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Theoretical prediction of β and τ_E in a hardcore Z-
pinch

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ABSTRACT

The energy confinement time and maximum achievable pressure are critical figures of merit for any proposed magnetic fusion concept. The present work focuses on these issues for a hardcore Z-pinch, which is the cylindrical limit of a large aspect ratio levitated dipole (e.g. LDX-Levitated Dipole Experiment). An analysis is presented that theoretically predicts both τ_E and β for this configuration. The model makes the optimistic assumption that transport is purely classical in the region of the profile that is magnetohydrodynamically (MHD) stable against interchange modes. In the interchange-unstable region use is made of the quasilinear theory described in the accompanying paper [1] which shows that the plasma pressure relaxes to the MHD marginally stable profile while the density evolves to $n \propto \left[\oint dl / B \right]^{-1}$. Analytic and numerical calculations lead to explicit scaling relations for τ_E and β which can be tested in future LDX experiments.

I. Introduction

The goal of this analysis is to obtain a theoretical prediction for the energy confinement time τ_E and maximum plasma β in a hardcore Z-pinch configuration, the cylindrical limit of the levitated dipole concept as embodied in the Levitated Dipole Experiment (LDX) [2,3]. These are critical figures of merit for any magnetic fusion concept. Their values depend on the magnetohydrodynamic (MHD) and micro stability properties of the magnetic geometry as well as on experimentally controlled parameters such as the heating power, particle density, coil current, and plasma-wall interface.

To put the problem in perspective recall that the tokamak community has been working for decades to obtain a first principles understanding of anomalous heat transport, an effort that has only recently come close to fruition. With a much newer and much smaller program one might ask how the LDX community expects to achieve the same end goal in such a short time. The answer is that in some ways the LDX physics, although anomalous, is simpler than that of the tokamak. Specifically, tokamaks typically operate in a regime that is MHD stable and the anomalies are due to the more complicated, nonlinear evolution of weaker instabilities such as the ion temperature gradient mode, the electron temperature gradient mode, and the trapped electron mode [4]. The operation of a tokamak too close to the MHD “Troyon stability limit” is inherently dangerous due to potential plasma disruptions, which can cause physical damage to the device. The stability of the levitated dipole on the other hand will likely be dominated by the ideal MHD interchange mode. It is believed that violation of the MHD interchange stability limit will result in a “soft-landing”, relaxing the pressure to its marginally stable profile [1]. If correct, this would give scientists the freedom to conduct experiments very close to the MHD stability limit. Therefore a simpler model that focuses on MHD rather than micro turbulence, accompanied by a simpler analysis may hopefully lead to a reasonably accurate prediction in a relatively short time.

In terms of comparative physics note that both the levitated dipole and the tokamak can be driven MHD unstable by unfavorable magnetic field line curvature. One specific tokamak instability, the localized interchange mode, is usually easily stabilized by magnetic shear. This effect is absent in a closed line configuration such as LDX.

Instead, interchange stability, which is the strictest limitation in LDX, is provided by plasma compressibility.

At present, no direct attempts have been made to predict the energy confinement time and beta in a dipole configuration, although several implicit attempts have been made by the LDX community by calculating expected plasma pressure profiles for point designs. It is worthwhile to review these calculations before describing the present calculation. One of the first calculations is due to Garnier *et.al.* [5]. This calculation assumed an interchange stable pressure profile which obeyed the marginal stability condition outside the peak of the heating source and was a polynomial function in the inner region of plasma. This approach is presently used by the LDX experimental team for reconstruction purposes [6].

Ricci, Rogers and Dorland have also recently run the nonlinear GS2 code to estimate particle transport in LDX [7]. Other examples of pressure profile calculations involve analytical equilibrium and stability properties in the ideal point dipole geometry, carried out by Krasheninnikov, Catto and Hazeltine [8,9]. Recently, Guazzotto and Freidberg have presented numerical calculations predicting the equilibrium beta limit for the actual LDX geometry with and without flow[10]

Along a different path, a nonlinear numerical simulation has been carried out by Pastukhov and Chudin [11,12]. They use a time dependent, reduced MHD model in cylindrical geometry to model a hard core Z-pinch. Their results show that a strongly heated plasma transports away all excess energy through chaotic large-scale convective cells. The end result is a quasi-static pressure profile that hovers near the theoretical marginally stable profile.

Each of these calculations sheds valuable insight into the behavior of transport in a levitated dipole. However, none actually derives simple analytic scaling relations showing the dependence of τ_E and β on plasma parameters, similar to the L-mode and H-mode scaling relations for a tokamak. That is the goal of the present work.

The approach taken to achieve this goal is as follows. The plasma profile is separated into two regions. In the first region, near the levitation coil, the plasma is MHD stable. Here, a simple transport model is used to determine the steady state pressure, density, and

magnetic field profiles. The transport coefficients correspond to their classical values as derived by Braginskii [13]. Clearly this is an optimistic assumption.

In the second outer region of the plasma beyond the pressure peak, the plasma is susceptible to the $m = 0$ interchange instability. Based on the quasilinear analysis of the interchange instability evolution as described in the accompanying paper [1] we assume the plasma relaxes to its marginal state in accordance with the stability criterion first given by Kadomstev [14]. The transition between the two regions is defined as that point in which the local pressure gradient resulting from classical transport just matches the critical pressure gradient predicted by Kadomstev. The quasilinear theory also makes a prediction for the density profile, $n \propto [\oint dl / B]^{-1}$, which is used in the present analysis.

This procedure leads to explicit expressions for the pressure, density, and magnetic field profiles which can then be easily used to evaluate τ_E and β . It is worth noting that the scaling relations obtained here should be qualitatively valid for the toroidal LDX configuration. However, there may be important quantitative differences because of the tight aspect ratio in the actual LDX experiment. With this in mind our results show that a hardcore Z-pinch model of LDX predicts an energy confinement time $\tau_E = 2.3 \times 10^{-2}$ seconds and a corresponding average $\beta = 2.3 \times 10^{-3}$ for an input heating power of 30 kW, a levitation coil current of 1.3 MA, an edge temperature of 10 eV, and an edge density of $1.73 \times 10^{17} m^{-3}$. Time will tell if the experimental performance matches the theoretical predictions.

The paper is organized as follows: In Section II the MHD-transport model including sources is defined for both the MHD stable and unstable region. Section III contains several simple reference calculations using standard classical and anomalous transport models ignoring the effect of MHD instabilities. In Section IV simple analytic expressions are derived for τ_E and β using a low β , localized heating source approximation. Finally, Section V presents the numerical calculation of τ_E and β , obtained using more realistic profiles from the actual LDX experiment. From these results we compute “empirical” scaling relations for τ_E and β as a function of

experimental parameters and geometry, which then can be tested in future LDX experiments

II. The MHD-Transport model

A. The MHD stable region

The starting point for the analysis is the definition of the MHD-Transport model. The geometry of interest is shown in Fig. 1. In the region of the profile that is MHD stable, near the levitation coil, we use a standard single fluid transport model. For simplicity viscosity is neglected and it is assumed that $T_e = T_i \equiv T$. The non-trivial transport coefficients are the perpendicular ion thermal conductivity χ and the ambipolar particle diffusion coefficient D . In the MHD stable region these are given by their classical values as derived by Braginskii [13]. We focus on the steady state behavior of the plasma. Therefore the non-trivial field quantities, $p, n, T, \mathbf{B} = B\mathbf{e}_\theta$ and $\mathbf{u} = u\mathbf{e}_r$, are only functions of r . With these assumptions the plasma behavior in the stable region is described by the Braginskii transport equations

$$\begin{aligned}
 \text{Mass:} \quad & \frac{1}{r} \frac{d}{dr}(rnu) = 0 \\
 \text{Ohm's law:} \quad & nu = -D \left(\frac{dn}{dr} + \frac{n}{4T} \frac{dT}{dr} \right) \\
 & \approx -D \frac{dn}{dr} \\
 \text{Momentum:} \quad & \frac{B}{\mu_0 r} \frac{d}{dr}(rB) + \frac{dp}{dr} = -\rho u \frac{du}{dr} \\
 & \approx 0 \\
 \text{Energy:} \quad & \frac{1}{r} \frac{d}{dr} \left(rn\chi \frac{dT}{dr} \right) + S_E = \frac{3}{2} u \frac{dp}{dr} + \frac{5}{2} \frac{p}{r} \frac{d}{dr}(ru) - \eta_\perp J_\perp^2 \\
 & \approx 0
 \end{aligned} \tag{1}$$

where

$$\begin{aligned}
\chi &= \frac{r_{Li}^2}{\tau_i} \\
D &= \left(\frac{m_e}{2m_i} \right)^{1/2} \chi \\
r_{Li}^2 &= \frac{2m_i T}{e^2 B^2} \\
\tau_i &= 12\pi^{3/2} \frac{\epsilon_0^2 m_i^{1/2} T^{3/2}}{n e^4 \ln \Lambda}
\end{aligned} \tag{2}$$

Note that several approximations have been made which can be justified as follows. In the expression for the particle flux we neglect the $(n/4T)(dT/dr)$ contribution. This term essentially forces $n \sim 1/T^{1/4}$ and leads to un-physically large values of the density near the coil and the wall where the temperature is low. In these regions other physics (i.e. ionization and recombination) becomes important resulting in much smoother profiles. Neglecting the $(n/4T)(dT/dr)$ term from the outset avoids this difficulty and results in density profiles much closer to practical experimental situations.

The next approximation involves the momentum equation. Here, we neglect inertia because of the slow transport time scale. Lastly, in the energy equation there are two approximations. First, ohmic heating is neglected since it is usually small compared to the external heating power (i.e. $\eta_{\perp} J_{\perp}^2 \ll S_E$) in a levitated dipole. Second, we neglect compression, convection, and electron thermal conduction losses, which for most plasmas are small compared to the ion thermal conduction losses (i.e. $D \sim \chi_e \ll \chi_i$)

Combining these approximations and eliminating the radial velocity u leads to the desired set of fluid transport equations in the MHD stable region.

$$\begin{aligned}
\frac{1}{r} \frac{d}{dr} \left(rD \frac{dn}{dr} \right) &= 0 && \text{(mass)} \\
\frac{B}{\mu_0 r} \frac{d}{dr} (rB) + \frac{dp}{dr} &= 0 && \text{(momentum)} \\
\frac{1}{r} \frac{d}{dr} \left(rn\chi \frac{dT}{dr} \right) + S_E &= 0 && \text{(energy)}
\end{aligned} \tag{3}$$

For boundary conditions we assume that at the surface of the levitation coil, $r = r_c$, there is complete recycling of the particle flux. In terms of maximizing the total number of particles in the plasma this is an optimistic assumption. The corresponding boundary condition is given by

$$\left. \frac{dn}{dr} \right|_{r_c} = 0 \quad (4)$$

For simplicity, the boundary condition on T at the levitation coil is modeled by a perfect heat sink.

$$T|_{r_c} = 0 \quad (5)$$

There is also a boundary condition on the magnetic field. On the surface of the levitation coil Ampere's law requires that

$$B|_{r_c} = \frac{\mu_0 I_c}{2\pi r_c} \quad (6)$$

Here, I_c is the current flowing in the levitation coil. The quantity I_c is an input parameter that appears in the scaling relations.

We also require jump conditions across the stable-unstable boundary in the plasma. These are discussed shortly, after the discussion of the model for the MHD unstable region.

B. The MHD unstable region

The model used in the MHD unstable region, beyond the peak pressure, is based on the quasilinear analysis of the interchange instability [1]. For background, recall that a linear Z-pinch is potentially unstable to two ideal MHD modes [14]: (1) the $m = 0$ interchange mode and (2) the $m = 1$ helical mode. The presence of the hard core has a strong stabilizing effect on the $m = 1$ mode so that in general its critical β for instability is higher than that of the $m = 0$ mode.

Furthermore, resistivity does not alter the $m = 0$ stability condition. There is no shear stabilization to be mitigated by resistivity. Also, resistive modes for $m \neq 0$ are excited only when the interchange mode is already unstable, as has been shown by

Simakov and Catto [15]. Lastly, for simplicity, all electrostatic and kinetic modes are neglected in this study, which is an optimistic assumption.

The conclusion is that the ideal MHD $m = 0$ interchange instability is the most dangerous mode in a levitated dipole and the analysis therefore focuses on this mode.

The quasilinear theory of the interchange mode shows that as the plasma is gradually heated from a cold initial state, its pressure profile evolves in such a way that it hovers around the marginally stable boundary. The marginal stability condition for interchange modes was first derived by Kadomstev [14] and is given below. In addition, the quasilinear theory predicts that in this marginally stable region the density profile evolves to a state in which the number of particles confined in any given flux tube is independent of radius, as first conjectured by Pastukhov and Chudin [11,12]. These two results from quasilinear theory, combined with the MHD pressure balance relation represent the profile model for the MHD unstable region.

$$\begin{aligned} \frac{r}{p} \frac{dp}{dr} + \frac{2\gamma B^2}{B^2 + \mu_0 \gamma p} &= 0 && \text{(Kadomstev criterion)} \\ \frac{r}{n} \frac{dn}{dr} + \frac{2B^2}{B^2 + \mu_0 \gamma p} &= 0 && \text{(density-flux tube relation)} \\ \frac{B}{\mu_0 r} \frac{d}{dr}(rB) + \frac{dp}{dr} &= 0 && \text{(pressure balance)} \end{aligned} \quad (7)$$

In deriving these results it has been assumed that the quasilinear diffusion coefficient

$$D_Q = \sum_n \omega_i(k_n) |\xi_n|^2 \quad \text{(sum over unstable modes)} \quad (8)$$

satisfies two inequalities. First, it must be large enough so that anomalous transport dominates classical transport: $D_Q \gg \chi$. Second, it must be small enough so that nonlinear mode coupling can be neglected.

The boundary conditions assume that a source injects particles at the outer edge of the plasma establishing any desired plasma density. This is equivalent to specifying

$$n|_{r_w} = n_w \quad (9)$$

The quantity n_w is an input parameter. Note that there is a maximum value for n_w determined by the requirement that the maximum density within the plasma must satisfy the condition $\omega_{pe}^2 \leq \omega_{ce}^2$ in order for electron cyclotron heating to be effective.

The edge temperature condition is set by the wall properties for a limiter type plasma-wall interaction or by the scrape-off layer properties when a divertor is used. In either case, the boundary condition can be written as

$$T|_{r_w} = T_w \quad (10)$$

where T_w is an input parameter. Its value will be low for a limiter and higher for a divertor. For marginal stability of the interchange mode it is critical that T_w be finite.

Lastly, Ampere's law imposes a boundary condition on the edge magnetic field that can be written as

$$B|_{r_w} = \frac{\mu_0(I_c + I_p)}{2\pi r_w} \quad (11)$$

where I_p is the plasma current.

To complete the model we now need to specify the jump conditions across the stable-unstable boundary.

C. The jump conditions

The quasilinear analysis shows, not unexpectedly, that across the stable-unstable boundary, defined as $r = r_s$, the density, pressure, and magnetic field must be continuous.

$$\begin{aligned} \llbracket n \rrbracket_{r_s} &= 0 \\ \llbracket p \rrbracket_{r_s} &= 0 \\ \llbracket B \rrbracket_{r_s} &= 0 \end{aligned} \quad (12)$$

The situation with respect to the continuity of the fluxes is slightly complicated. The actual quasilinear diffusion equations are second order differential equations, which then require that the fluxes be continuous across the boundary. Even so, the quasilinear analysis also shows that when $D_\perp \gg \chi$ the fluxes change rapidly within a narrow transition layer at the stability-instability boundary. The consequence is that an accurate approximation to the profiles is obtained by ignoring to leading order in χ/D_\perp the flux contributions from the stable region. This allows us to integrate the corresponding transport equations in the unstable region one time, resulting in the two first order

differential equations appearing in Eq. (7). These are the only conditions required for the complete solution. This important point is explicitly demonstrated in the accompanying paper describing the quasilinear analysis.

There is one last condition required to close the system, one that determines the location of the stability-instability boundary. The boundary location corresponds to the point where the pressure gradient in the classical region just equals the critical marginal stability value in the unstable region. Mathematically, the value of r_s is determined by requiring that

$$\left[\frac{r}{p} \frac{dp}{dr} + \frac{2\gamma B^2}{B^2 + 2\mu_0\gamma p} \right]_{r_s^-} = 0 \quad (13)$$

This condition implies that in the region $r_c \leq r \leq r_s$ the plasma is stable against the interchange mode and classical transport should apply. In the region $r_s \leq r \leq r_w$ the classical transport profiles would violate the MHD stability condition. Here, the profiles relax to their marginal stability profiles as determined by quasilinear theory. The actual situation is somewhat more complicated and is discussed in more detail as the calculation progresses. Looking ahead, the end result is that Eq. (13) is indeed correct but that there is a subsidiary constraint on the minimum heating power that must be satisfied. For practical situations this is usually accomplished quite easily.

III. The reference cases

In this section τ_E and β are calculated for two reference cases that serve as a basis for comparison with the quasilinear transport model. The two cases correspond to (1) classical transport over the entire plasma and (2) Bohm transport over the entire plasma. To make calculations analytically tractable, the localized energy source in all reference cases is approximated by a delta function.

A. Classical transport

Consider the case of classical transport. For analytic simplicity we focus on the low β regime where the magnetic field can be accurately approximated by

$$B_\theta(r) \approx \frac{\mu_0 I_c}{2\pi r} \quad (14)$$

Here, I_c is the current flowing in the levitation coil. The relevant equations and boundary conditions are given by

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left(r D_c \frac{dn}{dr} \right) &= 0 & \frac{dn(r_c)}{dr} &= 0, \quad n(r_w) = n_w \\ \frac{1}{r} \frac{d}{dr} \left(r n \chi_c \frac{dT}{dr} \right) &= -S_0 \delta(r - r_h) & T(r_c) &= T(r_w) = 0 \end{aligned} \quad (15)$$

The solution to the density equation is easily found:

$$n(r) = n_w = \text{const.} \quad (16)$$

The density is uniform across the plasma.

The energy equation can be simplified as follows. First note that the wall edge temperature appearing in the boundary conditions has been set to zero: $T_w = 0$. Even for very modest heating powers the profiles are only weakly dependent on this parameter. Thus, it can be set to zero with a negligible error. Next, the source amplitude S_0 is related to the total input power P by the relation

$$S_0 = \frac{P}{4\pi^2 R_0 r_h} \quad (17)$$

The Braginskii thermal diffusivity coefficient has already been specified in Eq. (2).

Lastly, we normalize the radius to the coil radius: $x = r/r_c$. A short calculation then yields a simplified form for the energy equation.

$$\begin{aligned} \frac{1}{x} \frac{d}{dx} \left(\frac{x^3}{T_V^{1/2}} \frac{dT_V}{dx} \right) &= -\alpha_c \delta(x - x_h) \\ \alpha_c &= 3.0 \times 10^4 \frac{P_K I_M^2}{R_0 r_h r_c n_{\max}^2} \end{aligned} \quad (18)$$

Here, we have converted to practical units as follows: T_V (eV), I_M (MA), P_K (kW), and n_{\max} ($10^{17} m^{-3}$).

The solution to Eq. (18) is easily found. Simple analytic expressions are obtained if we consider the reasonably realistic geometric regime $1 \ll x_h \ll x_w$. In this regime the solution can be expressed as

$$\begin{aligned} T_V &\approx \left(\frac{\alpha_C}{4x_h}\right)^2 \left(1 - \frac{1}{x^2}\right)^2 & 1 \leq x \leq x_h \\ T_V &\approx \left(\frac{\alpha_C x_h}{4}\right)^2 \left(\frac{1}{x^2} - \frac{1}{x_w^2}\right)^2 & x_h \leq x \leq x_w \end{aligned} \quad (19)$$

From the solution, three interesting, experimentally relevant quantities can be evaluated: (1) the energy confinement time τ_E , (2) the average plasma beta $\bar{\beta}$, and (3) the maximum temperature T_{\max} . These are defined and given by

$$\begin{aligned} \tau_E &\equiv \frac{3 \int n T d\mathbf{r}}{P} = 1.1 \times 10^5 \frac{P_K I_M^4}{R_0^2 r_h^2 n_{\max}^3} \quad \text{sec} \\ \bar{\beta} &\equiv \frac{2\mu_0 \langle p \rangle}{B_\theta^2(r_w)} = 2.3 \times 10^2 \frac{P_K^2 I_M^2}{R_0^2 r_h^2 n_{\max}^3} \\ T_{\max} &= 5.6 \times 10^7 \frac{P_K^2 I_M^4}{R_0^2 r_h^4 n_{\max}^4} \quad \text{eV} \end{aligned} \quad (20)$$

Observe, as expected, the very favorable scaling with power and current resulting from classical transport. For the anticipated LDX parameters listed in Table 1,

| r_c (m) | r_h (m) | r_w (m) | R_0 (m) | n_{wall} ($10^{17} m^{-3}$) | P_K (kW) | I_M (MA) | T_w (eV) |
|-----------|-----------|-----------|-----------|--|------------|------------|------------|
| 0.15 | 0.30 | 1.95 | 0.38 | 1.73 | 30 | 1.3 | 10 |

Table 1 Anticipated LDX parameters

we find that $\tau_E = 5.3 \times 10^7$ seconds. This is an enormously optimistic value. In fact the confinement is so good that $\bar{\beta} > 1$, clearly violating the original low β approximation.

The approximation is only valid for sufficiently low heating powers satisfying $P \leq 10W$. At higher values of the heating power, both the low β approximation fails and the interchange mode is excited. Clearly, this region of validity is uninteresting from the experimental standpoint. At a critical heating power $P \approx 10W$ the energy confinement time can be as large as 1.8×10^4 seconds. Even so, this optimistic result serves as a useful reference point for measuring how much reduction occurs in τ_E when classical confinement is replaced by quasilinear transport due to the interchange mode.

B. Bohm diffusion

As another point of reference we can redo the calculation assuming Bohm rather than classical diffusion. There is no justification for using Bohm diffusion. It simply serves as a second reference point. For Bohm diffusion the only change is to replace the classical value of χ_c with the Bohm value as follows [16].

$$\chi_B = \frac{1}{16} \frac{T}{eB_\theta} \quad (21)$$

The modified equation for the temperature reduces to

$$\frac{1}{x} \frac{d}{dx} \left(x^2 T_V \frac{dT_V}{dx} \right) = -\alpha_B \delta(x - x_h) \quad (22)$$

$$\alpha_B = 5.1 \times 10^3 \frac{P_K I_M}{R_0 r_h n_{\max}}$$

The solution is again easily found and in the limit $1 \ll x_h \ll x_w$ can be written as

$$T_V = \left[2\alpha_B \left(1 - \frac{1}{x} \right) \right]^{1/2} \quad 1 \leq x \leq x_h \quad (23)$$

$$T_V = \left[2\alpha_B x_h \left(\frac{1}{x} - \frac{1}{x_w} \right) \right]^{1/2} \quad x_h \leq x \leq x_w$$

For this case the experimental parameters of interest are given by

$$\begin{aligned}
\tau_E &= 7.5 \times 10^{-2} \left(\frac{R_0 r_w^3 I_M n_{\max}}{P_K} \right)^{1/2} \text{ sec} \\
\bar{\beta} &= 1.6 \times 10^{-4} \left(\frac{r_w^3 n_{\max} P_K}{R_0 I_M^3} \right)^{1/2} \\
T_{\max} &= 1.0 \times 10^2 \left(\frac{P_K I_M}{R_0 r_h n_{\max}} \right)^{1/2} \text{ eV}
\end{aligned} \tag{24}$$

For the LDX parameters given in Table 1 we find that $\tau_E = 3.4 \times 10^{-2}$ seconds, $\bar{\beta} = 3.4 \times 10^{-3}$, and $T_{\max} = 1.4$ keV. As expected, Bohm diffusion leads to a large reduction in confinement as compared to classical transport. Note also that the scaling with the applied power P_K is qualitatively different. Confinement improves with increasing power for classical transport while it degrades for Bohm diffusion.

IV. Quasilinear transport

With the reference cases established we next turn to the more interesting model that takes into account quasilinear transport due to interchange modes. The configuration of interest is illustrated in Fig. 2.

Note that from a mathematical point of view this is a three, possibly four, region problem. We now solve the problem region by region, starting from the levitation coil and working our way out to the surrounding wall. In the first region near the coil, $r_c < r < r_h$, we assume classical transport and as for the reference cases we require $T(r_c) = 0$ and $dn(r_c)/dr = 0$.

As before, the solution to the particle diffusion equation satisfying the boundary condition at $r = r_c$ implies that the density is uniform across the region.

$$n_1(x) = n_{\max} \quad 1 < x < x_h \quad (24)$$

Here $n_{\max} = \text{const.}$ is the maximum value of the density across the entire profile. Its value in LDX is usually determined by the accessibility requirement associated with electron cyclotron heating, $\omega_{pe}^2 \leq \omega_{ce}^2$. At the limit, this condition can be expressed as

$$n_{\max} = \frac{\epsilon_0 B_\theta^2(r_h)}{m_e} = 3.9 \frac{I_M^2}{r_h^2} \quad 10^{17} m^{-3} \quad (25)$$

The temperature in region 1 satisfies the classical diffusion equation:

$$\frac{1}{x} \frac{d}{dx} \left(\frac{x^3}{T_V^{1/2}} \frac{dT_V}{dx} \right) = -\alpha_c \delta(x - x_h) \quad (26)$$

where α_c is given in Eq. (18). The solutions satisfying the boundary at $r = r_c$ is given by

$$T_1(x) = \left[\frac{\alpha_c x_h}{4} \lambda \left(1 - \frac{1}{x^2} \right) \right]^2 \quad \text{eV} \quad 1 \leq x \leq x_h \quad (27)$$

In this expression λ is an as yet undetermined positive, dimensionless, free integration constant. Its value is ultimately found by matching onto the quasilinear solutions.

The second region, $r_h < r < r_s$, between the heating source and the quasilinear region is also characterized by classical transport. The solution here is connected to the first region by the jump conditions

$$\begin{aligned} \llbracket n \rrbracket_{r_h} &= 0, & \llbracket D_C (dn/dr) \rrbracket_{r_h=0} &= 0 \\ \llbracket T \rrbracket_{r_h} &= 0, & \llbracket n \chi_C (dT/dr) \rrbracket_{r_h} &= -S_0 \end{aligned} \quad (28)$$

The solutions for the density and temperature are easily found and can be written as

$$\begin{aligned} n_2(x) &= n_{\max} \\ T_2(x) &= \left[\frac{\alpha_c x_h}{4} \left(\frac{1-\lambda}{x^2} - \frac{1-\lambda x_h^2}{x_h^2} \right) \right]^2 \quad \text{eV} \quad x_h \leq x \leq x_s \end{aligned} \quad (29)$$

The solution in region 2 is valid as long as the interchange stability condition is not violated¹. Mathematically, the solution is valid when $K(x) > 0$, where $K(x)$ is the Kadomstev stability function. Specifically, stability requires that

$$K(x) \equiv \frac{2\gamma B_\theta^2}{B_\theta^2 + \mu_0 p} + \frac{r}{p} \frac{dp}{dr} \approx 2 \left[\gamma - \frac{2(1-\lambda)x_h^2}{\lambda x_h^2(x^2-1) - (x^2-x_h^2)} \right] > 0 \quad (30)$$

The approximate relation is valid in the low β limit. A detailed analysis of this relation combined with the quasilinear solutions in region 3 indicates that there is a wide range of possibilities for the existence of multiple classical and quasilinear regions in the pressure profile. To simplify the analysis we focus on the regime that is of most interest experimentally, corresponding to modest-to-high heating powers. It is shown shortly that in this regime $\lambda \ll 1$. The analysis is further simplified by the reasonable approximation $x_h^2 \gg 1$ and assuming that $\lambda x_h^2 \sim 1$ for a maximal ordering. Under these conditions we see that instability occurs just past the location of the heating source (i.e. $x \rightarrow x_h$) when

$$K(x) \approx 2 \left[\gamma - \frac{2}{\lambda x^2} \right] < 0 \quad x \rightarrow x_h \quad (31)$$

or

$$\lambda x_h^2 < \frac{2}{\gamma} \quad (32)$$

Hereafter, we assume the plasma is operating in a regime in which Eq. (32) is satisfied. The implication is that region 2 becomes vanishingly thin and the solution for $x > x_h^+$ (i.e. region 3) must satisfy the quasilinear diffusion equations. As shown in the accompanying paper the quasilinear solutions are connected to the classical solutions in region 2 by the following jump conditions.

$$\llbracket n \rrbracket_{r_h} = 0, \quad \llbracket T \rrbracket_{r_h} = 0 \quad (33)$$

The results of the quasilinear analysis predict that the solutions in region 3 should hover around the marginal stability boundary. Thus, when taking into account the jump conditions we see that these profiles can be written as

¹ Note that the solution in region 1 automatically satisfies the stability criterion since $dp_1/dr > 0$

$$\begin{aligned}
n_3(x) &= n_{\max} \left(\frac{x_h}{x} \right)^2 \\
T_3(x) &= T_{\max} \left(\frac{x_h}{x} \right)^{2(\gamma-1)}
\end{aligned} \tag{34}$$

where

$$T_{\max} = \left[\left(\frac{\alpha_C x_h \lambda}{4} \right) \left(1 - \frac{1}{x_h^2} \right) \right]^2 \approx \left(\frac{\alpha_C x_h \lambda}{4} \right)^2 \quad \text{eV} \tag{35}$$

The region of validity of the region 3 solutions is determined as follows. At any arbitrary point $x > x_h$ we must attempt to connect the region 3 solution to an additional region 4 classical solution and then re-evaluate the interchange stability criterion. If the stability condition is satisfied, then the region 4 solution is the one that must be used. If, on the other hand, the stability criterion is violated for all x in the range $x_h < x < x_w$ then no region 4 solution exists and the quasilinear solution is valid from the heating source out to the wall. This last situation is the one that is of experimental interest and occurs even for quite modest heating powers.

The next step in the analysis is to explicitly calculate the condition for no region 4 to exist. To do this we assume the opposite. A region 4 solution exists starting at a radius $x = x_s$ with $x_h < x_s < x_w$. In this region the density and temperature would be given by

$$\begin{aligned}
n_4(x) &= n_{\max} \left(\frac{x_h}{x_s} \right)^2 = \text{const.} \\
T_4(x) &= T_w \left[k_0 \left(\frac{1}{x^2} - \frac{1}{x_w^2} \right) + 1 \right]^2
\end{aligned} \tag{36}$$

where T_w is the known wall temperature and from the jump condition $[[T]]_{x_s} = 0$ the constant k_0 can be written as

$$k_0 = \left[\left(\frac{T_{\max}}{T_w} \right)^{1/2} \left(\frac{x_h}{x_s} \right)^{\gamma-1} - 1 \right] \left(\frac{x_s^2 x_w^2}{x_w^2 - x_s^2} \right) \tag{37}$$

The condition for the plasma to be interchange stable in region 4 can be easily evaluated. We find

$$K(x) = 2 \left[\gamma - \frac{2k_0 x_w^2}{k_0(x_w^2 - x^2) + x_w^2 x^2} \right] \geq 0 \quad (38)$$

The quantity $K(x)$ must be positive over the whole region with equality holding at the transition point: $K(x_s) = 0$.

The next step is to substitute k_0 from Eq. (37) into Eq. (38) evaluated at $x = x_s$. A short calculation then shows that a stable region 4 can exist only if

$$\left(\frac{T_{\max}}{T_h} \right)^{1/2} \left(\frac{x_h}{x_w} \right)^{\gamma-1} \leq \frac{2\xi^{\gamma-1}}{2-\gamma+\gamma\xi^2} \quad (39)$$

$$\xi \equiv \frac{x_s}{x_w}$$

Observe that the right hand side of Eq. (39) has a maximum when

$$\xi^2 = \frac{(\gamma-1)(2-\gamma)}{\gamma(3-\gamma)} = \frac{1}{10} \quad (40)$$

with the numerical value corresponding to $\gamma = 5/3$. It therefore follows that a sufficient condition for there *not* to exist a classical region 4 is that

$$\left(\frac{T_{\max}}{T_w} \right)^{1/2} \left(\frac{x_h}{x_w} \right)^{\gamma-1} > 4 \left(\frac{1}{10} \right)^{2/3} \approx 0.86 \quad (41)$$

Assuming that Eq. (41) is satisfied, we can now investigate the consequences of the quasilinear transport model. The main conclusion is that for all practical purposes the profiles represent a two region solution to the transport problem. The inner region from the coil up to the heating surface has classical transport. The outer region, from the heating region to the wall has quasilinear transport. There is a vanishingly thin classical region just beyond the heating surface up to the quasilinear region but the dimensions of this region are so small as to have no effect on the experimental quantities of interest.

The simplified forms of the profiles are summarized below.

$$\begin{aligned} \text{Region 1 } (r_c < r < r_h): \quad n &= n_{\max} & T &\approx T_{\max} \left[\left(\frac{r_h^2}{r^2} \right) \left(\frac{r^2 - r_c^2}{r_h^2 - r_c^2} \right) \right]^2 \\ \text{Region 3 } (r_h < r < r_w): \quad n &= n_{\max} \left(\frac{r_h}{r} \right)^2 & T &= T_{\max} \left(\frac{r_h}{r} \right)^{2(\gamma-1)} \end{aligned} \quad (42)$$

These profiles are illustrated in Fig. 3. The solutions are characterized by the following properties.

- (1) The wall density n_w must be adjusted such that the maximum density n_{\max} satisfies the electron cyclotron heating constraint.

$$n_w = n_{\max} \left(\frac{x_h}{x_w} \right)^2 = 3.9 \frac{I_M^2}{r_w^2} = 1.73 \times 10^{17} m^{-3} \quad (43)$$

Here and below all numerical values correspond to the anticipated LDX parameters given in Table 1.

- (2) With quasilinear transport the maximum temperature is completely determined by the wall temperature and the geometry. It does not depend directly on the values of the classical transport coefficient, the magnitude of the external heating power, or the coil current.

$$T_{\max} = T_w \left(\frac{r_w}{r_h} \right)^{2(\gamma-1)} = 121 \quad \text{eV} \quad (44)$$

- (3) We next check the ‘‘large power’’ assumptions made in the analysis to verify that the power levels involved are indeed reasonable. There are two assumptions to check. First note that the quasilinear analysis implies that the dimensionless parameter λ is given by

$$\begin{aligned} \lambda_{[30kW]} &= \frac{4}{\alpha_c x_h} (T_{\max})^{1/2} \\ &= 1.3 \times 10^{-4} \frac{R_0 r_c^2 n_{\max}^2 T_w^{1/2}}{P_K I_M^2} \left(\frac{r_w}{r_h} \right)^{\gamma-1} \\ &= 2.0 \times 10^{-3} \frac{R_0 r_c^2 T_w^{1/2} I_M^2}{P_K r_h^4} \left(\frac{r_w}{r_h} \right)^{\gamma-1} \\ &= 1.4 \times 10^{-3} \end{aligned} \quad (45)$$

We see that even if we lower the heating power to only $P_K = 1$ kW the value of λ is quite small: $\lambda_{[1kW]} = 4.0 \times 10^{-2}$. Thus, the inequality assumption given by Eq. (32) is well satisfied:

$$\lambda_{[30kW]} \left(\frac{r_h}{r_c} \right)^2 = 5.3 \times 10^{-3} \ll \frac{\gamma}{2} \quad (46)$$

The second “large power” assumption associated with Eq. (41) is automatically satisfied by virtue of the relationship between T_{\max} and T_w given by Eq. (44).

- (4) The critical experimental parameter, the energy confinement time, can be easily evaluated from the profiles. We obtain

$$\begin{aligned} \tau_E (\text{sec}) &\approx 9.5 \times 10^{-4} \left(\frac{\gamma}{\gamma-1} \right) \frac{R_0 r_h^2 n_{\max} T_{\max}}{P_K} \\ &= 9.3 \times 10^{-3} \left(\frac{r_w}{r_h} \right)^{4/3} \frac{R_0 I_M^2 T_w}{P_K} \\ &= 2.4 \times 10^{-2} s \end{aligned} \quad (47)$$

Observe that the confinement time scaling relation is qualitatively similar to that of Bohm diffusion in the sense that the same parameters appear in the numerators and denominators of both expressions. However, the quasilinear interchange scaling has a stronger dependence on the parameters. The comparable magnitude of τ_E in the quasilinear case is explained by the favorable behavior of the particle density in the plasma core. The particle density in Bohm diffusion remains flat over the entire profile while the quasilinear equations force the particle density to peak near the coil. A calculation with a fixed total number of particles in the chamber would result in a noticeably smaller value of τ_E in the quasilinear model.

- (5) The last parameter of interest is the volume averaged beta. Another short calculation leads to

$$\begin{aligned} \bar{\beta} &= 2.1 \times 10^{-6} \left(\frac{\gamma}{\gamma-1} \right) \frac{r_h^2 n_{\max} T_{\max}}{I_M^2} \\ &= 1.9 \times 10^{-5} \left(\frac{r_w}{r_h} \right)^{4/3} T_w \\ &= 2.4 \times 10^{-3} \end{aligned} \quad (48)$$

Interestingly, when the density is set to its maximum value consistent with electron cyclotron heating, the value of $\bar{\beta}$ is independent of both power and current. It

depends primarily on the geometry and the value of the edge temperature. Its value is relatively low thereby providing justification for the low β approximation used in much of the analysis. We can interpret the absence of a P_K dependence as follows. When the heating power is sufficiently large, the enhanced transport due to the interchange mode is so strong that any additional power supplied to the plasma is immediately lost by quasilinear heat diffusion. In other words increasing the power input does not increase the plasma pressure.

The results of the analysis are conveniently summarized in Fig. 4. Plotted here are curves of τ_E and $\bar{\beta}$ versus P_K . Also shown for comparison are the corresponding curves for classical and Bohm transport. The interchange mode does indeed have a dramatic impact on confinement.

V. Numerical results

A more accurate evaluation of the critical plasma scaling relations has been obtained by solving the MHD-Transport model numerically. Three main improvements are introduced with respect to the simple analytic model: (1) finite β is allowed, although this leads to only a small correction, (2) a more realistic distributed heating source is included, and (3) no assumptions are made with respect to the geometric dimensions of the system (i.e. we do not assume $r_c \ll r_h \ll r_w$).

The parameters for the base case correspond to those given in Table 1 and are close to the anticipated values for LDX when operation starts with the coil fully levitated [3]. We have also eased the restriction on the maximum particle density in the plasma core, allowing $\omega_{pe}^2 \leq 2\omega_{ce}^2$ reflecting the possibility of using the extraordinary wave to heat the plasma. The coil temperature is taken to be $T_c = 0.1\text{eV}$. The localized heating source used is modeled as a shifted Maxwellian, and can be written as

$$S_E(r) = \frac{P}{4\pi^{5/2}R_0r_h\Delta r} \exp\left[-\left(\frac{r-r_h}{\Delta r}\right)^2\right] \quad (49)$$

All parameters have already been specified except for the profile width, which is assumed to be $\Delta r = 3\text{cm}$.

The numerical solutions are obtained using a straightforward iterative procedure. The density and temperature profiles for the base case are illustrated in Fig. 5. They are quite similar in appearance to the simple analytic profiles illustrated in Fig. 3. The critical figures of merits for this case are as follows,

$$\begin{aligned}\tau_E &= 2.3 \times 10^{-2} \text{ s} \\ \bar{\beta} &= 2.3 \times 10^{-3} \\ T_{\max} &= 172 \text{ eV}\end{aligned}\tag{50}$$

The values obtained from the simple analytic model are quite similar to the numerical values. The numerical model predicts a higher core temperature due to the finite width of the energy source. A strong energy source deposits enough power in the inner region, just inside the pressure peak, to cause the onset of instability at $r \approx .25\text{m}$. This creates a more favorable temperature and higher particle density in the plasma core. These changes, however only marginally influence τ_E and $\bar{\beta}$ due to low volume of the affected plasma.

Variation of the input parameters around the base point leads to a set of “empirical” scaling relations for the critical plasma parameters. These represent one of the main results of the paper and are given by

$$\begin{aligned}\tau_{[\text{sec}]} &= 2.61 \times 10^{-4} \times P^{-0.96} r_c^{-0.45} r_w^{3.21} r_h^{-1.08} T_w^{0.98} n_w^{0.92} I_c^{0.08} \\ \bar{\beta} &= 1.53 \times 10^{-6} \times P^{0.04} r_c^{-0.43} r_w^{3.19} r_h^{-1.08} T_w^{0.98} n_w^{0.92} I_c^{-1.91} \\ T_{\max} &= 1.07 \times P^{0.05} r_c^{-0.18} r_w^{1.13} r_h^{-1.31} T_w^{0.99} n_w^{-0.09} I_c^{0.07}\end{aligned}\tag{52}$$

Similar to the tokamak scaling relation, the “natural” units are used. The power P is expressed in kilowatts (kW), all distances are in meters (m), the temperatures are in electron-volts (eV), the coil current is in mega-Amperes (MA) and the particle density is expressed in the units of 10^{17} m^{-3} . With all geometrical parameters of the machine fixed, the key drivers of the figures of merits are given below:

$$\begin{aligned}
\tau &\sim \frac{T_w^{0.98} n_w^{0.92}}{P^{0.96}} r_h^{-1.08} \\
\bar{\beta} &\sim \frac{T_w^{0.98} n_w^{0.92}}{I^{1.91}} r_h^{-1.08} \\
T_{\max} &\sim \frac{T_w^{0.99}}{r_h^{1.31}}
\end{aligned} \tag{53}$$

To test robustness of the scaling relation, a large number of numerical simulations have been carried out by varying the parameters about the base case. In particular, each parameter has been independently varied over the following ranges,

$$\begin{aligned}
3kW &< P < 90kW \\
1eV &< T_w < 50eV \\
0.01eV &< T_c < 1eV \\
1.7 \cdot 10^{16} &< n_{\max} < 1.7 \cdot 10^{18} \\
0.65MA &< I_{coil} < 6.5MA \\
10cm &< r_c < 20cm \\
r_c + 10cm &< r_h < r_c + 60cm \\
1m &< r_w < 3m
\end{aligned} \tag{54}$$

The graph of the calculated τ_E versus the predicted value from the scaling relation given by Eq. (52) is shown in Fig. 6. The empirical fits to the numerical data are accurate to within 8%.

Hopefully these results will serve as useful guidelines for the future operation of LDX.

VI. Conclusions

We have introduced an MHD-Transport model to predict the basic parameters of experimental interest for a hard-core Z-pinch, which is the cylindrical limit of the toroidal dipole concept as manifested in LDX. The model separates the transport essentially into two regions, an inner region near the coil, which is characterized by classical transport, and an outer region beyond the pressure peak characterized by quasilinear transport arising from the interchange mode. The analysis has led to the following conclusions.

- (1) The interchange mode dominates transport leading to relatively small values of τ_E , $\bar{\beta}$, and T_{\max} , comparable to Bohm transport.
- (2) Beyond a certain heating power ($P \approx 1kW$ for LDX) the model predicts that further increases will not lead to noticeable increases in the particle pressure. The anomalous transport due to the interchange mode is so strong that all excess energy is immediately lost from the system.
- (3) Within the model corresponding to a localized heating source, there are two strategies for improving performance, besides the obvious one of making the device larger. First, by increasing the coil current, the maximum achievable density consistent with ECH is raised and this leads to improved performance. The second strategy is to raise the edge temperature by improvements in the divertor design. How to do this is somewhat nebulous at present but all performance parameters improve linearly with edge temperature.
- (4) A more subtle possibility to improve machine performance arises in the toroidal LDX geometry, where the mod-B surfaces cross the field lines and broaden ECRH heating profile. With multiple frequency energy sources, the LDX team can potentially create a tailored low power heating profile that causes the pressure profile to be just below its marginal shape as determined by the simple localized heating source. This causes the energy confinement time to increase to its classical value since by definition the profile is everywhere stable to the interchange mode: there is a large increase in τ_E . Even so the peak temperature and average beta remain unchanged since their values are primarily determined by the edge conditions and the geometry, and not the magnitude of the heating power. Also, note that the system will not experience an inward particle diffusion. Therefore, in order to raise the particle density and the pressure in a plasma core, use should be made of volume particle sources, such as pellet injectors. Lastly, an ultimately successful ignition of the plasma will produce a strong localized heating source, thereby strongly limiting the possibility of external control of the heating profile.

Overall, even with these limitations, the MHD-Transport model predicts that LDX should achieve reasonable performance once the coil is levitated if an edge temperature of $T_w = 10$ eV is achievable.

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Figure captions

1. Hard core Z-pinch model of LDX
2. Different regions of stability in the quasilinear transport model
3. A base case: A low beta plasma with the quasilinear diffusion: particle density and ion temperature profiles
4. Analytical predictions of figures of merits for low beta plasmas with the classical, Bohm's and quasilinear diffusion: (a) Energy confinement time τ_E , (b) Average $\bar{\beta}$
5. The base case: particle density and ion temperature profiles predicted by the numerical model
6. Numerically calculated τ_E vs. the predicted value by the "empirical" scaling relation