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Gyrokinetic Equivalence

Felix I. Parra and Peter Catto

Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge MA 02139 USA

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Felix I Parra and Peter J Catto

Plasma Science and Fusion Center, MIT, Cambridge, MA 02139, USA

E-mail: `fparra@mit.edu`, `catto@psfc.mit.edu`

Abstract. We compare two different derivations of the gyrokinetic equation: the Hamiltonian approach in [Dubin D H E *et al* 1983 *Phys. Fluids* **26** 3524] and the recursive methodology in [Parra F I *et al* 2008 *Plasma Phys. Control. Fusion* **50** 065014]. We prove that both approaches yield the same result at least to second order in a Larmor radius over macroscopic length expansion. There are subtle differences in the definitions of some of the functions that need to be taken into account to prove the equivalence.

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1. Introduction

In this article we prove that the gyrokinetic results of reference [1] are completely consistent with the pioneering results by Dubin *et al* [2]. In reference [1], the recursive approach developed in [3] was generalized for nonlinear electrostatic gyrokinetics in a general magnetic field. In reference [2], the nonlinear electrostatic gyrokinetic equation was derived for a constant magnetic field and a collisionless plasma using Hamiltonian methods. The asymptotic expansion was carried out to higher order in [2] because the calculation is easier in a constant magnetic field. When the method proposed in [1] is extended to next order, the results are different in appearance, but we will prove that these differences are due to subtle differences in some definitions.

Both methods [1] and [2] are asymptotic expansions in the small parameter $\delta = \rho/L \ll 1$. Here L is a characteristic macroscopic length in the problem and $\rho = v_{\rm th}/\Omega$ is the gyroradius, with $\Omega = ZeB/Mc$ the gyrofrequency, $v_{\rm th} = \sqrt{T/M}$ the thermal velocity, T the temperature, Ze and M the charge and mass of the particle, $\mathbf{B}$ and $B = |\mathbf{B}|$ the magnetic field and its magnitude, and c the speed of light. In both methods, the phase space $\{\mathbf{r}, \mathbf{v}\}$, with $\mathbf{r}$ and $\mathbf{v}$ the position and velocity of the particles, is expressed in gyrokinetic variables, defined order by order in δ. In reference [2], the gyrokinetic variables are obtained by Hamiltonian methods and the gyrokinetic equation is found to second order in δ. In reference [1], the gyrokinetic variables are found by imposing that their time derivative is gyrophase independent. Here $d/dt \equiv \partial/\partial t + \mathbf{v}\cdot\nabla + (-Ze\nabla\phi/M + \Omega\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla_v$ is the Vlasov operator, with $\hat{\mathbf{b}} = \mathbf{B}/B$ the unit vector parallel to the magnetic field. In reference [1], the gyrokinetic equation was only found to first order. In this article, we will calculate the gyrokinetic equation and the gyrokinetic variables to higher order for a constant magnetic field, and we will compare the results with those in [2]. The orderings and assumptions are the same as those in [1], in particular, the pieces of the distribution function and the potential with short wavelengths scale as

$$f_k/f_s \sim e\phi_k/T \sim (k_\perp L)^{-1} \gtrsim \delta, \quad (1)$$

with $k_\perp L \gtrsim 1$. Here f_s is the lowest order distribution function with a slow variation in both $\mathbf{r}$ and $\mathbf{v}$. The wavenumber is characterized by its components $k_{||}$ and $k_\perp$, parallel and perpendicular to the magnetic field, respectively. The parallel wavelengths are assumed to be always comparable to the macroscopic scale, $k_{||}L \sim 1$.

2. Constant magnetic field results

The general gyrokinetic variables obtained in [1] are $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, $E = E_0 + E_1 + E_2$, $\mu = \mu_0 + \mu_1 + \mu_2$ and $\varphi = \varphi_0 + \varphi_1 + \varphi_2$, with $\mathbf{r}$ the position of the particle, $E_0 = v^2/2$ the kinetic energy, $\mu_0 = v_\perp^2/2B$ the magnetic moment, and φ_0 the gyrophase defined via $\mathbf{v}_\perp = v_\perp(\hat{\mathbf{e}}_1\cos\varphi_0 + \hat{\mathbf{e}}_2\sin\varphi_0)$. Here, $v_{||} = \mathbf{v}\cdot\hat{\mathbf{b}}$ and $\mathbf{v}_\perp = \mathbf{v} - v_{||}\hat{\mathbf{b}}$ are the velocities parallel and perpendicular to the magnetic field, and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are two unit vectors defined so that $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{b}}$ form an orthonormal system with $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. Since the unit vector $\hat{\mathbf{b}}$ is assumed constant in space and time, we can define $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ so that they are also constant, and we do so to ease the comparison with [2]. The corrections found in [1] specialized to constant magnetic field are, for the gyrocenter position $\mathbf{R}$, $\mathbf{R}_1 = \Omega^{-1}\mathbf{v} \times \hat{\mathbf{b}}$ and $\mathbf{R}_2 = -(c/B\Omega)\nabla_\mathbf{R}\widetilde{\Phi} \times \hat{\mathbf{b}}$; for the kinetic energy E, $E_1 = Ze\widetilde{\phi}/M$ and $E_2 = (c/B)(\partial\widetilde{\Phi}/\partial t)$; and for the magnetic moment μ and the gyrophase φ, $\mu_1 = Ze\widetilde{\phi}/MB$ and $\varphi_1 = -(Ze/MB)(\partial\widetilde{\Phi}/\partial\mu)$. The corrections μ_2 and φ_2 were not calculated because they were not needed to obtain the gyrokinetic equation to first order in δ under the assumptions in [1]. Here, $\widetilde{\phi} = \phi - \langle\phi\rangle$, where $\langle\phi\rangle$ is the gyroaverage holding the gyrokinetic variables $\mathbf{R}$, E, μ and t fixed, and $\widetilde{\Phi} = \int^\varphi d\varphi'\,\widetilde{\phi}(\mathbf{R}, E, \mu, \varphi', t)$, with $\langle\widetilde{\Phi}\rangle = 0$. We will see that the definitions of $\widetilde{\phi}$ and $\widetilde{\Phi}$ differ slightly from the definitions of similar functions in [2].

We require the gyroaverage of $d\mathbf{R}/dt$ and dE/dt to higher order than in [1], and we need the second order corrections μ_2 and φ_2. For constant magnetic fields, $\langle d\mathbf{R}/dt\rangle$, $\langle dE/dt\rangle$ and the correction μ_2 can be easily calculated by employing the methodology in reference [1]. We will define μ_2 so that the gyroaverage of $d\mu/dt$ is zero to order $\delta^2 v_{\rm th}^3/BL$. The correction φ_2 will not be necessary for our purposes. Once we have $\mathbf{R}$, E, μ and their derivatives to higher order, we will compare these results to both the gyrokinetic Vlasov equation and the gyrokinetic Poisson's equation in [2].

3. Time derivative of R

Employing the definitions of $\mathbf{R}_1$ and $\mathbf{R}_2$, we find

$$d\mathbf{R}/dt = v_{||}\hat{\mathbf{b}} - (c/B)\nabla\phi \times \hat{\mathbf{b}} + d\mathbf{R}_2/dt. \tag{2}$$

The gyroaverage of this expression is performed holding the gyrokinetic variables $\mathbf{R}$, E, μ and t fixed to obtain

$$\langle d\mathbf{R}/dt\rangle = u\hat{\mathbf{b}} - (c/B)\langle\nabla\phi\rangle \times \hat{\mathbf{b}}, \tag{3}$$

where $u = \langle v_{||}\rangle$. We have employed that our gyrokinetic variables are defined so that when the Vlasov operator is applied to a function with a zero gyroaverage, like $\mathbf{R}_2 = \mathbf{R}_2(\mathbf{R}, E, \mu, \varphi, t)$, the result also has a zero gyroaverage; namely $\langle d\mathbf{R}_2/dt\rangle = 0$.

The gradient $\nabla\phi$ is written in the gyrokinetic variables by using

$$\nabla\phi = \nabla\mathbf{R}\cdot\nabla_\mathbf{R}\phi + \frac{\partial\phi}{\partial\mu}\nabla\mu + \frac{\partial\phi}{\partial\varphi}\nabla\varphi \simeq \nabla_\mathbf{R}\phi + \nabla\mathbf{R}_2\cdot\nabla_\mathbf{R}\phi + \frac{\partial\phi}{\partial\mu}\nabla\mu_1 + \frac{\partial\phi}{\partial\varphi}\nabla\varphi_1. \tag{4}$$

Here, we neglect $\partial\phi/\partial E$ because the function $\mathbf{R}_1$ depends weakly on E. To obtain the second equality, we use that $\nabla\mathbf{R}_1 = 0 = \nabla\mu_0 = \nabla\varphi_0$. The gyroaverage of equation (4), obtained employing the definitions of $\mathbf{R}_2$, μ_1 and φ_1, gives

$$\langle\nabla\phi\rangle \simeq \nabla_\mathbf{R}\langle\phi\rangle - \frac{c}{B\Omega}\langle\nabla_\mathbf{R}\nabla_\mathbf{R}\widetilde{\Phi}\cdot(\hat{\mathbf{b}}\times\nabla_\mathbf{R}\phi)\rangle + \frac{Ze}{MB}\left\langle\frac{\partial\phi}{\partial\mu}\nabla_\mathbf{R}\widetilde{\phi} - \frac{\partial\phi}{\partial\varphi}\nabla_\mathbf{R}\left(\frac{\partial\widetilde{\Phi}}{\partial\mu}\right)\right\rangle. \tag{5}$$

This equation can be simplified by integrating by parts in φ to obtain

$$\left\langle\frac{\partial\phi}{\partial\mu}\nabla_\mathbf{R}\widetilde{\phi} - \frac{\partial\phi}{\partial\varphi}\nabla_\mathbf{R}\left(\frac{\partial\widetilde{\Phi}}{\partial\mu}\right)\right\rangle = \left\langle\frac{\partial\widetilde{\phi}}{\partial\mu}\nabla_\mathbf{R}\widetilde{\phi} + \widetilde{\phi}\nabla_\mathbf{R}\left(\frac{\partial\widetilde{\phi}}{\partial\mu}\right)\right\rangle = \frac{1}{2}\nabla_\mathbf{R}\left(\frac{\partial}{\partial\mu}\langle\widetilde{\phi}^2\rangle\right). \tag{6}$$

We next demonstrate that

$$\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \phi) \rangle = (1/2) \nabla_{\mathbf{R}} \langle \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\phi}) \rangle \tag{7}$$

by first noticing that

$$\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \phi) \rangle = \nabla_{\mathbf{R}} \langle \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\phi}) \rangle + \langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\Phi}) \rangle. \tag{8}$$

Integrating by parts in φ in the second term we find $\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\Phi}) \rangle = -\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\phi}) \rangle$, giving the result in (7).

Finally, substituting equations (6) and (7) into equation (5), and using the result in (3), we find

$$\langle d\mathbf{R}/dt \rangle = u\hat{\mathbf{b}} - (c/B) \nabla_{\mathbf{R}} \Psi \times \hat{\mathbf{b}}, \tag{9}$$

with

$$\Psi = \langle \phi \rangle + (Ze/2MB)(\partial \langle \widetilde{\phi}^2 \rangle / \partial \mu) + (c/2B\Omega) \langle \nabla_{\mathbf{R}} \widetilde{\phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\Phi}) \rangle. \tag{10}$$

To find u, we need $v_{||}$ as a function of the gyrokinetic variables. To do so, we use

$$v_{||}^2/2 = E_0 - \mu_0 B = E - \mu B - (E_2 - \mu_2 B), \tag{11}$$

where we employ $E_1 - \mu_1 B = 0$. According to this result, the difference between $u = \langle v_{||} \rangle$ and $v_{||}$ is necessarily of order $\delta^2 v_{\rm th}$. Once we calculate μ_2, we will be able to find u.

4. Time derivative of E

Employing the definitions of E_1 and E_2, and gyroaveraging, we find

$$\langle dE/dt \rangle = -(Ze/M) \langle \mathbf{v} \cdot \nabla \phi \rangle. \tag{12}$$

Here, we have used that $\langle dE_1/dt \rangle = 0 = \langle dE_2/dt \rangle$.

The term $\mathbf{v} \cdot \nabla \phi$ can be conveniently rewritten by employing

$$\frac{d\phi}{dt} = \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{r}} + \mathbf{v} \cdot \nabla \phi = \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} + \frac{d\mathbf{R}}{dt} \cdot \nabla_{\mathbf{R}} \phi + \frac{d\mu}{dt}\frac{\partial \phi}{\partial \mu} + \frac{d\varphi}{dt}\frac{\partial \phi}{\partial \varphi}. \tag{13}$$

Here, we neglect $\partial \phi / \partial E$ again. Solving for $\mathbf{v} \cdot \nabla \phi$ and gyroaveraging, we find

$$\langle \mathbf{v} \cdot \nabla \phi \rangle = \left\langle \frac{d\mathbf{R}}{dt} \cdot \nabla_{\mathbf{R}} \phi \right\rangle + \left\langle \frac{d\mu}{dt}\frac{\partial \phi}{\partial \mu} \right\rangle + \left\langle \frac{d\varphi}{dt}\frac{\partial \phi}{\partial \varphi} \right\rangle - \left\langle \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} \right\rangle. \tag{14}$$

To simplify the calculation, let us assume that we knew the corrections $\mathbf{R}_3$, $\mu_3 - \langle \mu_3 \rangle$, φ_2 and φ_3 (obtaining these corrections is straight forward following the procedure in [1] but will be unnecessary). With these corrections, we find that to the order needed, $d\mathbf{R}/dt = \langle d\mathbf{R}/dt \rangle$, given in (9), $d\mu/dt = \langle d\mu/dt \rangle \simeq 0$ and $d\varphi/dt = \langle d\varphi/dt \rangle$. Then, equation (14) simplifies to

$$\langle \mathbf{v} \cdot \nabla \phi \rangle = [u\hat{\mathbf{b}} - (c/B) \nabla_{\mathbf{R}} \Psi \times \hat{\mathbf{b}}] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle - \langle \partial \phi / \partial t|_{\mathbf{r}} - \partial \phi / \partial t|_{\mathbf{R},E,\mu,\varphi} \rangle, \tag{15}$$

with Ψ given in equation (10) and $\langle \partial \phi / \partial \varphi \rangle = 0$. Notice that assuming that we already have $\mathbf{R}_3$, $\mu_3 - \langle \mu_3 \rangle$, φ_2 and φ_3 is only a shortcut to find the result in (15). To obtain $\langle \phi \rangle$ to the order required, these higher order corrections are not needed, neither are they necessary for the difference between time derivatives, as we will prove next. The difference between time derivatives is

$$\left\langle \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} \right\rangle = \left\langle \left.\frac{\partial\mathbf{R}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \cdot \nabla_{\mathbf{R}}\phi + \left.\frac{\partial\mu}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \frac{\partial\phi}{\partial\mu} + \left.\frac{\partial\varphi}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \frac{\partial\phi}{\partial\varphi} \right\rangle. \quad (16)$$

The procedure for rewriting (16) is analogous to that used on (4). Using the definitions of $\mathbf{R}_1$, $\mathbf{R}_2$, μ_1 and φ_1, we find that $\partial\mathbf{R}/\partial t|_{\mathbf{r},\mathbf{v}} \simeq \partial\mathbf{R}_2/\partial t|_{\mathbf{r},\mathbf{v}}$, $\partial\mu/\partial t|_{\mathbf{r},\mathbf{v}} \simeq \partial\mu_1/\partial t|_{\mathbf{r},\mathbf{v}}$ and $\partial\varphi/\partial t|_{\mathbf{r},\mathbf{v}} \simeq \partial\varphi_1/\partial t|_{\mathbf{r},\mathbf{v}}$, giving

$$\langle \partial\phi/\partial t|_{\mathbf{r}} - \partial\phi/\partial t|_{\mathbf{R},E,\mu,\varphi} \rangle = \partial(\Psi - \langle\phi\rangle)/\partial t, \quad (17)$$

where we use the equivalent to equations (6) and (7) with $\partial/\partial t$ replacing $\nabla_{\mathbf{R}}$. The final result, obtained by combining equations (12), (15) and (17), is

$$\langle dE/dt \rangle = (Ze/M)\{\partial(\Psi - \langle\phi\rangle)/\partial t - [u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}}\Psi \times \hat{\mathbf{b}}] \cdot \nabla_{\mathbf{R}}\langle\phi\rangle\}. \quad (18)$$

5. Second order correction μ_2

The correction μ_2, according to [1], is given by

$$\mu_2 = \Omega^{-1} \int^{\varphi} d\varphi' \, [d(\mu_0 + \mu_1)/dt - \langle d(\mu_0 + \mu_1)/dt \rangle] + \langle\mu_2\rangle, \quad (19)$$

where $\langle\mu_2\rangle$ is found requiring that $\langle d\mu/dt \rangle = 0$ to order $\delta^2 v_{\mathrm{th}}^3/BL$.

The time derivative of $\mu_0 + \mu_1$ is given by

$$d(\mu_0 + \mu_1)/dt = (Ze/MB)(-\mathbf{v}_\perp \cdot \nabla\phi + d\widetilde{\phi}/dt). \quad (20)$$

To rewrite $\mathbf{v}_\perp \cdot \nabla\phi$ as a function of the gyrokinetic variables, we employ $\mathbf{v}_\perp \cdot \nabla\phi = \mathbf{v} \cdot \nabla\phi - v_{||}\hat{\mathbf{b}} \cdot \nabla\phi$ and equation (13) to find

$$-\mathbf{v}_\perp \cdot \nabla\phi + d\widetilde{\phi}/dt = -d\langle\phi\rangle/dt + \partial\phi/\partial t|_{\mathbf{r}} + v_{||}\hat{\mathbf{b}} \cdot \nabla\phi. \quad (21)$$

To the order we are interested in, $d\mathbf{R}/dt \simeq u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}}\langle\phi\rangle \times \hat{\mathbf{b}}$, giving

$$-\mathbf{v}_\perp \cdot \nabla\phi + d\widetilde{\phi}/dt = -\partial\langle\phi\rangle/\partial t|_{\mathbf{R},E,\mu,\varphi} - u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle + \partial\phi/\partial t|_{\mathbf{r}} + v_{||}\hat{\mathbf{b}} \cdot \nabla\phi. \quad (22)$$

According to equation (11), the difference between $u = \langle v_{||} \rangle$ and $v_{||}$ is higher order, and according to equation (17), the difference between $\partial\phi/\partial t|_{\mathbf{r}}$ and $\partial\phi/\partial t|_{\mathbf{R},E,\mu,\varphi}$ is negligible. Therefore, equations (20) and (22) give

$$d(\mu_0 + \mu_1)/dt = (Ze/MB)(\partial\widetilde{\phi}/\partial t + u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\phi}), \quad (23)$$

which in turn, using equation (19), yields

$$\mu_2 = (c/B^2)(\partial\widetilde{\Phi}/\partial t + u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\Phi}) + \langle\mu_2\rangle. \quad (24)$$

To find $\langle\mu_2\rangle$ we require that $\langle d\mu/dt \rangle = 0$ to order $\delta^2 v_{\mathrm{th}}^3/BL$. The gyroaverage of $d\mu/dt$ is given by

$$\langle d\mu/dt \rangle = \langle d(\mu_0 + \mu_1 + \mu_2)/dt \rangle = -(Ze/MB)\langle \mathbf{v}_\perp \cdot \nabla\phi \rangle + d\langle\mu_2\rangle/dt, \quad (25)$$

where the gyroaverages of $d\mu_1/dt$ and $d(\mu_2 - \langle\mu_2\rangle)/dt$ vanish. The term $\langle \mathbf{v}_\perp \cdot \nabla\phi\rangle$ can be conveniently rewritten to higher order than in (23) by employing equation (15) to find

$$\langle \mathbf{v}_\perp \cdot \nabla\phi\rangle = [u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}}\Psi \times \hat{\mathbf{b}}] \cdot \nabla_{\mathbf{R}}\langle\phi\rangle - \partial(\Psi - \langle\phi\rangle)/\partial t - \langle v_{||}\hat{\mathbf{b}} \cdot \nabla\phi\rangle, \tag{26}$$

where we used equation (17). Employing equation (4) and the fact that the difference between $u = \langle v_{||}\rangle$ and $v_{||}$ is order $\delta^2 v_{\rm th}$ (11), we find

$$\langle v_{||}\hat{\mathbf{b}} \cdot \nabla\phi\rangle \simeq \langle v_{||}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\phi\rangle + u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}(\Psi - \langle\phi\rangle) \simeq u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Psi. \tag{27}$$

To obtain the second equality, we employ $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle \gg \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\phi}$, which means that $\langle v_{||}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\phi\rangle \simeq \langle v_{||}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle\rangle = u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle$ to order $\delta^2 T v_{\rm th}/eL$. Then, equation (26) becomes $\langle \mathbf{v}_\perp \cdot \nabla\phi\rangle = -d(\Psi - \langle\phi\rangle)/dt$, where to this order $d/dt = \partial/\partial t + [u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}}\langle\phi\rangle \times \hat{\mathbf{b}}] \cdot [\nabla_{\mathbf{R}} - (Ze/M)\nabla_{\mathbf{R}}\langle\phi\rangle(\partial/\partial E)]$ and $\partial(\Psi - \langle\phi\rangle)/\partial E = 0$. Finally, imposing $\langle d\mu/dt\rangle = 0$ on equation (25), we find

$$\langle\mu_2\rangle = -(Ze/MB)\,(\Psi - \langle\phi\rangle)\,. \tag{28}$$

6. Comparisons with Dubin *et al*

To compare with reference [2], we first need to write the gyrokinetic equation in the same variables that are used in that reference, i.e., we need to employ u instead of E. The change is easy to carry out. We substitute E_2 and (24) into (11) to write

$$v_{||} = \sqrt{2[E - (\mu - \langle\mu_2\rangle)B]} + (c/B)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\Phi}, \tag{29}$$

where we Taylor expand $E_2 - (\mu_2 - \langle\mu_2\rangle)B = -(c/B)u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\Phi}$. Then, gyroaveraging this equation we find

$$u^2/2 = E - (\mu - \langle\mu_2\rangle)B. \tag{30}$$

Applying the Vlasov operator to this expression and gyroaveraging, we find

$$\langle du/dt\rangle = -(Ze/M)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Psi, \tag{31}$$

where we used equations (18), (28) and $\langle d\mu/dt\rangle = 0$. With this equation, equation (9) and the fact that $\langle d\mu/dt\rangle = 0$, we find the same gyrokinetic Vlasov equation as in reference [2], namely

$$\partial f/\partial t + [u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}}\Psi \times \hat{\mathbf{b}}] \cdot \nabla_{\mathbf{R}} f - (Ze/M)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Psi(\partial f/\partial u) = 0, \tag{32}$$

with $f(\mathbf{R}, u, \mu, t)$. The subtle differences between our function Ψ of (10) and the function ψ in reference [2], given in their equation (19b), come from their introduction of the potential function $\phi(\mathbf{R} + \boldsymbol{\rho}, t) \neq \phi(\mathbf{r}, t)$, leading to subtle differences in the definitions of $\langle\phi\rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$. Here, the vector $\boldsymbol{\rho}(\mu, \theta)$ is

$$\boldsymbol{\rho} = (\sqrt{2\mu B}/\Omega)(\hat{\mathbf{e}}_1 \cos\theta - \hat{\mathbf{e}}_2 \sin\theta), \tag{33}$$

with θ the gyrokinetic gyrophase as defined in [2]. The relation between the gyrophase θ and our gyrophase is $\theta = -\pi/2 - \varphi$. From now on, we will denote the functions $\langle\phi\rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ as they are defined in [2] with the subindex D. The definitions in [2] are then

$$\overline{\phi}_D = \overline{\phi}_D(\mathbf{R}, \mu, t) \equiv \frac{1}{2\pi}\oint d\theta\, \phi(\mathbf{R} + \boldsymbol{\rho}, t), \tag{34}$$

$$\widetilde{\phi}_D = \widetilde{\phi}_D(\mathbf{R}, \mu, \theta, t) \equiv \phi(\mathbf{R} + \boldsymbol{\rho}, t) - \overline{\phi}_D \tag{35}$$

and

$$\widetilde{\Phi}_D = \widetilde{\Phi}_D(\mathbf{R}, \mu, \theta, t) \equiv \int^{\theta} d\theta'\, \widetilde{\phi}_D(\mathbf{R}, \mu, \theta', t) \tag{36}$$

such that $\langle\widetilde{\Phi}\rangle = 0$. Note that these definitions coincide with ours to order $\delta T/e$, except for $\widetilde{\Phi}_D$, for which $\widetilde{\Phi}_D \simeq -\widetilde{\Phi}$. The sign is due to the definition of the gyrophase θ. To second order, however, Taylor expanding $\phi(\mathbf{r}, t) = \phi(\mathbf{R} + \boldsymbol{\rho} - \boldsymbol{\rho} - \mathbf{R}_1 - \mathbf{R}_2, t)$ gives

$$\phi \simeq \phi(\mathbf{R} + \boldsymbol{\rho}, t) - (\boldsymbol{\rho} + \mathbf{R}_1 + \mathbf{R}_2) \cdot \nabla_{\mathbf{R}}\phi, \tag{37}$$

where

$$\boldsymbol{\rho} + \mathbf{R}_1 \simeq -(Mc\mu_1/Zev_\perp^2)\mathbf{v} \times \hat{\mathbf{b}} - (\varphi_1/\Omega)\mathbf{v}_\perp = O(\delta\rho). \tag{38}$$

To obtain equation (38), we Taylor expand $\mu \simeq \mu_0 + \mu_1$ and $\varphi \simeq \varphi_0 + \varphi_1$ around μ_0 and φ_0 in equation (33). Employing the lowest order results $\partial\phi/\partial\varphi \simeq -\Omega^{-1}\mathbf{v}_\perp \cdot \nabla\phi$ and $\partial\phi/\partial\mu \simeq -(Mc/Zev_\perp^2)(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi$, we write equation (37) as

$$\phi \simeq \phi(\mathbf{R} + \boldsymbol{\rho}, t) - \frac{Ze}{MB}\left(\widetilde{\phi}\frac{\partial\phi}{\partial\mu} - \frac{\partial\widetilde{\Phi}}{\partial\mu}\frac{\partial\phi}{\partial\varphi}\right) + \frac{c}{B\Omega}(\nabla_{\mathbf{R}}\widetilde{\Phi} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\phi, \tag{39}$$

where we used the definitions of $\mathbf{R}_2$, μ_1 and φ_1. Then, gyroaveraging, we find

$$\langle\phi\rangle \simeq \overline{\phi}_D - (Ze/MB)\partial\langle\widetilde{\phi}^2\rangle/\partial\mu + (c/B\Omega)\langle(\nabla_{\mathbf{R}}\widetilde{\Phi} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\widetilde{\phi}\rangle. \tag{40}$$

Substituting this equation into the definition (10) of Ψ and employing that to lowest order $\widetilde{\phi} \simeq \widetilde{\phi}_D$ and $\widetilde{\Phi} \simeq -\widetilde{\Phi}_D$, we find

$$\Psi = \overline{\phi}_D - (Ze/2MB)\partial\langle\widetilde{\phi}_D^2\rangle/\partial\mu - (c/2B\Omega)\langle(\nabla_{\mathbf{R}}\widetilde{\Phi}_D \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\widetilde{\phi}_D\rangle, \tag{41}$$

exactly as in equation (19b) of reference [2].

Finally, we will compare the quasineutrality equations in both methods. Taylor expanding the ion distribution function around $\mathbf{R}_g = \mathbf{r} + \Omega^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, $v_{||}$, μ_0 and φ_0, we find

$$f_i(\mathbf{R}, u, \mu, t) \simeq f_{ig} + \mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig} - \frac{c}{B}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\Phi}\frac{\partial f_{ig}}{\partial v_{||}} + (\mu_1 + \mu_2)\frac{\partial f_{ig}}{\partial \mu_0} + \frac{\mu_1^2}{2}\frac{\partial^2 f_{ig}}{\partial \mu_0^2}, \tag{42}$$

where $f_{ig} = f_i(\mathbf{R}_g, v_{||}, \mu_0, t)$. Here we have used equations (29) and (30) to obtain that $u \simeq v_{||} - (c/B)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\widetilde{\Phi}$. The ion density is given by

$$n_i = \int d^3v\, f_i \simeq \int d^3v\, \left[f_{ig} + \mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig} + (\mu_1 + \langle\mu_2\rangle)\frac{\partial f_{ig}}{\partial \mu_0} + \frac{\mu_1^2}{2}\frac{\partial^2 f_{ig}}{\partial \mu_0^2}\right]. \tag{43}$$

Here, the integrals of $(c/B)(\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\widetilde{\Phi})(\partial f_{ig}/\partial v_{||})$ and $(\mu_2 - \langle\mu_2\rangle)(\partial f_{ig}/\partial\mu_0)$ vanish because $\oint d\varphi_0\widetilde{\Phi} = 0$ and the only gyrophase dependence is in $\widetilde{\Phi}$ since f_{ig} is assumed to be a smooth function of $\mathbf{r}$ and $\mathbf{v}$ to lowest order, giving $f_{ig} = f_i(\mathbf{R}_g, v_{||}, \mu_0, t) \simeq f_i(\mathbf{r}, v_{||}, \mu_0, t)$. The integral $\oint d\varphi_0\widetilde{\Phi}$ is performed holding $\mathbf{r}$, $v_{||}$, μ_0 and t fixed, and it vanishes to lowest order as proven in the Appendix. On the other hand, the integral of $\mathbf{R}_2\cdot\nabla_{\mathbf{R}_g} f_{ig}$ does not vanish. Here the gyrophase dependence of f_{ig} due to the short wavelength pieces becomes important due to the steep gradient [recall the ordering in (1)].

In equation (43), we can employ $\widetilde{\phi} \simeq \widetilde{\phi}_D$ and $\widetilde{\Phi} \simeq -\widetilde{\Phi}_D$ in the higher order terms. However, for μ_1 we need the difference between $\widetilde{\phi}$ and $\widetilde{\phi}_D$. Subtracting (40) from (39), we find

$$\widetilde{\phi} \simeq \widetilde{\phi}_D - \frac{Ze}{MB}\left(\widetilde{\phi}\frac{\partial\phi}{\partial\mu} - \frac{\partial\widetilde{\Phi}}{\partial\mu}\frac{\partial\widetilde{\phi}}{\partial\varphi}\right) + \frac{c}{B\Omega}(\nabla_{\mathbf{R}}\widetilde{\Phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\phi + \frac{Ze}{MB}\frac{\partial}{\partial\mu}\langle\widetilde{\phi}^2\rangle - \frac{c}{B\Omega}\langle(\nabla_{\mathbf{R}}\widetilde{\Phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\widetilde{\phi}\rangle. \tag{44}$$

In this equation, $\widetilde{\phi}_D = \widetilde{\phi}_D(\mathbf{R}, \mu, \varphi, t)$, but for equation (43), it is better to use $\widetilde{\phi}_{Dg} = \widetilde{\phi}_D(\mathbf{R}_g, \mu_0, \varphi_0, t)$. By Taylor expanding, we find that

$$\widetilde{\phi}_D \simeq \widetilde{\phi}_{Dg} + \mathbf{R}_2\cdot\nabla_{\mathbf{R}_g}\widetilde{\phi}_{Dg} + \mu_1\frac{\partial\widetilde{\phi}_{Dg}}{\partial\mu_0} + \varphi_1\frac{\partial\widetilde{\phi}_{Dg}}{\partial\varphi_0}. \tag{45}$$

This equation, combined with equation (44) and the definitions of $\mathbf{R}_2$, μ_1 and φ_1, leads to

$$\widetilde{\phi} \simeq \widetilde{\phi}_{Dg} - \frac{Ze\widetilde{\phi}_{Dg}}{MB}\frac{\partial\overline{\phi}_{Dg}}{\partial\mu} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\overline{\phi}_{Dg} + \frac{Ze}{MB}\frac{\partial}{\partial\mu}\langle\widetilde{\phi}^2_{Dg}\rangle + \frac{c}{B\Omega}\langle(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\widetilde{\phi}_{Dg}\rangle, \tag{46}$$

where we have used that, to lowest order, $\widetilde{\phi}_D \simeq \widetilde{\phi}_{Dg}$, $\overline{\phi}_D \simeq \overline{\phi}_{Dg} \equiv \overline{\phi}_D(\mathbf{R}_g, \mu_0, t)$ and $\widetilde{\Phi}_D \simeq \widetilde{\Phi}_{Dg} \equiv \widetilde{\Phi}_D(\mathbf{R}_g, \mu_0, \varphi_0, t)$. Substituting equation (46) into (43) yields

$$n_i \simeq \int d^3v\left\{f_{ig} + \frac{Ze\widetilde{\phi}_{Dg}}{MB}\frac{\partial f_{ig}}{\partial\mu_0} + \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g} f_{ig} + \frac{Z^2e^2\widetilde{\phi}^2_{Dg}}{2M^2B^2}\frac{\partial^2 f_{ig}}{\partial\mu_0^2} + \frac{Ze}{MB}\left[-\frac{Ze\widetilde{\phi}_{Dg}}{MB}\frac{\partial\overline{\phi}_{Dg}}{\partial\mu_0} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\overline{\phi}_{Dg} + \frac{Ze}{2MB}\frac{\partial}{\partial\mu_0}\langle\widetilde{\phi}^2_{Dg}\rangle + \frac{c}{2B\Omega}\langle(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\widetilde{\phi}_{Dg}\rangle\right]\frac{\partial f_{ig}}{\partial\mu_0}\right\}, \tag{47}$$

where we have used the definitions of $\mathbf{R}_2$ and $\langle\mu_2\rangle$. This result is exactly the same as in equation (20) in reference [2]. For comparison, we give n_i to order $\delta^2 n_i$ with the definitions of $\langle\phi\rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ in [1],

$$n_i \simeq \int d^3v\left\{f_{ig} + \frac{Ze\widetilde{\phi}_g}{MB}\frac{\partial f_{ig}}{\partial\mu_0} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_g\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g} f_{ig} + \frac{Z^2e^2\widetilde{\phi}^2_g}{2M^2B^2}\frac{\partial^2 f_{ig}}{\partial\mu_0^2} + \frac{Ze}{MB}\left[\frac{Ze\widetilde{\phi}_g}{MB}\frac{\partial\widetilde{\phi}_g}{\partial\mu_0} - \frac{Ze\widetilde{\Phi}_g}{MB}\frac{\partial\widetilde{\phi}_g}{\partial\varphi_0} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_g\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\widetilde{\phi}_g - \frac{Ze}{2MB}\frac{\partial}{\partial\mu_0}\langle\widetilde{\phi}^2_g\rangle + \frac{c}{2B\Omega}\langle(\nabla_{\mathbf{R}_g}\widetilde{\Phi}_g\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\widetilde{\phi}_g\rangle\right]\frac{\partial f_{ig}}{\partial\mu_0}\right\}. \tag{48}$$

We have found this equation substituting μ_1, $\mathbf{R}_2$ and $\langle\mu_2\rangle$ into (43). From the functions $\widetilde{\phi}(\mathbf{R},\mu,\varphi,t)$ and $\widetilde{\Phi}(\mathbf{R},\mu,\varphi,t)$, we have defined $\widetilde{\phi}_g = \widetilde{\phi}(\mathbf{R}_g,\mu_0,\varphi_0,t)$ and $\widetilde{\Phi}_g = \widetilde{\Phi}(\mathbf{R}_g,\mu_0,\varphi_0,t)$. The relationships between $\widetilde{\phi}$ and $\widetilde{\phi}_g$ and between $\widetilde{\Phi}$ and $\widetilde{\Phi}_g$ are similar to the one given in (45).

7. Summary

The methodology and results of reference [1] are completely consistent with the results of [2] since they give the same gyrokinetic equation (32), generalized potential Ψ (41) and quasineutrality condition (47).

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Appendix

To prove that $\oint d\varphi_0 \widetilde{\Phi} = 0$ vanishes, we Fourier analyze $\phi = (2\pi)^{-3}\int d^3k\,\tilde{\phi}_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})$, giving to lowest order

$$\phi(\mathbf{r},t) \simeq \phi(\mathbf{R} - \Omega^{-1}\mathbf{v}\times\hat{\mathbf{b}},t) = \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp[i\mathbf{k}\cdot\mathbf{R} - iz\sin(\varphi_0 - \varphi_{\mathbf{k}})], \tag{A.1}$$

where $z = k_\perp v_\perp/\Omega$. Here we employ $\mathbf{r} \simeq \mathbf{R} - \Omega^{-1}\mathbf{v}\times\hat{\mathbf{b}}$ and we define $\varphi_{\mathbf{k}}$ such that $\mathbf{k}_\perp = k_\perp(\hat{\mathbf{e}}_1\cos\varphi_{\mathbf{k}} + \hat{\mathbf{e}}_2\sin\varphi_{\mathbf{k}})$ to write $\mathbf{k}\cdot\mathbf{r} \simeq \mathbf{k}\cdot\mathbf{R} - z\sin(\varphi_0 - \varphi_{\mathbf{k}})$. Then, we use

$$\exp(iz\sin\varphi) = \sum_{m=-\infty}^{\infty} J_m(z)\exp(im\varphi), \tag{A.2}$$

with $J_m(z)$ the Bessel function of the first kind, to find

$$\phi \simeq \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{R})\sum_{m=-\infty}^{\infty} J_m(z)\exp[-im(\varphi_0 - \varphi_{\mathbf{k}})]. \tag{A.3}$$

Employing this expression, we obtain $\widetilde{\phi}$ by subtracting the average in φ_0 (component $m = 0$), and we find $\widetilde{\Phi}$ by integrating $\widetilde{\phi}$ over φ_0, giving

$$\widetilde{\Phi} \simeq \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{R})\sum_{m\neq 0}\frac{i}{m}J_m(z)\exp[-im(\varphi_0 - \varphi_{\mathbf{k}})], \tag{A.4}$$

where the summation includes every positive and negative m different from 0. To rewrite $\widetilde{\Phi}$ as a function of $\mathbf{r}$, $v_{||}$ μ_0 and φ_0, we need the expression

$$\exp(i\mathbf{k}\cdot\mathbf{R}) \simeq \exp(i\mathbf{k}\cdot\mathbf{r}) \sum_{p=-\infty}^{\infty} J_p(z) \exp[ip(\varphi_0 - \varphi_{\mathbf{k}})], \tag{A.5}$$

deduced from $\mathbf{R} \simeq \mathbf{r} + \Omega^{-1}\mathbf{v} \times \hat{\mathbf{b}}$ and (A.2). Then, we find

$$\widetilde{\Phi} \simeq \frac{1}{(2\pi)^3} \int d^3k\, \phi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}) \sum_{m\neq 0,p} \frac{i}{m} J_m(z) J_p(z) \exp[i(p-m)(\varphi_0 - \varphi_{\mathbf{k}})]. \tag{A.6}$$

Finally, integrating in φ_0, we obtain

$$\frac{1}{2\pi} \oint d\varphi_0\, \widetilde{\Phi} = \frac{1}{(2\pi)^3} \int d^3k\, \phi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}) \sum_{m\neq 0} \frac{i}{m} [J_m(z)]^2 = 0 \tag{A.7}$$

since $J_{-m}(z) = (-1)^m J_m(z)$.

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