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Transversely-Flowing Strongly-Magnetized Plasma**

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Ion collection by oblique surfaces of an object in a transversely-flowing strongly-magnetized plasma

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The equations governing a collisionless obliquely-flowing plasma around an ion-absorbing object in a strong magnetic field are shown to have an exact analytic solution even for arbitrary (two-dimensional) object-shape, when temperature is uniform, and diffusive transport can be ignored. The solution has an extremely simple geometric embodiment. It shows that the ion collection flux density to a convex body's surface depends only upon the orientation of the surface, and provides the theoretical justification and calibration of oblique 'Mach-probes'. The exponential form of this exact solution helps explain the approximate fit of this function to previous numerical solutions.

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Ion collection by solid objects is key to a variety of phenomena including the charging of dust[1] and spacecraft[2], and the measurement of plasma parameters by electric probes[3]. In strongly magnetized situations, where the ion Larmor radius is much smaller than the object, the ion collection is highly anisotropic. The ion motion parallel to the magnetic field can be approximated as governed by free-flowing compressible hydrodynamic equations, while perpendicular motion is slower, diffusive or, as in the present work, arising from impressed transverse flow. For convenience the object is here considered to be stationary and the plasma flowing, but this is obviously equivalent to motion of the object in a stationary plasma. A remarkable theoretical result is derived here consisting of a direct analytic solution of the equations governing collisionless ion collection when the plasma has fixed temperature, fixed transverse velocity, negligible transverse diffusion, and arbitrary parallel velocity in the external, uniform unperturbed region. This solution helps resolve remaining questions concerning the calibration of so-called Mach-probes, for measuring plasma flow, and provides analytic expressions for the collection flux density as a function of position on a (two-dimensional) object of essentially arbitrary shape.

When, as is typical, the Debye length is short compared with other lengths in the problem, the interesting plasma region is quasineutral. The charged thin sheath surrounding the object can be ignored except in so far as it imposes a boundary condition upon the plasma 'presheath' solution: the 'Bohm condition'[4]. If the ion Larmor radius is also negligibly small, the 'magnetized presheath' boundary layer where ions are accelerated across the magnetic field can also be ignored. The plasma transport anisotropy arising from the magnetic field causes the presheath perturbation to the (uniform) background plasma to elongate along the magnetic field until cross-field divergence makes up the parallel flow to the object[3]. Full solutions to the non-linear equations for *diffusive* transverse transport have been obtained numerically[5, 6]. The scaled solution, and hence the

collected flux, depends only upon the ratio of momentum diffusivity to particle diffusivity. For given ratio (usually unity), it yields the dependence of the collected ion flux density on the parallel Mach-number, which amounts to the calibration factor for a (parallel) Mach-probe, when the external perpendicular velocity is ignorable.

To measure the perpendicular plasma velocity, when it is not negligible, one can use oblique collection faces, which show sensitivity to that flow. The calibration of that sensitivity was calculated[7] using an extension of the one-dimensional approximation of [5, 6], still treating the perpendicular transport as diffusive, but accounting for the boundary-condition modification[8] that the transverse flow causes. This solution can be shown to be a Galilean transformation of the solution for zero transverse flow[3]. So it is appropriate for an oblique surface of effectively infinite dimension in the transverse flow direction, but finite in the direction perpendicular to both flow and magnetic field. Practical Mach-probes, in contrast, are more often multi-faceted 'Gundestrup' type[9, 10], where many short adjacent collectors are used with different orientations. So it is not obvious theoretically that the diffusive solution should apply. And indeed the opposite limit is probably more appropriate, where diffusion (and viscosity) is ignored in comparison with the transverse flow. That alternative is solved here.

The continuity and parallel momentum equations for the ion fluid in steady state are

$$\begin{aligned} M_{\perp} \frac{\partial n}{\partial y} + \frac{\partial}{\partial x} (nM) &= 0 \quad , \\ nM_{\perp} \frac{\partial M}{\partial y} + nM \frac{\partial M}{\partial x} + \frac{\partial n}{\partial x} &= 0 \quad , \end{aligned} \quad (1)$$

where M and M_{\perp} are respectively the parallel (x -) and perpendicular (y -) velocities normalized to the sound speed, c_s , (i.e. the Mach numbers), and n is the density. The z -coordinate is ignorable. The perpendicular velocity M_{\perp} is taken as a constant (positive without loss of generality) which is appropriate for uniform external flow. This perpendicular velocity assumption is the

only essential difference between this problem and the *unmagnetized* quasineutral fluid-plasma problem, which can also be analysed analytically for supersonic flow past a flat plate or wedge[11, 12]. In the outer unperturbed region far from the object, $n = n_\infty$ and $M = M_\infty$. The assumption of uniform temperature is physically reasonable when the pressure term mostly arises from the electrons, which are assumed Boltzmann-distributed.

These homogeneous equations can be placed into the following form which displays their ‘characteristics’:

$$\begin{cases} \left[M_\perp \frac{\partial}{\partial y} + (M + 1) \frac{\partial}{\partial x} \right] (\ln n + M) = 0, \\ \left[M_\perp \frac{\partial}{\partial y} + (M - 1) \frac{\partial}{\partial x} \right] (\ln n - M) = 0. \end{cases} \quad (2)$$

The solution thus requires both

$$(\ln n + M) = \text{const} \quad \text{along } dx = dy(M + 1)/M_\perp, \quad (3)$$

and

$$(\ln n - M) = \text{const} \quad \text{along } dx = dy(M - 1)/M_\perp. \quad (4)$$

These will be referred to respectively as the positive and negative characteristics.

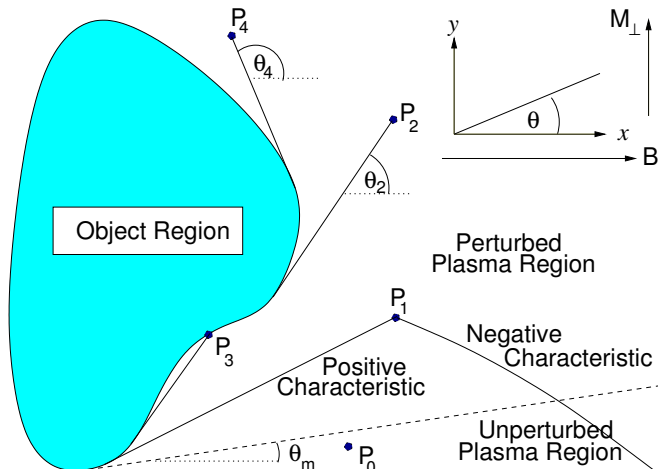


FIG. 1: Construction of the solutions at different points. P_0 is in the unperturbed region. For P_1 the characteristics are shown. P_3 is in a concave region and so its positive characteristic is not tangent at P_3 . A value $\theta > \pi/2$ such as for P_4 is not problematic.

For definiteness, we now consider plasma that is on the higher- x side (to the right) of the object. Figure 1 will be used for illustration. For any point in the plasma, a positive and a negative characteristic pass through it. If we track the characteristics backward and find that they both originate in the unperturbed plasma at $y \rightarrow -\infty$, then we know that both the conditions $\ln n + M = \ln n_\infty + M_\infty$ and $\ln n - M = \ln n_\infty - M_\infty$ (where ∞ indicates values in the unperturbed region) are satisfied at that point. These simultaneous equations

have only one solution: $n = n_\infty$, $M = M_\infty$. The point is in the unperturbed region if both its characteristics originate there; for example: P_0 . The characteristics for such points are straight lines with slopes $M_\perp/(M_\infty \pm 1)$.

Next we consider the possibility that just one of the characteristics originates not at $y = -\infty$, but on the surface of the object. We notice that since the positive characteristic (3) always has more positive dz than the negative (4), the positive characteristic is always to the left of the negative characteristic, when tracing backward from a common point. Thus the positive characteristic will always encounter the object first; so the characteristic that originates on the object is the positive one. On that characteristic, $(\ln n + M)$ is constant, but not equal to the unperturbed value. However, because of the negative characteristics, the parameters do satisfy $\ln n - M = \ln n_\infty - M_\infty$ both at the point under consideration (e.g. P_1), and along the full length of its positive characteristic. The only way to satisfy these two requirements is that, along the positive characteristic, $M = \text{const}$ and $n = \text{const}$. If $M = \text{const}$, then the slope of the characteristic, $M_\perp/(M + 1)$ is constant. It is a straight line.

The remaining question is, which straight-line is the positive characteristic? This can be answered only by taking into account the condition at the boundary between the plasma and the object. The physics of the ion absorption, when the object is negatively charged, is approximated by taking ions to flow into it as if they were flowing into a vacuum. This requires the ions to flow into the surface as fast as they can, consistent with existence of an overall solution. In other words, at the object boundary, M must be as negative as possible consistent with the overall solution. The most negative M is equivalent to the greatest slope-angle $\theta \equiv \arctan[M_\perp/(M + 1)]$ (even perhaps such that $\theta > \pi/2$). Thus for points close to the boundary, the positive characteristic selected is the line with largest θ that intersects the boundary (when tracked backwards). Such an extremum is always *tangential* to the object boundary where it intersects it. Thus all positive characteristics that originate on the boundary do so as tangents, and for any point in the perturbed plasma region *the positive characteristic is that straight line passing through the point which has greatest θ and originates as a tangent on the object*. Once that line is determined geometrically, its slope determines M and hence n at all points along it. If the steepest tangent angle is less than $\theta_m = M_\perp/(M_\infty + 1)$, then the positive characteristic does not intersect the object, but has slope $\theta = \theta_m$; and the point is in unperturbed plasma. This specifies the entire solution for an arbitrary-shaped object:

$$n = n_\infty \exp(M - M_\infty), \quad M = M_\perp \cot \theta - 1. \quad (5)$$

Because the argument concerning the boundary condition is subtle, and to connect this derivation to prior

more specific solutions, an alternative approach giving the same solution is now presented. Consider first an object which presents to the plasma a single plane face at angle θ . The beginning of this face is taken as the origin. [We can suppose that the face which connects to it in an edge at the origin is parallel to the x -axis, although that is not important.] It can be shown, as a straightforward generalization of [13], that this system (with an absorbing boundary) is self-similar. The solution depends only on the ratio x/y . Actually two self-similar forms satisfy the equations: either the trivial uniform solution $M = \text{const}$, $n = \text{const}$; or the non-uniform solution, $M = M_{\perp}x/y \mp 1$, $\ln n = \pm M + \text{const}$ (with the upper sign the correct choice for our configuration). The full solution with the appropriate boundary conditions is obtained by matching these forms by a slope-discontinuity along the line $x/y = x/y|_m = (M_{\infty} + 1)/M_{\perp}$. Thus for this single-faced problem, for $x/y \leq x/y|_m$,

$$n = n_{\infty} \exp(M - M_{\infty}), \quad M = M_{\perp}x/y - 1, \quad (6)$$

while for $x/y > x/y|_m$, $M = M_{\infty}$, $n = n_{\infty}$.

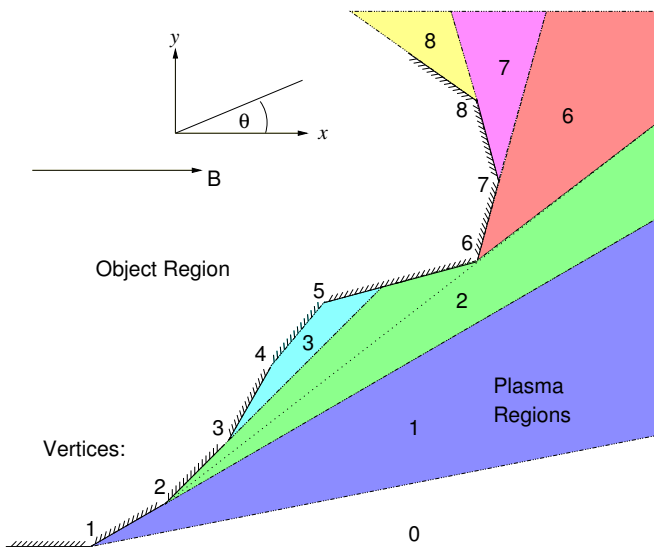


FIG. 2: Geometric construction of the regions of applicability of the solutions for a multifaceted polygonal surface.

Now consider a multi-faceted object, as illustrated in figure 2, where each (planar) facet and its leading (lower) vertex is numbered. For the first facet (from vertices 1 to 2), the solution (6) just obtained applies, in region 1. The region 0 is the unperturbed, uniform plasma, and the line separating regions 0 and 1 is the matching line $x/y = (M_{\infty} + 1)/M_{\perp}$. Because the y -velocity is fixed and positive, influence cannot propagate downwards. So the solution on the first facet is unaffected by the presence of different facets at higher y . At the trailing edge of that facet (vertex 2) a transition to a different solution must occur.

Regard vertex 2 as the new origin, relative to which the coordinates are $(x_2, y_2) = (x - p_2, y - q_2)$ (where (p_i, q_i) are the coordinates of vertex i). We can then, by just the same approach, construct a self-similar solution to the governing equations (1) in the region adjacent to facet 2; that solution depends only on the ratio x_2/y_2 . This new solution must be matched by a slope discontinuity to the solution adjacent to the first facet. There is just one line where that matching can be made, because on its continuity of M requires

$$M_1 = M_{\perp}x_1/y_1 - 1 = M_2 = M_{\perp}x_2/y_2 - 1, \quad (7)$$

which shows $x_1/y_1 = x_2/y_2$. The matching line is the projection of facet 1 beyond its upper vertex. If facet 2 were of lesser slope than facet 1, this match line would not be in the plasma region; it would be inside the object. In that case there would in fact be no second solution, the solution from facet 1 would run directly into facet 2.

Clearly the process of matching solutions can be continued to an arbitrary number of facets. The projections of the facets define wedge-shaped regions in which the solution corresponding to each facet terminates in a matching line. In region i , the complete solution is

$$n = n_{\infty} \exp(M - M_{\infty}), \quad M = M_{\perp}x_i/y_i - 1, \quad (8)$$

where $(x_i, y_i) \equiv (x - p_i, y - q_i)$. If a facet lies in a concave region of the object, like facets 4 and 5 of figure 2, then it does not possess its own plasma region. Instead the solution(s) of the earlier region(s) apply right up to the respective fractions of that facet.

There is no difficulty with the angle of the facets exceeding $\pi/2$. The solution (8) applies equally well in regions 7 and 8 of figure 2, where x_i/y_i is negative. However, if the angle of the facets exceeds π , this solution does not apply because then the plasma region would lie to the left of the object, and the negative characteristic could not reach to the surface from the right-hand side of the object. Normally, such facets are simply part of the left-hand boundary of the object and are thus governed by the appropriate left-hand version of eq (8) tracked round the left side of the object. If (part of a) facet cannot be reached from either side of the object without passing through an angle of π , then it corresponds to a region that has no plasma flowing to it either from right or left. The density there is zero. No characteristics originating in plasma pass through it. Finally, plasma in regions such that y exceeds all points on the object are said to be in the 'wake'. Normally the wake begins on the upper-most (rear-most) point of the object. At that point the solutions from the right and the left collide, forming a shock, and the simplified equations we are considering cannot be satisfied. For example, the temperature cannot be the same in the shocked region. Since the wake does not affect the ion collection by the object (in so far as equations (1) apply), it is not further analysed here.

Obviously, the continuous curved-boundary solution (5) is the straightforward limit of the multifaceted solution (8) as the number of facets tends to infinity. The discussion about facet angles, etc., applies directly to tangents for the continuous case.

The solution (5) provides an extremely simple formula for the ion flux to a surface not affected by concavity. Adding the perpendicular and parallel components, the total flux density is $nc_s(M_\perp \cos \theta - M \sin \theta) = nc_s \sin \theta$. Written as flux per unit area perpendicular to the magnetic field, this is

$$\Gamma_{\parallel} = n_{\infty} c_s \exp[-1 - (M_{\infty} - M_{\perp} \cot \theta)]. \quad (9)$$

This form indicates, first, that for points not in a concave region of the object, the collected flux depends only on the angle of the surface there, and not on the shape of the object at smaller y . This is an important, but not obvious, result for establishing Mach-probe performance. Second, the exponential dependence upon M_{∞} is within 10% of the dependence, $\exp[-1 - 1.1(M_{\infty} - M_{\perp} \cot \theta)]$, that fits the diffusive solution[3, 7]; the only difference being to replace 1.1 with 1. This agreement, well within most experiments' uncertainties, is an indication that the calibration is not strongly dependent upon the assumed cross-field transport regime. Moreover, the present solution helps to explain why the exponential form is such a good fit to the prior numerical solutions, and to give confidence that it is reasonable to extend the form to supersonic velocities, which in the present context are unproblematic. Furthermore the present solution shows unambiguously that leading faces, for which $\theta < \theta_m$ receive simply the unperturbed flux [$\Gamma_{\parallel} = n_{\infty} c_s (M_{\perp} \cot \theta - M_{\infty})$], while trailing faces, even those for which $M_{\infty} - M_{\perp} \cot \theta > 1$, are governed by the formula (9). The boundary conditions at the magnetic presheath edge arise here naturally from the analysis of the quasi-neutral equations. No prior assumption about the boundary velocity is required, although the condition is, of course, consistent with prior treatments.

In a concave region of the object, e.g. at P_3 , the surface

angle (local tangent) θ_s , is smaller than the characteristic's angle θ , and the distinction must be retained. This leads to an enhanced ion flux, equal to equation (9) times the extra factor $(M_{\perp} \cot \theta_s - M_{\perp} \cot \theta + 1)$.

For the equations used, the present solutions are exact. The formulation is, of course, approximate. Modifications both to the boundary conditions and to the equations may be appropriate in some practical applications. Nevertheless, the present analysis captures the dominant physics of the situation and helps to clarify both the operation of oblique Mach-probes and the ion collection by objects of essentially arbitrary shape when the ion Larmor radius is negligible.

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