathbf{V}^{(p)} \cdot \mathbf{\Gamma}^{(p,c_n)} - \mathbf{V}^{(w)} \cdot \mathbf{\Gamma}^{(w,c_n)} - \mathbf{V}^{(c_1)} \cdot \mathbf{\Gamma}^{(c_1,c_n} - \cdots + \mathbf{V}^{(c_n)} \cdot \mathbf{C}_n &= \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(c_n)}
\end{aligned}
\tag{2.143}
$$

where

$$\mathbf{\Psi}^{(p)} \stackrel{def}{=} \mathbf{I} + \mathbf{\Gamma}^{(p,p)}, \tag{2.144}$$

$$\mathbf{\Omega}^{(w)} \stackrel{def}{=} \mathbf{I} - \mathbf{\Gamma}^{(w,w)}, \tag{2.145}$$

$$\mathbf{C}_1 \stackrel{def}{=} \mathbf{I} - \mathbf{\Gamma}^{(c_1,c_1)}, \tag{2.146}$$

$$\vdots$$

$$\mathbf{C}_n \stackrel{def}{=} \mathbf{I} - \mathbf{\Gamma}^{(c_n,c_n)}. \tag{2.147}$$

This represents the solution to the general problem in closed form. Whether it is useful in the sense of being tractable in practical terms, has yet to be determined.

2.7 An Algorithm for Solving the Matrix System of Equations

For convenience we redefine the matrix coefficients in eq.(2.143) and write

$$
\begin{aligned}
\mathbf{V}_1 \cdot \mathbf{A}_{11}^{(0)} + \mathbf{V}_2 \cdot \mathbf{A}_{12}^{(0)} + \cdots + \mathbf{V}_m \cdot \mathbf{A}_{1m}^{(0)} &= \boldsymbol{\xi} \cdot \mathbf{B}_1^{(0)} \\
\mathbf{V}_1 \cdot \mathbf{A}_{21}^{(0)} + \mathbf{V}_2 \cdot \mathbf{A}_{22}^{(0)} + \cdots + \mathbf{V}_m \cdot \mathbf{A}_{2m}^{(0)} &= \boldsymbol{\xi} \cdot \mathbf{B}_2^{(0)} \\
&\vdots \\
\mathbf{V}_1 \cdot \mathbf{A}_{m1}^{(0)} + \mathbf{V}_2 \cdot \mathbf{A}_{m2}^{(0)} + \cdots + \mathbf{V}_m \cdot \mathbf{A}_{mm}^{(0)} &= \boldsymbol{\xi} \cdot \mathbf{B}_m^{(0)},
\end{aligned}
$$

where the connection to the coefficients of eq.(2.143) is straight-forward. First we multiply these equations from the right by the inverse of the last matrix coefficient in each equation, then take the first equation and subtract the second equation, then subtract the third equation from the first equation and so on. This way we eliminate the last variable $\mathbf{V}_m$, and we arrive at a reduced system of order $(m-1)\times(m-1)$, which we write as

$$\begin{aligned}
\mathbf{V}_1\cdot\mathbf{A}^{(1)}_{1\,1}+\mathbf{V}_2\cdot\mathbf{A}^{(1)}_{1\,2}+\quad\ldots\quad+\mathbf{V}_{m-1}\cdot\mathbf{A}^{(1)}_{1\,m-1} &= \boldsymbol{\xi}\cdot\mathbf{B}^{(1)}_1\\
\mathbf{V}_1\cdot\mathbf{A}^{(1)}_{2\,1}+\mathbf{V}_2\cdot\mathbf{A}^{(1)}_{2\,2}+\quad\ldots\quad+\mathbf{V}_{m-1}\cdot\mathbf{A}^{(1)}_{2\,m-1} &= \boldsymbol{\xi}\cdot\mathbf{B}^{(1)}_2\\
&\vdots\\
\mathbf{V}_1\cdot\mathbf{A}^{(1)}_{m-1\,1}+\mathbf{V}_2\cdot\mathbf{A}^{(1)}_{m-1\,2}+\ldots+\mathbf{V}_{m-1}\cdot\mathbf{A}^{(1)}_{m-1\,m-1} &= \boldsymbol{\xi}\cdot\mathbf{B}^{(1)}_{m-1}\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A}^{(1)}_{1\,1} &= \mathbf{A}^{(0)}_{1\,1}\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\mathbf{A}^{(0)}_{2\,1}\cdot\left(\mathbf{A}^{(0)}_{2\,m}\right)^{-1}\\
\mathbf{A}^{(1)}_{1\,2} &= \mathbf{A}^{(0)}_{1\,2}\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\mathbf{A}^{(0)}_{2\,2}\cdot\left(\mathbf{A}^{(0)}_{2\,m}\right)^{-1}\\
&\vdots\\
\mathbf{A}^{(1)}_{1\,m-1} &= \mathbf{A}^{(0)}_{1\,m-1}\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\mathbf{A}^{(0)}_{2\,m-1}\cdot\left(\mathbf{A}^{(0)}_{2\,m}\right)^{-1}\,,
\end{aligned}$$

and in general we have

$$\mathbf{A}^{(1)}_{k\,l}=\mathbf{A}^{(0)}_{1\,l}\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\mathbf{A}^{(0)}_{(k+1)\,l}\cdot\left(\mathbf{A}^{(0)}_{(k+1)\,m}\right)^{-1}\,. \tag{2.148}$$

For the left-hand side we find

$$\begin{aligned}
\boldsymbol{\xi}\cdot\mathbf{B}^{(1)}_1 &= \boldsymbol{\xi}\cdot\mathbf{B}^{(0)}_1\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\boldsymbol{\xi}\cdot\mathbf{B}^{(0)}_2\cdot\left(\mathbf{A}^{(0)}_{2\,m}\right)^{-1}\\
&\vdots\\
\boldsymbol{\xi}\cdot\mathbf{B}^{(1)}_k &= \boldsymbol{\xi}\cdot\mathbf{B}^{(0)}_1\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\boldsymbol{\xi}\cdot\mathbf{B}^{(0)}_{k+1}\cdot\left(\mathbf{A}^{(0)}_{k+1\,m}\right)^{-1}\,,
\end{aligned} \tag{2.149}$$

which determines $\mathbf{B}^{(1)}_k$ as

$$\mathbf{B}^{(1)}_k=\mathbf{B}^{(0)}_1\cdot\left(\mathbf{A}^{(0)}_{1\,m}\right)^{-1}-\mathbf{B}^{(0)}_{k+1}\cdot\left(\mathbf{A}^{(0)}_{k+1\,m}\right)^{-1}\,. \tag{2.150}$$

We continue the procedure by multiplying each equation again from the right by the inverse of the matrix coefficient of $\mathbf{V}_{m-1}$, and obtain

$$
\begin{aligned}
\mathbf{V}_1 \cdot \mathbf{A}^{(2)}_{11} + \mathbf{V}_2 \cdot \mathbf{A}^{(2)}_{12} + \quad \dots \quad + \mathbf{V}_{m-2} \cdot \mathbf{A}^{(2)}_{1\,m-2} &= \boldsymbol{\xi} \cdot \mathbf{B}^{(2)}_1 \\
\mathbf{V}_1 \cdot \mathbf{A}^{(2)}_{21} + \mathbf{V}_2 \cdot \mathbf{A}^{(2)}_{22} + \quad \dots \quad + \mathbf{V}_{m-2} \cdot \mathbf{A}^{(2)}_{2\,m-2} &= \boldsymbol{\xi} \cdot \mathbf{B}^{(2)}_2 \\
&\ \vdots \\
\mathbf{V}_1 \cdot \mathbf{A}^{(2)}_{m-2\,1} + \mathbf{V}_2 \cdot \mathbf{A}^{(2)}_{m-2\,2} + \dots + \mathbf{V}_{m-2} \cdot \mathbf{A}^{(2)}_{m-2\,m-2} &= \boldsymbol{\xi} \cdot \mathbf{B}^{(2)}_{m-2},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{A}^{(2)}_{11} &= \mathbf{A}^{(1)}_{11} \cdot \left(\mathbf{A}^{(1)}_{1\,m-1}\right)^{-1} - \mathbf{A}^{(1)}_{21} \cdot \left(\mathbf{A}^{(1)}_{2\,m-1}\right)^{-1} \\
&\ \vdots \\
\mathbf{A}^{(2)}_{kl} &= \mathbf{A}^{(1)}_{1l} \cdot \left(\mathbf{A}^{(1)}_{1\,m-1}\right)^{-1} - \mathbf{A}^{(1)}_{k+1\,l} \cdot \left(\mathbf{A}^{(1)}_{k+1\,m-1}\right)^{-1} \\
&\ \vdots \\
\mathbf{A}^{(2)}_{m-2\,l} &= \mathbf{A}^{(1)}_{1l} \cdot \left(\mathbf{A}^{(1)}_{1\,m-1}\right)^{-1} - \mathbf{A}^{(1)}_{m-1\,l} \cdot \left(\mathbf{A}^{(1)}_{m-1\,m-1}\right)^{-1}.
\end{aligned}
$$

The general recurrence relation is given by

$$
\mathbf{A}^{(\alpha+1)}_{kl} = \mathbf{A}^{(\alpha)}_{1l} \cdot \left(\mathbf{A}^{(\alpha)}_{1\,m-\alpha}\right)^{-1} - \mathbf{A}^{(\alpha)}_{k+1\,l} \cdot \left(\mathbf{A}^{(\alpha)}_{k+1\,m-\alpha}\right)^{-1}. \tag{2.151}
$$

And for the left-hand side we obtain

$$
\boldsymbol{\xi} \cdot \mathbf{B}^{(\alpha+1)}_k = \boldsymbol{\xi} \cdot \mathbf{B}^{(\alpha)}_1 \cdot \left(\mathbf{A}^{(\alpha)}_{1\,m-\alpha}\right)^{-1} - \boldsymbol{\xi} \cdot \mathbf{B}^{(\alpha)}_{k+1} \cdot \left(\mathbf{A}^{(\alpha)}_{k+1\,m-\alpha}\right)^{-1}, \tag{2.152}
$$

or

$$
\mathbf{B}^{(\alpha+1)}_k = \mathbf{B}^{(\alpha)}_1 \cdot \left(\mathbf{A}^{(\alpha)}_{1\,m-\alpha}\right)^{-1} - \mathbf{B}^{(\alpha)}_{k+1} \cdot \left(\mathbf{A}^{(\alpha)}_{k+1\,m-\alpha}\right)^{-1}. \tag{2.153}
$$

Notice that $k,\ l < m - \alpha$. Thus, having carried out the iteration to the point where $\alpha = m - 2$, we have the final result

$$
\mathbf{V}_1 \cdot \mathbf{A}^{(m-1)}_{11} = \boldsymbol{\xi} \cdot \mathbf{B}^{(m-1)}_1, \tag{2.154}
$$

$$
\mathbf{V}_1 = \boldsymbol{\xi} \cdot \mathbf{B}^{(m-1)}_1 \cdot \left(\mathbf{A}^{(m-1)}_{11}\right)^{-1} = \boldsymbol{\xi} \cdot \boldsymbol{\Gamma}, \tag{2.155}
$$

and

$$\Gamma = \mathbf{B}_1^{(m-1)} \cdot \left(\mathbf{A}_{1\,1}^{(m-1)}\right)^{-1} . \tag{2.156}$$

$\mathbf{A}_{k\,l}^{(0)}$ and $\boldsymbol{\xi} \cdot \mathbf{B}_k^{(0)}$, where $k \leq m$ and $l \leq m$ are given, and from the knowledge of these quantities we determine $\mathbf{A}_{k\,l}^{(\alpha)}$ and $\mathbf{B}_k^{(\alpha)}$ recursively by eqs.(2.151) and (2.153), and the problem in principle is solved.

CPU - time Estimates

According to current available information on MF-Cray systems the cpu time for a typical matrix inversion of a 64 × 64 matrix is less than a millisecond, addition of two numbers is one instruction set and takes of the order 10^{-8}s, regarding multiplications we make an estimate and consider that to be equivalent to 5 instruction sets.

We can then make the following table for the number of operations envolved

Iteration	Inversions	multiplications	additions
Iteration no. 1	m	$2(m-1)^2$	$(m-1)^2$
Iteration no. 2	$m-1$	$2(m-2)^2$	$(m-2)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$ iteration	1	$2 \cdot 1$	1
Sum	$\frac{m(m+1)}{2}$	$2\frac{m(m-1)(2m-1)}{6}$	$\frac{m(m-1)(2m-1)}{6}$

If we consider a case of 8 conductors, we have $m = 10$, and for the total amount of cputime we obtain

$$55 \cdot 10^{-3} + 28.5 \cdot 10^{-6} + 28.5 \cdot 10^{-7}\, s \sim 0.06 s .$$

From this little exercise we conclude that the most time consuming operation by far is the inversion of matrices. The other operations are negligible in comparison.

2.8 The Vacuum Energy δW_V^b

In principle we have now obtained $\mathbf{V}_1$ at the plasma vacuum interface given by eq.(2.155). We are therefore ready to compute the vacuum contribution to δW, δW_V^b, in the presence of an axisymmetric infinitely conducting wall and with an arbitrary number of circular cross section conductors

$$\frac{1}{2\mu_0}\int_{S_p} dSV^*\mathbf{n}\cdot\nabla V\,. \tag{2.157}$$

We start by listing the main results obtained. From eq.(2.35) we have

$$\mathbf{n}\cdot\nabla V(\mathbf{r}) = \frac{A_n(v)}{\epsilon RQ}e^{in\phi}\,,$$

and as in eq.(2.115) we represent

$$A_n(v) = iR_0B_0\sum_{k,l}\{2\pi(k+l)\bar{B}_l + n\bar{B}_l\}\bar{\xi}_k e^{2\pi i(k+l)v}\,,$$

and from eq.(2.107)

$$V_n(v) = iR_0B_0e^{in\phi}\sum_{m=-\infty}^{\infty}\bar{V}_m e^{2\pi imv}\,.$$

From eq.(2.15) we have

$$dS = Rd\phi dl_p = \epsilon RQR_0d\phi dv\,.$$

We substitute from these relations in eq.(2.157) to obtain

$$\begin{aligned}
\delta W_V^b &= \frac{1}{2\mu_0}\int_{S_p}\epsilon R_0RQd\phi dv\frac{A_n(v)}{\epsilon RQ}e^{in\phi}(iR_0B_0e^{in\phi}\sum_{m=-\infty}^{\infty}\bar{V}_m e^{2\pi imv})^* \\
&= \frac{\pi R_0^2B_0^2}{\mu_0}\sum_{k,l,m}\left((k+l)\bar{B}_l + \frac{n}{2\pi}\bar{B}_l\right)\bar{\xi}_k\bar{V}_m^*\delta_{(k+l-m)0} \\
&= \frac{\pi R_0^2B_0^2}{\mu_0}\sum_{l,m}\left(m\bar{B}_l + \frac{n}{2\pi}\bar{B}_l\right)\bar{\xi}_{m-l}\bar{V}_m^*\,.
\end{aligned}$$

Let $m - l = k \quad \longrightarrow \quad l = m - k$, and we obtain

$$\delta W_V^b = \frac{\pi R_0^2 B_0^2}{\mu_0} \sum_{k,m} \left(m\tilde{B}_{m-k} + \frac{n}{2\pi} \bar{B}_{m-k} \right) \bar{\xi}_k V_m^{\prime *} . \tag{2.158}$$

It is now convenient to define a matrix $\mathbf{B}$ with elements B_{mk} as

$$B_{mk} = \frac{\pi R_0^2 B_0^2}{\mu_0} \left(m\tilde{B}_{m-k} + \frac{n}{2\pi} \bar{B}_{m-k} \right) . \tag{2.159}$$

This way we may write

$$\delta W_V^b = \mathbf{V}^* \cdot \mathbf{B} \cdot \boldsymbol{\xi} = \boldsymbol{\xi}^* \cdot \boldsymbol{\Gamma}^* \cdot \mathbf{B} \cdot \boldsymbol{\xi} \stackrel{def}{=} \boldsymbol{\xi}^* \cdot \mathbf{W}_V^b \cdot \boldsymbol{\xi} , \tag{2.160}$$

where we have used eq.(2.155) in the last step, and

$$\mathbf{W}_V^b \stackrel{def}{=} \boldsymbol{\Gamma}^* \cdot \mathbf{B} . \tag{2.161}$$

This finally determines the vacuum part of the perturbation in the energy for the case of a conducting wall and conductors in the vacuum region, and $\boldsymbol{\Gamma}$ is given by eq.(2.156).

2.9 Numerical Scheme

In order to apply eqs.(1.93) and (1.94) there are two main obstacles. First, one has to determine δW_b, i.e., the perturbation in energy with an infinitely conducting wall and conductors present. Second, one has to determine the energy integral over the interior of the conductors. In this Chapter we summarize the results for numerical computation.

2.9.1 Numerical Evaluation of δW_b

The vacuum-part of the perturbation in energy is given by

$$\delta W_V^b \stackrel{def}{=\!=} \boldsymbol{\xi}^* \cdot \mathbf{W}_V^b \cdot \boldsymbol{\xi}\,, \tag{2.162}$$

$$\mathbf{W}_V^b \stackrel{def}{=} \boldsymbol{\Gamma}^* \cdot \mathbf{B}\,. \tag{2.163}$$

B− Matrix

For this case the surface is restricted to the plasma-vacuum interface, thus $v, v' \in S_p$.

$$\begin{aligned}
[1+\epsilon x(v')]\hat{B}_p(v') &= B_0 \sum_l \tilde{B}_l e^{2\pi i l v'}\,,\\
\epsilon Q(v')\hat{B}_\phi(v') &= B_0 \sum_m \bar{B}_m e^{2\pi i m v'}\,,
\end{aligned}$$

$$B_{mk} = \frac{\pi R_0^2 B_0^2}{\mu_0}\left(m\tilde{B}_{m-k} + \frac{n}{2\pi}\bar{B}_{m-k}\right). \tag{2.164}$$

This determines the matrix **B**.

2.9.2 Γ-Matrices

Functions and Recurrency Relations

$$\alpha = \frac{R'^2 + R^2 + (Z'-Z)^2}{2R'R}\,,$$

$$k = \alpha - \sqrt{\alpha^2 - 1} < 1\,,$$

$$h = \sqrt{\frac{RR'}{k}}\,,$$

$$\bar{K}(k) = K(k) + \frac{1}{2}\ln|\sin\pi(v - v')|$$

$$\begin{aligned} M_0(k) &= K(k)\,, && (2.165)\\ \bar{M}_0(k) &= \bar{K}(k)\,, && (2.166)\\ M_2(k) &= \frac{1}{2k}\{E(k) - (1-k)K(k)\}\,, && (2.167)\\ M_0'(k) &= \frac{1}{k}\left\{\frac{E(k)}{(1-k^2)} - K(k)\right\}\,, && (2.168)\\ M_2'(k) &= -\frac{1}{2k^2}\left\{\frac{E(k)}{1+k} - (1-k)K(k)\right\}\,. && (2.169) \end{aligned}$$

$$L_{2n}(k) = 2^{2n-1}M_{2n}(k) + \sum_{m=1}^{n}(-1)^m\frac{n}{m}\binom{2n-(m+1)}{m-1}2^{2n-2m}M_{2n-2m}(k)\,. \quad (2.170)$$

$$L'_{2n}(k) = 2^{2n-1}M'_{2n}(k) + \sum_{m=1}^{n}(-1)^m\frac{n}{m}\binom{2n-(m+1)}{m-1}2^{2n-2m}M'_{2n-2m}(k)\,. \quad (2.171)$$

$$\bar{L}_{2n}(v, v') = L_{2n}(v, v') + \frac{1}{2}\ln|\sin\pi(v' - v)|$$

$$M_{2n}(k) = \frac{2n-2}{2n-1}f(k)M_{2n-2}(k) + \frac{2n-3}{2n-1}g(k)M_{2n-4}\,, \quad (2.172)$$

$$\begin{aligned} M_{2n}'(k) &= \frac{2n-2}{2n-1}f(k)M_{2n-2}'(k) + \frac{2n-3}{2n-1}g(k)M_{2n-4}' \\ &+ \frac{2n-2}{2n-1}f'(k)M_{2n-2}(k) + \frac{2n-3}{2n-1}g'(k)M_{2n-4}\,, \quad (2.173) \end{aligned}$$

$$f(k) = \frac{-1+6k-k^2}{4k}, \qquad f'(k) = \frac{1-k^2}{4k^2}, \tag{2.174}$$

$$g(k) = \frac{(1-k)^2}{4k}, \qquad g'(k) = -\frac{1-k^2}{4k^2}. \tag{2.175}$$

$$M_4(k) = \frac{1}{3k^2}\left\{f_4(k)E(k) + g_4(k)(1-k)K(k)\right\},$$

$$f_4(k) = \frac{1}{4}\{-1+6k-k^2\}, \qquad g_4(k) = \frac{1}{4}\{1-5k\},$$

$$M_4'(k) = \frac{1}{3k^3}\left\{\hat{f}_4(k)E(k) + \hat{g}_4(k)(1-k)K(k)\right\},$$

$$\hat{f}_4(k) = \frac{2-4k-k^2-k^3}{4(1+k)}, \qquad \hat{g}_4(k) = k - \frac{1}{2},$$

Integrals

The basic elements we need to compute the $\mathbf{\Gamma}^{(S_i,S_j)}$ matrices are

$$\bar{I}_n \stackrel{def}{=} (-1)^n \frac{R_0}{h} \bar{L}_{2n}(v,v'), \qquad \bar{\bar{I}}_n \stackrel{def}{=} -\frac{R_0}{2h} \ln|\sin\pi(v'-v)|, \tag{2.176}$$

$$\hat{I}_n = (-1)^n \frac{R_0^3}{h^3(1-k^2)}\{2kL'_{2n}(k) + L_{2n}(k)\}, \tag{2.177}$$

Fourier transforms

$$F^{(S_i,S_j)}(v,v') = \frac{\epsilon}{R_0^2}\left\{\dot{y}(v')[R'^2 - R^2 - (Z-Z')^2] - 2R'\dot{x}(v')(Z'-Z)\right\}\Bigg|_{\substack{R,Z\in S_i \\ R',Z'\in S_j}}$$

$$\frac{\epsilon}{2\pi}\dot{y}(v')\bar{I}_n(v,v') = \sum_{k,m} H_{km} e^{2\pi i(kv'+mv)}, \tag{2.178}$$

$$\frac{1}{2\pi}F^{(S_i,S_j)}(v,v')\hat{I}_n(v,v') = \sum_{k,m} F_{km} e^{2\pi i(kv'+mv)}, \tag{2.179}$$

$$V_n(v') = iR_0B_0\sum_l \bar{V}_l e^{2\pi i l v'} . \tag{2.180}$$

$$-\frac{R_0}{4\pi h(v,v')}\epsilon\dot{y}(v') = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} Y_{k\,l}e^{2\pi i(kv'+lv)} . \tag{2.181}$$

$$f(q) \stackrel{def}{=} \begin{cases} -\frac{1}{2q} & (q \neq 0) \\ -\ln 2 & (q = 0) \end{cases} . \tag{2.182}$$

$$\Gamma_{m\,l} = 2\{H_{-m\,l} + F_{-m\,l} + \sum_k f(m+k)Y_{k\,(l-k-m)}\} , \tag{2.183}$$

Matrices

In the following the superscripts (α,β) means $v \in S_\alpha$, $v' \in S_\beta$. Exsample: $\mathbf{\Gamma}^{(c_1,w)}$ means that $v \in S_{c_1}$ (conductor surface S_{c_1}) and $v' \in S_w$ (wall surface). With the apropriate specification of surfaces eq.(2.183) provides the means of determining the following matrices:

$$\begin{array}{lllll}
\mathbf{\Gamma}^{(p,p)} = \quad , & \mathbf{\Gamma}^{(w,p)} = \quad , & \mathbf{\Gamma}^{(c_1,p)} = \quad , & \mathbf{\Gamma}^{(c_2,p)} = \quad , & \cdots \\
\mathbf{\Gamma}^{(p,w)} = \quad , & \mathbf{\Gamma}^{(w,w)} = \quad , & \mathbf{\Gamma}^{(c_1,w)} = \quad , & \mathbf{\Gamma}^{(c_2,w)} = \quad , & \cdots \\
\mathbf{\Gamma}^{(p,c_1)} = \quad , & \mathbf{\Gamma}^{(w,c_1)} = \quad , & \mathbf{\Gamma}^{(c_1,c_1)} = \quad , & \mathbf{\Gamma}^{(c_2,c_1)} = \quad , & \cdots \\
\mathbf{\Gamma}^{(p,c_2)} = \quad , & \mathbf{\Gamma}^{(w,c_2)} = \quad , & \mathbf{\Gamma}^{(c_1,c_2)} = \quad , & \mathbf{\Gamma}^{(c_2,c_2)} = \quad , & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

The next step is to use the information above to determine the following set of matrices:

$$\begin{array}{lll}
\mathbf{A}_{11}^{(0)} = \mathbf{I} + \mathbf{\Gamma}^{(p,p)} , & \mathbf{A}_{21}^{(0)} = \mathbf{\Gamma}^{(p,w)} , & \mathbf{A}_{31}^{(0)} = \mathbf{\Gamma}^{(p,c_1)} , \\
\mathbf{A}_{12}^{(0)} = \mathbf{\Gamma}^{(w,p)} , & \mathbf{A}_{22}^{(0)} = \mathbf{I} - \mathbf{\Gamma}^{(w,w)} , & \mathbf{A}_{32}^{(0)} = -\mathbf{\Gamma}^{(w,c_1)} , \\
\mathbf{A}_{13}^{(0)} = \mathbf{\Gamma}^{(c_1,p)} , & \mathbf{A}_{23}^{(0)} = -\mathbf{\Gamma}^{(c_1,w)} , & \mathbf{A}_{33}^{(0)} = \mathbf{I} - \mathbf{\Gamma}^{(c_1,c_1)} , \\
\mathbf{A}_{14}^{(0)} = \mathbf{\Gamma}^{(c_2,p)} , & \cdots & \cdots \\
\vdots & \vdots & \vdots \\
\mathbf{A}_{1m}^{(0)} = \mathbf{\Gamma}^{(c_l,p)} , & \mathbf{A}_{2m}^{(0)} = -\mathbf{\Gamma}^{(c_l,w)} , & \mathbf{A}_{3m}^{(0)} = -\mathbf{\Gamma}^{(c_l,c_1)} ,
\end{array}$$

$$
\begin{aligned}
&\mathbf{A}_{41}^{(0)} = \mathbf{\Gamma}^{(p,c_2)}\,, && \mathbf{A}_{51}^{(0)} = \mathbf{\Gamma}^{(p,c_3)}\,,\\
&\mathbf{A}_{42}^{(0)} = -\mathbf{\Gamma}^{(w,c_2)}\,, && \mathbf{A}_{52}^{(0)} = -\mathbf{\Gamma}^{(w,c_3)}\,,\\
&\mathbf{A}_{43}^{(0)} = -\mathbf{I} - \mathbf{\Gamma}^{(c_1,c_2)}\,, && \mathbf{A}_{53}^{(0)} = -\mathbf{\Gamma}^{(c_1,c_3)}\,,\\
&\mathbf{A}_{44}^{(0)} = \mathbf{I} - \mathbf{\Gamma}^{(c_2,c_2)}\,, && \mathbf{A}_{54}^{(0)} = \mathbf{\Gamma}^{(c_2,c_3)}\,,\\
&\mathbf{A}_{45}^{(0)} = -\mathbf{\Gamma}^{(c_3,c_2)}\,, && \mathbf{A}_{55}^{(0)} = -\mathbf{I} - \mathbf{\Gamma}^{(c_3,c_3)}\,,\\
&\vdots && \vdots\\
&\mathbf{A}_{4m}^{(0)} = --\mathbf{\Gamma}^{(c_l,c_2)}\,, && \mathbf{A}_{5m}^{(0)} = -\mathbf{\Gamma}^{(c_l,c_3)}\,,
\end{aligned}
$$

Given the matrices above we are now ready to start the iteration using the following formulas

$$
\mathbf{A}_{k\,l}^{(\alpha+1)} = \mathbf{A}_{1\,l}^{(\alpha)} \cdot \left(\mathbf{A}_{1\,m-\alpha}^{(\alpha)}\right)^{-1} - \mathbf{A}_{k+1\,l}^{(\alpha)} \cdot \left(\mathbf{A}_{k+1\,m-\alpha}^{(\alpha)}\right)^{-1}. \tag{2.184}
$$

$$
\mathbf{B}_{k}^{(\alpha+1)} = \mathbf{B}_{1}^{(\alpha)} \cdot \left(\mathbf{A}_{1\,m-\alpha}^{(\alpha)}\right)^{-1} - \mathbf{B}_{k+1}^{(\alpha)} \cdot \left(\mathbf{A}_{k+1\,m-\alpha}^{(\alpha)}\right)^{-1}. \tag{2.185}
$$

Notice that $k,\ l < m - \alpha$. Thus, having carried out the iteration to the point where $\alpha = m - 2$, we have the final result

$$
\mathbf{V}_1 \cdot \mathbf{A}_{1\,1}^{(m-1)} = \boldsymbol{\xi} \cdot \mathbf{B}_{1}^{(m-1)}\,, \tag{2.186}
$$

$$
\mathbf{V}_1 = \boldsymbol{\xi} \cdot \mathbf{B}_{1}^{(m-1)} \cdot \left(\mathbf{A}_{1\,1}^{(m-1)}\right)^{-1} = \boldsymbol{\xi} \cdot \mathbf{\Gamma}\,, \tag{2.187}
$$

$$
\mathbf{\Gamma} = \mathbf{B}_{1}^{(m-1)} \cdot \left(\mathbf{A}_{1\,1}^{(m-1)}\right)^{-1}. \tag{2.188}
$$

2.9.3 Conductor Integral

$$
\mathbf{A} = \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(p)} - \mathbf{U}^{(p)} \cdot \mathbf{\Lambda}^{(p,c)}\,,
$$

$$
\mathbf{B} = \left\{\{\mathbf{\Gamma}^{(c,p)} \cdot (\mathbf{\Lambda}^{(c,p)})^{-1}\} \cdot \{\mathbf{I} - \mathbf{\Lambda}^{(c,c)}\} - \mathbf{\Gamma}^{(c,c)}\right\}^{-1}.
$$

$$
\hat{\mathbf{U}}^{(c)} = \mathbf{A} \cdot \mathbf{B}^{-1}
$$

$$k_0^2 = \frac{1}{R_c^2}(n^2 - \alpha R_c^2) > 0\,. \tag{2.189}$$

$$x_0 = k_0 r_c = \frac{r_c}{R_c}\sqrt{n^2 - \alpha R_c^2}\,,$$

$$c_m = -\frac{f_m}{2kr_c}\frac{1}{I'_m(k_0 r_c)}\,,$$

Notice that the components of the vector $\hat{\mathbf{U}}^{(c)}$ is f_m. Thus c_m is determined in terms of these components.

$$\begin{aligned}
\hat{C}_{lm} &\stackrel{def}{=} \int_0^{x_0} I_l(x) I_m(x) x dx\,,\\
C_{lm} &\stackrel{def}{=} \hat{C}_{l+1\,m+1} + \hat{C}_{l-1\,m-1}\,,\\
\mathbf{c} &\stackrel{def}{=} \{c_1 \cdots c_n\}\,.
\end{aligned}$$

Then we may write

$$\int_{V_c} \left(|A_r|^2 + |A_\theta|^2\right) Rr d\theta d\phi dr = \frac{8\pi^2 R_c}{k_0^2}\mathbf{c}\cdot\mathbf{C}\cdot\mathbf{c}^t\,. \tag{2.190}$$

Alternative approximate evaluaton of the same integral;

$$\begin{aligned}
\int_0^{r_c} A_r^2 r dr + \int_0^{r_c} \hat{A}_\theta^2 r dr &\sim 2\int_0^{r_c}\left(\sum_m \frac{f_m}{kr_c}\left(\frac{r}{r_c}\right)^{|m|-1}\right)^2 r dr\\
&= \sum_{m,n}\frac{2f_m f_n}{k^2}\frac{1}{|m|+|n|}\,.
\end{aligned}$$

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Appendix A

Variation of δW_F

In this section we shall compute the variation in δW_F

$$\delta W_F = \frac{1}{2}\int_{V_p} dV\{\frac{\mathbf{Q}^2}{\mu_0} + \gamma p(\nabla\cdot\boldsymbol{\xi})^2 - \boldsymbol{\xi}\cdot\mathbf{J}\times\mathbf{Q} + \boldsymbol{\xi}\cdot\nabla p\nabla\cdot\boldsymbol{\xi}\}dS\,, \qquad (A.1)$$

$$\begin{aligned}
\delta(\delta W_F) &= \int_{V_p} dV\{\frac{\mathbf{Q}\cdot\delta\mathbf{Q}}{\mu_0} + \gamma p(\nabla\cdot\boldsymbol{\xi})\nabla\cdot\delta\boldsymbol{\xi} \\
&\quad -\frac{1}{2}\delta\boldsymbol{\xi}\cdot\mathbf{J}\times\mathbf{Q} - \frac{1}{2}\boldsymbol{\xi}\cdot\mathbf{J}\times\delta\mathbf{Q} \\
&\quad +\frac{1}{2}\delta\boldsymbol{\xi}\cdot\nabla p\nabla\cdot\boldsymbol{\xi} + \frac{1}{2}\delta\boldsymbol{\xi}\cdot\nabla p\nabla\cdot\delta\boldsymbol{\xi}\}dS\,.
\end{aligned}$$

We find

$$\begin{aligned}
\frac{1}{\mu_0}\mathbf{Q}\cdot\delta\mathbf{Q} &= \frac{1}{\mu_0}\mathbf{Q}\cdot\nabla\times(\delta\boldsymbol{\xi}\times\mathbf{B}) \\
&= \frac{1}{\mu_0}\nabla\cdot\{(\delta\boldsymbol{\xi}\times\mathbf{B})\times\mathbf{Q}\} + \frac{1}{\mu_0}\delta\boldsymbol{\xi}\cdot\mathbf{B}\times(\nabla\times\mathbf{Q})\,,
\end{aligned}$$

and

$$\gamma p\nabla\cdot\boldsymbol{\xi}\,\nabla\cdot\delta\boldsymbol{\xi} = \nabla\cdot\{\delta\boldsymbol{\xi}\gamma p\nabla\cdot\boldsymbol{\xi}\} - \delta\boldsymbol{\xi}\cdot\nabla\{\gamma p\nabla\cdot\boldsymbol{\xi}\}\,,$$

$$-\frac{1}{2}\boldsymbol{\xi}\cdot\mathbf{J}\times\delta\mathbf{Q} = -\frac{1}{2\mu_0}\boldsymbol{\xi}\cdot(\nabla\times\mathbf{B})\times\delta\mathbf{Q}$$

$$
\begin{aligned}
= \; & -\frac{1}{2\mu_0}\nabla\cdot\{\delta\mathbf{Q}\,\boldsymbol{\xi}\cdot\mathbf{B} - \boldsymbol{\Phi}\cdot\mathbf{v}\} \\
& -\delta\boldsymbol{\xi}\cdot\{\frac{1}{2}\nabla(\boldsymbol{\xi}\cdot\nabla p) - \frac{1}{2}\nabla\mathbf{B}\cdot(\boldsymbol{\xi}\times\mathbf{J}) \\
& +\frac{1}{2}\mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\xi}\times(\mathbf{B}\cdot\nabla)\mathbf{J}\}, \qquad \text{(A.2)}
\end{aligned}
$$

Here

$$
\boldsymbol{\Phi} \stackrel{def}{=} \mathbf{B}\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi}\mathbf{B}, \qquad \text{is a dyadic}
$$

and

$$
\mathbf{v} = \boldsymbol{\xi}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\boldsymbol{\xi} + \mathbf{B}\times(\nabla\times\boldsymbol{\xi}) = \nabla(\mathbf{B}\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\times(\nabla\times\mathbf{B}).
$$

Details on this step can be found in the last section of this appendix. We also have

$$
\frac{1}{2}\boldsymbol{\xi}\cdot\nabla p\,\nabla\cdot\delta\boldsymbol{\xi} = \frac{1}{2}\nabla\cdot\{\delta\boldsymbol{\xi}\,\boldsymbol{\xi}\cdot\nabla p\} - \frac{1}{2}\delta\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla p).
$$

We collect the expanded terms and find

$$
\begin{aligned}
\delta(\delta W_F) \; = \; & \int_{V_p} dV \Bigg[\delta\boldsymbol{\xi}\cdot\Bigg\{\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{Q}) - \nabla(\gamma p\nabla\cdot\boldsymbol{\xi}) \\
& -\frac{1}{2}\mathbf{J}\times\mathbf{Q} - \frac{1}{2}\nabla(\boldsymbol{\xi}\cdot\nabla p) + \frac{1}{2}\nabla\mathbf{B}\cdot(\boldsymbol{\xi}\times\mathbf{J}) \\
& -\frac{1}{2}\mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\xi}\times(\mathbf{B}\cdot\nabla)\mathbf{J} \\
& +\frac{1}{2}\nabla p\,\nabla\cdot\boldsymbol{\xi} - \frac{1}{2}\nabla(\boldsymbol{\xi}\cdot\nabla p)\Bigg\} \\
& +\nabla\cdot\Bigg\{\frac{1}{\mu_0}(\delta\boldsymbol{\xi}\times\mathbf{B})\times\mathbf{Q} + \delta\boldsymbol{\xi}\gamma p\nabla\cdot\boldsymbol{\xi} \\
& \quad -\frac{1}{2\mu_0}\delta\mathbf{Q}\boldsymbol{\xi}\cdot\mathbf{B} + \frac{1}{2\mu_0}\boldsymbol{\Phi}\cdot\mathbf{v} + \frac{1}{2}\delta\boldsymbol{\xi}\boldsymbol{\xi}\cdot\nabla p\Bigg\}\Bigg] \\
= \; & \int_{V_p} dV \Bigg[\delta\boldsymbol{\xi}\cdot\Bigg\{-\frac{1}{\mu_0}(\nabla\times\mathbf{Q})\times\mathbf{B} - \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{Q} \\
& -\nabla\{\gamma p\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\mu_0}\mathbf{Q}\times(\nabla\times\mathbf{B})+\frac{1}{2}\nabla p\nabla\cdot\boldsymbol{\xi}+\frac{1}{2}\nabla\mathbf{B}\cdot(\boldsymbol{\xi}\times\mathbf{J})\\
&-\frac{1}{2}\mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}+\frac{1}{2}\boldsymbol{\xi}\times(\mathbf{B}\cdot\nabla)\mathbf{J}\Big\}\\
&+\nabla\cdot\Bigg\{\frac{1}{\mu_0}(\delta\boldsymbol{\xi}\times\mathbf{B})\times\mathbf{Q}+\delta\boldsymbol{\xi}\gamma p\nabla\cdot\boldsymbol{\xi}\\
&\quad-\frac{1}{2\mu_0}\delta\mathbf{Q}\boldsymbol{\xi}\cdot\mathbf{B}+\frac{1}{2\mu_0}\boldsymbol{\Phi}\cdot\mathbf{v}+\frac{1}{2}\delta\boldsymbol{\xi}\boldsymbol{\xi}\cdot\nabla p\Bigg\}\Bigg]\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
-\frac{1}{\mu_0}\mathbf{Q}\times(\nabla\times\mathbf{B}) &= \mathbf{J}\times\mathbf{Q}=\mathbf{J}\times\{\nabla\times(\boldsymbol{\xi}\times\mathbf{B})\}\\
&= \mathbf{J}\times\{\mathbf{B}\cdot\nabla\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla\mathbf{B}+\boldsymbol{\xi}\nabla\cdot\mathbf{B}-\mathbf{B}\nabla\cdot\boldsymbol{\xi}\}\\
&= \mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}-\mathbf{J}\times(\boldsymbol{\xi}\cdot\nabla)\mathbf{B}-\mathbf{J}\times\mathbf{B}\nabla\cdot\boldsymbol{\xi}\,.
\end{aligned}
$$

Since

$$\nabla p=\mathbf{J}\times\mathbf{B}\,,$$

we also have

$$-\frac{1}{2\mu_0}\mathbf{Q}\times(\nabla\times\mathbf{B})+\frac{1}{2}\nabla p\nabla\cdot\boldsymbol{\xi}=\frac{1}{2}\mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}-\frac{1}{2}\mathbf{J}\times(\boldsymbol{\xi}\cdot\nabla)\mathbf{B}\,,$$

and also

$$\mathbf{F}(\boldsymbol{\xi})\stackrel{def}{=}-\frac{1}{\mu_0}\mathbf{Q}\times(\nabla\times\mathbf{B})-\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{Q})+\nabla\{\boldsymbol{\xi}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{\xi}\}\,.$$

We may now write

$$
\begin{aligned}
\delta(\delta W_F) \;=\; \int_{V_p} dV \Bigg[\quad & \delta\boldsymbol{\xi}\cdot\{\, -\mathbf{F}(\boldsymbol{\xi}) \\
& +\frac{1}{2}\mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi} - \frac{1}{2}\mathbf{J}\times(\boldsymbol{\xi}\cdot\nabla)\mathbf{B} \\
& +\frac{1}{2}(\nabla\mathbf{B})\cdot\boldsymbol{\xi}\times\mathbf{J} - \frac{1}{2}\mathbf{J}\times(\mathbf{B}\cdot\nabla)\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\xi}\times(\mathbf{B}\cdot\nabla)\mathbf{J}\Big\} \\
& +\nabla\cdot\Big\{\frac{1}{\mu_0}(\delta\boldsymbol{\xi}\times\mathbf{B})\times\mathbf{Q} + \delta\boldsymbol{\xi}\gamma p\nabla\cdot\boldsymbol{\xi} \\
& \quad -\frac{1}{2\mu_0}\delta\mathbf{Q}\boldsymbol{\xi}\cdot\mathbf{B} + \frac{1}{2\mu_0}\boldsymbol{\Phi}\cdot\mathbf{v} + \frac{1}{2}\delta\boldsymbol{\xi}\boldsymbol{\xi}\cdot\nabla p\Big\}\Bigg] .
\end{aligned}
$$

Notice also that

$$
\frac{1}{2}(\nabla\mathbf{B})\cdot\boldsymbol{\xi}\times\mathbf{J} + \frac{1}{2}\boldsymbol{\xi}\times(\mathbf{J}\cdot\nabla)\mathbf{B} - \frac{1}{2}\mathbf{J}\times(\boldsymbol{\xi}\cdot\nabla)\mathbf{B} \equiv 0 .
$$

A proof of this is provided elsewhere [4]. Since $\mathbf{B}\cdot\nabla\mathbf{J} = \mathbf{J}\cdot\nabla\mathbf{B}$ we obtain

$$
\delta(\delta W_F) = -\int_{V_p} dV\delta\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}) + \int_{S_p}\mathbf{n}\cdot\mathbf{T}, \tag{A.3}
$$

where

$$
\begin{aligned}
\mathbf{T} \;\stackrel{def}{=}\; & \frac{1}{\mu_0}(\delta\boldsymbol{\xi}\times\mathbf{B})\times\mathbf{Q} + \delta\boldsymbol{\xi}\gamma p\nabla\cdot\boldsymbol{\xi} \\
& \quad -\frac{1}{2\mu_0}\delta\mathbf{Q}\boldsymbol{\xi}\cdot\mathbf{B} + \frac{1}{2\mu_0}\boldsymbol{\Phi}\cdot\mathbf{v} + \frac{1}{2}\delta\boldsymbol{\xi}\boldsymbol{\xi}\cdot\nabla p .
\end{aligned}
$$

We need a few more vector identities, which we are now providing

$$
\begin{aligned}
\mathbf{n}\cdot(\delta\boldsymbol{\xi}\times\mathbf{B})\times\mathbf{Q} &= -\mathbf{n}\cdot\delta\boldsymbol{\xi}\,\mathbf{B}\cdot\mathbf{Q}, \\
\mathbf{n}\cdot\delta\mathbf{Q} &= \mathbf{n}\cdot\nabla\times(\delta\boldsymbol{\xi}\times\mathbf{B}) = -\nabla_s\cdot\mathbf{V}, \\
\mathbf{V}\stackrel{def}{=}\mathbf{n}\times(\delta\boldsymbol{\xi}\times\mathbf{B}) &= -\mathbf{B}\mathbf{n}\cdot\delta\boldsymbol{\xi}, \\
\mathbf{n}\cdot\delta\mathbf{Q}\,\boldsymbol{\xi}\cdot\mathbf{B} &= -(\nabla_s\cdot\mathbf{V})\,\boldsymbol{\xi}\cdot\mathbf{B} = -\nabla_s\cdot\{\boldsymbol{\xi}\cdot\mathbf{B}\,\mathbf{V}\} + \mathbf{V}\cdot\nabla_s\{\boldsymbol{\xi}\cdot\mathbf{B}\}, \\
\mathbf{V}\cdot\nabla_s\{\boldsymbol{\xi}\cdot\mathbf{B}\} &= -\mathbf{n}\cdot\delta\boldsymbol{\xi}\,\mathbf{B}\cdot\nabla\{\boldsymbol{\xi}\cdot\mathbf{B}\} \\
&= -\mathbf{n}\cdot\delta\boldsymbol{\xi}\,\mathbf{B}\cdot\{\boldsymbol{\xi}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\boldsymbol{\xi} + \boldsymbol{\xi}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\boldsymbol{\xi})\} \\
&= -\mathbf{n}\cdot\delta\boldsymbol{\xi}\{\mathbf{B}\,\boldsymbol{\xi}:\nabla\mathbf{B} + \mathbf{B}\,\mathbf{B}:\nabla\boldsymbol{\xi}\} - \mathbf{n}\cdot\delta\boldsymbol{\xi}\mu_0\mathbf{J}\times\mathbf{B}\cdot\boldsymbol{\xi},
\end{aligned}
$$

thus

$$
\begin{aligned}
-\frac{1}{2\mu_0}\mathbf{n}\cdot\delta\mathbf{Q}\boldsymbol{\xi}\cdot\mathbf{B} &= \frac{1}{2\mu_0}\nabla_s\cdot\{\boldsymbol{\xi}\cdot\mathbf{B}\mathbf{V}\} \\
&\quad +\frac{1}{2\mu_0}\mathbf{n}\cdot\delta\boldsymbol{\xi}\{\mathbf{B}\boldsymbol{\xi}:\nabla\mathbf{B}+\mathbf{B}\mathbf{B}:\nabla\boldsymbol{\xi}\}+\frac{1}{2}\mathbf{n}\cdot\delta\boldsymbol{\xi}\,\mathbf{J}\times\mathbf{B}\cdot\boldsymbol{\xi},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2\mu_0}\mathbf{n}\cdot\boldsymbol{\Phi}\cdot\mathbf{v} &= -\frac{1}{2\mu_0}\mathbf{n}\cdot\delta\boldsymbol{\xi}\,\mathbf{B}\cdot\mathbf{v} \\
&= -\frac{1}{2\mu_0}\mathbf{n}\cdot\delta\boldsymbol{\xi}\{\mathbf{B}\mathbf{B}:\nabla\boldsymbol{\xi}+\mathbf{B}\boldsymbol{\xi}:\nabla\mathbf{B}\}.
\end{aligned}
$$

By adding the two last equations we obtain

$$
\begin{aligned}
&-\frac{1}{2\mu_0}\mathbf{n}\cdot\delta\mathbf{Q}\boldsymbol{\xi}\cdot\mathbf{B}+\frac{1}{2\mu_0}\mathbf{n}\cdot\boldsymbol{\Phi}\cdot\mathbf{v} \\
&\qquad =\frac{1}{2\mu_0}\nabla_s\cdot\{\boldsymbol{\xi}\cdot\mathbf{B}\,\mathbf{V}\}+\frac{1}{2}\mathbf{n}\cdot\delta\boldsymbol{\xi}\,\boldsymbol{\xi}\cdot\nabla p.
\end{aligned}
$$

And finally

$$
\mathbf{n}\cdot\mathbf{T}=-\mathbf{n}\cdot\delta\boldsymbol{\xi}\left\{\frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{Q}-\gamma p\nabla\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla p\right\}+\frac{1}{2\mu_0}\nabla_s\cdot\{\boldsymbol{\xi}\cdot\mathbf{B}\mathbf{V}\}.
$$

The last term in this expression integrates to zero by using the formula given in eq.(1.9), and noticing the fact that we are integrating over a closed surface where $\mathbf{n}\cdot\mathbf{V}=0$.

Summarizing the results, we now have

$$
\begin{aligned}
\delta(\delta W_F) &= -\int_{V_p} dV\delta\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}) \qquad \text{(A.4)} \\
&\quad -\int_{S_p}\mathbf{n}\cdot\delta\boldsymbol{\xi}\{\frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{Q}-\gamma p\nabla\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla p\}, \qquad \text{(A.5)}
\end{aligned}
$$

and this is the desired result.

A.1 Variation of δW_F, details

We consider the term

$$\boldsymbol{\xi} \cdot \mathbf{J} \times \delta\mathbf{Q} = \frac{1}{\mu_0}\boldsymbol{\xi} \cdot (\nabla \times \mathbf{B}) \times \delta\mathbf{Q} = \frac{1}{\mu_0}\delta\mathbf{Q} \cdot \boldsymbol{\xi} \times (\nabla \times \mathbf{B}) . \tag{A.6}$$

We have

$$\boldsymbol{\xi} \times (\nabla \times \mathbf{B}) = \nabla(\boldsymbol{\xi} \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \boldsymbol{\xi}) - \boldsymbol{\xi} \cdot \nabla\mathbf{B} - \mathbf{B} \cdot \nabla\boldsymbol{\xi} , \tag{A.7}$$

thus

$$\delta\mathbf{Q} \cdot [\boldsymbol{\xi} \times (\nabla \times \mathbf{B})] = \delta\mathbf{Q} \cdot \nabla(\boldsymbol{\xi} \cdot \mathbf{B}) - \delta\mathbf{Q} \cdot \{\boldsymbol{\xi} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\boldsymbol{\xi} + \mathbf{B} \times (\nabla \times \boldsymbol{\xi})\} . \tag{A.8}$$

We also have

$$\delta\mathbf{Q} = \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}) = \mathbf{B} \cdot \nabla\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi} \cdot \nabla\mathbf{B} - \mathbf{B}\nabla \cdot \delta\boldsymbol{\xi} = \nabla \cdot \{\mathbf{B}\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi}\mathbf{B}\} , \tag{A.9}$$

since $\nabla \cdot \mathbf{B} = \mathbf{0}$. We already have

$$\boldsymbol{\Phi} \stackrel{def}{=} \mathbf{B}\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi}\mathbf{B} . \tag{A.10}$$

It then follows that $\boldsymbol{\Phi} = -\boldsymbol{\Phi}_c$, i.e., $\boldsymbol{\Phi}$ is an anti-symmetric dyadic. Here $\boldsymbol{\Phi}_c$ is the conjugate of $\boldsymbol{\Phi}$. We may therefore write

$$\delta\mathbf{Q} = \nabla \cdot \boldsymbol{\Phi} . \tag{A.11}$$

By combining these results we obtain

$$\delta\mathbf{Q} \cdot [\boldsymbol{\xi} \times (\nabla \times \mathbf{B})] = \nabla \cdot (\delta\mathbf{Q}\boldsymbol{\xi} \cdot \mathbf{B}) - (\nabla \cdot \boldsymbol{\Phi}) \cdot \mathbf{v} . \tag{A.12}$$

Notice that since $\delta\mathbf{Q}$ is the curl of some vector, the divergence of $\delta\mathbf{Q}$ is identically zero. We have introduced

$$\mathbf{v} \stackrel{def}{=} \boldsymbol{\xi} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\boldsymbol{\xi} + \mathbf{B} \times (\nabla \times \boldsymbol{\xi}) = \nabla(\mathbf{B} \cdot \boldsymbol{\xi}) - \boldsymbol{\xi} \times (\nabla \times \mathbf{B}) , \tag{A.13}$$

and furthermore

$$(\nabla \cdot \mathbf{\Phi}) \cdot \mathbf{v} = \nabla \cdot (\mathbf{\Phi} \cdot \mathbf{v}) - \mathbf{\Phi}_c : \nabla \mathbf{v} . \tag{A.14}$$

Since $\mathbf{v}$ may be written as

$$\mathbf{v} = \nabla \phi + \mathbf{u} , \tag{A.15}$$

with $\mathbf{u} = \mu_0 \mathbf{J} \times \boldsymbol{\xi}$. We notice that $\mathbf{\Phi} : \nabla\nabla\phi = 0$ since $\mathbf{\Phi}$ is anti-symmetric, where ϕ may be any scalar function. Thus

$$(\nabla \cdot \mathbf{\Phi}) \cdot \mathbf{v} = \nabla \cdot (\mathbf{\Phi} \cdot \mathbf{v}) - \mu_0 \mathbf{\Phi} : \nabla(\boldsymbol{\xi} \times \mathbf{J}) . \tag{A.16}$$

Elaborating on the last term we have

$$\mathbf{\Phi} : \nabla(\boldsymbol{\xi} \times \mathbf{J}) = \{\mathbf{B}\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi}\mathbf{B}\} : \nabla(\boldsymbol{\xi} \times \mathbf{J}) .$$

We first consider

$$\begin{aligned} \mathbf{B}\delta\boldsymbol{\xi} : \nabla(\boldsymbol{\xi} \times \mathbf{J}) &= \mathbf{B} \cdot (\delta\boldsymbol{\xi} \cdot \nabla)\boldsymbol{\xi} \times \mathbf{J} \\ &= \delta\boldsymbol{\xi} \cdot \nabla\{\mathbf{B} \cdot \boldsymbol{\xi} \times \mathbf{J}\} - (\delta\boldsymbol{\xi} \cdot \nabla\mathbf{B}) \cdot \boldsymbol{\xi} \times \mathbf{J} \\ &= \delta\boldsymbol{\xi} \cdot \nabla\{\boldsymbol{\xi} \cdot \nabla p\} - (\delta\boldsymbol{\xi} \cdot \nabla\mathbf{B}) \cdot \boldsymbol{\xi} \times \mathbf{J} . \end{aligned}$$

Then we consider

$$-\delta\boldsymbol{\xi}\mathbf{B} : \nabla(\boldsymbol{\xi} \times \mathbf{J}) = -\delta\boldsymbol{\xi} \cdot \{(\mathbf{B} \cdot \nabla\boldsymbol{\xi}) \times \mathbf{J} + \boldsymbol{\xi} \times (\mathbf{B} \cdot \nabla\mathbf{J})\} ,$$

from which we obtain

$$\begin{aligned} (\nabla \cdot \mathbf{\Phi}) \cdot \mathbf{v} &= \nabla \cdot (\mathbf{\Phi} \cdot \mathbf{v}) - \mu_0 \mathbf{\Phi} : \nabla(\boldsymbol{\xi} \times \mathbf{J}) \\ &= \nabla \cdot (\mathbf{\Phi} \cdot \mathbf{v}) - \mu_0 \delta\boldsymbol{\xi} \cdot \nabla\{\boldsymbol{\xi} \cdot \nabla p\} + \mu_0 (\delta\boldsymbol{\xi} \cdot \nabla\mathbf{B}) \cdot \boldsymbol{\xi} \times \mathbf{J} \\ &\quad + \mu_0 \delta\boldsymbol{\xi} \cdot \{(\mathbf{B} \cdot \nabla\boldsymbol{\xi}) \times \mathbf{J} + \boldsymbol{\xi} \times (\mathbf{B} \cdot \nabla\mathbf{J})\} . \end{aligned}$$

Finally we then obtain

$$\begin{aligned} \delta\mathbf{Q} \cdot \boldsymbol{\xi} \times \mathbf{J} &= \frac{1}{\mu_0} \delta\mathbf{Q} \cdot \boldsymbol{\xi} \times (\nabla \times \mathbf{B}) \\ &= \frac{1}{\mu_0} \nabla \cdot \{\delta\mathbf{Q}\boldsymbol{\xi} \cdot \mathbf{B} - \mathbf{\Phi} \cdot \mathbf{v}\} \\ &\quad + \delta\boldsymbol{\xi} \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p) - (\delta\boldsymbol{\xi} \cdot \nabla\mathbf{B}) \cdot \boldsymbol{\xi} \times \mathbf{J} \\ &\quad - \delta\boldsymbol{\xi} \cdot \{(\mathbf{B} \cdot \nabla\boldsymbol{\xi}) \times \mathbf{J} + \boldsymbol{\xi} \times (\mathbf{B} \cdot \nabla\mathbf{J})\} . \end{aligned} \tag{A.17}$$

This result is employed in eq.(A.2)

Appendix B

Alternative Formulae (derivation)

We consider the boundary condition

$$\nabla_s(\phi_i - \phi_0) + \mu_0\sigma\gamma d\mathbf{n} \times \mathbf{A}_{w0} = 0\,, \tag{B.1}$$

and notice that this boundary condition is a natural boundary condition for the variational problem. As such, it is not satisfied by the trial functions in the general case. However, if we disregard this fact here and assume that the trial functions also satisfies this boundary conditon. (We shall discuss the meaning of this assumption elswhere.) Then let the trial functions be given as

$$\phi_i = c_1\phi_\infty + c_2\phi_b\,, \qquad \phi_0 = c_3\phi_\infty\,, \qquad (c_3 = c_1) \tag{B.2}$$

and

$$\mathbf{A}_{w0} = c_3\hat{\mathbf{A}}_{w0}\,. \tag{B.3}$$

We then find

$$\mathbf{n} \times \hat{\mathbf{A}}_0 = -\frac{c_2}{c_1}(\mu_0\sigma\gamma d)^{-1}\nabla_s\phi_b\,. \tag{B.4}$$

Taking the square of this relation and then integrating the resulting equation over the conductor surface we obtain

$$\int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_0|^2 dS = \frac{c_2^2}{c_1^2}(\mu_0\sigma\gamma d)^{-2}\int_{S_b} |\nabla_s\phi_b|^2 dS\,, \tag{B.5}$$

or

$$\delta \hat{W}_w = \frac{\sigma\gamma d}{2} \int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_0|^2 dS = \frac{c_2^2}{2c_1^2} \frac{1}{\mu_0^2 \sigma\gamma d} \int_{S_b} |\nabla_s \phi_b|^2 dS \,. \tag{B.6}$$

Moreover from the solution for c_2, eq.(1.62) we find

$$\frac{c_2^2}{c_1^2} = \left(\frac{c_2}{1-c_2}\right)^2 = \left(\frac{\alpha}{\beta+\alpha}\right)^2 \frac{1}{\left(1-\frac{\alpha}{\alpha+\beta}\right)^2} = \frac{\alpha^2}{\beta^2}\,, \tag{B.7}$$

with $\alpha = \delta\hat{W}_w$ and $\beta = \delta W_b - \delta W_\infty$. Now substituting back in eq.(B.6), we obtain

$$\alpha = \frac{\alpha^2}{\beta^2} \frac{1}{\mu_0^2 \sigma\gamma d} \frac{1}{2} \int_{S_b} |\nabla_s \phi_b|^2 dS \,, \tag{B.8}$$

or

$$\alpha = \frac{\beta^2 \mu_0^2 \sigma\gamma d}{\frac{1}{2}\int_{S_b} |\nabla_s \phi_b|^2 dS}\,, \tag{B.9}$$

which may be rewritten as

$$\alpha = \frac{\beta^2 \mu_0^2 \sigma\gamma d}{\frac{1}{2}\int_{S_b} |\nabla_s \phi_b|^2 dS} = \frac{\sigma\gamma d}{2} \int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_0|^2 dS \,, \tag{B.10}$$

and we obtain

$$\frac{1}{2\mu_0} \int_{S_b} |\mathbf{n} \times \hat{\mathbf{A}}_0|^2 dS = \frac{(\delta W_b - \delta W_\infty)^2}{\frac{1}{2\mu_0}\int_{S_b} |\nabla_s \phi_b|^2 dS} \,. \tag{B.11}$$

Finally we may now write the expression for $\bar{b}$ as

$$\bar{b} = \frac{\frac{1}{2\mu_0}\int_{S_b} |\mathbf{n} \times \mathbf{A}_{w0}|^2 dS}{\delta W_b - \delta W_\infty} = \frac{\delta W_b - \delta W_\infty}{\frac{1}{2\mu_0}\int_{S_b} |\nabla_s \phi_b|^2 dS} \,. \tag{B.12}$$

Notice that on S_b : $\nabla_s \phi_b = \nabla \phi_b - \mathbf{n}\mathbf{n} \cdot \nabla \phi_b = \nabla \phi_b$, since $\mathbf{n} \cdot \nabla \phi_b = 0$ on this surface.

In summary we have

$$\gamma\tau_D = -\frac{\delta W_\infty}{\delta W_b}, \tag{B.13}$$

$$\tau_D = \mu_0 \sigma d \bar{b}, \tag{B.14}$$

$$\bar{b} = \frac{\delta W_b - \delta W_\infty}{\frac{1}{2\mu_0}\int_{S_b} |\nabla_s \phi_b|^2 dS}, \tag{B.15}$$

which is the result given in eq.(1.73).

Appendix C

Boundary Condition on a Conductor

In order to solve for the perturbation in the magnetic field inside a resistive conductor, we need the information about the normal component of the magnetic field from outside. This is in our case given as $\mathbf{n} \cdot \nabla \phi_\infty|_{S_b}$. Notice that our basefunctions for expansion are ϕ_∞ and ϕ_b. Since $\mathbf{n} \cdot \nabla \phi_b|_{S_b} = 0$, where S_b is any bounding conductor surface, i.e., wall or conductor surfaces, this part gives no contribution from conductors or wall.

The problem we need to solve is therefore only related to ϕ_∞. We shall now in a couple of different ways obtain the solution to this problem.

C.1 Numerical Differentiation

In this approach we consider a solution ϕ_∞ obtained by a Green's function technique. Having obtained this function we then evaluate ϕ_∞ at two or more points (dependent on accuracy required) along a normal at the surface under consideration. From this information we then compute the directional derivative in the usual way, which gives us the value of $\mathbf{n} \cdot \nabla \phi_b|_{S_b}$ at the point of the surface considered.

We now outline a procedure for finding ϕ_∞ at an arbitrary point in the vacuum region. First compute ϕ_∞ at the plasma-vacuum interface by solving the integral equation

$$\sigma\phi_\infty(\mathbf{r}) = -\int_{S_p} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\} dS_p\,, \qquad \text{(C.1)}$$

where σ takes the value $\frac{1}{2}$ when the observation point moves onto the surface and otherwise $\sigma = 1$. In the case considered the argument $\mathbf{r}$ for the function $\phi_\infty(\mathbf{r})$ is evaluated on the surface S_p ($\sigma = \frac{1}{2}$). Similar to eq.(2.107) we represent ϕ_∞ at the plasma-vacuum interface as

$$\phi_\infty|_{S_b} = iR_0B_0e^{in\phi}\sum_m U_m^{(p)}e^{2\pi imv}\,.$$

Notice that ϕ in the exponent refers to the toroidal angle variable. We solve eq.(C.1) as before by Fourier expansion and obtain

$$\mathbf{U}^{(p)} + \mathbf{U}^{(p)}\cdot\mathbf{\Gamma}^{(p,p)} = \boldsymbol{\xi}\cdot\mathbf{\Lambda}^{(p)}\,,$$

or

$$\mathbf{U}^{(p)} = \boldsymbol{\xi}\cdot\mathbf{\Lambda}^{(p)}\cdot\{\mathbf{I} + \mathbf{\Gamma}^{(p,p)}\}^{-1}\,, \qquad \text{(C.2)}$$

which is also the result we obtain by replacing $\mathbf{V}$ by $\mathbf{U}$ in eq.(2.135), and only keeping the first two terms on the left-hand side of the equation. Thus having obtained ϕ_∞ at the plasma-vacuum interface, we are now in a position to determine ϕ_∞ at an arbitrary point $\mathbf{r}$ by again using eq.(C.1), but now with $\mathbf{r}$ being an arbitrary point not on the plasma-vacuum interface. This way we obtain

$$\begin{aligned}\phi_\infty(\mathbf{r}) &= -\int_{S_p} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\} dS_p \\ &= -I_1 + I_2\,. \end{aligned} \qquad \text{(C.3)}$$

Since the observation point is not on the surface of integration, the integrals I_1 and I_2 as given by eqs.(2.28) and (2.34) are not singular. Moreover these integrals are now determined for any given value of the observation point $\mathbf{r} = (R, Z)$. Notice that there is no reference to the poloidal angle. Because of the axisymmetry the toroidal angle only appear in the exponential for a given mode number.

We also notice that the observation point located on the surface S_j in these integrals can now be located anywhere except on the surface S_p. In accordance with this it is convenient to change the notation so we write

$$I_n^{(S_i,S_j)}(v,v') \quad \Rightarrow \quad I_n^{(\hat{\mathbf{r}},S_j)}(\hat{\mathbf{r}},v') \tag{C.4}$$
$$I_n^{(S_i,S_j)}(v,v') \quad \Rightarrow \quad I_n^{(\hat{\mathbf{r}},S_j)}(\hat{\mathbf{r}},v')\,. \tag{C.5}$$

The solution can now be written as

$$\phi_\infty(\bar{\mathbf{r}}) = -\mathbf{U}^{(p)} \cdot \mathbf{\Gamma}(\bar{\mathbf{r}}) + \boldsymbol{\xi} \cdot \mathbf{\Lambda}(\bar{\mathbf{r}})\,.$$

Here $\mathbf{U}^{(p)}$ is given by eq.(C.2) and $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ (to be determined) are now vectors instead of matrices, and they are functions of $\bar{\mathbf{r}} = (R, Z)$. For a given value of (R, Z) the left-hand side of the above equation is given, thus determining $\phi_\infty(R, Z)$. Notice that the left-hand side of eq.(C.3) is determined, in contrast to the other cases we have considered, where we ended up with an implicit equation for the unknown vector.

C.1.1 Determining $\mathbf{\Gamma}(R, Z)$ and $\mathbf{\Lambda}(R, Z)$

We have

$$I_1 = \int_{S_p} \phi_\infty(\mathbf{r}')\mathbf{n} \cdot \nabla G(\bar{\mathbf{r}}, \mathbf{r}') dS_p \stackrel{def}{=} I_1^{\bar{\mathbf{r}},S_j}(\bar{\mathbf{r}}, v') \tag{C.6}$$

$$I_2 = \int_{S_p} G(\bar{\mathbf{r}}, \mathbf{r}')\mathbf{n} \cdot \nabla \phi_\infty(\mathbf{r}')\} dS_p \stackrel{def}{=} \hat{I}_2^{\bar{\mathbf{r}},S_j}(\bar{\mathbf{r}}, v') \tag{C.7}$$

$$\begin{aligned}
\phi_\infty(\bar{\mathbf{r}}) &= -I_1^{\bar{\mathbf{r}},S_j}(\bar{\mathbf{r}}, v') + \hat{I}_2^{\bar{\mathbf{r}},S_j}(\bar{\mathbf{r}}, v') \\
&= -\frac{e^{in\phi}}{2\pi} \int_0^1 V_n(v')\{\epsilon \dot{y}(v') I_n^{(\bar{\mathbf{r}},S_p)}(\bar{\mathbf{r}}, v') \\
&\qquad + F^{(\bar{\mathbf{r}},S_p)}(\bar{\mathbf{r}}, v') \hat{I}_n^{(\bar{\mathbf{r}},S_p)}(\bar{\mathbf{r}}, v')\} dv' \\
&\qquad - \frac{e^{in\phi}}{2\pi} \int_0^1 A_n(v') I_n^{(\bar{\mathbf{r}},S_p)} dv'\,,
\end{aligned} \tag{C.8}$$

where

$$
\begin{aligned}
I_n^{(\bar{\mathbf{r}},S_p)}(\bar{\mathbf{r}},v') &= (-1)^n \frac{R_0}{h}\int_0^{\frac{\pi}{2}} \frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{1}{2}}}\,, \\
&= (-1)^n \frac{R_0}{h} L_{2n}(k) \qquad \text{(C.9)} \\
\hat{I}_n^{(\bar{\mathbf{r}},S_p)}(\bar{\mathbf{r}},v') &= (-1)^n \frac{R_0^3}{h^3}\int_0^{\frac{\pi}{2}} \frac{\cos 2n\theta}{\{(1+k)^2 - 4k\sin^2\theta\}^{\frac{3}{2}}} \\
&= (-1)^n \frac{R_0^3}{h^3} \hat{L}_{2n}(k)\,. \qquad \text{(C.10)}
\end{aligned}
$$

Here

$$
\hat{L}_{2n}(k) = \frac{2k}{1-k^2}\frac{d}{dk}L_{2n} + \frac{1}{1-k^2}L_{2n}\,,
$$

$$
h = \sqrt{\frac{RR'}{k}}\,, \qquad k = \alpha - \sqrt{\alpha^2 - 1}\,, \qquad \alpha = \frac{R'^2 + R^2 + (Z'-Z)^2}{2RR'}\,.
$$

To continue we follow the procedure outlined in eqs.(2.118) to (2.131). The only difference now is that we have to specify a point (R, Z) in the poloidal plane, instead of taking double Fourier transforms. Thus instead of matrices as given by Γ in eq.(2.125) and by Λ in eq.(2.131) we now end up with vectors for these same quantities.

C.2 Direct computation of $\mathbf{n} \cdot \nabla\phi_\infty|_{S_c}$

We have that ϕ_∞ has already been determined on the surface S_p as a Fourier series in eq.(C.2). We shall now outline a procedure for direct computation of $\mathbf{n} \cdot \nabla\phi_\infty|_{S_c}$, that is, on a given closed surface S_c in the vacuum region, which may be arbitrarily chosen at this point, we shall determine $\mathbf{n} \cdot \nabla\phi_\infty|_{S_c}$ or rather the Fourier transform of this quantity. Here $\mathbf{n}$ is the surface normal to the surface S_c, pointing into the volume enclosed by S_c.

In order to achieve this we shall again make use of Green's third identity, a modified form of eq(2.4), which now becomes

$$
\frac{1}{2}\phi_\infty(\mathbf{r}) = -\int_{S_p} \{\phi_\infty(\mathbf{r}')\mathbf{n} \cdot \nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n} \cdot \nabla\phi_\infty(\mathbf{r}')\} dS_p
$$

$$+ \int_{S_c} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\}dS_p \qquad \mathbf{r}\in S_p. \tag{C.11}$$

$$\begin{aligned}\frac{1}{2}\phi_\infty(\mathbf{r}) &= -\int_{S_p} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\}dS_p \\ &+ \int_{S_c} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\}dS_p \\ &\qquad \mathbf{r}\in S_c.\end{aligned} \tag{C.12}$$

From eq.(C.1) with $\sigma = \frac{1}{2}$ we have

$$\frac{1}{2}\phi_\infty(\mathbf{r}) = -\int_{S_p} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\}dS_p. \qquad \mathbf{r}\in S_p$$

Equation (C.11) reduces to

$$\int_{S_c} \{\phi_\infty(\mathbf{r}')\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')\}dS_p = 0, \qquad \mathbf{r}\in S_p \tag{C.13}$$

In principle $\phi_\infty(\mathbf{r}')$ is determined by eq.(C.3), and by using this information we can then determine $\mathbf{n}\cdot\nabla\phi|_{S_c}$ from eq.(C.13), however, we shall use a different approach and solve eq.(C.13) in conjunction with eq.(C.12). Again using Fourier transforms we write

$$\phi_\infty(\mathbf{r}')|_{\mathbf{r}'\in S_c} = \sum_k U_k^{(c)} e^{2\pi i k v'}, \tag{C.14}$$

$$\mathbf{n}\cdot\nabla\phi_\infty(\mathbf{r}')|_{\mathbf{r}'\in S_c} = \sum_k \hat{U}_k^{(c)} e^{2\pi i k v'}, \tag{C.15}$$

$$G(\mathbf{r},\mathbf{r}')dS|_{\substack{\mathbf{r}\in S_c\\ \mathbf{r}'\in S_p}} \rightarrow \sum_k \Gamma_{kl}^{(c,p)} e^{2\pi i(kv+lv')}dv', \tag{C.16}$$

$$\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}')dS|_{\substack{\mathbf{r}\in S_c\\ \mathbf{r}'\in S_p}} \rightarrow \sum_k \Lambda_{kl}^{(c,p)} e^{2\pi i(kv+lv')}dv', \tag{C.17}$$

$$G(\mathbf{r},\mathbf{r}')dS|_{\substack{\mathbf{r}\in S_c\\ \mathbf{r}'\in S_c}} \rightarrow \sum_k \Gamma_{kl}^{(c,c)} e^{2\pi i(kv+lv')}dv', \tag{C.18}$$

$$\mathbf{n}\cdot\nabla G(\mathbf{r},\mathbf{r}')dS|_{\substack{\mathbf{r}\in S_c\\ \mathbf{r}'\in S_c}} \rightarrow \sum_k \Lambda_{kl}^{(c,c)} e^{2\pi i(kv+lv')}dv'. \tag{C.19}$$

In terms of the appropriate Fourier vectors and matricies eq.(C.12) and eq.(C.13) can now be written as

$$\mathbf{U}^{(c)} \cdot \mathbf{\Lambda}^{(p,c)} - \hat{\mathbf{U}}^{(c)} \cdot \mathbf{\Gamma}^{(c,p)} = 0 \tag{C.20}$$

$$\begin{aligned} \mathbf{U}^{(c)} &= -\mathbf{U}^{(p)} \cdot \mathbf{\Lambda}^{(c,p)} + \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(p)} \\ &\quad + \mathbf{U}^{(c)} \cdot \mathbf{\Lambda}^{(c,c)} - \hat{\mathbf{U}}^{(c)} \cdot \mathbf{\Gamma}^{(c,c)} \end{aligned} \tag{C.21}$$

or

$$\hat{\mathbf{U}}^{(c)} \cdot \mathbf{\Gamma}^{(c,p)} - \mathbf{U}^{(c)} \cdot \mathbf{\Lambda}^{(c,p)} = 0 \tag{C.22}$$

$$\hat{\mathbf{U}}^{(c)} \cdot \mathbf{\Gamma}^{(c,c)} + \mathbf{U}^{(c)} \cdot \{\mathbf{I} - \mathbf{\Lambda}^{(c,c)} = \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(p)} - \mathbf{U}^{(p)} \cdot \mathbf{\Lambda}^{(c,p)} . \tag{C.23}$$

Notice that the factor $\frac{1}{2}$ multiplying $\phi_\infty(\mathbf{r})$ in eq.(C.12) has been accounted for in the definitions of $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$. From eq.(C.22) we find

$$\mathbf{U}^{(c)} = \hat{\mathbf{U}}^{(c)} \cdot \mathbf{\Gamma}^{(c,p)} \cdot (\mathbf{\Lambda}^{(c,p)})^{-1} . \tag{C.24}$$

We substitute for $\mathbf{U}^{(c)}$ by eq.(C.24) in eq.(C.23) and obtain

$$\hat{\mathbf{U}}^{(c)} = \mathbf{A} \cdot \mathbf{B}^{-1} \tag{C.25}$$

where

$$\mathbf{A} = \boldsymbol{\xi} \cdot \mathbf{\Lambda}^{(p)} - \mathbf{U}^{(p)} \cdot \mathbf{\Lambda}^{(p,c)} , \tag{C.26}$$

$$\mathbf{B} = \left\{ \{\mathbf{\Gamma}^{(c,p)} \cdot (\mathbf{\Lambda}^{(c,p)})^{-1}\} \cdot \{\mathbf{I} - \mathbf{\Lambda}^{(c,c)}\} - \mathbf{\Gamma}^{(c,c)} \right\}^{-1} . \tag{C.27}$$

The result given in eq.(C.25) is the solution to our problem. The matrices $\mathbf{\Gamma}^{(\alpha,\beta)}$ and $\mathbf{\Lambda}^{(\alpha,\beta)}$ are determined in a similar way to the procedure used in eqs.(2.122) to (2.131). The solution is obtained as a Fourier series, which is convenient for our purpose.

Appendix D

The Limit of $F(v, v')$

We shall study the limit $|v - v'| \to 0$ of the expression

$$F(v,v') \stackrel{def}{=} \frac{\epsilon}{R_0^2}\left\{\dot{y}(v')[(R'+R)(R'-R)-(Z'-Z)^2]-2R'\dot{x}(v')(Z'-Z)\right\}.$$

First we consider

$$\begin{aligned}
\dot{y}(v')(R'+R)(R'-R) &= \{\dot{y}(v)+\ddot{y}(v)(v'-v)+\cdots\}R_0^2\{2[1+\epsilon x(v)]+\epsilon\dot{x}(v)(v'-v)+\cdots\} \\
&\quad \times\epsilon\{\dot{x}(v)(v'-v)+\frac{1}{2}\ddot{x}(v)(v'-v)^2+\cdots\} \\
&= 2\epsilon R_0^2[1+\epsilon x(v)]\dot{x}(v)\dot{y}(v)(v'-v) \\
+\epsilon R_0^2\{\epsilon\dot{x}^2(v)\dot{y}(v) &+ 2\dot{x}(v)\ddot{y}(v)[1+\epsilon x(v)]+\ddot{x}(v)\dot{y}(v)[1+\epsilon x(v)]\}(v'-v)^2+\cdots \\
&= 2\epsilon R_0^2[1+\epsilon x(v)]\dot{x}(v)\dot{y}(v)(v'-v) \\
&+ \epsilon R_0^2\left\{\epsilon\dot{x}^2(v)\dot{y}(v)+[2\dot{x}(v)\ddot{y}(v)+\ddot{x}(v)\dot{y}(v)][1+\epsilon x(v)]\right\}(v'-v)^2+\cdots
\end{aligned}$$

Then consider

$$\begin{aligned}
\dot{y}(v')(Z'-Z)^2 &= \{\dot{y}(v)+\ddot{y}(v)(v'-v)+\cdots\}R_0^2\{1+\epsilon y(v')-1-\epsilon y(v)\}^2 \\
&= \epsilon^2R_0^2\{\dot{y}(v)+\ddot{y}(v)(v'-v)+\cdots\}\{\dot{y}(v)(v'-v)+\cdots\}^2 \\
&= \epsilon^2R_0^2\dot{y}^3(v)(v'-v)^2+\cdots.
\end{aligned}$$

And then we consider the last term

$$
\begin{aligned}
2R'\dot{x}(v)(Z'-Z) &= 2R_0^2[1+\epsilon x(v')]\dot{x}(v')[\epsilon y(v')-\epsilon y(v)] \\
&= 2\epsilon R_0^2\{1+\epsilon x(v)+\epsilon\dot{x}(v)(v'-v)+\cdots\}\{\dot{x}(v)+\ddot{x}(v)(v'-v)+\cdots\} \\
&\qquad \times\{\dot{y}(v)(v'-v)+\frac{1}{2}\ddot{y}(v)(v'-v)^2+\cdots\} \\
&= 2\epsilon R_0^2\left\{[1+\epsilon x(v)]\dot{x}(v)\dot{y}(v)(v'-v)\right. \\
&+ \left.\{[1+\epsilon x(v)][\ddot{x}(v)\dot{y}(v)+\frac{1}{2}\dot{x}(v)\ddot{y}(v)]+\epsilon\dot{x}^2(v)\dot{y}(v)\}(v'-v)^2+\cdots\right\}.
\end{aligned}
$$

We notice that the first order terms cancel, since

$$
2\epsilon R_0^2[1+\epsilon x(v)]\dot{x}(v)\dot{y}(v)-2\epsilon R_0^2[1+\epsilon x(v)]\dot{x}(v)\dot{y}(v)=0\,.
$$

Then we obtain

$$
\begin{aligned}
F(v,v') &= \frac{\epsilon}{R_0^2}\epsilon R_0^2\{\epsilon\dot{x}^2(v)\dot{y}(v)+[1+\epsilon x(v)][2\dot{x}(v)\ddot{y}(v)+\ddot{x}(v)\dot{y}(v)] \\
&- \epsilon\dot{y}^3(v)-[1+\epsilon x(v)][2\ddot{x}(v)\dot{y}(v)+\dot{x}(v)\ddot{y}(v)]-2\epsilon\dot{x}^2(v)\dot{y}(v)\}(v'-v)^2+\cdots \\
&= \epsilon^2\{[1+\epsilon x(v)][-\ddot{x}(v)\dot{y}(v)+\dot{x}(v)\ddot{y}(v)]-\epsilon\dot{y}(v)[\dot{x}^2(v)+\dot{y}^2(v)]\}(v'-v)^2+\mathcal{O}(v'-v) \\
&= [1+\epsilon x(v)]Q^3\kappa-\epsilon^3\dot{y}^2(v)Q^2+\mathcal{O}(v'-v)^3 \\
&= Q^2\{[1+\epsilon x(v)]Q\kappa-\epsilon^3\dot{y}^2(v)\}+\mathcal{O}(v'-v)^3\,,
\end{aligned}
$$

where

$$
\kappa \overset{def}{=} \epsilon^2\frac{\dot{x}(v)\ddot{y}(v)-\dot{y}(v)\ddot{x}(v)}{\{\dot{x}^2(v)+\dot{y}^2(v)\}^{\frac{3}{2}}}\,,
$$

is the curvature in the poloidal direction.