Stability criterion for edge-localized, high-n external modes in tokamaks

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January, 2001

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Abstract

The ideal-magnetohydrodynamics (MHD) energy principle is used to derive a necessary stability criterion for high-toroidal-number (n) external modes in axisymmetric equilibria. The corresponding trial functions are expressed in the ballooning representation, but have a finite amplitude at the plasma boundary and can apply to equilibria where the conventional, high-n internal ballooning criterion predicts stability. These trial functions are constructed by solving the standard local ballooning equation at the plasma boundary flux surface with the radial wave number parameter as a complex eigenvalue, such that the radial envelope of the mode is an exponential decaying into the plasma. The resulting stability criterion includes the surface and vacuum contributions to the MHD potential energy associated with the mode finite edge amplitude, and provides a framework for analyzing free-boundary ballooning and peeling modes.

I. INTRODUCTION.

Tokamak plasmas in the H-mode confinement regime are characterized by significantly large edge values of the pressure gradient and the parallel current density. These can be sources of free energy to drive ideal-magnetohydrodynamics (MHD) instabilities, and are presumed to have a key influence on edge localized fluctuations and general tokamak performance[1,2]. For low values of the toroidal wavenumber n, the MHD stability analysis requires two-dimensional numerical simulations. However, for high-n, the ballooning formalism[3-6] provides a powerful tool that allows one, under certain circumstances, to reduce the analysis to a series of one-dimensional problems. In its original formulation[6] as well as in a recent generalization [7], the ballooning formalism applies only to pressure-driven internal modes: their amplitude is required to vanish at the plasma boundary either because the instability drive is internally localized[6] or because a perfectly conducting wall at the plasma boundary is assumed[7]. For edge-localized instabilities it is important to relax the conducting wall boundary condition and consider external modes. These result in a non-vanishing perturbation of the plasma boundary and the vacuum magnetic field, and can tap the instability drive associated with the equilibrium parallel current density. The work of $Ref.7$ included also a study of such external modes at moderate to high values of n for the simplified $s - \alpha$ equilibrium model[3] but, like in the conventional low-n analyses, this relied on a two-dimensional numerical code.

In this work we present a generalization of the ballooning formalism that is applicable to high-n, edge-localized external modes. Based on the ideal-MHD energy principle[8], we derive a necessary criterion for stability that involves only one-dimensional calculations. Its expression is valid for general axisymmetric equilibria and applies to configurations where the conventional high-n internal ballooning criterion predicts stability. Thus our criterion extends the applicability of the one-dimensional ballooning representation techniques to a new class of instabilities including the surface kink or peeling modes[7,9,10].

II. POTENTIAL ENERGY MINIMIZATION.

The ideal-MHD energy principle[8] states that an equilibrium configuration is linearly stable if and only if, for any small perturbation compatible with the ideal-MHD constraints, the incremental potential energy is positive. Assuming an equilibrium without surface currents and an incompressible

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perturbation displacement ξ, the incremental potential energy associated with such displacement is

$$W[\xi_\perp] = W_F[\xi_\perp] + W_V[\xi_\perp], \tag{1}$$

$$W_F[\xi_\perp] = \frac{1}{2} \int_{plasma} d^3\mathbf{x} \left[\mid \mathbf{Q}_\perp \mid^2 + B^2(\nabla \cdot \xi_\perp + 2\xi \cdot \kappa)^2 - 2(\xi \cdot \nabla p)(\xi \cdot \kappa) + \mathbf{j}_\parallel \cdot (\xi_\perp \times \mathbf{Q}_\perp) \right], \tag{2}$$

$$W_V[\xi_\perp] = \frac{1}{2} \int_{vacuum} d^3\mathbf{x} \mid \mathbf{Q}_V \mid^2 . \tag{3}$$

Here $\mathbf{B}$, κ, $\mathbf{j}$ and p stand respectively for the equilibrium magnetic field, magnetic curvature, current density and pressure, and the subscripts $\parallel$ and $\perp$ refer to the components parallel and perpendicular to the equilibrium magnetic field. The perturbed magnetic fields in the plasma and vacuum regions are $\mathbf{Q} = \nabla \times (\xi \times \mathbf{B})$, and $\mathbf{Q}_V$ which is determined by the conditions that its curl vanishes and its normal component equals that of $\mathbf{Q}$ at the plasma-vacuum interface, plus regularity at infinity.

We use the following two representations for the equilibrium magnetic field:

$$\mathbf{B} = \nabla\psi \times \nabla\varphi + F(\psi)\nabla\varphi = \nabla\psi \times \nabla S, \tag{4}$$

where $2\pi\psi$ is the poloidal flux, $2\pi F$ is the poloidal current, φ is the toroidal angle and ψ, S are a pair of Clebsch potentials. Then, the component of the perturbed magnetic field normal to the equilibrium flux surfaces is

$$\mathbf{Q} \cdot \nabla\psi = \mathbf{B} \cdot \nabla(\xi \cdot \nabla\psi). \tag{5}$$

After standard application of Green's theorem, the vacuum energy integral can be reduced to a double surface integral at the plasma-vacuum interface S_a:

$$W_V[\xi \cdot \nabla\psi] = \frac{1}{8\pi} \int_{S_a} dS_a \int_{S_{a+}} dS'_{a+} \left[\frac{\mathbf{B} \cdot \nabla(\xi \cdot \nabla\psi)}{\mid \nabla\psi \mid} \right](\mathbf{x}) \ G(\mathbf{x}; \mathbf{x}'_+) \left[\frac{\mathbf{B} \cdot \nabla(\xi \cdot \nabla\psi)}{\mid \nabla\psi \mid} \right](\mathbf{x}'_+), \tag{6}$$

where $G(\mathbf{x}; \mathbf{x}')$ is the Green's function satisfying

$$\nabla^2 G(\mathbf{x}; \mathbf{x}') = -4\pi \ \delta(\mathbf{x} - \mathbf{x}') \tag{7}$$

with the boundary conditions that the normal component of its $\mathbf{x} - gradient$ vanishes at the plasma-vacuum interface plus regularity at infinity. For axisymmetric configurations, the following Fourier expansion in cylindrical coordinates holds:

$$G(\mathbf{x}; \mathbf{x}') = \sum_{n=-\infty}^{\infty} cos[n(\varphi - \varphi')] \ G_n(R, Z; R', Z'). \tag{8}$$

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The plasma contribution to the incremental potential energy $W_F[\xi_\perp]$ can be minimized with respect to the component $\xi \cdot (\nabla\psi \times \mathbf{B})$ by perturbative expansion in the limit of large toroidal mode numbers, exactly as in the conventional theory of internal ballooning modes. We introduce the representation

$$\xi \cdot \nabla\psi = \Re\Big[e^{-in\varphi}Y(\psi,\chi)\Big], \tag{9}$$

where ψ, χ, φ are orthogonal flux coordinates with volume element $d^3\mathbf{x} = J\, d\psi\, d\chi\, d\varphi$, and assume the orderings

$$iR^2\nabla\varphi \cdot \nabla = n \gg 1, \tag{10}$$

$$i\nabla\psi \cdot \nabla = O(n), \tag{11}$$

$$\mathbf{B} \cdot \nabla = O(1). \tag{12}$$

The result is

$$\xi \cdot (\nabla\psi \times \mathbf{B}) = \Re\Big[e^{-in\varphi}\Big[\frac{i}{n}\nabla\psi \cdot \nabla Y(\psi,\chi) + O(\frac{1}{n})\Big]\Big], \tag{13}$$

where the correction of order $1/n$ is given explicitly in $Ref.6$. Substituting this minimizing solution into $W_F[\xi_\perp]$ and integrating over the toroidal angle we obtain a plasma energy functional $W_F[Y]$ which, like the vacuum energy, depends only on the component of the displacement ξ normal to the equilibrium flux surfaces.

As in the theory of internal ballooning modes, we adopt the following norm functional:

$$N[Y] = \frac{\pi}{2}\int_{plasma} J\, d\psi\, d\chi\Big[\frac{1}{\mid\nabla\psi\mid^2}\mid Y \mid^2 + \frac{\mid\nabla\psi\mid^2}{B^2}\mid\frac{i}{n}\frac{\partial Y}{\partial\psi}\mid^2\Big]. \tag{14}$$

Then, for any positive λ^2 to be considered here as a variational parameter, we obtain after integration by parts:

$$W_F[Y] + \lambda^2 N[Y] = W_{BT}[Y] + \lambda^2 N_{BT}[Y] - \frac{\pi}{2}\int_{plasma} J\, d\psi\, d\chi\, Y^*\,(LY + \lambda^2 MY). \tag{15}$$

The boundary terms $W_{BT}[Y]$ and $N_{BT}[Y]$ are surface integrals at the plasma-vacuum interface that we must retain since we will be considering external modes. Their expressions, to leading order in $n \gg 1$, are:

$$W_{BT}[Y] = \frac{\pi}{2ni}\oint_{S_a} d\chi\, Y^*\Big(\frac{\partial}{\partial\chi} - in\frac{JF}{R^2}\Big)\Big[\Big(\frac{\mathbf{j}\cdot\mathbf{B}}{B^2}\Big)Y - \frac{\mid\nabla\psi\mid^2}{JB^2}\frac{i}{n}\frac{\partial}{\partial\psi}\Big[\Big(\frac{\partial}{\partial\chi} - in\frac{JF}{R^2}\Big)Y\Big]\Big], \tag{16}$$

$$N_{BT}[Y] = \frac{\pi}{2ni}\oint_{S_a} J\, d\chi\, Y^*\frac{\mid\nabla\psi\mid^2}{B^2}\Big(\frac{i}{n}\frac{\partial Y}{\partial\psi}\Big). \tag{17}$$

The linear differential operators L and M are given explicitly in *Ref*.6, and

$$LY + \lambda^2 MY = 0 \tag{18}$$

is the Euler equation to minimize $W_F[Y]$ subject to the constraint $N[Y] = constant$, λ^2 being the corresponding Lagrange multiplier.

Our next step, at which we depart from the conventional theory of internal ballooning modes, is to consider trial functions $Y_t(\psi, \chi)$ that are solutions of the Euler equation (18) subject only to regularity throughout the plasma domain but free of any boundary condition at the plasma-vacuum interface. Such solutions can be found for a continuum of values of the variational parameter $\lambda^2 > 0$, which remains otherwise unconstrained and makes Eq.(18) non-singular. Furthermore we restrict ourselves to the class of such trial functions that can be realized by the ballooning representation[3-6]:

$$Y_t(\psi, \chi) = e^{in[\varphi - S - S_0(\psi)]} \sum_{l=-\infty}^{\infty} e^{-2\pi i l n q(\psi)} \, y(\psi, \chi - 2\pi l), \tag{19}$$

with

$$\left| \frac{i}{n} \frac{\partial y}{\partial \psi} \right| \ll \left| \frac{i}{n} \frac{\partial Y_t}{\partial \psi} \right| = O(1). \tag{20}$$

We have chosen to define the multivalued Clebsch potential S as

$$S(\psi, \chi, \varphi) = \varphi - \int_0^\chi d\chi' \left(\frac{JF}{R^2} \right)(\psi, \chi'), \tag{21}$$

and to write down explicitly the still arbitrary flux function $S_0(\psi)$. The latter can be reprsented as

$$S_0(\psi) = -\int_\psi^{\psi_a} d\psi' \, k(\psi'), \tag{22}$$

so that $k(\psi)$ can be interpreted as a wavenumber for the fast variation of the perturbation perpendicular to the equilibrium flux surfaces. The equilibrium inverse rotational transform is

$$2\pi q(\psi) = \oint d\chi \left(\frac{JF}{R^2} \right)(\psi, \chi). \tag{23}$$

Then, $Y_t(\psi, \chi)$ is a solution of the Euler equation (18):

$$L\left[\frac{i}{n} \frac{\partial}{\partial \psi}, \left(\frac{\partial}{\partial \chi} - in\frac{JF}{R^2} \right), \psi, \chi; \frac{1}{n} \right] Y_t(\psi, \chi) + \lambda^2 M\left[\frac{i}{n} \frac{\partial}{\partial \psi}, \psi, \chi \right] Y_t(\psi, \chi) = 0, \tag{24}$$

provided $y(\psi, \eta)$, with $-\infty < \eta = \chi - 2\pi l < \infty$ and $y(\psi, \eta \to \pm\infty) \to 0$, is a solution of

$$L\left[\left(k + \frac{\partial S}{\partial \psi} + \frac{i}{n} \frac{\partial}{\partial \psi} \right), \frac{\partial}{\partial \eta}, \psi, \eta; \frac{1}{n} \right] y(\psi, \eta) + \lambda^2 M\left[\left(k + \frac{\partial S}{\partial \psi} + \frac{i}{n} \frac{\partial}{\partial \psi} \right), \psi, \eta \right] y(\psi, \eta) = 0. \tag{25}$$

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Once the above described trial function $Y_t(\psi, \chi)$ has been obtained, the corresponding incremental potential energy, whose positivity is a necessary condition for stability, is

$$W[Y_t] = W_V[Y_t] + W_{BT}[Y_t] + \lambda^2 N_{BT}[Y_t] - \lambda^2 N[Y_t], \tag{26}$$

where W_V, W_{BT}, N_{BT} and N are as defined by Eqs. (6), (16), (17) and (14) respectively. For a given equilibrium, Y_t and $W[Y_t]$ depend on the parameter λ, which should be varied to seek the minimum of $W[Y_t]$.

III. STABILITY CRITERION FOR EDGE-LOCALIZED MODES.

The Euler equation in ballooning representation variables (25) is readily amenable to a perturbative solution for small $1/n$. To lowest order, one gets the $n = \infty$ local ballooning equation[3-6]:

$$L\Big[\Big(k + \frac{\partial S}{\partial \psi}\Big), \frac{\partial}{\partial \eta}, \psi, \eta; 0\Big] y_0(\psi, \eta) + \lambda^2 M\Big[\Big(k + \frac{\partial S}{\partial \psi}\Big), \psi, \eta\Big] y_0(\psi, \eta) = 0. \tag{27}$$

This is an ordinary differential equation in the extended poloidal variable η, where ψ enters only as a parameter. If at some flux surface ψ, and for some real positive λ^2 and real k, a solution of Eq.(27) exists satisfying the boundary conditions

$$y_0(\psi, \eta \to \pm\infty) \to 0, \tag{28}$$

then the equilibrium is unstable against $n = \infty$ internal ballooning modes.

We are interested in the situation where no such solutions exist, namely the equilibrium is stable against the $n = \infty$ internal ballooning criterion. In this case we may solve Eq.(27) subject to the boundary conditions (28) at the plasma-vacuum interface $\psi = \psi_a$, with λ^2 as a real positive parameter and $k(\psi_a) = k_{aR} + ik_{aI}$ as a complex eigenvalue. If one such solution $y_0(\psi_a, \eta)$ exists with $k_{aI} > 0$, we can construct, based on it, a regular and physically acceptable external mode trial function $Y_t(\psi, \chi)$. To leading order in $n \gg 1$ this is

$$Y_t(\psi, \chi) = e^{nk_{aI}(\psi - \psi_a) - in[k_{aR}(\psi - \psi_a) + S - \varphi]} \sum_{l=-\infty}^{\infty} e^{-2\pi iln[q_a + q_a'(\psi - \psi_a)]} y_0(\psi_a, \chi - 2\pi l), \tag{29}$$

where $q_a = q(\psi_a)$ and $q_a' = dq(\psi_a)/d\psi$. This function has a radial envelope that decays exponentially into the plasma and retains a finite value at the plasma-vacuum interface.

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Taking Eq.(29) to Eqs. (6), (16), (17) and (14), we can evaluate the different contributions to the potential energy (26) generated by our trial function. To leading order in $n \gg 1$ and up to an overall positive constant we get:

$$W_V = \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\eta' \; cos[nS(\eta) - nS(\eta')] \; nG_n(\psi_a, \eta; \psi_{a+}, \eta') \; \frac{\partial y_0(\eta)}{\partial \eta} \; \frac{\partial y_0^*(\eta')}{\partial \eta'}, \qquad (30)$$

$$W_{BT} = 2i \sum_{l=-\infty}^{\infty} e^{-2\pi i l \Delta_q} \int_{-\infty}^{\infty} d\eta \; y_0^*(\eta + 2\pi l) \frac{\partial}{\partial \eta} \left[-\left(\frac{\mathbf{j} \cdot \mathbf{B}}{B^2} \right) y_0(\eta) \; + \; \frac{| \nabla\psi |^2}{JB^2} \left(k_{aR} + ik_{aI} + \frac{\partial S}{\partial \psi} \right) \frac{\partial y_0(\eta)}{\partial \eta} \right], \qquad (31)$$

$$N_{BT} = -2i \sum_{l=-\infty}^{\infty} e^{-2\pi i l \Delta_q} \int_{-\infty}^{\infty} J \; d\eta \; y_0^*(\eta + 2\pi l) \; y_0(\eta) \frac{| \nabla\psi |^2}{B^2} \left(k_{aR} + ik_{aI} + \frac{\partial S}{\partial \psi} \right), \qquad (32)$$

$$N = \sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i l \Delta_q}}{k_{aI} - \pi i l q_a'} \int_{-\infty}^{\infty} J \; d\eta \; y_0^*(\eta + 2\pi l) \; y_0(\eta) \times \qquad (33)$$

$$\times \left[\frac{1}{| \nabla\psi |^2} + \frac{| \nabla\psi |^2}{B^2} \left(k_{aR} - ik_{aI} - 2\pi l q_a' + \frac{\partial S}{\partial \psi} \right) \left(k_{aR} + ik_{aI} + \frac{\partial S}{\partial \psi} \right) \right].$$

In the above Eqs.(30-33), all quantities are evaluated at the plasma boundary flux surface ψ_a. There is an explicit dependence on the equilibrium parameter

$$\Delta_q = nq_a - m_0, \qquad (34)$$

where m_0 is the integer nearest to nq_a, so that $-1/2 \le \Delta_q \le 1/2$. As functions of Δ_q, W_V, W_{BT}, N_{BT} and N are periodic in the interval $[-1/2, 1/2]$. This is a reflection of the translational invariance of the system with respect to the fast radial variable nq, since, for $n \gg 1$, there are many equivalent

mode rational surfaces within the scale of the equilibrium variation. Despite the explicit appearance of n in the expression for the vacuum potential energy, the latter is of order unity and comparable to the other terms as $n \to \infty$ because $G_n = O(1/n)$ and the integrals are dominated by the contribution from the region $\eta \simeq \eta'$ where the Green's function peaks. The boundary term that arises from the plasma potential energy, W_{BT}, consists of two pieces: the first one represents the kink instability drive due to the equilibrium parallel current, and the second one is a residual contribution from the magnetic field line bending energy. We note that the evaluation of the norm N, using Eqs.(14) and (29), required an integration over ψ that could be carried out analytically to leading order in $n \gg 1$ by virtue of the exponential dependence of $Y_t(\psi, \chi)$ on ψ. Finally, it is worth pointing out that, although not immediately obvious, the expressions (30-33) for W_V, W_{BT}, N_{BT} and N are real and N is positive definite as they should.

We can now state our stability criterion for high-n, edge-localized modes. In terms of the quantities W_V, W_{BT}, N_{BT} and N given by Eqs.(30-33), the necessary condition for stability is that, for any positive value of λ^2,

$$\frac{W}{N} = \frac{W_V + W_{BT} + \lambda^2 N_{BT}}{N} - \lambda^2 > 0. \tag{35}$$

If, for some range of λ, W/N becomes negative, then the equilibrium is unstable and our variational estimate of the instability growth rate γ is:

$$-\rho_m \, \gamma^2 = Min_\lambda \left[\left(\frac{W}{N} \right)(\lambda) \right], \tag{36}$$

where ρ_m is some average of the mass density over the region where the mode is localized, namely the plasma edge.

IV. CONCLUDING REMARKS.

Our stability criterion for edge localized, high-n modes has some features that are worthy of note. It applies to external modes in general two-dimensional equilibria, but involves only one-dimensional analysis. It applies also to the full range of values of the equilibrium parameter Δ_q, and shows manifestly the periodic dependence on this parameter due to the presence of multiple mode resonant surfaces in the toroidal equilibrium. For finite values of Δ_q, the effect of the coupling of multiple poloidal harmonics in two-dimensional toroidal equilibria cannot be neglected, and our ballooning formalism approach incorporates automatically such mode coupling effect with complete generality. This is in contrast with the analytic peeling mode criterion[7,9,10]. The latter is based on single poloidal

mode analysis, and therefore should be applicable only for small values of Δ_q, when the perturbation is dominated by the single poloidal harmonic whose resonant surface approaches the plasma boundary. Finally, our criterion allows evaluation of instability growth rates, albeit in a variational sense. This limitation is due to the fact that we restrict ourselves to trial functions that can be represented with the ballooning formalism, and this might not cover the full range of allowable plasma perturbations. Also, in order that our trial function be physically acceptable, the radial wavenumber k, obtained as a complex eigenvalue of the local ballooning equation, must have a positive imaginary part. While we do not have a general proof that such a solution will always exist, we have found this to be the case for the $s - \alpha$ equilibrium model[3]. The application of our theory to the $s - \alpha$ model will be reported in detail in a forthcoming publication.

ACKNOWLEDGMENTS.

REFERENCES.

[1] T. Ozeki, M.S. Chu, L.L. Lao, T.S. Taylor, M.S. Chance, S. Kinoshita, K.H. Burrell and R.D. Stambaugh, Nucl. Fusion **30**, 1425 (1990).

[2] H.R. Wilson, J.W. Connor, A.R. Field, R.J. Hastie, R.L. Miller and J.B. Taylor, Nucl. Fusion **40**, 713 (2000).

[3] J.W. Connor, R.J. Hastie and J.B. Taylor, Phys. Rev. Letters **40**, 396 (1978).

[4] A.H. Glasser, in Proc. of the Finite Beta Theory Workshop, Varenna, Italy, 1977 (U.S. Dept. of Energy, Washington DC, 1979) p.55.

[5] Y.C. Lee and J.W. Van Dam, ibid. p.93.

[6] J.W. Connor, R.J. Hastie and J.B. Taylor, Proc. R. Soc. A **365**, 1 (1979).

[7] J.W. Connor, R.J. Hastie, H.R. Wilson and R.L. Miller, Phys. Plasmas **5**, 2687 (1998).

[8] I.B. Bernstein, E.A. Frieman, M.D. Kruskal and R.M. Kulsrud, Proc. R. Soc. A **244**, 17 (1958).

[9] D. Lortz, Nucl. Fusion **15**, 49 (1975).

[10] J.A. Wesson, Nucl. Fusion **18**, 97 (1978).