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MHD Stability Comparison Theorems Revisited

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Abstract

Magnetohydrodynamic (MHD) stability comparison theorems are presented for several different plasma models, each one corresponding to a different level of collisionality: a collisional fluid model (ideal MHD), a collisionless kinetic model (kinetic MHD), and two intermediate collisionality hybrid models (Vlasov-fluid and kinetic MHD-fluid). Of particular interest is a re-examination of the often quoted statement that ideal MHD makes the most conservative predictions with respect to stability boundaries. Some of the models have already been investigated in the literature and we clarify and generalize these results. Other models are essentially new and for them we derive new comparison theorems. Three main conclusions can be drawn: 1) It is crucial to distinguish between ergodic and closed field line systems; 2) In the case of ergodic systems, ideal MHD does indeed make conservative predictions compared to the other models; 3) In closed line systems undergoing perturbations that maintain the closed line symmetry this is no longer true. Specifically, when the ions are collisionless and their gyro radius is finite, as in the Vlasov-fluid model, there is no compressibility stabilization. The Vlasov-fluid model is more unstable than ideal MHD. The reason for this is related to the wave-particle resonance associated with the perpendicular precession drift motion of the particles (i.e. the $\mathbf{E}\times\mathbf{B}$ drift and magnetic drifts), combined with the absence of any truly toroidally trapped particles. The overall conclusion is that to determine macroscopic stability boundaries for any magnetic geometry using a simple, conservative approach, one should analyze the ideal MHD energy principle for incompressible displacements.

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1 Introduction

Ideal magnetohydrodynamics (Ideal MHD) is a model often used to design and analyze fusion confinement devices. In this model the plasma is described as a single magnetized fluid. Within the framework of ideal MHD the question of the linear stability against fast macroscopic modes can be cast in the convenient form of an energy principle [1]. Here, "fast" implies that $\omega \sim v_{Ti} / L$, where ω is the frequency of the mode, v_{Ti} is the ion thermal velocity, and L is the typical size of the device. The energy principle requires the evaluation of the potential energy δW_{MHD} due to any linear perturbation $\boldsymbol{\xi}$ of the MHD fluid. The principle states that the plasma will be stable to MHD modes *iff* $\delta W_{MHD} \geq 0$ for any allowable displacement $\boldsymbol{\xi}$. Using the energy principle to evaluate the stability property of a fusion device with respect to MHD modes is therefore equivalent to evaluating the sign of δW_{MHD} for the perturbations of concern.

Unfortunately, ideal MHD is based on the assumption that both the ions and electrons are highly collisional on the MHD time scale, $\tau_{MHD} \sim L / v_{Ti}$. This assumption is not valid for the ions in a fusion-grade plasma and is only marginally valid for the electrons. The violation of high collisionality can be seen by examining Fig. 1, in which the lines $\omega = \nu_{ee}$ and $\omega = \nu_{ii}$ are plotted in T, n (temperature, density) space assuming $\omega = v_{Ti} / L$. The electron-electron collision frequency ν_{ee} and the ion-ion collision frequency ν_{ii} are given by Braginskii [2]. Also shown in Fig. 1 is the rectangle corresponding to the region of fusion interest: $10^{18}\ \mathrm{m}^{-3} < n < 10^{20}\ \mathrm{m}^{-3}$ and $0.5\ \mathrm{keV} < T < 50\ \mathrm{keV}$. Observe that for plasmas of fusion interest the ions are indeed collisionless on the MHD time scale, while the electrons are borderline collisionless.

The point here is that unstable MHD modes are known experimentally to frequently result in violent plasma behavior. It is thus critical to assess the reliability of the ideal MHD predictions using more accurate models applicable to fusion-grade plasmas. This important problem was originally studied by Kruskal and Oberman [3] and Rosenbluth and Rostoker [4] many years ago. The results have been since refined in [5], [6], [7]. In each of these articles, a new, zero gyro radius, collisionless kinetic model is employed for both species, a model now known as kinetic

MHD [8]. The most significant result is the well-known stability comparison theorem described by the inequality $\delta W_{MHD} \le \delta W_{KK}$, where δW_{KK} is the change of potential energy in the kinetic MHD model. According to this inequality, ideal MHD stability implies kinetic MHD stability. Or in other words, the predictions of ideal MHD, the simpler but physically invalid model, can be trusted in the sense that they are more conservative than those of kinetic MHD, the more accurate model.

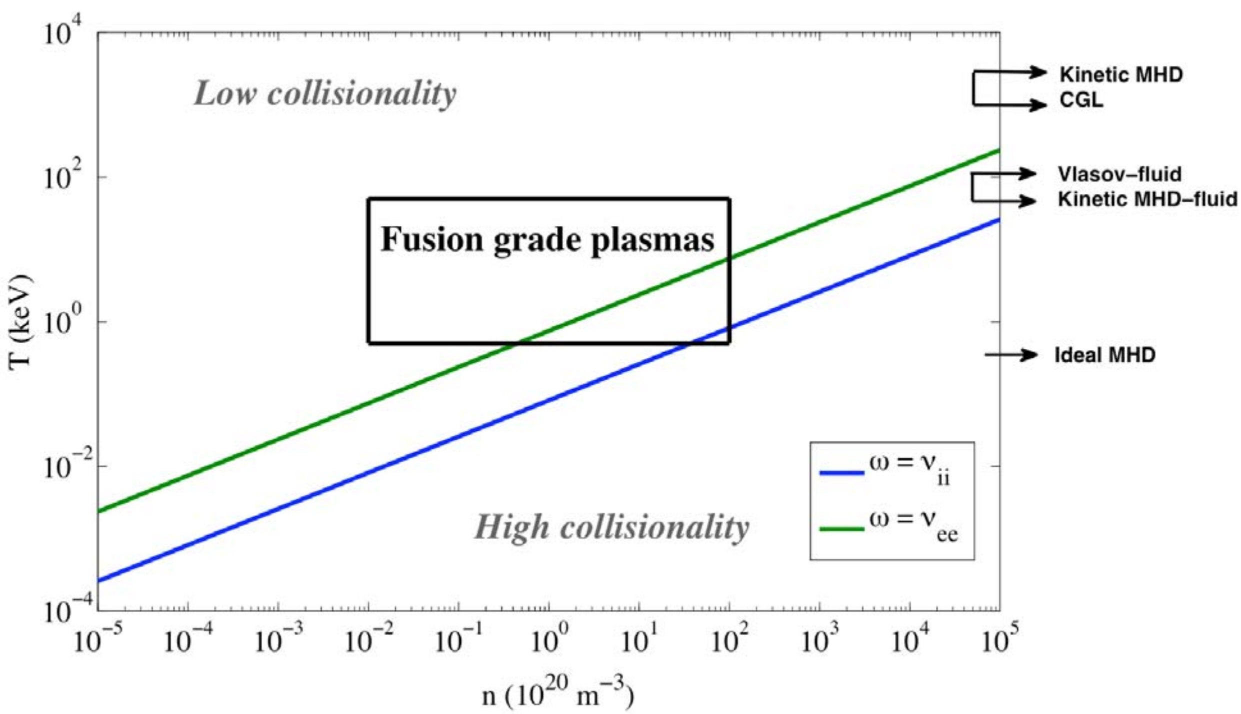

Fig. 1. Electron and ion collisionalities in plasmas of fusion interest

What then are the issues that have motivated the present work? There are three issues. First, these early pioneering studies did not explicitly distinguish between ergodic and closed line systems. In effect, the comparison theorems that were derived were focused on the ideal MHD compressibility stabilization term, and its effect on closed field line systems such as the levitated dipole concept [9], where compressibility plays an important role [9], [10]. Comparison theorems concerning ergodic systems, such as the tokamak, were not explicitly formulated.

Second, the zero gyro radius mathematical expansion used in kinetic MHD allows wave-particle resonances, but only due to the parallel motion of the particles, corresponding to

$\omega - k_{\|} v_{\|} = 0$. For closed line systems, however, the important interchange mode is characterized by $k_{\|} = 0$. Thus, for this mode resonant particle effects vanish in the kinetic MHD model. Even so, when $k_{\|} = 0$ there remains the physical possibility of an alternate wave-particle resonance due to the perpendicular guiding center particle drifts (i.e. for $\mathbf{v}_d = \mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \mathbf{v}_{\nabla B} + \mathbf{v}_{\boldsymbol{\kappa}}$) which occurs when $\omega - \mathbf{k}_{\perp} \cdot \mathbf{v}_d = 0$, and whose physics is not treated in the kinetic MHD model. To capture this effect a model is thus needed that allows for a finite ion gyro radius – the Vlasov-fluid model.

Third, the kinetic MHD model shows that in ergodic systems there is a collisionless compressibility stabilization that results from the exact periodicity of the motion of the trapped particles [11], [12]; the particle orbits oscillate but have zero guiding center drift. This periodicity can again be traced back to the zero gyro radius assumption. In the finite gyro radius Vlasov-fluid model, the trapped particles actually precess either poloidally or toroidally, and their motion is therefore no longer exactly periodic. This implies that "trapped particle compressibility stabilization" should vanish in the Vlasov-fluid model.

Because of these issues we have been motivated to re-examine the general question of MHD stability comparison theorems. Our new contributions are as follows. For the kinetic MHD model we have clarified and generalized the early results. Specifically we have shown the differences in stability criteria that arise between ergodic and closed line systems. We have also generalized early results to include electromagnetic effects. Lastly, we have derived stability criteria for ergodic systems without having to make the original assumption that $\omega^2 = 0$ at marginal stability.

For the Vlasov-fluid model we have derived an important generalization. The early studies focused solely on ergodic systems. We have generalized these studies to include closed line systems ultimately obtaining a comparison theorem that is valid for both ergodic and closed line systems.

We have also introduced a new hybrid model where the ions are treated by kinetic MHD and the electrons by a generalized fluid description. Here too we derive comparison theorems for ergodic and closed line systems. This model helps bridge the gap between the pure kinetic MHD and Vlasov-fluid models.

Under the assumption that the Vlasov-fluid model provides the most accurate description of MHD stability in a fusion grade plasma we arrive at the following high level conclusions: (1) ideal MHD predicts the correct (not conservative) stability boundaries in ergodic systems, (2) the Vlasov-fluid model is more unstable than ideal MHD for closed line systems because of the vanishing of compressibility stabilization, and (3) the most accurate test for MHD stability for any magnetic geometry is equivalent to testing ideal MHD stability for incompressible displacements.

The structure of the paper is as follows. In Section 2, we present a brief review of the well-known ideal MHD potential energy, δW_{MHD} with a focus on plasma compressibility. Section 3 reviews and generalizes the results for kinetic MHD resulting in the derivation a potential energy δW_{KK}. In Section 4, we derive the potential energy δW_{KF} for the hybrid model in which the electrons are treated as a fluid and the ions are described by kinetic MHD. This result allows us to compare the kinetic MHD predictions (i.e. δW_{KK}) to the hybrid predictions (i.e. δW_{KF}) where electrons are collision-dominated which is the case, for instance, near the edge of present day fusion experiments. Finally, in Section 5 we review and generalize the stability predictions of the Vlasov-fluid model, δW_{VF}, to include closed line configurations.

For the sake of readability, only the key starting equations and end results are presented in the main body of the text. The amount of detail involved in the analysis is to put it mildly, “large”. These details are presented in three appendices which can be accessed on the Physics of Plasmas webpage, along with the online version of this article.

2 Energy relations for comparison theorems: ideal MHD

In this section, the well-known ideal MHD energy principle [1] is reviewed with a focus on the role of plasma compressibility and its dependence on magnetic geometry.

As is well known, in ideal MHD, the question of the stability for static equilibrium can be expressed in the following variational form:

$$\omega^2 = \frac{\delta W_{MHD}(\boldsymbol{\xi},\boldsymbol{\xi}^*)}{K_{MHD}(\boldsymbol{\xi},\boldsymbol{\xi}^*)} \tag{1}$$

Here, ω is the eigenfrequency, $\boldsymbol{\xi}$ is the perturbed MHD fluid displacement, and δW_{MHD}, K_{MHD} are the perturbed potential and kinetic energies of the plasma respectively. The potential energy can be written as

$$\begin{aligned}\delta W_{MHD}(\boldsymbol{\xi},\boldsymbol{\xi}^*) &= \delta W_{\perp}(\boldsymbol{\xi}_{\perp},\boldsymbol{\xi}_{\perp}^*) + \delta W_{C}(\boldsymbol{\xi},\boldsymbol{\xi}^*) \\ \delta W_{\perp} &= -\int \boldsymbol{\xi}_{\perp}^* \cdot \left[(\tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \tilde{\mathbf{B}}) + \nabla(\boldsymbol{\xi}_{\perp} \cdot \nabla p)\right] d\mathbf{r} \\ \delta W_{C} &= \int \gamma p \mid \nabla \cdot \boldsymbol{\xi} \mid^2 \, d\mathbf{r}\end{aligned} \tag{2}$$

while the kinetic energy has the form

$$K_{MHD} = \int \rho \mid \boldsymbol{\xi} \mid^2 \, d\mathbf{r} \tag{3}$$

In these expressions $\mathbf{J}$ is the plasma current, $\mathbf{B}$ is the magnetic field, p is the plasma pressure, ρ is the mass density, $\gamma = 5/3$ is the coefficient of adiabatic compression, and the notation $\tilde{Q}$ refers to a perturbed quantity. The subscript "0" is suppressed from all equilibrium quantities. The perturbed magnetic field and current density are given by the well-known relations $\tilde{\mathbf{B}} = \nabla \times \left(\boldsymbol{\xi}_{\perp} \times \mathbf{B}\right)$ and $\mu_0 \tilde{\mathbf{J}} = \nabla \times \nabla \times \left(\boldsymbol{\xi}_{\perp} \times \mathbf{B}\right)$. To simplify the analysis attention is focused on internal modes as evidenced by the absence of a boundary term in Eq. (2). However, it is important to note that all the comparison theorems derived here have been generalized to cover external modes as well. Key features to observe from Eq. (2) are that $\delta W_{\perp}$ depends only on $\boldsymbol{\xi}_{\perp}$ and that the only appearance of $\boldsymbol{\xi}_{\parallel}$ is in the $\nabla \cdot \boldsymbol{\xi}$ stabilizing plasma compression term in δW_C.

Ideal MHD stability theory states that a mode is stable *iff* $\delta W_{MHD} \geq 0$ for all allowable plasma displacements. Therefore, stability is determined by minimizing δW_{MHD} with respect to $\boldsymbol{\xi}$, and then calculating the value of δW_{MHD} for the minimizing $\boldsymbol{\xi}$. If $\delta W_{MHD} \geq 0$ the plasma is stable, whereas if $\delta W_{MHD} < 0$ the plasma is unstable.

Now, since $\xi_{\|}$ appears only in the plasma compressibility term, it is convenient to first perform a universal minimization with respect to $\xi_{\|}$. As is well known (e.g. [13]) this minimization leads to the general minimizing condition

$$\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi})=0 \tag{4}$$

To solve this equation, two different cases have to be distinguished: (1) systems where $\mathbf{B}\cdot\nabla\neq 0$ which include ergodic field line geometries as well as closed line systems undergoing perturbations that break the closed line symmetry, and (2) closed line systems undergoing perturbations that do not break the closed line symmetry.

For the first case of interest where $\mathbf{B}\cdot\nabla$ is not singular, the equation $\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi})=0$ is trivially solved yielding $\nabla\cdot\boldsymbol{\xi}=0$. The minimization with respect to $\xi_{\|}$ thus implies that

$$\delta W_{MHD}(\boldsymbol{\xi},\boldsymbol{\xi}^{*})=\delta W_{\perp}(\boldsymbol{\xi}_{\perp},\boldsymbol{\xi}_{\perp}^{*}) \tag{5}$$

Ideal MHD stability for ergodic systems is incompressible: $\delta W_{C}=0$.

For the second case there is a periodicity constraint $\xi_{\|}(l)=\xi_{\|}(l+L)$, where l is any point along the arc length of the field line and L is the length of the line. Minimizing with respect to $\xi_{\|}$ then leads to the following expression for the compressibility contribution:

$$\delta W_{C}=\int\gamma p\,|\,\nabla\cdot\boldsymbol{\xi}\,|^{2}\,d\mathbf{r}=\int\gamma p\,|\,\langle\nabla\cdot\boldsymbol{\xi}_{\perp}\rangle\,|^{2}\,d\mathbf{r} \tag{6}$$

where Q indicates the field line average of the quantity Q:

$$\left\langle Q\right\rangle=\frac{\oint Q\frac{dl}{B}}{\oint\frac{dl}{B}} \tag{7}$$

For this case the plasma compressibility term must be maintained and included in the final minimization with respect to $\boldsymbol{\xi}_\perp$. δW_C is a positive definite quantity, and represents plasma compressibility stabilization, a term which plays a crucial role in the MHD stability of closed field line configurations, such as the levitated dipole, and the field-reversed configuration. It is this term that needs to be carefully examined as more sophisticated physics models are introduced. Indeed, the difficulty with the validity of the ideal MHD model is easily observed in δW_C, which depends explicitly on the ratio of specific heats γ. This factor arises from the adiabatic energy equation, $d/dt\left(p\rho^{-\gamma}\right) = 0$ which is only valid when the plasma is highly collisional, a condition that is never satisfied in fusion grade plasmas, certainly not for the ions.

3 Energy relations for comparison theorems: Kinetic MHD ions, Kinetic MHD electrons

Kinetic MHD has been extensively studied [3] - [7], and an excellent derivation of the basic model can be found in [8]. Several stability results for kinetic MHD have already been derived in the literature and for the sake of brevity we simply summarize these results when appropriate. In the present work, the results are generalized to include electromagnetic effects. Also the basic energy relation is re-derived without the need for taking the limit $\omega^2 \to 0$.

3.1 The model

The kinetic MHD model consists of the exact moments of mass and momentum derived from the Vlasov equation combined with the low frequency form of Maxwell equations. Introducing the small gyro radius assumption $\rho_i / L \ll 1$, averaging over the gyro phase angle, and keeping only the leading order terms in ρ_i / L, leads to a simplified form of the Vlasov equation for each species. Electron inertia can also be neglected since $m_e / m_i \ll 1$. The starting nonlinear equations for the kinetic MHD model reduce to

$$
\begin{aligned}
&\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{v})=0\\
&m_i n\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=\mathbf{J}\times\mathbf{B}-\sum_j \nabla\cdot\mathsf{P}_j\\
&\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})\\
&\nabla\times\mathbf{B}=\mu_0\mathbf{J}\\
&\nabla\cdot\mathbf{B}=0\\
&E_\parallel\equiv-\frac{\mathbf{B}\cdot\nabla\phi}{B}-\frac{\partial A_\parallel}{\partial t}=-\frac{\mathbf{B}\cdot(\nabla\cdot\mathsf{P}_e)}{enB}\\
&n_i=n_e\equiv n
\end{aligned}
\tag{8}
$$

In Eq. (8), n_j is the density of the species j, $\mathbf{v}$ is the plasma velocity, m_i is the ion mass, P_j is the pressure tensor of the species j, $E_\parallel$ is the parallel component of the electric field, ϕ is the electrostatic potential, and $A_\parallel$ is the parallel component of the vector potential. The pressure tensor for each species j is of the form

$$
\mathsf{P}=\begin{pmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{pmatrix}
\tag{9}
$$

a consequence of keeping only zeroth order terms in ρ_i / L in the kinetic equation. This implies that

$$
\nabla\cdot\mathsf{P}=\nabla p_\perp+\left(p_\parallel-p_\perp\right)\boldsymbol{\kappa}+\mathbf{b}\mathbf{B}\cdot\nabla\left(\frac{p_\parallel-p_\perp}{B}\right)
\tag{10}
$$

The solution for the distribution function is needed only to calculate the density and pressure moments, n, $p_\perp$, and $p_\parallel$, which for each species are given by

$$
\begin{aligned}
n &= \int f\, d\mathbf{w} = \frac{2^{1/2}\pi B}{m^{3/2}} \int \frac{1}{\left(\varepsilon - \mu B\right)^{1/2}} f\, d\varepsilon\, d\mu \\
p_\perp &= \int \frac{m w_\perp^2}{2} f\, d\mathbf{w} = \frac{2^{1/2}\pi B}{m^{3/2}} \int \frac{\mu B}{\left(\varepsilon - \mu B\right)^{1/2}} f\, d\varepsilon\, d\mu \\
p_\parallel &= \int m w_\parallel^2 f\, d\mathbf{w} = \frac{2^{3/2}\pi B}{m^{3/2}} \int \left(\varepsilon - \mu B\right)^{1/2} f\, d\varepsilon\, d\mu
\end{aligned}
\tag{11}
$$

Note that $\mathbf{w}$ is the random component of particle velocity while here and below $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is the macroscopic plasma velocity. Also, $\varepsilon = (m / 2)(w_\perp^2 + w_\parallel^2)$ and $\mu = m w_\perp^2 / 2B$ are the basic velocity variables describing the kinetic MHD distribution function $f(\mathbf{r}, \varepsilon, \mu, t)$. The distribution function itself satisfies the gyro averaged kinetic equation which can be written as

$$
\frac{\partial f}{\partial t} + (\mathbf{v} + w_\parallel \mathbf{b})\cdot\nabla f + \bar{\dot{\varepsilon}}\, \frac{\partial f}{\partial \varepsilon} = 0 \tag{12}
$$

where $\mathbf{b} = \mathbf{B} / B$. The over-bar operator indicates an average over a gyro period while $\dot{\varepsilon}$ denotes $d\varepsilon / dt$ with d / dt representing the full Vlasov operator. One of the main results of kinetic MHD theory is a derivation of the gyro averaged value of $\dot{\varepsilon}$ in the limit of vanishing gyro radius, which is given by

$$
\bar{\dot{\varepsilon}} = q w_\parallel E_\parallel - m w_\parallel \mathbf{b} \cdot \frac{d\mathbf{v}}{dt} - \frac{m w_\perp^2}{2} \nabla \cdot \mathbf{v} + m \left(\frac{w_\perp^2}{2} - w_\parallel^2 \right) \mathbf{b} \cdot \left(\mathbf{b} \cdot \nabla \right) \mathbf{v} \tag{13}
$$

The above kinetic equations apply to both electrons and ions with the appropriate choice of the mass m and charge q in the expression for $\bar{\dot{\varepsilon}}$. The same macroscopic velocity $\mathbf{v}$ appears in both the electron and ion equations since $\mathbf{v}_e \approx \mathbf{v}_i \equiv \mathbf{v}$. A final point to note is that although $E_\parallel = -\mathbf{b} \cdot \nabla \phi - \partial A_\parallel / \partial t$ is formally a first order quantity in the gyro radius expansion, it explicitly appears in (13) and must be maintained for a self-consistent closure. It is ultimately calculated by a combination of the charge neutrality condition and the parallel component of the electron fluid momentum equation. This completes the specification of the kinetic MHD model.

3.2 Equilibrium

Consider now equilibrium in the kinetic MHD model. In order to compare macroscopic stability thresholds with those of ideal MHD we choose equilibrium distribution functions that are independent of the adiabatic invariant μ; that is both the electron and ion equilibrium distribution functions are of the form $f(\mathbf{r},\varepsilon,\mu) \to f(\psi,\varepsilon)$ where $\psi(\mathbf{r})$ is the usual flux function satisfying $\mathbf{b}\cdot\nabla\psi = 0$. The equilibrium pressure tensor is then isotropic: $p_{\perp}(\psi) = p_{\parallel}(\psi) = p_{e,i}(\psi)$. For static equilibria (i.e. $\mathbf{v} = 0$) the plasma momentum equation therefore becomes

$$\begin{aligned} &\mathbf{J}\times\mathbf{B} = \nabla p \\ &p = p(\psi) = p_i(\psi) + p_e(\psi) \end{aligned} \tag{14}$$

Furthermore, since $E_{\parallel} = -\mathbf{B}\cdot(\nabla\cdot\mathsf{P}_e)/enB = -\mathbf{B}\cdot\nabla p_e/enB$ it follows that $E_{\parallel} = 0$. Similarly for static equilibria $\mathbf{E}_{\perp} = 0$ implying that $\mathbf{v}_E = \mathbf{E}\times\mathbf{B}/B^2 = 0$. The conclusion is that the equilibria of interest are identical to those in ideal MHD.

3.3 Stability – ergodic systems

Consider next the kinetic MHD energy relation. As in ideal MHD this relation is a quadratic integral obtained by linearizing about the equilibrium solution. All perturbed quantities are written as $\tilde{Q}(\mathbf{r},\varepsilon,\mu,t) = \tilde{Q}(\mathbf{r},\varepsilon,\mu)\exp(-i\omega t)$. The energy integral is expressed in terms of the displacement $\boldsymbol{\xi}$ defined by $\tilde{\mathbf{v}} \equiv -i\omega\boldsymbol{\xi}$. The precise form of this integral depends upon the geometry (i.e. ergodic vs. closed line). These forms can be deduced by combining the results from various papers in the literature (e.g. [7], [15], [16]). For convenience, a self-contained derivation is presented in Appendix A. Importantly, the derivation generalizes previous investigations by allowing a non-zero value for $\tilde{A}_{\parallel}$ representing electromagnetic effects.

The first energy relation of interest corresponds to either ergodic systems or closed line systems undergoing symmetry breaking perturbations. These systems are characterized by the condition $\mathbf{B}\cdot\nabla \neq 0$ (except perhaps on isolated surfaces). In this case the energy relation can be written as

$$\left|\omega\right|^2 = -\frac{\delta W_\perp(\boldsymbol{\xi}_\perp,\boldsymbol{\xi}_\perp^*) + \delta W_{kk}(\boldsymbol{\xi},\boldsymbol{\xi}^*)}{K_\perp(\boldsymbol{\xi}_\perp,\boldsymbol{\xi}_\perp^*)} \tag{15}$$

where $\delta W_\perp(\boldsymbol{\xi}_\perp,\boldsymbol{\xi}_\perp^*)$ is identical to that corresponding to ideal MHD as given in Eq. (2) and

$$K_\perp(\boldsymbol{\xi}_\perp,\boldsymbol{\xi}_\perp^*) = \int \rho\left|\boldsymbol{\xi}_\perp\right|^2 d\mathbf{r} \tag{16}$$

The modification to the potential energy δW_{kk} is evaluated for arbitrary equilibrium distribution functions $f(\varepsilon,\psi)$ that need only satisfy the constraint:

$$\frac{\partial f}{\partial \varepsilon} < 0 \tag{17}$$

The result is a complicated expression which has the form

$$\begin{aligned}
\delta W_{kk}(\boldsymbol{\xi},\boldsymbol{\xi}^*) &= \left|\omega\right|^2 \int \frac{d\mathbf{r}}{n}(U_i + U_e + U_h) \\
U_i &= \hat{T}_i\left[\int \frac{\partial f_i}{\partial \varepsilon}\, d\mathbf{w} \int \frac{\partial f_i}{\partial \varepsilon}\left|\tilde{s}_i\right|^2 d\mathbf{w} - \left|\int \frac{\partial f_i}{\partial \varepsilon}\,\tilde{s}_i\, d\mathbf{w}\right|^2\right] \\
U_e &= \hat{T}_e\left[\int \frac{\partial f_e}{\partial \varepsilon}\, d\mathbf{w} \int \frac{\partial f_e}{\partial \varepsilon}\left|\tilde{s}_e\right|^2 d\mathbf{w} - \left|\int \frac{\partial f_e}{\partial \varepsilon}\,\tilde{s}_e\, d\mathbf{w}\right|^2\right] \\
U_h &= \frac{1}{\left(\hat{T}_i + \hat{T}_e\right)}\left|\hat{T}_i \int \frac{\partial f_i}{\partial \varepsilon}\,\tilde{s}_i d\mathbf{w} + \hat{T}_e \int \frac{\partial f_e}{\partial \varepsilon}\,\tilde{s}_e d\mathbf{w}\right|^2
\end{aligned} \tag{18}$$

where $\hat{T}(\psi)$ and the orbit integral $\tilde{s}$ for each species are given by

$$\frac{1}{\hat{T}} = -\frac{1}{n}\int \frac{\partial f}{\partial \varepsilon} d\mathbf{w} > 0$$
$$\tilde{s}(\varepsilon,\mu,\mathbf{r},t) = \int_{-\infty}^{t} \left[\frac{m w_\perp^2}{2} \nabla \cdot \boldsymbol{\xi}_\perp + m\left(\frac{w_\perp^2}{2} - w_\parallel^2 \right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} - q\left(\tilde{\phi} - w_\parallel \tilde{A}_\parallel \right) \right] e^{-i\omega t'} dt' \tag{19}$$

The quantity $\hat{T}(\psi)$ has the dimensions of temperature and is indeed equal to the temperature for the case of a local Maxwellian distribution function. Also, while the trajectory integrals enter the energy integrals in a positive definite form, they contain the unknowns $\tilde{\phi} \text{ and } \tilde{A}_\parallel$. These quantities can be expressed in terms of the plasma displacement $\boldsymbol{\xi}$ although the relations involve a set of coupled integral equations. Fortunately, these complicated relations are not required for the analysis, as is shown in Appendix A.

A simple application of Schwarz's inequality implies that $\delta W_{kk} \geq 0$. This allows us to draw two conclusions. First, assume the system is ideal MHD stable:

$$\delta W_{MHD} = \delta W_\perp (\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) > 0 \tag{20}$$

where $\boldsymbol{\xi}_{\perp MHD}$ is the ideal MHD eigenfunction. It immediately follows that

$$\delta W_{KK} \equiv \delta W_\perp + \delta W_{kk} \geq \delta W_\perp (\boldsymbol{\xi}^*_{\perp KK}, \boldsymbol{\xi}_{\perp KK}) \geq \delta W_\perp (\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) > 0 \tag{21}$$

Here, $\boldsymbol{\xi}_{\perp KK}$ is the kinetic MHD eigenfunction and the last inequality holds because of the minimizing energy principle associated with the ideal MHD potential energy. Equation (21), however, leads to a contradiction in Eq. (15); that is, $\left|\omega\right|^2 < 0$. The contradiction arises because the assumption $\mathrm{Im}(\omega) > 0$ has been made in the derivation of Eq. (18) when integrating back to $t' = -\infty$ in the orbit integrals. The resolution of the contradiction is that $\mathrm{Im}(\omega) \leq 0$ which implies that the system is linearly stable. The first conclusion, therefore, is that ideal MHD linear stability guarantees kinetic MHD linear stability.

A second conclusion is related to the fact that δW_{kk} appears to be proportional to $|\omega|^2$. This suggests that $\delta W_{kk} = 0$ when $\omega^2 = 0$, implying from Eq. (15) that the condition for marginal stability in the kinetic MHD model is given by $\delta W_\perp = 0$ [2]. In other words the conditions for marginal stability in ideal MHD and Kinetic MHD seem to be identical. This conclusion is indeed true, but only for a straight cylindrical plasma with circular cross section. In toroidal systems there are trapped particles and it has been shown [11], [12] that these particles produce a contribution to the integral in Eq. (18) that is proportional to $1/|\omega|^2$. The end result is that δW_{kk} is finite as $\omega \to 0$.

This can be demonstrated explicitly by examining the parallel motion in the trajectory integral. Qualitatively the integrand is proportional to $\exp[-i\omega t + ik_{\|} l(t')]$ where $l(t')$ is the parallel trajectory of a particle. For a passing particle $l(t') \approx w_{\|} t'$ and the trajectory integral $s \propto 1/(\omega - k_{\|} w_{\|})$ which is finite in the limit of $\omega^2 \to 0$.

However, for the periodic motion of a trapped particle $l(t') \approx l_0 \cos(\omega_B t')$ where $\omega_B(\varepsilon, \mu, \mathbf{r})$ is the bounce frequency of the orbit. The integrand in Eq. (19) can therefore be expanded in a Fourier series and the zeroth harmonic yields a contribution proportional to $1/\omega$. Specifically, as $\omega^2 \to 0$ the orbit integral reduces to

$$\begin{aligned} \tilde{s} &\to \frac{i}{\omega \tau_B} \overline{s} \\ \overline{s} &= \int_0^{\tau_B} \left[\frac{m w_\perp^2}{2} \nabla \cdot \boldsymbol{\xi}_\perp + m \left(\frac{w_\perp^2}{2} - w_\|^2 \right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} - q \left(\tilde{\phi} - w_\| \tilde{A}_\| \right) \right] dt' \end{aligned} \tag{22}$$

[2] The statement that $\omega^2 = 0$ corresponds to marginal stability in kinetic MHD for equilibrium distribution functions satisfying $\partial f / \partial \varepsilon < 0$ has been proven for certain geometries [7], [15] and conjectured to be true for general geometry. However, the discussion above makes use of this conjecture only when discussing the role of trapped particles. It is worthwhile to stress that the conjecture has not been used to derive the sufficient condition for stability which shows that ideal MHD stability implies kinetic MHD stability.

Here, $\tau_B = 2\pi / \omega_B$ is the bounce period. The $1/\omega$ factor in the denominator leads to a finite value for δW_{kk} given by

$$
\begin{aligned}
\delta W_{kk}(\boldsymbol{\xi},\boldsymbol{\xi}^*) &= \frac{1}{4\pi^2}\int \frac{d\mathbf{r}}{n}(U_i + U_e + U_h) \\
U_i &= \hat{T}_i\left[\int \frac{\partial f_i}{\partial \varepsilon}\,d\mathbf{w}\int_T \frac{\partial f_i}{\partial \varepsilon}\left|\omega_{Bi}\overline{s}_i\right|^2 d\mathbf{w} - \left|\int_T \frac{\partial f_i}{\partial \varepsilon}\,\omega_{Bi}\overline{s}_i\,d\mathbf{w}\right|^2\right] \\
U_e &= \hat{T}_e\left[\int \frac{\partial f_e}{\partial \varepsilon}\,d\mathbf{w}\int_T \frac{\partial f_e}{\partial \varepsilon}\left|\omega_{Be}\overline{s}_e\right|^2 d\mathbf{w} - \left|\int_T \frac{\partial f_e}{\partial \varepsilon}\,\omega_{Be}\overline{s}_e\,d\mathbf{w}\right|^2\right] \\
U_h &= \frac{1}{\left(\hat{T}_i + \hat{T}_e\right)}\left|\hat{T}_i\int_T \frac{\partial f_i}{\partial \varepsilon}\,\omega_{Bi}\overline{s}_i\,d\mathbf{w} + \hat{T}_e\int_T \frac{\partial f_e}{\partial \varepsilon}\,\omega_{Be}\overline{s}_e\,d\mathbf{w}\right|^2
\end{aligned}
\tag{23}
$$

where the subscript T on the integrals denotes integration over the region of velocity space corresponding to trapped particles.

Thus, the second conclusion is that a toroidal kinetic MHD system is positively stable when the ideal MHD system is marginally stable. This behavior corresponds to trapped particle compressibility stabilization [11], [12], an effect obviously not present in a straight cylinder. The results for ergodic kinetic MHD systems in the limit $\omega^2 \to 0$ can be conveniently summarized as follows:

$$
\begin{aligned}
&\delta W_{KK} = \delta W_\perp = \delta W_{MHD} && \text{straight cylinder} \\
&\delta W_{KK} = \delta W_\perp + \delta W_{kk} > \delta W_{MHD} && \text{torus}
\end{aligned}
\tag{24}
$$

3.4 Stability – closed line systems

The second energy relation of interest corresponds to closed line systems undergoing perturbations that maintain the closed line symmetry. The analysis presented in Section 3.3 also applies to this case but is not directly useful for determining MHD stability comparison theorems. The reason is that for closed line systems $\delta W_{MHD} = \delta W_\perp + \delta W_C$ and there is no obvious way to show analytically whether δW_{kk} in its present form is bigger or smaller than the MHD compression stabilization term δW_C. What is needed is a quantitative estimate of δW_{kk}, and not just a determination of its sign. This requires a substantial amount of analysis which is made possible by two previously derived results: (1) a proof that for closed line systems marginal stability occurs for $\omega^2 \to 0$ [7], and (2) the derivation of an elegant procedure for determining the sign of certain integrals [4]. The analysis differs from the ergodic case in that the periodicity requirements imposed by the closed line symmetry must be taken into account. The details are presented in Appendix A. The end result is an inequality expression for δW_{KK}, valid in the limit of marginal stability $\omega^2 \to 0$.

$$\delta W_{KK} = \delta W_\perp(\boldsymbol{\xi}_\perp, \boldsymbol{\xi}_\perp^*) + \delta W_{kk}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) \qquad (25)$$

In this case

$$\delta W_{kk} \geq \int \frac{5}{3} p \left| \left\langle \nabla \cdot \boldsymbol{\xi}_\perp \right\rangle \right|^2 d\mathbf{r} - e^2 \sum_s \int \int \frac{\partial f}{\partial \varepsilon} \left[\left\langle \left| \tilde{\phi} \right|^2 \right\rangle_b - \left| \left\langle \tilde{\phi} \right\rangle_b \right|^2 \right] d\mathbf{w}\, d\mathbf{r} \qquad (26)$$

The first term on the right hand side is just δW_C, the ideal MHD compressibility effect. In the second term the notation $\left\langle Q \right\rangle_b$ denotes the average over the periodic orbit,

$$\left\langle Q \right\rangle_b = \frac{\int Q \frac{dl}{\left| w_\parallel \right|}}{\int \frac{dl}{\left| w_\parallel \right|}} \qquad (27)$$

The second term is positive by virtue of Schwarz's inequality and the assumption $\partial f / \partial \varepsilon < 0$.

For a system that is ideal MHD stable, $\delta W_{MHD} = \delta W_\perp + \delta W_C > 0$. It then follows that

$$\delta W_{KK} \equiv \delta W_\perp + \delta W_{kk} \geq \delta W_{MHD}(\boldsymbol{\xi}^*_{\perp KK}, \boldsymbol{\xi}_{\perp KK}) \geq \delta W_{MHD}(\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) > 0 \qquad (28)$$

Thus, if a plasma is stable in ideal MHD it is even more stable in kinetic MHD.

Additional insight can be gained about closed field line systems by examining the special case when $\mathbf{b}\cdot\nabla \equiv 0$ corresponding for instance to the $m = 0$ mode in a cylindrical Z-pinch. In this case, the kinetic equation for the perturbed distribution functions takes a particularly simple form because $\tilde{E}_{\|} = 0$ and all the terms in f are now fluid-like:

$$\tilde{f} = -\boldsymbol{\xi}_\perp \cdot \nabla f - i\omega m w_\| \xi_\| \frac{\partial f}{\partial \varepsilon} + m\left[\frac{w_\perp^2}{2}\nabla\cdot\boldsymbol{\xi}_\perp + \left(\frac{w_\perp^2}{2} - w_\|^2\right)\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}\right]\frac{\partial f}{\partial \varepsilon} \tag{29}$$

The perturbed pressures are easily obtained by integration over velocity space. We find

$$\begin{aligned} \tilde{p}_\perp &= -\boldsymbol{\xi}_\perp \cdot \nabla p - 2p\nabla\cdot\boldsymbol{\xi}_\perp - p\,\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \\ \tilde{p}_\| &= -\boldsymbol{\xi}_\perp \cdot \nabla p - p\nabla\cdot\boldsymbol{\xi}_\perp + 2p\,\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \end{aligned} \tag{30}$$

Using these expressions, we obtain an exact energy integral given by

$$\begin{aligned} \omega^2 &= \frac{\delta W_{KK}}{K_\perp} \\ \delta W_{KK} &= \delta W_{MHD} + 3\int p\left|\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} + \frac{1}{3}\nabla\cdot\boldsymbol{\xi}_\perp\right|^2 d\mathbf{r} \end{aligned} \tag{31}$$

It is clear in this case that $\delta W_{KK}(\boldsymbol{\xi},\boldsymbol{\xi})$ is always a purely real quantity, so that marginal stability does indeed occur at $\omega = 0$. The system is unstable iff $\delta W_{KK}(\boldsymbol{\xi},\boldsymbol{\xi}^*) \le 0$. The expression for $\delta W_{KK}(\boldsymbol{\xi},\boldsymbol{\xi})$ in Eq. (31), clearly demonstrates that ideal MHD stability implies kinetic MHD stability.

3.5 The Chew-Goldberger-Low double adiabatic model

We close this section by briefly reviewing the Chew-Goldberger-Low (CGL) double adiabatic model [17]. This is one of the earliest models that attempts to take into account plasma anisotropy. The model is relatively simple although not well justified mathematically or physically. We have made no improvements or generalizations of the model and present it here primarily for the sake of completeness.

The basic idea used to derive the CGL model is to calculate the perpendicular and parallel energy moments of the kinetic MHD model. In these energy equations the perpendicular and parallel heat fluxes are neglected as well as temperature equilibration. These assumptions, which cannot be justified in fusion grade plasmas, lead to a closure of the model described by two fluid-like energy relations for $p_\perp$ and $p_\parallel$.

$$\begin{aligned} &\frac{d}{dt}\left(\frac{p_\parallel B^2}{n^3}\right) = 0 \\ &\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = 0 \end{aligned} \tag{32}$$

Here, $p_\perp = p_{\perp e} + p_{\perp i}$ and $p_\parallel = p_{\parallel e} + p_{\parallel i}$. From Eq. (32) it is clear why the model is sometimes referred to as the "double adiabatic model".

Isotropic equilibria in the CGL model can be easily calculated and correspond to the choice $p_\perp = p_\parallel = p(\psi)$. The equilibria are identical to those of ideal MHD. Stability is greatly simplified because of the fluid treatment of the pressures. In particular, simple expressions for the perturbed $p_\perp$ and $p_\parallel$ are obtained:

$$\begin{aligned} \tilde{p}_\parallel &= -\boldsymbol{\xi}_\perp \cdot \nabla p - p\nabla \cdot \boldsymbol{\xi} - 2p\,\mathbf{bb} : \nabla\boldsymbol{\xi} \\ \tilde{p}_\perp &= -\boldsymbol{\xi}_\perp \cdot \nabla p - 2p\nabla \cdot \boldsymbol{\xi} + p\,\mathbf{bb} : \nabla\boldsymbol{\xi} \end{aligned} \tag{33}$$

From these relations it is straightforward to carry out the stability analysis using the standard procedure. This leads to the following form for the CGL energy relation

$$\omega^2 = \frac{\delta W_{CGL}}{K_{MHD}} = \frac{\delta W_{\perp} + \delta W_{cgl}}{K_{MHD}} \quad (34)$$

where

$$\delta W_{cgl} = \frac{5}{3}\int p\left|\nabla\cdot\boldsymbol{\xi}\right|^2 d\mathbf{r} + 3\int p\left|\frac{1}{3}\nabla\cdot\boldsymbol{\xi} - \mathbf{bb}:\nabla\boldsymbol{\xi}\right|^2 d\mathbf{r} \quad (35)$$

Equation (35) is valid for both ergodic and closed line systems since with a fluid treatment there are no trapped or resonant particle effects. Consider first the implications for ergodic systems. Unlike ideal MHD, minimizing with respect to $\xi_{\|}$ does not lead to the condition $\nabla\cdot\boldsymbol{\xi} = 0$ because of the $\mathbf{bb}:\nabla\boldsymbol{\xi}$ term. Therefore $\delta W_{cgl} > 0$ and the system is more stable than ideal MHD. For closed line systems it immediately follows from Eq. (35) that $\delta W_{cgl} > \delta W_C$. Again the CGL model predicts greater stability than ideal MHD.

Earlier studies [3], [4] that mainly focused on closed line systems also showed that the CGL model is more stable than kinetic MHD. Even so, because of unjustified assumptions used in the CGL model, we do not dwell on deriving these results.

The overall conclusion is that for both ergodic and closed line systems the CGL model is more stable than ideal MHD as $\omega^2 \to 0$: $\delta W_{CGL} > \delta W_{MHD}$.

4 Energy relations for comparison theorems: kinetic MHD ions, fluid electrons

In this section we describe a hybrid model where the ions are treated with the kinetic MHD description and the electrons are treated as a fluid. This corresponds to the regime of collisionless ions (always valid for fusion plasmas) and collision dominated electrons (marginally valid for fusion plasmas). The kinetic MHD ion – fluid electron plasma is a new model for MHD modes, which serves as a transition between the fully kinetic MHD and Vlasov-fluid descriptions. Since the kinetic MHD model has already been described in Sec. 2, what is needed here to provide closure is a fluid description of the electrons.

4.1 The electron fluid model

We assume that the electrons are collision dominated and thus describe them by a fluid model with isotropic pressure. The electron momentum equation is separated into perpendicular and parallel components. The first non-vanishing contribution to the perpendicular component is zeroth order in the gyro radius expansion. The parallel component is first order but still must be maintained in certain parts of the analysis (e.g. the evaluation of $\bar{\dot{\varepsilon}}$) for self-consistency. The relevant equations are $\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B} = 0$ and $E_\parallel = -(\mathbf{b}\cdot\nabla p_e)/en$ where $\mathbf{v}(\mathbf{r},t)$ is the (ion) fluid velocity which is approximately equal to the electron fluid velocity in the limit of small gyro radius, and where we have neglected the terms associated with friction in the parallel component. In contrast to the evaluation of $\bar{\dot{\varepsilon}}$, $E_\parallel$ can be neglected in Faraday's law thus implying the frozen-in law $\partial\mathbf{B}/\partial t = \nabla\times\left(\mathbf{v}\times\mathbf{B}\right)$.

The closure of the electron fluid model is slightly more complicated than the usual simple adiabatic energy relation. The reason is that it is the parallel electron thermal conductivity that often dominates the behavior, except for the case of closed lines. The desired closure relation for electrons starts with the following form of the energy equation which includes parallel thermal conductivity, convection and compression

$$\frac{1}{\gamma-1}\left(\frac{\partial p_e}{\partial t} + \mathbf{v}\cdot\nabla p_e + \gamma p_e \nabla\cdot\mathbf{v}\right) = \mathbf{B}\cdot\nabla\left(\frac{\kappa_\parallel}{B^2}\mathbf{B}\cdot\nabla T_e\right) \tag{36}$$

Here $\gamma = 5/3$ and we want to consider the limit $\kappa_\parallel \to \infty$.

In ergodic systems for which the operator $\mathbf{B}\cdot\nabla \neq 0$, the solution to Eq. (36) is simply $\mathbf{B}\cdot\nabla T_e = \mathbf{B}\cdot\nabla(p_e/n) = 0$. If we now form the operation $d[\mathbf{B}\cdot\nabla(p_e/n)]/dt = 0$ then a short calculation that makes use of Faradays law and the conservation of mass leads to the following energy equation

$$\frac{\partial p_e}{\partial t} + \mathbf{v}\cdot\nabla p_e + p_e\nabla\cdot\mathbf{v} = 0 \tag{37}$$

corresponding to the isothermal condition $\gamma = 1$. See Appendix B for details. For closed line systems undergoing perturbations that do not break the symmetry there is an additional periodicity constraint that must be satisfied. In this case Eq. (37) is generalized as follows

$$\frac{\partial p_e}{\partial t} + \mathbf{v}\cdot\nabla p_e + p_e\nabla\cdot\mathbf{v} + (\gamma - 1)\frac{n}{\langle n\rangle}\langle p_e\nabla\cdot\mathbf{v}\rangle = 0 \tag{38}$$

where the averages are taken over each magnetic line as defined in Eq. (7). The details are also given in Appendix B. Equation (38) can be viewed as the general closure relation for the electron fluid if we keep in mind that $\langle p_e\nabla\cdot\mathbf{v}\rangle = 0$ for ergodic systems.

Combining these results leads to the following set of nonlinear equations describing the kinetic MHD ion – fluid electron model.

$$\begin{aligned}
&\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\\
&m_i n\frac{d\mathbf{v}}{dt} = \mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla p_{\perp i} - \left(p_{\parallel i} - p_{\perp i}\right)\boldsymbol{\kappa} - \mathbf{b}\mathbf{B}\cdot\nabla\left(\frac{p_{\parallel i} - p_{\perp i}}{B}\right)\\
&\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})\\
&\nabla\times\mathbf{B} = \mu_0\mathbf{J}\\
&\nabla\cdot\mathbf{B} = 0\\
&E_\parallel = -(\mathbf{b}\cdot\nabla p_e)/en\\
&n_i = n_e \equiv n
\end{aligned} \tag{39}$$

In these equations the ion density and pressures are calculated from the solution to the ion kinetic MHD equation and the electron pressure is obtained from the solution to Eq. (38).

4.2 Equilibrium

The equilibria of interest are assumed to be static (i.e. $\mathbf{v} = 0$) with isotropic pressure. This corresponds to requiring the ion distribution function to be of the form $f_i = f_i(\psi, \varepsilon)$ and assuming $p_e = p_e(\psi)$. For these choices it follows that $E_\| = \mathbf{E}_\perp = 0$. The relevant equilibrium equations are then given by

$$\begin{aligned} &\mathbf{J} \times \mathbf{B} = \nabla p \\ &p = p(\psi) = p_i(\psi) + p_e(\psi) \end{aligned} \tag{40}$$

which are identical to ideal MHD. Once again we are comparing the stability of identical equilibria using different dynamical models.

4.3 Stability – ergodic systems

The stability analysis for the kinetic MHD ion – fluid electron model is quite similar to that of the fully kinetic MHD model and the details are presented in Appendix B. There are two main differences and both help to simplify the analysis. First, because of the electron energy equation it is straightforward to derive a direct relationship between the perturbed pressure and the plasma displacement. Importantly, no complicated trajectory integrals are involved. For the case of ergodic systems or closed line systems undergoing symmetry breaking perturbations the relation between $\tilde{p}_e$ and $\boldsymbol{\xi}$ can be written as

$$\tilde{p}_e = -\boldsymbol{\xi}_\perp \cdot \nabla p_e - p_e \nabla \cdot \boldsymbol{\xi} \tag{41}$$

The second simplification arises from the parallel component of the electron momentum equation. The hybrid model yields an explicit relation between $\tilde{E}_\|$ and $\boldsymbol{\xi}$ thereby eliminating the need to introduce the scalar and vector potentials. The desired relation is given by

$$\tilde{E}_{\parallel} = \mathbf{b} \cdot \nabla \left(\frac{p_e \nabla \cdot \boldsymbol{\xi}}{en} \right) \tag{42}$$

Using these results we have again derived an energy relation (see Appendix B) which has a similar form to Eq. (15):

$$\left|\omega\right|^2 = -\frac{\delta W_{\perp}(\boldsymbol{\xi}_{\perp}, \boldsymbol{\xi}_{\perp}^*) + \delta W_{kf}(\boldsymbol{\xi}, \boldsymbol{\xi}^*)}{K_{\perp}(\boldsymbol{\xi}_{\perp}, \boldsymbol{\xi}_{\perp}^*)} \tag{43}$$

Here, $\delta W_{kf} \geq 0$ is the kinetic contribution to the potential energy:

$$\begin{aligned} \delta W_{kf}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) &= \left|\omega\right|^2 \int \frac{d\mathbf{r}}{n} (U_i + U_h) \\ U_i &= \hat{T}_i \left[\int \frac{\partial f_i}{\partial \varepsilon} d\mathbf{w} \int \frac{\partial f_i}{\partial \varepsilon} \left| \tilde{s}_i \right|^2 d\mathbf{w} - \left| \int \frac{\partial f_i}{\partial \varepsilon} \tilde{s}_i \, d\mathbf{w} \right|^2 \right] \\ U_h &= \frac{1}{\left(\hat{T}_i + T_e \right)} \left| \hat{T}_i \int \frac{\partial f_i}{\partial \varepsilon} \tilde{s}_i d\mathbf{w} \right|^2 \end{aligned} \tag{44}$$

where $T_e = p_e / n$ and

$$\begin{aligned} \frac{1}{\hat{T}_i} &= -\frac{1}{n} \int \frac{\partial f_i}{\partial \varepsilon} d\mathbf{w} > 0 \\ \tilde{s}_i(\varepsilon, \mu, \mathbf{r}, t) &= \int_{-\infty}^{t} \left[\frac{m_i w_{\perp}^2}{2} \nabla \cdot \boldsymbol{\xi}_{\perp} + m_i \left(\frac{w_{\perp}^2}{2} - w_{\parallel}^2 \right) \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa} + T_e \nabla \cdot \boldsymbol{\xi} \right] e^{-i\omega t'} dt' \end{aligned} \tag{45}$$

We again can deduce two conclusions from the energy relation. First ideal MHD stability implies kinetic MHD-fluid stability:

$$\delta W_{KF} \equiv \delta W_{\perp} + \delta W_{kf} \geq \delta W_{\perp}(\boldsymbol{\xi}_{\perp KF}^*, \boldsymbol{\xi}_{\perp KF}) \geq \delta W_{\perp}(\boldsymbol{\xi}_{\perp MHD}^*, \boldsymbol{\xi}_{\perp MHD}) > 0 \tag{46}$$

The quantity $\boldsymbol{\xi}_{\perp KF}$ is the exact eigenfunction for the hybrid model. As in kinetic MHD Eq. (44) leads to a contradiction in Eq. (43) which can only be resolved by assuming that $\mathrm{Im}(\omega) \le 0$ which implies stability.

The second conclusion involves the limit $\omega^2 = 0$ which we assume corresponds to marginal stability. In this limit $\delta W_{kf} = 0$ for a straight cylinder and is positive in a torus because of trapped particle compressibility stabilization; that is, $\delta W_{kf} > 0$ and is given by

$$
\begin{aligned}
\delta W_{kf}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) &= \frac{1}{4\pi^2}\int \frac{d\mathbf{r}}{n}(U_i + U_h) \\
U_i &= \hat{T}_i\left[\int \frac{\partial f_i}{\partial \varepsilon} d\mathbf{w} \int_T \frac{\partial f_i}{\partial \varepsilon}\left|\omega_{Bi}\overline{s}_i\right|^2 d\mathbf{w} - \left|\int_T \frac{\partial f_i}{\partial \varepsilon}\omega_{Bi}\overline{s}_i\, d\mathbf{w}\right|^2\right] \\
U_h &= \frac{1}{\left(\hat{T}_i + T_e\right)}\left|\hat{T}_i \int_T \frac{\partial f_i}{\partial \varepsilon}\omega_{Bi}\overline{s}_i\, d\mathbf{w}\right|^2
\end{aligned}
\tag{47}
$$

The overall conclusions at $\omega^2 = 0$ can be summarized as follows:

$$
\begin{aligned}
&\delta W_{KF} = \delta W_{\perp} = \delta W_{MHD} && \text{straight cylinder} \\
&\delta W_{KF} = \delta W_{\perp} + \delta W_{kf} > \delta W_{MHD} && \text{torus}
\end{aligned}
\tag{48}
$$

Finally, we note that it is tempting and quite plausible to assert that the hybrid model is less stable than the fully kinetic MHD model since trapped particle compressibility stabilization arises only from the ions and not both species. However, since the marginal eigenfunctions are not the same this is not a rigorous conclusion, only a likely conjecture.

4.4 Stability – closed line systems

We next consider the energy relation for the hybrid model corresponding to closed line systems undergoing perturbations that maintain the closed line symmetry. Again, to obtain the desired result, an estimate is needed for δW_{kf} that can be compared to the ideal MHD compressibility term δW_C. The analysis is similar but simpler than that of the pure kinetic MHD model because of the fluid treatment for the electrons. For the closed line case this fluid treatment allows us to express many of the perturbed quantities directly in terms of the displacement vector.

$$\begin{aligned}
\tilde{n}_e &= -\boldsymbol{\xi}\cdot\nabla n - n\nabla\cdot\boldsymbol{\xi} \\
\tilde{p}_e &= -\boldsymbol{\xi}\cdot\nabla p_e - p_e\nabla\cdot\boldsymbol{\xi} - \left(\gamma-1\right)p_e\left\langle\nabla\cdot\boldsymbol{\xi}_\perp\right\rangle \\
\tilde{E}_\parallel &= \mathbf{b}\cdot\nabla\left[\left(1/en\right)\left(p_e\nabla\cdot\boldsymbol{\xi}+\left(\gamma-1\right)p_e\left\langle\nabla\cdot\boldsymbol{\xi}_\perp\right\rangle\right)\right]
\end{aligned} \tag{49}$$

Using this information, and following the procedure described in Section 3.4, we have derived an energy relation analogous to Eq. (26) which is valid at marginal stability, i.e. in the limit $\omega^2 \to 0$. The details are presented in Appendix B. The energy relation is given by

$$\delta W_{KF} = \delta W_\perp + \delta W_{kf} \tag{50}$$

where (for $\gamma = 5/3$)

$$\delta W_{kf} \geq \delta W_C = \int \frac{5}{3} p\left|\left\langle\nabla\cdot\boldsymbol{\xi}_\perp\right\rangle\right|^2 d\mathbf{r} \tag{51}$$

As for pure kinetic MHD, a system that is ideal MHD stable (i.e. $\delta W_{MHD} = \delta W_\perp + \delta W_C > 0$) is even more stable in the hybrid model. Specifically,

$$\delta W_{KF} \equiv \delta W_\perp + \delta W_{kf} \geq \delta W_{MHD}(\boldsymbol{\xi}^*_{\perp KF},\boldsymbol{\xi}_{\perp KF}) \geq \delta W_{MHD}(\boldsymbol{\xi}^*_{\perp MHD},\boldsymbol{\xi}_{\perp MHD}) > 0 \tag{52}$$

For the special case when $\mathbf{b}\cdot\nabla \equiv 0$, corresponding for instance to the $m=0$ mode in a cylindrical Z-pinch, the energy integral simplifies considerably. It can be explicitly evaluated for arbitrary ω^2 and has the form

$$\omega^2 = \frac{\delta W_{KF}}{K_\perp}$$

$$\delta W_{KF} = \delta W_{MHD} + 3\int p_i \left| \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} + \frac{1}{3}\nabla \cdot \boldsymbol{\xi}_\perp \right|^2 d\mathbf{r} \tag{53}$$

This is identical to the equivalent form for pure kinetic MHD given by Eq. (31) except that it is only the ions that contribute to the integral. Marginal stability occurs at $\omega^2 = 0$ and the system is unstable iff $\delta W_{KF}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) \le 0$.

5 Energy relations for comparison theorems: Vlasov ions, fluid electrons

The last model of interest is a hybrid model with Vlasov ions and fluid electrons. The motivation for using the Vlasov equation for the ions is to allow us to consider stability for arbitrary $k_\perp$ including both $k_\perp a \sim 1$ and $k_\perp \rho_i \sim 1$. The regime $k_\perp \rho_i \sim 1$ is important for closed line systems such as the levitated dipole and the field reversed configuration as well as ballooning and interchange modes in ergodic systems. The crucial feature included in the Vlasov, but not the kinetic MHD, description is the possibility of particle resonances with the perpendicular guiding center velocity as well as the parallel velocity. Specifically the resonance condition changes from $\omega - k_\| v_\| = 0$ to $\omega - k_\| v_\| - \mathbf{k}_\perp \cdot \mathbf{v}_d = 0$, where $\mathbf{v}_d$ includes the $\mathbf{E} \times \mathbf{B}$, curvature, and grad-B guiding center drifts.

An alternative model for the ions that also contains the desired physics is gyrokinetics [18], [19]. Perhaps surprisingly, if we take as our final goal the derivation of an energy relation, then the exact Vlasov description is actually simpler to analyze than the approximate gyrokinetic description, for reasons that will become apparent in the remainder of this article.

Ideally we would like to be able to treat the electrons with the Vlasov equation but this becomes too complicated mathematically. The basic difficulty is that a dual Vlasov model contains far more physics than just MHD behavior. Thus some simplifications are needed to restrict the physical content of the overall model such that attention can be focused on MHD phenomena. A fluid model for electrons meets this purpose. It is also possible to treat the electrons as collisionless by using the simpler kinetic MHD description. This, however, is deceptive and corresponds to an inconsistent mathematical ordering. The reason is that even in the limit $m_e \to 0$ the perpendicular guiding center drifts of the electrons (for $T_e \sim T_i$) are important when $k_\parallel \approx 0$ and $k_\perp \rho_i \sim 1$.

In carrying out the analysis there are three issues that arise that are worth noting. First, a simplified energy equation must be used for the electrons in order to focus on MHD modes which are defined as modes in which the magnetic field is frozen into the plasma. Second, a special choice must be made for the form of the equilibrium ion distribution function in order to guarantee zero macroscopic fluid velocity, corresponding to static equilibrium. This choice also has the feature of making the analysis valid for arbitrary 3-D geometries. Third, the analysis is carried out using a procedure which is traditionally and wisely thought to be highly inefficient and mathematically complex when applied to models that make use of a gyro radius expansion (e.g. gyrokinetics and kinetic MHD). The "forbidden" approach that we use directly calculates the perpendicular ion current from the distribution function rather than using moments. There is no problem doing this with the Vlasov equation since no gyro radius expansion is used and, as is shown, it leads to a simplified analysis if attention is focused solely on obtaining an energy integral. Each of these points is discussed in more detail as the analysis progresses.

5.1 The electron model

The electrons are treated as a massless isotropic fluid. The mass and momentum equations are given by their standard form:

$$
\begin{aligned}
&\frac{\partial n_e}{\partial t} + \nabla \cdot \left(n_e \mathbf{v}_e \right) = 0 \\
&\mathbf{E} + \mathbf{v}_e \times \mathbf{B} + \frac{\nabla p_e}{e n_e} = 0
\end{aligned}
\tag{54}
$$

where $\mathbf{v}_e$ is the electron fluid velocity, and where for simplicity we have neglected the parallel thermal gradient force in the momentum equation. It is the energy equation that raises a problem. This can be seen by substituting the momentum equation into Faraday's law.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v}_e \times \mathbf{B} + \frac{\nabla p_e}{en_e} \right) \tag{55}$$

The problem is that in order to focus on MHD modes, we require, by definition, that the magnetic field be frozen into the plasma. This requires that the $\nabla \times (\nabla p_e / en_e)$ term be zero or small. However, when $k_\perp \rho_i \sim 1$, the term is comparable in magnitude to the other terms. We could assume an intermediate ordering such as $k_\perp \rho_i \ll 1 \ll k_\perp L$ but this leaves us in the awkward position of making a gyro radius expansion in Faraday's law but not the ion Vlasov equation.

Our approach is to postulate an alternative energy equation which must have three desirable properties: (1) it must be mathematically simple, (2) it must include electron plasma compressibility effects, and (3) it must guarantee that the magnetic field is tied to the electron fluid. A model which has these features is as follows.

$$p_e = K n_e^{\gamma_e} \tag{56}$$

Our model looks very similar to the usual adiabatic energy relation but there is one important difference. In our model both the equilibrium and perturbed pressure satisfy the same relation. In the usual adiabatic relation, $d(p_e / n_e^{\gamma_e}) / dt = 0$ the equilibrium pressure and density profiles are independent of each other and it is only the perturbations that are non-trivially governed by Eq. (56). Thus, our model is a special case of the more general adiabatic relation. The main consequence of Eq. (56) is that in the stability analysis only the pressure gradient can drive instabilities. In contrast, for the general adiabatic relation the parameter $\eta_e = d \ln T_e / d \ln n_e$ also appears which can drive instabilities such as the entropy mode. Specifically, when $\gamma_e = 5/3$, then our model implies that $\eta_e = 2/3$ and for this value the entropy mode is always stable, as shown in cylindrical and point-dipole geometries in references [20] and [21]. Thus, choosing Eq. (56) as the energy relation for electrons allows us to focus on MHD modes, which is the topic of interest.

5.2 The Vlasov-fluid model

The basic equations describing the Vlasov-fluid model are obtained by evaluating the quantity $\mathbf{J}\times\mathbf{B}$ with the electron current calculated from $\mathbf{v}_e$ and the ion current by the usually inefficient process of integrating over the distribution function. A short calculation leads to the following model.

$$\begin{aligned}
&\mathbf{J}\times\mathbf{B} = \nabla p_e + e\int\Big(\mathbf{E}+\mathbf{u}\times\mathbf{B}\Big) f_i\, d\mathbf{u}\\
&\frac{\partial f_i}{\partial t} + \mathbf{u}\cdot\nabla f_i + \frac{e}{m_i}\Big(\mathbf{E}+\mathbf{u}\times\mathbf{B}\Big)\cdot\nabla_u f_i = 0\\
&\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{v_e}) = 0\\
&p_e = K n_e^{\gamma_e}\\
&\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v_e}\times\mathbf{B})\\
&\nabla\times\mathbf{B} = \mu_0\mathbf{J}\\
&\nabla\cdot\mathbf{B} = 0\\
&E_\parallel = -(\mathbf{b}\cdot\nabla p_e)\,/\,en\\
&n_i = n_e \equiv n
\end{aligned} \tag{57}$$

Here, $\mathbf{u}$ represents the total (not random) particle velocity.

5.3 Equilibrium

An exact analytic equilibrium satisfying the Vlasov-fluid equations can be found that is valid for arbitrary 3-D geometries. The key point to recognize is that our primary interest is in *static* equilibria. The motivation for focusing on static equilibria is to enable a mathematically consistent comparison with static ideal MHD equilibria which is the usual “gold” standard for macroscopic stability analyses. We emphasize that equilibria with flow are possible and often necessary when comparing with detailed experimental data. However, when comparing with other theoretical models it is necessary to focus on the identical class of equilibria - those that have zero equilibrium flow.

The condition of identically zero macroscopic equilibrium flow implies that the equilibrium ion distribution function be of the form

$$f_i = f(\varepsilon)$$
$$\varepsilon = \frac{m_i u^2}{2} + e\phi\left(\mathbf{r}\right) \tag{58}$$

where $\phi\left(\mathbf{r}\right)$ is the electrostatic potential. From Eq. (58) it follows that the ions are electrostatically confined and that the ion pressure is isotropic. A short calculation also shows that the pressure and density are related by

$$p_i\left(\phi\right) = \int \frac{m_i u^2}{3} f \, d\mathbf{u}$$
$$n_i\left(\phi\right) = -\frac{1}{e}\frac{dp_i}{d\phi} \tag{59}$$

Now, since there is no equilibrium ion flow,

$$\mathbf{J} \times \mathbf{B} = \nabla p_e + en\mathbf{E} = \nabla p_e - en\nabla\phi \tag{60}$$

Here, we have set $n_e = n_i \equiv n$. Substituting Eq. (59) into Eq. (60) then yields

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{61}$$

where $p = p_e + p_i$. Moreover, since $p_e = Kn^\gamma = K[n(\phi)]^\gamma$, the total pressure also has the form $p = p(\phi)$. The condition $\mathbf{B} \cdot \nabla p = (dp / d\phi)\mathbf{B} \cdot \nabla\phi = -(dp / d\phi)\mathbf{B} \cdot \mathbf{E} = 0$ then implies that $E_\| = 0$ in equilibrium. The overall conclusion is that the choice $f_i = f(\varepsilon)$ leads to Vlasov-fluid equilibria that are identical to ideal MHD equilibria.

5.4 Stability

Linear stability in the Vlasov-fluid model is carried out in terms of the electron displacement vector $\boldsymbol{\xi}$. The relationship between $\tilde{\mathbf{v}}_e$ and $\boldsymbol{\xi}$ in a system in which there is an equilibrium flow $\mathbf{v}_e = -\mathbf{J}/en$ is given by $\tilde{\mathbf{v}}_e = -i\omega\boldsymbol{\xi} + \mathbf{v}_e \cdot \nabla\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\mathbf{v}_e$ [22]. Using this definition it follows that most of the perturbed quantities can be easily expressed in terms of $\boldsymbol{\xi}$.

$$
\begin{aligned}
&\tilde{n}_e = -\boldsymbol{\xi}_\perp \cdot \nabla n - n\nabla \cdot \boldsymbol{\xi} \\
&\tilde{p}_e = -\boldsymbol{\xi}_\perp \cdot \nabla p_e - \gamma_e p_e \nabla \cdot \boldsymbol{\xi} \\
&\tilde{\mathbf{B}} = \nabla \times \left(\boldsymbol{\xi}_\perp \times \mathbf{B}\right) \\
&\tilde{\mathbf{E}} = i\omega\boldsymbol{\xi}_\perp \times \mathbf{B} - \nabla\left[\left(\boldsymbol{\xi}_\perp \cdot \nabla p_i - \gamma_e p_e \nabla \cdot \boldsymbol{\xi}\right)/en\right]
\end{aligned}
\tag{62}
$$

The remaining unknown is the perturbed distribution function which, as shown in Appendix C, can be written as

$$
\begin{aligned}
&\tilde{f}_i = \left[\frac{1}{n}\left(\boldsymbol{\xi}_\perp \cdot \nabla p_i - \gamma_e p_e \nabla \cdot \boldsymbol{\xi}\right) + i\omega\tilde{s}\right]\frac{\partial f_i}{\partial \varepsilon} \\
&\tilde{s} = \int_{-\infty}^{t}\left[e(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot \boldsymbol{\xi}_\perp - (\gamma_e p_e / n)\nabla \cdot \boldsymbol{\xi}\right]dt'
\end{aligned}
\tag{63}
$$

As for the other models, an energy integral can be obtained for the Vlasov-fluid model. The details are presented in Appendix C. A critical point regarding this energy integral is that unlike for the other models, there is no need to distinguish between ergodic and closed field line geometries. The reason is that the orbit integral $\tilde{s}$ does not have any terms that are proportional to $1/\omega$. It is the $1/\omega$ terms in $\tilde{s}$ that yield a finite contribution in the product $i\omega\tilde{s}$, giving rise to trapped particle compressibility stabilization and closed line periodicity stabilization.

The vanishing of the $1/\omega$ terms occurs because the resonant denominator arising from the trajectory integral is modified from its kinetic MHD form $\omega - k_\parallel w_\parallel$ to its Vlasov-fluid form $\omega - k_\parallel u_\parallel - \mathbf{k}_\perp \cdot \mathbf{v}_d$ where $\mathbf{v}_d$ is the guiding center drift velocity comprised of the grad-B, curvature, and $\mathbf{E}\times\mathbf{B}$ drifts. Thus, even when $k_{\|}=0$ the resonant denominator in the Vlasov-fluid model does not vanish as $\omega^2 \to 0$ because there is always a non-zero precession drift.

This behavior can be seen explicitly by examining the Vlasov-fluid energy integral

$$\left|\omega\right|^2 = -\frac{\delta W_\perp}{K_{VF}} \tag{64}$$

where

$$\begin{aligned} K_{VF} &= \int d\mathbf{r}\,\frac{\hat{T}_i}{n}\left(V_1+V_2\right) \\ V_1 &= \int \frac{\partial f_i}{\partial \varepsilon}\,d\mathbf{u}\int \frac{\partial f_i}{\partial \varepsilon}\left|\tilde{s}\right|^2 d\mathbf{u} - \left|\int \frac{\partial f_i}{\partial \varepsilon}\,\tilde{s}\,d\mathbf{u}\right|^2 \\ V_2 &= \frac{\gamma_i p_i}{\gamma_i p_i + \gamma_e p_e}\left|\int \frac{\partial f_i}{\partial \varepsilon}\,\tilde{s}\,d\mathbf{u}\right|^2 \end{aligned} \tag{65}$$

and $\gamma_{e,i}(\psi) = d\ln p_{e,i} / d\ln n$. Clearly, $K_{VF} > 0$ when $\partial f_i / \partial\varepsilon < 0$, by Schwarz's inequality.

A sufficient condition for stability can now easily be obtained. Assume the plasma, for any type of geometry, is ideal MHD stable for incompressible displacements: $\delta W_\perp \geq 0$. Then, Eq. (64) is a contradiction, similar to that derived for the other models, which can only be resolved by recognizing that the original assumption $\mathrm{Im}(\omega) > 0$ is violated. In other words the system is linearly stable. This conclusion makes use of the fact that K_{VF} remains finite as $\omega \to 0$. Therefore, incompressible stability in ideal MHD implies stability in the Vlasov-fluid model for any type of geometry.

Consider next displacements corresponding to $\delta W_\perp < 0$. Since the Vlasov-fluid operator is not self adjoint it is not possible to rigorously conclude that the plasma is unstable in this model. However, there is strong motivation to conjecture that this is indeed the case. The reason is that the incompressible ideal MHD eigenfunction at marginal stability is also an exact eigenfunction of the Vlasov-fluid model. Then, once any plasma parameter, for example β, is changed, the presence of resonant particles strongly suggests that the resulting eigenvalue will be complex.

Changing β in the appropriate direction (presumably by increasing it) should then produce a positive growth rate. Assuming the conjecture to be correct, then the stability results as $\omega^2 \to 0$ can be summarized as follows

$$\begin{aligned} &\delta W_{VF} = \delta W_{\perp}(\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) = \delta W_{MHD}(\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) \qquad \text{ergodic systems} \\ &\delta W_{VF} \equiv \delta W_{\perp}(\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) \le \delta W_{MHD}(\boldsymbol{\xi}^*_{\perp MHD}, \boldsymbol{\xi}_{\perp MHD}) \qquad \text{closed line systems} \end{aligned} \tag{66}$$

Equations (65) and (66) indicate that the Vlasov-fluid model does not exhibit any form of compressibility stabilization. The absence of compressibility stabilization is likely to be more important for closed line configurations such as the levitated dipole and the field reversed configuration which depend on this effect for good plasma performance. Even so, we point out that the nonlinear effects may be very important since modifications to the distribution function may lead to stabilization without the severe consequences usually associated with ideal MHD. This is an area that needs further investigation. Even if so, it is still very worthwhile to understand the predictions of linear stability as is contained in each of the models under consideration.

6 Summary

We have derived a series of MHD stability comparison theorems corresponding to different plasma physics models, varying from collisional to collisionless in their physical content. Some of the results are generalizations and clarifications of existing results. Other results involve the introduction of new models and the derivation of new comparison theorems. In general we have shown that it is necessary to distinguish between ergodic systems and closed line systems. Also, cylindrical systems must sometimes be distinguished from toroidal systems.

Below, we summarize in the form of two tables the results of our analysis. Specifically, we present the comparison results for each energy relation in the marginal stability limit $\omega^2 \to 0$. The first table corresponds to ergodic systems including closed line systems undergoing symmetry breaking perturbations. The second table corresponds to closed line systems undergoing perturbations that maintain the closed line symmetry. In both tables the entries are arranged in (the most probable) ascending order with the most conservative model appearing first. The comparisons for ergodic systems are made against the reference model corresponding to the ideal MHD potential energy for incompressible displacements $\delta W_{\perp}$. For closed line systems the comparisons are made with respect to the compressible ideal MHD potential energy $\delta W_{MHD} = \delta W_{\perp} + \delta W_{C}$.

Model	Comparison Theorem
Ideal MHD	$\delta W_{MHD} = \delta W_{\perp}$
Vlasov-fluid	$\delta W_{VF} = \delta W_{\perp}$
Kinetic ion-fluid electron	$\delta W_{KF} = \delta W_{\perp}$ cylindrical $\delta W_{KF} > \delta W_{\perp}$ toroidal
Kinetic ion-kinetic electron	$\delta W_{KK} = \delta W_{\perp}$ cylindrical $\delta W_{KK} > \delta W_{KF} > \delta W_{\perp}$ toroidal
Double adiabatic CGL	$\delta W_{CGL} > \delta W_{KK} > \delta W_{KF} > \delta W_{\perp}$

Table 1. Summary of comparison theorems for ergodic systems

Model	Comparison Theorem
Vlasov-fluid	$\delta W_{VF} = \delta W_{\perp}$
Ideal MHD	$\delta W_{MHD} = \delta W_{\perp} + \delta W_{C} > \delta W_{\perp}$
Kinetic ion-fluid electron	$\delta W_{KF} > \delta W_{MHD}$ cylindrical $\delta W_{KF} > \delta W_{MHD}$ toroidal
Kinetic ion-kinetic electron	$\delta W_{KK} > \delta W_{MHD}$ cylindrical $\delta W_{KK} > \delta W_{KF} > \delta W_{MHD}$ toroidal
Double adiabatic CGL	$\delta W_{CGL} > \delta W_{KK} > \delta W_{KF} > \delta W_{MHD}$

Table 2. Summary of comparison theorems for closed line systems

The overall conclusions are as follows. For ergodic systems stability boundaries are accurately predicted by the ideal MHD energy principle for incompressible displacements: $\delta W_\perp = 0$. The trapped particle compressibility stabilization arising in the kinetic MHD model may be an artifact since the more accurate (in terms of gyro radius approximations) Vlasov-fluid model also predicts marginal stability when $\delta W_\perp = 0$.

For closed line systems, the usual statement that ideal MHD represents the most conservative stability estimate is incorrect. While the statement is true with respect to kinetic MHD models it fails for the Vlasov-fluid model. In this model resonant particles moving with the perpendicular precession drift velocity eliminate all compressibility stabilization effects so that the stability boundary is again given by $\delta W_\perp = 0$. There is no compressibility stabilization.

The results presented here may be more important for closed line configurations such as the levitated dipole and the field reversed configuration where MHD compressibility stabilization plays an important role in predicted plasma performance. Even so, the comparisons theorems only apply to linear stability and the nonlinear MHD behavior may not be catastrophic, particularly for modes driven by a small class of resonant particles.

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