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Non-physical momentum sources in slab geometry gyrokinetics

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Abstract. We investigate momentum transport in the Hamiltonian electrostatic gyrokinetic formulation in Dubin D H E *et al* [1983 *Phys. Fluids* **26** 3524]. We prove that the long wavelength electric field obtained from the gyrokinetic quasineutrality introduces a non-physical momentum source in the low flow ordering.

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1. Introduction

The evolution of the long wavelength electric field on transport time scales depends exclusively on the transport of momentum from one flux surface to the next [1, 2, 3, 4, 5, 6, 7]. For electrostatic turbulence in statistical steady state, the transport of momentum is expected to be slow, on the order of the gyroBohm estimate. This simple estimate gives the momentum flux

$$\Pi \sim D_{gB} \times \nabla (n_i M V_i) \sim \delta_i^2 (V_i/v_i) p_i, \tag{1}$$

with $D_{gB} = \delta_i \rho_i v_i$ the gyroBohm diffusion coefficient, $\delta_i = \rho_i/L \ll 1$ the ratio between the ion gyroradius and a macroscopic scale length L, $v_i = \sqrt{2T_i/M}$ and $\rho_i = Mcv_i/ZeB$ the ion thermal velocity and gyroradius, and n_i, T_i, $p_i = n_i T_i$ and V_i the ion density, temperature, pressure and average velocity. Here, M and Ze are the ion mass and charge, B is the magnetic field strength, and e and c are the electron charge magnitude and the speed of light. The size of the momentum flux depends on the ordering of the average velocity V_i. In the high flow ordering, the ion velocity is assumed to be sonic, making the $\mathbf{E} \times \mathbf{B}$ drift dominate over any other contribution to the ion flow and giving a momentum flux of order $\Pi_{\rm hf} \sim \delta_i^2 p_i$. In the low flow or drift ordering, the $\mathbf{E} \times \mathbf{B}$ drift competes with the magnetic drifts and the diamagnetic flow, giving $V_i \sim \delta_i v_i$ and $\Pi_{\rm lf} \sim \delta_i^3 p_i$. Employing these estimates, we can obtain to which order in δ_i the ion distribution function is needed to determine the correct transport of momentum and hence the correct long wavelength electric field. In this article we will restrict ourselves to the low flow limit that we consider more relevant to the core, and only comment briefly on the high flow limit.

The requirements on the distribution function imposed by the self-consistent calculation of the long wavelength electric field have become very important due to recent developments in gyrokinetic simulations of turbulence. Generally, gyrokinetic simulations are based on δf formulations [8, 9, 10, 11] in which the calculation of turbulence saturation and the long time scale evolution of the radial profiles of density, temperature and rotation are effectively separated. However, several groups have been working on full f gyrokinetic simulations [12, 13, 14] that do not use an explicit equation for the transport of momentum, but solve a quasineutrality equation to obtain the electric field at all wavelengths. In reference [7], we showed that the lowest order gyrokinetic Fokker-Planck and quasineutrality equations are insufficient to determine the evolution of the long wavelength electric field. The argument is based on the vorticity or current conservation equation, equivalent to quasineutrality. The perpendicular current is obtained from the total momentum equation, giving

$$\mathbf{J}_\perp = \frac{c}{B} \hat{\mathbf{b}} \times \nabla p_\perp + \frac{c}{B}(p_{||} - p_\perp)(\hat{\mathbf{b}} \times \boldsymbol{\kappa}) - \frac{\partial}{\partial t}\left(\frac{Zen_i}{\Omega_i} \mathbf{V}_i \times \hat{\mathbf{b}} \right) + \frac{c}{B} \hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i), \tag{2}$$

with $p_\perp = \int d^3v\, (Mf_i + mf_e) v_\perp^2/2$ and $p_{||} = \int d^3v\, (Mf_i + mf_e) v_{||}^2$ the total perpendicular and parallel pressures, m the electron mass, $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$ the curvature of the magnetic field lines, $\Omega_i = ZeB/Mc$ the ion gyrofrequency, and $\overleftrightarrow{\boldsymbol{\pi}}_i = M \int d^3v\, f_i [\mathbf{vv} - (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}}$

$-\hat{\mathbf{b}}\hat{\mathbf{b}}) - v_{||}^2\hat{\mathbf{b}}\hat{\mathbf{b}}]$ the ion viscosity. The ion viscosity $\overleftrightarrow{\boldsymbol{\pi}}_i$ includes both the Reynolds stress and the neoclassical perpendicular viscosity, and its off-diagonal components determine the transport of momentum from one flux surface to the next. For this reason, the only piece of the current density that contributes to the determination of the long wavelength electric field is

$$\frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i) \sim \delta_i(\Pi/p_i)en_e v_i \sim \delta_i^3(V_i/v_i)en_e v_i. \tag{3}$$

Thus, in the low flow ordering, the current density must be calculated to order $\delta_i^4 en_e v_i$ to determine the long wavelength electric field. Most gyrokinetic codes employ the first order $\mathbf{E}\times\mathbf{B}$ and magnetic drifts that only give self-consistent current densities of order $\delta_i en_e v_i$, where the drifts are of order $\delta_i v_i$. Some derivations, among them the work of Dubin *et al* [15], are performed to higher order in δ_i, keeping corrections to the drifts of order $\delta_i^2 v_i$, but they are often restricted to simplified magnetic geometry. In any case, the highest order to which the current density can be found is $\delta_i^2 en_e v_i$; too low to self-consistently determine the long wavelength electric field.

In this article, we use the simplified geometry employed in the pioneering work by Dubin *et al* [15] to explicitly obtain the non-physical momentum sources introduced by gyrokinetic Fokker-Planck and quasineutrality equations valid only to order δ_i^2. This exercise illustrates the problem pointed out in [6, 7, 16], and demonstrates that this issue affects equally gyrokinetics based on recursive methods [16] and Lie transform approaches [17]. The results in [15] can be obtained using both procedures [18], and in this article we prove that even these higher order descriptions are unable to avoid non-physical sources of momentum.

The rest of the article is organized as follows. In section 2, we derive a higher order momentum conservation equation that determines the long wavelength electric field in the low flow ordering by employing moments of the full Vlasov equation. The resulting equation is the one against which the results of the approximate gyrokinetic quasineutrality equation must be compared. In section 3, we describe the results of [15] for completeness. Employing the time derivative of the gyrokinetic quasineutrality equation, we find a vorticity equation equivalent to quasineutrality. In section 4, we show that in the long wavelength limit the results obtained from the vorticity equation introduce a non-physical source of momentum. Finally, in section 5, we discuss the implications of this result for tokamak geometries.

2. Transport of momentum in a slab

In this section, we derive the transport of momentum in a slab. First, we present the geometry and assumptions. Then, based on these assumptions we obtain a momentum conservation equation in which the transport of momentum is of gyroBohm order. This is the equation that any gyrokinetic formulation should satisfy. In the next sections we will prove that modern formulations do not satisfy it even in the simple slab limit.

In accordance with [15], we assume a constant magnetic field $\mathbf{B}$, with $\hat{\mathbf{b}} = \mathbf{B}/B$

the unit vector parallel to the magnetic field. The plane perpendicular to $\hat{\mathbf{b}}$ is spanned by two unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ such that $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{b}}$. The macroscopic gradients of density, temperature and flow $\mathbf{V}_i \cdot \hat{\mathbf{y}}$ are in the direction $\hat{\mathbf{x}}$. To ease the comparison with tokamak physics, we assume that the total current in the $\hat{\mathbf{x}}$ direction $\langle \mathbf{J} \cdot \hat{\mathbf{x}} \rangle_x$ vanishes, with $\langle \ldots \rangle_x = A_{yz}^{-1} \int dy\, dz\, (\ldots)$ the flux surface average and $A_{yz} = \int dy\, dz$ the area of the flux surface. Here x, y and z are the coordinates along $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{b}}$.

The orderings are the same as those in [7, 16]. We allow perpendicular wavelengths as short as the ion gyroradius. The short wavelength pieces of the distribution function and the potential scale as

$$\frac{f_{i,k}}{f_{si}} \sim \frac{f_{e,k}}{f_{se}} \sim \frac{e\phi_k}{T_e} \sim \frac{1}{k_\perp L} \lesssim 1, \tag{4}$$

with $k_\perp \rho_i \lesssim 1$. Here f_{si} and f_{se} are the lowest order ion and electron distribution functions with a slow variation in both $\mathbf{r}$ and $\mathbf{v}$. These lowest order distribution functions are not necessarily Maxwellians. According to the orderings in (4), pieces of the distribution function and the potential with wavelengths on the order of the ion gyroradius are small in the expansion parameter δ_i. The perpendicular gradients of pieces with different wavelengths are comparable, i.e., $\nabla_\perp f_{i,k} \sim k_\perp f_{i,k} \sim f_{si}/L \sim \nabla_\perp f_{si}$ and $\nabla_\perp \phi_k \sim k_\perp \phi_k \sim T_e/eL$. With this ordering, the $\mathbf{E} \times \mathbf{B}$ drift is of order $\delta_i v_i$. The parallel wavelengths are assumed to be comparable to the macroscopic scale, $k_{||} L \sim 1$.

The transport of y-momentum in the x direction is given by

$$\frac{\partial}{\partial t} \langle n_i M \mathbf{V}_i \cdot \hat{\mathbf{y}} \rangle_x = -\frac{\partial}{\partial x} \langle \hat{\mathbf{x}} \cdot \overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot \hat{\mathbf{y}} \rangle_x. \tag{5}$$

For long wavelengths and the orderings in (4), it is possible to find a convenient expression for $\langle \hat{\mathbf{x}} \cdot \overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot \hat{\mathbf{y}} \rangle_x$ employing the $\mathbf{vv}$ moment of the Vlasov equation. According to the estimates in (1), $\langle \hat{\mathbf{x}} \cdot \overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot \hat{\mathbf{y}} \rangle_x$ must be calculated to order $\delta_i^3 p_i$. The $\mathbf{vv}$ moment of the Vlasov equation is

$$\Omega_i (\overset{\leftrightarrow}{\boldsymbol{\pi}}_i \times \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \overset{\leftrightarrow}{\boldsymbol{\pi}}_i) = \frac{\partial \overset{\leftrightarrow}{\mathbf{P}}_i}{\partial t} + \nabla \cdot \left(M \int d^3v\, f_i \mathbf{vvv} \right) + Zen_i (\mathbf{V}_i \nabla \phi + \nabla \phi \mathbf{V}_i), \tag{6}$$

with $\overset{\leftrightarrow}{\mathbf{P}}_i = M \int d^3v\, f_i \mathbf{vv}$ the stress tensor. The flux surface averaged yy component of tensor equation (6) gives

$$\langle \hat{\mathbf{x}} \cdot \overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot \hat{\mathbf{y}} \rangle_x = -\frac{1}{2\Omega_i} \frac{\partial}{\partial t} \langle \hat{\mathbf{y}} \cdot \overset{\leftrightarrow}{\mathbf{P}}_i \cdot \hat{\mathbf{y}} \rangle_x - \frac{1}{2\Omega_i} \frac{\partial}{\partial x} \left\langle M \int d^3v\, f_i (\mathbf{v} \cdot \hat{\mathbf{x}}) (\mathbf{v} \cdot \hat{\mathbf{y}})^2 \right\rangle_x - \left\langle \frac{c}{B} \frac{\partial \phi}{\partial y} n_i M \mathbf{V}_i \cdot \hat{\mathbf{y}} \right\rangle_x. \tag{7}$$

Since we are interested in transport time scales, we consider only the fast time average of the viscosity in (7), giving $\partial \langle \hat{\mathbf{y}} \cdot \overset{\leftrightarrow}{\mathbf{P}}_i \cdot \hat{\mathbf{y}} \rangle_x / \partial t \simeq \partial p_{i\perp} / \partial t$, with $p_{i\perp} = M \int d^3v\, f_i (v_\perp^2/2)$. Here, only the transport time scale variation of $p_{i\perp}$, with $\partial/\partial t \sim D_{gB}/L^2 \sim \delta_i^2 v_i / L$, contributes to the final answer. The Reynolds stress $\langle (c/B)(\partial_y \phi) n_i M \mathbf{V}_i \cdot \hat{\mathbf{y}} \rangle_x$ is formally larger than $\delta_i^3 p_i$. However, its fast time average must give the gyroBohm contribution in (1), i.e., $O(\delta_i^3 p_i)$. Finally, the term $\langle M \int d^3v\, f_i(\mathbf{v}\cdot\hat{\mathbf{x}})(\mathbf{v}\cdot\hat{\mathbf{y}})^2\rangle_x$ can be found using the **vvv** moment of the Vlasov equation. Its flux surface averaged yyy component is

$$\left\langle M \int d^3v\, f_i(\mathbf{v}\cdot\hat{\mathbf{x}})(\mathbf{v}\cdot\hat{\mathbf{y}})^2 \right\rangle_x = -\left\langle \frac{c}{B}\frac{\partial\phi}{\partial y}\hat{\mathbf{y}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\mathbf{y}} \right\rangle_x - \frac{1}{3\Omega_i}\frac{\partial}{\partial t}\left\langle M \int d^3v\, f_i(\mathbf{v}\cdot\hat{\mathbf{y}})^3 \right\rangle_x$$
$$-\frac{1}{3\Omega_i}\frac{\partial}{\partial x}\left\langle M \int d^3v\, f_i(\mathbf{v}\cdot\hat{\mathbf{x}})(\mathbf{v}\cdot\hat{\mathbf{y}})^3 \right\rangle_x . \quad (8)$$

In this equation, the fast time average makes the time derivative term negligible. The integral $\int d^3v\, f_i(\mathbf{v}\cdot\hat{\mathbf{x}})(\mathbf{v}\cdot\hat{\mathbf{y}})^3$ is also negligible because to first order the gyrophase dependent piece of the long wavelength component of f_i is proportional to $\mathbf{v}_\perp$. Therefore, the final result is

$$\left\langle M \int d^3v\, f_i(\mathbf{v}\cdot\hat{\mathbf{x}})(\mathbf{v}\cdot\hat{\mathbf{y}})^2 \right\rangle_x \simeq -\left\langle \frac{c}{B}\frac{\partial\phi}{\partial y}\hat{\mathbf{y}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\mathbf{y}} \right\rangle_x . \quad (9)$$

Substituting this result into equation (7) finally gives

$$\langle \hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x = -\frac{1}{2\Omega_i}\frac{\partial p_{i\perp}}{\partial t} + \frac{1}{2\Omega_i}\frac{\partial}{\partial x}\left\langle \frac{c}{B}\frac{\partial\phi}{\partial y}\hat{\mathbf{y}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\mathbf{y}} \right\rangle_x - \left\langle \frac{c}{B}\frac{\partial\phi}{\partial y} n_i M \mathbf{V}_i\cdot\hat{\mathbf{y}} \right\rangle_x . \quad (10)$$

This result is correct to $O(\delta_i^3 p_i)$ for long wavelengths and transport time scales. Any model that attempts to obtain the self-consistent long wavelength electric field must reproduce equation (10). We prove in the next sections that current formulations of gyrokinetics, even in the simple slab limit, are unable to do so.

3. Gyrokinetics in a slab

In this section, we describe the collisionless gyrokinetic formulation in a slab formulated by Dubin *et al* [15] and revisited in [18]. The gyrokinetic variables are the gyrocenter position $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, the parallel velocity $u = v_{||} + u_2$, the magnetic moment $\mu = \mu_0 + \mu_1 + \mu_2$ and the gyrophase $\varphi = \varphi_0 + \varphi_1 + \varphi_2$. Here, $\mu_0 = v_\perp^2/2B$ is the lowest order magnetic moment, and φ_0 is the zeroth order gyrophase, defined by $\mathbf{v}_\perp = v_\perp(\hat{\mathbf{x}}\cos\varphi_0 + \hat{\mathbf{y}}\sin\varphi_0)$. The exact definitions of the higher order corrections to the gyrokinetic variables do not concern us here. The important results are the Vlasov equation for $f_i(\mathbf{R}, u, \mu, t)$ and the quasineutrality equation to determine $\phi(\mathbf{r}, t)$. The Dubin *et al* [15] gyrokinetic Vlasov equation is

$$\frac{\partial f_i}{\partial t} + \left(u\hat{\mathbf{b}} - \frac{c}{B}\nabla_{\mathbf{R}}\Psi\times\hat{\mathbf{b}}\right)\cdot\nabla_{\mathbf{R}} f_i - \frac{Ze}{M}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\Psi\frac{\partial f_i}{\partial u} = 0, \quad (11)$$

with

$$\Psi = \Psi(\mathbf{R}, \mu, t) \equiv \overline{\phi} + \Psi^{(2)} \quad (12)$$

and

$$\Psi^{(2)} = -\frac{Ze}{2MB}\frac{\partial}{\partial\mu}\langle\widetilde{\phi}^2\rangle - \frac{c}{2B\Omega_i}\langle(\nabla_{\mathbf{R}}\widetilde{\phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\widetilde{\Phi}\rangle \sim \delta_i^2\frac{T_e}{e}. \quad (13)$$

Here, $\langle\ldots\rangle$ is the gyroaverage holding $\mathbf{R}$, u, μ and t fixed, and we use definitions for $\overline{\phi}$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ similar to those by Dubin *et al* [15], i.e.,

$$\overline{\phi} = \overline{\phi}(\mathbf{R}, \mu, t) \equiv \frac{1}{2\pi}\oint d\varphi\, \phi(\mathbf{R} + \boldsymbol{\rho}, t) \sim \frac{1}{k_\perp L}\frac{T_e}{e}, \quad (14)$$

$$\widetilde{\phi} = \widetilde{\phi}(\mathbf{R}, \mu, \varphi, t) \equiv \phi(\mathbf{R} + \boldsymbol{\rho}, t) - \overline{\phi} \sim \delta_i \frac{T_e}{e} \tag{15}$$

and

$$\widetilde{\Phi} = \widetilde{\Phi}(\mathbf{R}, \mu, \varphi, t) \equiv \int^{\varphi} d\varphi' \, \widetilde{\phi}(\mathbf{R}, \mu, \varphi', t) \sim \delta_i \frac{T_e}{e} \tag{16}$$

such that $\langle \widetilde{\Phi} \rangle = 0$. The gyroradius vector $\boldsymbol{\rho}$ is

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{R}, \mu, \varphi) \equiv -\frac{\sqrt{2\mu B}}{\Omega_i}(\hat{\mathbf{x}} \sin\varphi - \hat{\mathbf{y}} \cos\varphi) \neq -\frac{1}{\Omega_i} \mathbf{v} \times \hat{\mathbf{b}}. \tag{17}$$

The difference between $-\Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$ and $\boldsymbol{\rho}$ is due to the differences between $\mathbf{R}$, μ and φ, and $\mathbf{r}$, μ_0 and φ_0. Notice that our definition of $\widetilde{\Phi}$ differs from Dubin's definition [15, 18] in the sign because Dubin's gyrophase θ is related to ours by $\theta = -\varphi - \pi/2$. The sizes of the functions $\overline{\phi}$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ are related to the orderings in (4). The function $\overline{\phi}$ scales as the potential itself, i.e., $e\overline{\phi}/T_e \sim (k_\perp L)^{-1}$. The functions $\widetilde{\phi}$ and $\widetilde{\Phi}$ are always small in δ_i. For wavelengths on the order of the ion gyroradius this is obvious because $e\phi_k/T_e \sim \delta_i$. For longer wavelengths, even though the amplitude of the potential fluctuations is large, the ion gyroradius is small compared to the wavelength and the difference between $\phi(\mathbf{R})$ and $\phi(\mathbf{R} + \boldsymbol{\rho})$ is small in δ_i, giving $e\widetilde{\phi}/T_e \sim e\widetilde{\Phi}/T_e \sim \delta_i$. These order of magnitude estimates lead to $\Psi^{(2)} \sim \delta_i^2 T_e/e$, with $\Psi^{(2)}$ given in (13).

The quasineutrality condition is given by

$$Zen_{ip} = en_e - Ze\hat{N}_i, \tag{18}$$

with $n_e = \int d^3v \, f_e$ the electron density, $\hat{N}_i = \int d^3v \, f_{ig}$ the ion gyrocenter density and $n_{ip} = \int d^3v \, f_{ip}$ the ion polarization density. The only pieces of the ion distribution function that contribute to the ion density and hence the quasineutrality equation are f_{ig} and f_{ip} [15, 18], where

$$f_{ig} \equiv f_i(\mathbf{R}_g, v_{||}, \mu_0, t) \tag{19}$$

is found by replacing $\mathbf{R}$, u and μ in $f_i(\mathbf{R}, u, \mu, t)$ by $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$, $v_{||}$ and μ_0, and $f_{ip} = f_{ip}^{(1)} + f_{ip}^{(2)}$ is composed of

$$f_{ip}^{(1)} = \frac{Ze\widetilde{\phi}_g}{MB} \frac{\partial f_{ig}}{\partial \mu_0} \sim \delta_i f_{si} \tag{20}$$

and

$$f_{ip}^{(2)} = \frac{Z^2 e^2 \widetilde{\phi}_g^2}{2M^2 B^2} \frac{\partial^2 f_{ig}}{\partial \mu_0^2} + \frac{Ze}{MB}\left[-\frac{Ze\widetilde{\phi}_g}{MB} \frac{\partial \overline{\phi}_g}{\partial \mu_0} + \frac{c}{B\Omega_i}(\nabla_{\mathbf{R}_g} \widetilde{\Phi}_g \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_g} \overline{\phi}_g + \frac{Ze}{2MB} \frac{\partial}{\partial \mu_0} \langle \widetilde{\phi}_g^2 \rangle \right.$$
$$\left. + \frac{c}{2B\Omega_i} \langle (\nabla_{\mathbf{R}_g} \widetilde{\phi}_g \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_g} \widetilde{\Phi}_g \rangle \right] \frac{\partial f_{ig}}{\partial \mu_0} - \frac{c}{B\Omega_i} (\nabla_{\mathbf{R}_g} \widetilde{\Phi}_g \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_g} f_{ig} \sim \delta_i^2 f_{si}. \tag{21}$$

The functions $\overline{\phi}_g \equiv \overline{\phi}(\mathbf{R}_g, \mu_0, t)$, $\widetilde{\phi}_g \equiv \widetilde{\phi}(\mathbf{R}_g, \mu_0, \varphi_0, t)$ and $\widetilde{\Phi}_g \equiv \widetilde{\Phi}(\mathbf{R}_g, \mu_0, \varphi_0, t)$ are found by replacing $\mathbf{R}$, μ and φ by $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$, μ_0 and φ_0 in the functions $\overline{\phi}(\mathbf{R}, \mu, t)$, $\widetilde{\phi}(\mathbf{R}, \mu, \varphi, t)$ and $\widetilde{\Phi}(\mathbf{R}, \mu, \varphi, t)$.

Importantly, in the quasineutrality equation (18), only the function $f_{ig} \equiv f_i(\mathbf{R}_g, v_{||}, \mu_0, t)$ enters. This function has the same functional dependence on $\mathbf{R}_g$, $v_{||}$

and μ_0 as the function f_i on $\mathbf{R}$, u and μ. To calculate f_{ig} simply replace $\mathbf{R}$, u and μ by $\mathbf{R}_g$, $v_{||}$ and μ_0 in (11). Moreover, the gradient $\nabla_{\mathbf{R}_g}$ with respect to $\mathbf{R}_g$ holding $v_{||}$, μ_0, φ_0 and t fixed is equal to the gradient $\overline{\nabla}$ with respect to $\mathbf{r}$ holding $v_{||}$, μ_0, φ_0 and t fixed because $\mathbf{R} - \mathbf{r} = \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$ is independent of position in a slab. Using this property in (11), the final equation for f_{ig} becomes

$$\frac{\partial f_{ig}}{\partial t} + \overline{\nabla} \cdot \left[f_{ig} \left(v_{||}\hat{\mathbf{b}} - \frac{c}{B}\overline{\nabla}\Psi_g \times \hat{\mathbf{b}} \right) \right] - \frac{\partial}{\partial v_{||}} \left(f_{ig} \frac{Ze}{M} \hat{\mathbf{b}} \cdot \overline{\nabla}\Psi_g \right) = 0, \quad (22)$$

with $\Psi_g \equiv \Psi(\mathbf{R}_g, \mu_0, t)$. This result is useful to derive a vorticity equation from the quasineutrality condition (18). The time derivative of equation (18) can be found by employing (22) to obtain $\partial \hat{N}_i/\partial t$, and a similar drift kinetic equation for $\partial n_e/\partial t$. The final result is

$$Ze \frac{\partial n_{ip}}{\partial t} = \nabla \cdot \left(J_{g||}\hat{\mathbf{b}} + \tilde{\mathbf{J}}_E \right), \quad (23)$$

with the parallel current

$$J_{g||} = Ze \int d^3v\, f_{ig} v_{||} - e \int d^3v\, f_e v_{||} \quad (24)$$

and the polarization current

$$\tilde{\mathbf{J}}_E = -\frac{Zec}{B} \int d^3v \left[f_{ig}(\overline{\nabla}\Psi_g \times \hat{\mathbf{b}}) - (f_{ig} + f_{ip})(\nabla\phi \times \hat{\mathbf{b}}) \right]. \quad (25)$$

The vorticity equation (23) is equivalent to the quasineutrality equation (18). However, form (23) is advantageous because it allows us to study the transport of momentum that results from retaining the long wavelength electric field in the gyrokinetic quasineutrality equation. In the next section we prove that (23) and thereby (18) introduce non-physical momentum sources.

4. Gyrokinetic transport of momentum

In this section, we derive the cross field transport of y-momentum from vorticity equation (23). We do so by flux surface averaging its long wavelength piece. The contribution of the turbulence enters in the nonlinear beating of short wavelengths to give long wavelength results. We might expect to find equation (5) with the viscosity of (10), but the final result will have a non-physical source of momentum due to the higher order terms neglected in the gyrokinetic equation.

First, the long wavelength limit of $n_{ip} = \int d^3v\, f_{ip}$ is obtained to order $\delta_i^2 n_e$. Using this result, we show that vorticity equation (23) gives the evolution in time of the y component of the $\mathbf{E} \times \mathbf{B}$ flow as a function of the polarization current $\tilde{\mathbf{J}}_E \cdot \hat{\mathbf{x}}$. Next, by taking the long wavelength limit of the polarization current we prove that the cross field transport of y-momentum differs from the result in (5) by a non-physical momentum source.

4.1. Polarization density at $k_\perp L \sim 1$

In this subsection, we find the long wavelength limit of $n_{ip} = \int d^3v\, (f_{ip}^{(1)} + f_{ip}^{(2)})$ to order $\delta_i^2 n_e$. To order $\delta_i n_e$, only $f_{ip}^{(1)}$ from (20) contributes to n_{ip}. In this term there is nonlinear beating between $\widetilde{\phi}_g$ and f_{ig} and the short wavelength components must be kept. This beating gives a long wavelength result that we can evaluate by Taylor expanding $f_{ip}^{(1)}(\mathbf{R}_g, v_{||}, \mu_0, \varphi_0, t)$ around $\mathbf{r}$ to $O(\delta_i^2 f_{si})$ to find

$$f_{ip}^{(1)}(\mathbf{R}_g, v_{||}, \mu_0, \varphi_0) \simeq f_{ip}^{(1)}(\mathbf{r}, v_{||}, \mu_0, \varphi_0) + \frac{1}{\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} f_{ip}^{(1)} = \frac{Ze\widetilde{\phi}_0}{MB}\frac{\partial f_{i0}}{\partial \mu_0} + \overline{\nabla} \cdot \left[\frac{c}{B^2}(\mathbf{v} \times \hat{\mathbf{b}})\widetilde{\phi}_g \frac{\partial f_{ig}}{\partial \mu_0}\right], \qquad (26)$$

where $\widetilde{\phi}_0 \equiv \widetilde{\phi}(\mathbf{r}, \mu_0, \varphi_0, t)$ and $f_{i0} \equiv f_i(\mathbf{r}, v_{||}, \mu_0, t)$ are obtained by replacing $\mathbf{R}$, u, μ and φ by $\mathbf{r}$, $v_{||}$, μ_0 and φ_0 in $\widetilde{\phi}(\mathbf{R}, \mu, \varphi, t)$ and $f_i(\mathbf{R}, u, \mu, t)$. The integral over velocity of $\widetilde{\phi}_0 \partial_{\mu_0} f_{i0}$ vanishes because this term has vanishing gyroaverage. Then, the only term left is

$$\int d^3v\, f_{ip}^{(1)} \simeq \nabla \cdot \left[\frac{c}{B^2}\int d^3v\, (\mathbf{v} \times \hat{\mathbf{b}})\widetilde{\phi}_g \frac{\partial f_{ig}}{\partial \mu_0}\right]. \qquad (27)$$

This result is of order $\delta_i^2 n_e$, so only the lowest order pieces of $\widetilde{\phi}_g$ and f_{ig} must be kept. According to (4), the lowest order piece of f_{ig} has wavelengths on the order of the macroscopic length L. Consequently, to obtain a long wavelength contribution, only the long wavelength result $\widetilde{\phi}_g \simeq -\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi$ is needed, giving $\int d^3v\, (\mathbf{v} \times \hat{\mathbf{b}})\widetilde{\phi}_g(\partial_{\mu_0} f_{ig}) \simeq -\int d^3v\, (v_\perp^2/2\Omega_i)\nabla_\perp \phi(\partial_{\mu_0} f_{ig})$. Finally, integrating by parts in μ_0 leads to

$$\int d^3v\, f_{ip}^{(1)} \simeq \nabla \cdot \left(\frac{cn_i}{B\Omega_i}\nabla_\perp \phi\right). \qquad (28)$$

The contribution of $\int d^3v\, f_{ip}^{(2)}$, formally of order $\delta_i^2 n_e$, is in reality negligible. Since we are only interested in long wavelength pieces, we can expand around $\mathbf{r}$ and replace $\overline{\phi}_g$, $\widetilde{\phi}_g$, $\widetilde{\Phi}_g$ and f_{ig} by $\overline{\phi}_0 \equiv \overline{\phi}(\mathbf{r}, \mu_0, t)$, $\widetilde{\phi}_0 \equiv \widetilde{\phi}(\mathbf{r}, \mu_0, \varphi_0, t)$, $\widetilde{\Phi}_0 \equiv \widetilde{\Phi}(\mathbf{r}, \mu_0, \varphi_0, t)$ and $f_{i0} \equiv f_i(\mathbf{r}, v_{||}, \mu_0, t)$. As a result, many terms gyroaverage to zero. Moreover, one of the terms that does not vanish, $(Zec/2MB^2\Omega_i)(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0(\partial_{\mu_0} f_{i0})$, is negligible. The vector product $(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0$ can be written as $\overline{\nabla} \cdot [\widetilde{\Phi}_0(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}})]$. Then, using that $\widetilde{\Phi}_0(c/B)(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \sim \delta_i^2 v_i T_e/e$ and its divergence at long wavelengths is of order $\delta_i^2 v_i T_e/eL$, we find that $(Zec/2MB^2\Omega_i)(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0(\partial_{\mu_0} f_{i0}) \sim \delta_i^3 f_{si}$ and thus negligible. The remaining terms give

$$\int d^3v\, f_{ip}^{(2)} \simeq \frac{Z^2e^2}{2M^2B^2}\int d^3v\, \left(\widetilde{\phi}_0^2 \frac{\partial^2 f_{i0}}{\partial \mu_0^2} + \frac{\partial \widetilde{\phi}_0^2}{\partial \mu_0}\frac{\partial f_{i0}}{\partial \mu_0}\right) = \frac{Z^2e^2}{2M^2B^2}\int d^3v\, \frac{\partial}{\partial \mu_0}\left(\widetilde{\phi}_0^2 \frac{\partial f_{i0}}{\partial \mu_0}\right) = 0. \qquad (29)$$

Finally, combining (28) and (29), the long wavelength piece of n_{ip} is

$$n_{ip} \simeq \nabla \cdot \left(\frac{cn_i}{B\Omega_i}\nabla_\perp \phi\right), \qquad (30)$$

and at long wavelengths the flux surface average of vorticity equation (23) can be written as

$$\frac{\partial}{\partial t}\left\langle \frac{Mcn_i}{B}\frac{\partial \phi}{\partial x}\right\rangle_x = \frac{B}{c}\langle \tilde{\mathbf{J}}_E\cdot\hat{\mathbf{x}}\rangle_x. \tag{31}$$

This equation gives the evolution of the y-momentum of the $\mathbf{E}\times\mathbf{B}$ flow. According to the estimate in (1), this equation must be found to order $\delta_i^3 p_i/L$, since we would need $(B/c)\langle\tilde{\mathbf{J}}_E\cdot\hat{\mathbf{x}}\rangle_x$ to correspond to $\partial\Pi/\partial x$. In the next subsection we take the long wavelength limit of $(B/c)\langle\tilde{\mathbf{J}}_E\cdot\hat{\mathbf{x}}\rangle_x$ to order $\delta_i^3 p_i/L$ and show that equation (31) differs from (5) by a non-physical momentum source.

4.2. Polarization current at $k_\perp L\sim 1$

In this subsection, we obtain the long wavelength limit of $(B/c)\langle\tilde{\mathbf{J}}_E\cdot\hat{\mathbf{x}}\rangle_x$ to prove that vorticity equation (23) and hence quasineutrality equation (18) introduce non-physical sources of momentum. From (25), we obtain

$$\frac{B}{c}\langle\tilde{\mathbf{J}}_E\cdot\hat{\mathbf{x}}\rangle_x = Ze\left\langle\int d^3v\,(f_{ip}^{(1)}+f_{ip}^{(2)})\frac{\partial\phi}{\partial y}\right\rangle_x + Ze\left\langle\int d^3v\,f_{ig}\left(\frac{\partial\widetilde{\phi}_g}{\partial y}-\frac{\partial\Psi_g^{(2)}}{\partial y}\right)\right\rangle_x. \tag{32}$$

Here, the terms $Ze\langle\int d^3v\,f_{ip}^{(1)}\partial_y\phi\rangle_x$ and $Ze\langle\int d^3v\,f_{ig}\partial_y\widetilde{\phi}_g\rangle_x$ are formally of order $\delta_i p_i/L$, while $Ze\langle\int d^3v\,f_{ip}^{(2)}\partial_y\phi\rangle_x$ and $Ze\langle\int d^3v\,f_{ig}\partial_y\Psi_g^{(2)}\rangle_x$ are formally of order $\delta_i^2 p_i/L$. In the following paragraphs, we obtain the long wavelength limit of all these terms to order $\delta_i^3 p_i/L$.

Term $Ze\langle\int d^3v\,f_{ip}^{(1)}\partial_y\phi\rangle_x$. Since we are only interested in the long wavelength limit of this term, we can Taylor expand around $\mathbf{r}$ to write

$$\begin{aligned} Ze\left\langle\int d^3v\,f_{ip}^{(1)}\frac{\partial\phi}{\partial y}\right\rangle_x &\simeq \frac{Z^2e^2}{MB}\left\langle\int d^3v\,\frac{\partial f_{i0}}{\partial\mu_0}\widetilde{\phi}_0\frac{\partial\widetilde{\phi}_0}{\partial y}\right\rangle_x \\ &+Ze\left\langle\int d^3v\,\frac{1}{\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\overline{\nabla}\left(f_{ip}^{(1)}\frac{\partial\phi}{\partial y}\right)\right\rangle_x \\ &-Ze\left\langle\int d^3v\,\frac{1}{2\Omega_i^2}(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}):\overline{\nabla}\,\overline{\nabla}\left(f_{ip}^{(1)}\frac{\partial\phi}{\partial y}\right)\right\rangle_x. \end{aligned} \tag{33}$$

The reason for the sign in the last, higher order term is that we perform a second expansion in the middle term about $\mathbf{R}_g$ using $(\mathbf{v}\times\hat{\mathbf{b}})\cdot\overline{\nabla}f_{ip}^{(1)}(\mathbf{r},v_{||},\mu_0,\varphi_0,t)\simeq(\mathbf{v}\times\hat{\mathbf{b}})\cdot[\overline{\nabla}f_{ip}^{(1)}(\mathbf{R}_g,v_{||},\mu_0,\varphi_0,t)-\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\overline{\nabla}\,\overline{\nabla}f_{ip}^{(1)}]$. This subtlety is important for the final result. In the first term, we integrate by parts in μ_0 to finally obtain

$$\begin{aligned} Ze\left\langle\int d^3v\,f_{ip}^{(1)}\frac{\partial\phi}{\partial y}\right\rangle_x &\simeq -\frac{Z^2e^2}{2MB}\left\langle\int d^3v\,f_{i0}\frac{\partial^2\widetilde{\phi}_0^2}{\partial y\partial\mu_0}\right\rangle_x + \frac{\partial}{\partial x}\left\langle\int d^3v\,f_{ip}^{(1)}M(\mathbf{v}\cdot\hat{\mathbf{y}})\frac{c}{B}\frac{\partial\phi}{\partial y}\right\rangle_x \\ &-\frac{1}{2\Omega_i}\frac{\partial^2}{\partial x^2}\left\langle\int d^3v\,f_{ip}^{(1)}M(\mathbf{v}\cdot\hat{\mathbf{y}})^2\frac{c}{B}\frac{\partial\phi}{\partial y}\right\rangle_x. \end{aligned} \tag{34}$$

Term $Ze\langle\int d^3v\, f_{ig}\partial_y\widetilde{\phi}_g\rangle_x$**.** Employing the same procedure as for $Ze\langle\int d^3v\, f_{ip}^{(1)}\partial_y\phi\rangle_x$, we find

$$Ze\left\langle\int d^3v\, f_{ig}\frac{\partial\widetilde{\phi}_g}{\partial y}\right\rangle_x \simeq \frac{\partial}{\partial x}\left\langle\int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})\frac{c}{B}\frac{\partial\widetilde{\phi}_g}{\partial y}\right\rangle_x - \frac{1}{2\Omega_i}\frac{\partial^2}{\partial x^2}\left\langle\int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})^2\frac{c}{B}\frac{\partial\widetilde{\phi}_g}{\partial y}\right\rangle_x. \quad (35)$$

Here, the integral over velocity of $f_{i0}\partial_y\widetilde{\phi}_0$ vanishes because its gyroaverage is zero.

Term $Ze\langle\int d^3v\, f_{ip}^{(2)}\partial_y\phi\rangle_x$**.** In this higher order term, of order $\delta_i^2 p_i/L$, the Taylor expansion around $\mathbf{r}$ is only carried out to first order. Discarding terms that gyroaverage to zero leaves

$$\begin{aligned}
Ze\left\langle\int d^3v\, f_{ip}^{(2)}\frac{\partial\phi}{\partial y}\right\rangle_x \simeq{}& \frac{Z^3e^3}{M^2B^2}\left\langle\int d^3v\left[\frac{\partial^2 f_{i0}}{\partial\mu_0^2}\frac{1}{2}\widetilde{\phi}_0^2\left(\frac{\partial\overline{\phi}_0}{\partial y}+\frac{\partial\widetilde{\phi}_0}{\partial y}\right)\right.\right.\\
&\left.\left.+\frac{\partial f_{i0}}{\partial\mu_0}\left(-\widetilde{\phi}_0\frac{\partial\overline{\phi}_0}{\partial\mu_0}\frac{\partial\widetilde{\phi}_0}{\partial y}+\frac{1}{2}\frac{\partial\widetilde{\phi}_0^2}{\partial\mu_0}\frac{\partial\overline{\phi}_0}{\partial y}\right)\right]\right\rangle_x\\
&+\frac{Z^2e^2c}{MB^2\Omega_i}\left\langle\int d^3v\,\frac{\partial f_{i0}}{\partial\mu_0}\left[(\nabla\widetilde{\Phi}_0\times\hat{\mathbf{b}})\cdot\nabla\,\overline{\phi}_0\frac{\partial\widetilde{\phi}_0}{\partial y}\right.\right.\\
&\left.\left.+\frac{1}{2}(\nabla\widetilde{\phi}_0\times\hat{\mathbf{b}})\cdot\nabla\widetilde{\Phi}_0\frac{\partial\overline{\phi}_0}{\partial y}\right]\right\rangle_x\\
&-\frac{Zec}{B\Omega_i}\left\langle\int d^3v\,(\nabla\widetilde{\Phi}_0\times\hat{\mathbf{b}})\cdot\nabla f_{i0}\frac{\partial\widetilde{\phi}_0}{\partial y}\right\rangle_x\\
&+\frac{\partial}{\partial x}\left\langle\int d^3v\, f_{ip}^{(2)}M(\mathbf{v}\cdot\hat{\mathbf{y}})\frac{c}{B}\frac{\partial\phi}{\partial y}\right\rangle_x. \quad (36)
\end{aligned}$$

Integrating the term $(\partial^2_{\mu_0}f_{i0})\widetilde{\phi}_0^2(\partial_y\overline{\phi}_0)$ by parts once in μ_0, the first integral in (36) can be written as

$$\begin{aligned}
&\left\langle\int d^3v\left[\frac{\partial^2 f_{i0}}{\partial\mu_0^2}\frac{1}{2}\widetilde{\phi}_0^2\left(\frac{\partial\overline{\phi}_0}{\partial y}+\frac{\partial\widetilde{\phi}_0}{\partial y}\right)+\frac{\partial f_{i0}}{\partial\mu_0}\left(-\widetilde{\phi}_0\frac{\partial\overline{\phi}_0}{\partial\mu_0}\frac{\partial\widetilde{\phi}_0}{\partial y}+\frac{1}{2}\frac{\partial\widetilde{\phi}_0^2}{\partial\mu_0}\frac{\partial\overline{\phi}_0}{\partial y}\right)\right]\right\rangle_x =\\
&\qquad\left\langle\int d^3v\left[\frac{\partial^2 f_{i0}}{\partial\mu_0^2}\frac{1}{6}\frac{\partial\widetilde{\phi}_0^3}{\partial y}-\frac{1}{2}\frac{\partial f_{i0}}{\partial\mu_0}\frac{\partial}{\partial y}\left(\widetilde{\phi}_0^2\frac{\partial\overline{\phi}_0}{\partial\mu_0}\right)\right]\right\rangle_x. \quad (37)
\end{aligned}$$

The second term in (36) vanishes. To see this, we write it as

$$\begin{aligned}
&\left\langle\int d^3v\,\frac{\partial f_{i0}}{\partial\mu_0}\left[(\nabla\widetilde{\Phi}_0\times\hat{\mathbf{b}})\cdot\nabla\,\overline{\phi}_0\frac{\partial\widetilde{\phi}_0}{\partial y}+\frac{1}{2}(\nabla\widetilde{\phi}_0\times\hat{\mathbf{b}})\cdot\nabla\widetilde{\Phi}_0\frac{\partial\overline{\phi}_0}{\partial y}\right]\right\rangle_x =\\
&\qquad\left\langle\int d^3v\,\frac{\partial f_{i0}}{\partial\mu_0}\left[\frac{\partial\widetilde{\Phi}_0}{\partial y}\frac{\partial\widetilde{\phi}_0}{\partial y}\frac{\partial\overline{\phi}_0}{\partial x}-\frac{1}{2}\frac{\partial\overline{\phi}_0}{\partial y}\left(\frac{\partial\widetilde{\Phi}_0}{\partial x}\frac{\partial\widetilde{\phi}_0}{\partial y}+\frac{\partial\widetilde{\phi}_0}{\partial x}\frac{\partial\widetilde{\Phi}_0}{\partial y}\right)\right]\right\rangle_x =\\
&\qquad\frac{1}{2}\left\langle\int d^3v\,\frac{\partial f_{i0}}{\partial\mu_0}\left[\frac{\partial\overline{\phi}_0}{\partial x}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\widetilde{\Phi}_0}{\partial y}\right)^2-\frac{\partial\overline{\phi}_0}{\partial y}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\widetilde{\Phi}_0}{\partial x}\frac{\partial\widetilde{\Phi}_0}{\partial y}\right)\right]\right\rangle_x = 0. \quad (38)
\end{aligned}$$

The integrand gyroaverage vanishes because f_{i0} and $\overline{\phi}_0$ do not depend on φ_0.

Finally, the third term in (36) can be simplified by employing $(\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} f_{i0} = \overline{\nabla} \cdot [f_{i0}(\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}})]$ to write

$$\left\langle \int d^3v \, (\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} f_{i0} \frac{\partial \widetilde{\phi}_0}{\partial y} \right\rangle_x = \frac{\partial}{\partial x} \left\langle \int d^3v \, f_{i0} \frac{\partial \widetilde{\Phi}_0}{\partial y} \frac{\partial \widetilde{\phi}_0}{\partial y} \right\rangle_x - \left\langle \int d^3v \, f_{i0} (\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \left(\frac{\partial \widetilde{\phi}_0}{\partial y} \right) \right\rangle_x . \quad (39)$$

Here, the integral of $f_{i0}(\partial_y \widetilde{\Phi}_0)(\partial_y \widetilde{\phi}_0) = (1/2)\partial_{\varphi_0}[f_{i0}(\partial_y \widetilde{\Phi}_0)^2]$ vanishes because its gyroaverage vanishes. The integral of $f_{i0}(\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}(\partial_y \widetilde{\phi}_0)$ is simplified by realizing that integration by parts in φ_0 gives

$$\int d^3v \, f_{i0} (\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \left(\frac{\partial \widetilde{\phi}_0}{\partial y} \right) = - \int d^3v \, f_{i0} (\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \left(\frac{\partial \widetilde{\Phi}_0}{\partial y} \right) = - \int d^3v \, f_{i0} \frac{\partial}{\partial y} [(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0] - \int d^3v \, f_{i0} (\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \left(\frac{\partial \widetilde{\phi}_0}{\partial y} \right) . \quad (40)$$

From this equation, we find $\int d^3v \, f_{i0}(\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}(\partial_y \widetilde{\phi}_0) = -(1/2) \int d^3v \, f_{i0} \partial_y [(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0]$. Using this result, equation (39) becomes

$$\left\langle \int d^3v \, (\overline{\nabla}\widetilde{\Phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla} f_{i0} \frac{\partial \widetilde{\phi}_0}{\partial y} \right\rangle_x = \frac{1}{2} \left\langle \int d^3v \, f_{i0} \frac{\partial}{\partial y} [(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0] \right\rangle_x . \quad (41)$$

Substituting the results in (37), (38) and (41) into equation (36) and integrating by parts in μ_0 gives

$$Ze \left\langle \int d^3v \, f_{ip}^{(2)} \frac{\partial \phi}{\partial y} \right\rangle_x \simeq -\frac{Zec}{2B\Omega_i} \left\langle \int d^3v \, f_{i0} \frac{\partial}{\partial y} [(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0] \right\rangle_x + Ze \left\langle \int d^3v \, f_{i0} \frac{\partial \Psi_0^{(3)}}{\partial y} \right\rangle_x + \frac{\partial}{\partial x} \left\langle \int d^3v \, f_{ip}^{(2)} M(\mathbf{v} \cdot \hat{\mathbf{y}}) \frac{c}{B} \frac{\partial \phi}{\partial y} \right\rangle_x , \quad (42)$$

where we define the new quantity

$$\Psi_0^{(3)} = \frac{Z^2e^2}{6M^2B^2} \frac{\partial^2}{\partial \mu_0^2} \langle \widetilde{\phi}_0^3 \rangle_0 + \frac{Z^2e^2}{2M^2B^2} \frac{\partial}{\partial \mu_0} \left(\langle \widetilde{\phi}_0^2 \rangle_0 \frac{\partial \overline{\phi}_0}{\partial \mu_0} \right) \sim \delta_i^3 \frac{T_e}{e} . \quad (43)$$

Here, $\langle \ldots \rangle_0$ is the gyroaverage holding $\mathbf{r}$, $v_{||}$, μ_0 and t fixed.

Term $-Ze\langle \int d^3v \, f_{ig} \partial_y \Psi_g^{(2)} \rangle_x$. This term is higher order, and has to be expanded only to first order in δ_i, giving

$$-Ze \left\langle \int d^3v \, f_{ig} \frac{\partial \Psi_g^{(2)}}{\partial y} \right\rangle_x \simeq \left\langle \int d^3v \, f_{i0} \left\{ \frac{Z^2e^2}{2MB} \frac{\partial^2 \widetilde{\phi}_0^2}{\partial y \partial \mu_0} + \frac{Zec}{2B\Omega_i} \frac{\partial}{\partial y} [(\overline{\nabla}\widetilde{\phi}_0 \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\widetilde{\Phi}_0] \right\} \right\rangle_x - \frac{\partial}{\partial x} \left\langle \int d^3v \, f_{ig} M(\mathbf{v} \cdot \hat{\mathbf{y}}) \frac{c}{B} \frac{\partial \Psi_g^{(2)}}{\partial y} \right\rangle_x . \quad (44)$$

Substituting the results in (34), (35), (42) and (44) into equation (32) for the x component of the polarization current, we find

$$\frac{B}{c}\langle \tilde{\mathbf{J}}_E \cdot \hat{\mathbf{x}} \rangle_x = \frac{\partial}{\partial x}\left\langle \int d^3v\, f_i M(\mathbf{v}\cdot\hat{\mathbf{y}})\frac{c}{B}\frac{\partial \phi}{\partial y} - \int d^3v\, f_{ig} M(\mathbf{v}\cdot\hat{\mathbf{y}})\frac{c}{B}\frac{\partial \Psi_g}{\partial y}\right\rangle_x$$
$$-\frac{1}{2\Omega_i}\frac{\partial^2}{\partial x^2}\left\langle \int d^3v\, f_i M(\mathbf{v}\cdot\hat{\mathbf{y}})^2\frac{c}{B}\frac{\partial \phi}{\partial y} - \int d^3v\, f_{ig} M(\mathbf{v}\cdot\hat{\mathbf{y}})^2\frac{c}{B}\frac{\partial \overline{\phi}_g}{\partial y}\right\rangle_x$$
$$+Ze\left\langle \int d^3v\, f_{i0}\frac{\partial \Psi_0^{(3)}}{\partial y}\right\rangle_x . \qquad (45)$$

Employing the moments $(\mathbf{v}\cdot\hat{\mathbf{y}})$ and $(\mathbf{v}\cdot\hat{\mathbf{y}})^2$ of gyrokinetic equation (22), we find $\partial_x\langle \int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})(c/B)\partial_y\Psi_g\rangle_x = \partial_t\langle \int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})\rangle_x$ and $\partial_x\langle \int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})^2(c/B)\partial_y\overline{\phi}_g\rangle_x \simeq \partial_t\langle \int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})^2\rangle_x$ to the order of interest. Then, after fast time averaging, equation (45) becomes

$$\frac{B}{c}\langle \tilde{\mathbf{J}}_E \cdot \hat{\mathbf{x}} \rangle_x = \frac{\partial}{\partial x}\left\langle n_i M(\mathbf{V}_i\cdot\hat{\mathbf{y}})\frac{c}{B}\frac{\partial \phi}{\partial y}\right\rangle_x - \frac{1}{2\Omega_i}\frac{\partial^2}{\partial x^2}\left\langle \hat{\mathbf{y}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\mathbf{y}}\frac{c}{B}\frac{\partial \phi}{\partial y}\right\rangle_x$$
$$-\frac{1}{2\Omega_i}\frac{\partial^2 p_{i\perp}}{\partial t\partial x} + Ze\left\langle \int d^3v\, f_{i0}\frac{\partial \Psi_0^{(3)}}{\partial y}\right\rangle_x , \qquad (46)$$

where we have used that the long wavelength contributions to $\int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})$ and $\int d^3v\, f_{ig}M(\mathbf{v}\cdot\hat{\mathbf{y}})^2$ are $\Omega_i^{-1}\partial_x p_{i\perp}$ and $p_{i\perp}$ to lowest order.

4.3. Transport of y-momentum

Comparing the preceding results to the more accurate results of equation (5) and (10), we see that equations (31) and (46) give an incorrect transport of momentum equation for long wavelengths and long time scales, namely

$$\frac{\partial}{\partial t}\langle n_i M\mathbf{V}_i\cdot\hat{\mathbf{y}}\rangle_x = -\frac{\partial}{\partial x}\langle \hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x + Ze\left\langle \int d^3v\, f_{i0}\frac{\partial \Psi_0^{(3)}}{\partial y}\right\rangle_x , \qquad (47)$$

where $\langle \hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x$ is as given in (10), and we have used that at long wavelengths $\mathbf{V}_i\cdot\hat{\mathbf{y}} \simeq (c/B)\partial_x\phi + (n_iM\Omega_i)^{-1}\partial_x p_{i\perp}$. Notice that the momentum equation (47) derived from the Dubin *et al* [15] gyrokinetic equation has resulted in an unphysical source term $Ze\langle \int d^3v\, f_{i0}\partial_y\Psi_0^{(3)}\rangle_x \sim \delta_i^3 p_i/L$ that does not appear in the correct momentum equation (5). This extra third order source is equivalent to the Lorentz force due to a current density $J_x = \langle \int d^3v\, f_{i0}(c/B)\partial_y\Psi_0^{(3)}\rangle_x \sim \delta_i^4 e n_e v_i$. Such a small current density might seem negligible, but its effect is as large as any other term in (47). As a result, it leads to incorrect predictions for the y component of the velocity and, thereby, for the long wavelength electric field.

This non-physical source of momentum would have vanished if the third order correction to Ψ had been kept. More importantly, if we had neglected the second order correction $\Psi^{(2)}$ in (11), the source of momentum would have been much larger, i.e., $Ze\langle \int d^3v\, f_{i0}\partial_y\Psi_0^{(2)}\rangle_x \sim \delta_i^2 p_i/L$. In this case, after a period of time of the order of the transport time scale $t_E \sim L^2/D_{gB} \sim \delta_i^{-2} L/v_i$, the plasma would tend to acquire velocity on the order of the thermal velocity when trying to respond to the unphysical source of momentum!

5. Discussion

We have shown that the Hamiltonian gyrokinetic formulation of Dubin *et al* [15] results in a non-physical velocity profile in the low flow ordering unless a proper momentum description is employed. If quasineutrality or vorticity are used, it is necessary to keep some third order corrections to Ψ in (12) to recover the correct transport of momentum. Employing the lowest order version of the same procedure, as is done in full f gyrokinetic codes [12, 13, 14], it is easy to derive that for $\Psi \simeq \overline{\phi}$ the non-physical source of momentum becomes large enough to drive the velocity to the high flow ordering.

Notice that in a slab, it is necessary to calculate the gyrokinetic drifts up to $O(\delta_i^3 v_i)$ to recover the correct momentum equation in the low flow ordering, while $O(\delta_i^2 v_i)$ is sufficient for the high flow ordering. It might be surprising that the drifts are only needed up to order $\delta_i^3 v_i$ in the drift ordering whereas in section 1 we argued that $\delta_i^4 v_i$ terms were required. This simplification is a result of the special geometry of the slab. In a collisionless slab, the flux surface averaged current density due to the $O(\delta_i^4 v_i)$ drift is to $O(\delta_i^4 e n_e v_i)$

$$Ze \left\langle \int d^3v \, f_{i0}^{(0)} \frac{c}{B} \frac{\partial \Psi_0^{(4)}}{\partial y} \right\rangle_x = 0, \tag{48}$$

since the lowest order piece of the distribution function $f_{i0}^{(0)}$ is independent of y. In a tokamak, on the other hand, there are magnetic geometry effects that may prevent such a cancellation from happening.

In conclusion, solving the quasineutrality equation for all the pieces of the electric field, including the long wavelength pieces, in a tokamak requires a gyrokinetic formulation that keeps the corrections to the drifts up to order $\delta_i^4 v_i$ in the low flow ordering, and to order $\delta_i^3 v_i$ in the high flow ordering. Lagrangian formulations keep drifts to order $\delta_i^2 v_i$ at most. We have shown for a slab that next order corrections are required. This is not surprising since Lagrangian perturbation theory ensures conservation of an approximate form of the energy, but does not necessarily guarantee the correct transport of momentum. The slab case shows how the electric field obtained from quasineutrality introduces an artificial momentum source that will accelerate the plasma in the y direction. The higher order corrections to the drifts studied in this article appear in general geometries, but in addition there are magnetic geometry effects that make the equations almost intractable to order $\delta_i^2 v_i$, and hopelessly complicated to order $\delta_i^3 v_i$ and $\delta_i^4 v_i$. Therefore, trying to calculate all the contributions to the electric field employing a gyrokinetic quasineutrality equation is impractical. Instead, the momentum transport equation should be explicitly solved to determine the long wavelength velocity profile.

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