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# Dynamics of charged particles in spatially chaotic magnetic fields


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## Abstract

The spatial topology of magnetic field lines can be chaotic for fields generated by simple current configurations. This is illustrated for a system consisting of a circular current loop and a straight current wire. An asymmetric configuration of the current system leads to three-dimensional spatially chaotic magnetic fields. The motion of charged particles in these fields is not necessarily chaotic and exhibits intriguing dynamical properties. Particles having initial velocities closely aligned with the direction of the local magnetic field are likely to follow chaotic orbits in phase space. Other particles follow coherent and periodic orbits - these orbits being the same as in the symmetric current configuration for which the field lines are not chaotic. An important feature of particles with chaotic motion is that they undergo spatial transport across magnetic field lines. The cross-field diffusion is of interest in a variety of magnetized plasmas including laboratory and astrophysical plasmas.



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## I. INTRODUCTION

An intricate part of laboratory, space, and astrophysical plasmas are the magnetic fields. The fields can be induced externally as in controlled fusion devices or generated naturally as occurs in space. In the natural occurring plasmas, magnetic fields are ubiquitous and play a crucial role in microscopic and macroscopic scales. As noted by Parker [1], "It appears that the radical element responsible for the continuing thread of cosmic unrest is the magnetic field." In this paper we study the motion of charged particles in spatially chaotic magnetic fields. The chaotic magnetic fields are generated by uniform currents in a simple system composed of a circular current loop and a straight current wire. The magnetic field of such a system can be analytically obtained and is textbook material [2]. Since the magnetic field is divergence free, the magnetic field lines are completely deterministic and can be described by a Hamiltonian. For asymmetric configurations of the loop-wire system, when the magnetic field lines do not lie on a two-dimensional surface, we find that the field lines can become chaotic in space [3–5]. Subsequently, we study the dynamics of charged particles in such chaotic magnetic fields and compare it with the dynamics in a symmetric configuration where the field lines lie on a surface and are not chaotic. The particle dynamics in a magnetic field is a Hamiltonian system. So there is no randomness that is extrinsically included in either the description of the magnetic field or in the dynamics of the particles. From numerical simulations we demonstrate that in a chaotic magnetic field the particle motion is chaotic for a limited part of its available phase space. An interesting and new result that emerges from our studies is that the motion of a particle in a three-dimensional chaotic magnetic field is not necessarily chaotic. Previously, it has been shown that stochastic instability of magnetic field lines can induce anomalous diffusion across the confining magnetic field line [6]. We show that, in the presence of chaotic magnetic fields, cross-field diffusion occurs only for a restricted part of the dynamical phase space in which the particle motion is chaotic. Particles in most of the phase space do not undergo spatial diffusion in the presence of chaotic magnetic fields.

The idea to study properties of magnetic field lines using current systems is not new. It has been pointed out by Alfvén that a description in terms of currents gives important new aspects of cosmical electrodynamics and a different approach to the generation of magnetic fields [7]. In his book on mathematical problems, Ulam notes that topological properties of magnetic field lines in space can be studied simply by considering fields due to steady state currents flowing in wires along prescribed curves [8]. In this paper we take an additional step by studying the motion of particles in such magnetic fields.

The effect of magnetic fields on the motion of charged particles is of considerable interest. In cosmic rays the transport of particles across the magnetic field is believed to be responsible for regulating the high-energy particle density in galactic magnetic fields [9, 10]. The propagation of solar cosmic rays in the interplanetary space cannot be explained unless there is transport of energetic particles across the interplanetary magnetic field [11]. The existence of chaotic magnetic field lines in space is postulated as a means for explaining diffusion of charged particles [9, 12]. In these studies *ad hoc* random perturbations to the magnetic fields are included in the modeling to explain the diffusion of particles. In our case the chaotic magnetic field is a natural consequence of three-dimensional asymmetry of the current configuration.

The assumption of steady currents in the loop-wire system eliminates the need to include electric fields associated with time-varying magnetic fields. Consequently, the energy of the particles is a constant of the motion. Furthermore, we can compare general features of the particle dynamics with analytical studies of particle motion in magnetic fields [13]

While collective effects are important in the study of magnetized plasmas, a wealth of physical insight into the plasma behavior can be obtained from the dynamics of charged particles in the relevant magnetic fields. A kinetic description of the plasma is then formulated based on the test particle dynamics. The diffusion coefficients associated with particle transport that are incorporated, for example, into the Fokker-Planck description are obtained from single particle dynamics [14].

The paper is organized as follows. We show that chaotic magnetic field lines are generated by spatially asymmetric configurations of the loop-wire current system. We then study the dynamics of charged particles in these chaotic magnetic fields.

## II. MAGNETIC FIELD DUE TO A CURRENT LOOP AND A STRAIGHT CURRENT WIRE

We consider a current loop with its center located at the origin of a Cartesian coordinate system $(x, y, z)$ and lying in the x-y plane. In spherical coordinates, defined with the standard convention, the vector potential of the current loop is given by [2]

$$A_{L\phi}(r,\theta) \;=\; \frac{\mu_0 I_L}{2\pi}\;\frac{a}{\left(a^2+r^2+2ar\sin\theta\right)^{1/2}}\;F(k) \tag{1}$$

where $\mu_0$ is the free space permeability, $I_L$ is the current in the loop, $r$ is the radial coordinate, $\theta$ is the zenith angle, $\phi$ is the azimuthal angle, $a$ is the radius of the loop,

$$F(k) \;=\; \frac{1}{k^2}\;\left[\left(2-k^2\right)\mathrm{K}(k)-2\mathrm{E}(k)\right] \tag{2}$$

$$k^2 \;=\; \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta} \tag{3}$$

and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively [15]. The vector potential is along the azimuthal direction so the azimuthal component of the magnetic field $B_{L\phi}=0$. The other two components of the magnetic field due to the current loop are

$$B_{Lr} \;=\; \frac{1}{r\sin\theta}\;\frac{\partial}{\partial\theta}\left(\sin\theta\;A_{L\phi}\right) \tag{4}$$

$$B_{L\theta} \;=\; -\frac{1}{r}\;\frac{\partial}{\partial r}\left(r\;A_{L\phi}\right) \tag{5}$$

For a current carrying straight wire along the $z$-axis and perpendicular to the plane of the loop, the vector potential has just the $z$-component and depends only on the radial distance from the wire [2]

$$A_{Wz}(r) = \frac{\mu_0 I_W}{2\pi}\ln\left(r\right) \tag{6}$$

where $I_W$ is the current in the wire and $r=\sqrt{x^2+y^2}$ is the radial distance from the wire. The magnetic field is along the direction of the azimuthal angle

$$B_{W\phi} \;=\; -\frac{\partial}{\partial r}A_{Wz} \;=\; \frac{\mu_0 I_W}{2\pi}\;\frac{1}{r} \tag{7}$$

The other two components of the magnetic field due to the wire are $B_{Wr}=0$ and $B_{W\theta}=0$.

## III. MAGNETIC FIELD LINES FOR THE LOOP-WIRE SYSTEM

The equations for the magnetic field lines $\vec{B}=(B_r,B_\theta,B_\phi)$ in spherical coordinates are

$$\frac{dr}{B_r} \;=\; \frac{rd\theta}{B_\theta} \;=\; \frac{r\sin\theta\;d\phi}{B_\phi} \;=\; \frac{ds}{|B|} \tag{8}$$

where $s$ is the path length along the field line. The magnetic field line components are the sum of magnetic field components of the loop, (4) and (5), and the wire (7). We study the field lines for the loop-wire system by numerically solving Eq. (8). While the current loop is kept fixed in space, we allow for the wire to be displaced relative to the center of the loop. A simple translation of the wire relative to the origin can be easily incorporated into the formula for the magnetic field in Eq. (7). Due to the azimuthal symmetry of the current loop only the distance of the wire from the $z$-axis matters. For our computational results we normalize all distances to $a$, the radius of the loop, and the magnetic fields to $\mu_0 I_L/(2\pi a)$. In what follows, we will assume that the $I_W/I_L = 0.2$. Let $\Delta r$ be the normalized distance by which the wire is displaced with respect to the center of the loop.

Figure 1 shows the Poincaré surface-of-section, in the $x - z$ plane, of the magnetic field line in the case where the wire is passing through the center of the loop, i.e., $\Delta r = 0$. The flux surfaces defined by the Poincaré surface-of-sections are similar to those one would expect for a dipolar magnetic field due to a current loop.

Figure 2 shows the flux surfaces, in the surface-of-section plot, when the wire is displaced by $\Delta r = 0.001$. The initial conditions for the magnetic field lines in this figure are the same as in Fig. 1. The inner two surfaces are still well defined as in Fig. 1 but the outer surfaces have broken up into islands. The order of the islands increases as the field lines move away from the outer edge of the current loop. The formation of islands is indicative of the fact that the field lines are no longer confined to a surface but occupy a finite volume in space.

Figure 3 is the Poincaré surface-of-section for two field lines with $\Delta r = 0.01$. Here we see that the inner most flux surface is still well preserved but the outer flux surfaces have disintegrated to occupy large volumes of space. The outer magnetic field lines are chaotic.

In the case of Fig. 1, the loop-wire system is azimuthally symmetric so that the field lines lie on a two-dimensional flux surface. A displacement of the wire relative to the center of the loop breaks this azimuthal symmetry leading to field lines spanning a three-dimensional volume in space. The chaotic field lines are a result of symmetry breaking. There is no externally induced randomness as the currents in the loop and wire are kept constant and the geometry of the current carriers is preserved. Since we can create chaotic magnetic field lines with such a simple configuration, it is likely that, in general, three-dimensional field lines are chaotic. It is only under very special circumstances that one can preserve symmetries that do not lead to chaotic magnetic field lines.

## IV. CHARGED PARTICLE ORBITS IN THE LOOP-WIRE SYSTEM

The orbits of charged particles in a magnetic field are given by the Lorentz equation

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v} \times \vec{B}\left(\vec{r}\right) \quad (9)$$

where $\vec{r}$ and $\vec{v}$ are the position and velocity, respectively, of a particle of charge $q$ and mass $m$. The magnetic field $\vec{B}$ is composed of Eqs. (4), (5), and (7), with the appropriate spatial translations of the wire included in (7). We introduce dimensionless variables by multiplying time with the frequency $\Omega_0 = (q/m)(\mu_o I_L/2\pi a)$, and velocities by $\Omega_0/a$. The energy of a particle, $E = v^2/2 = \left(v_\perp^2 + v_\parallel^2\right)/2$, moving in just a spatially dependent magnetic field is conserved. Here $v_\perp$ and $v_\parallel$ are components of the particle velocity perpendicular and parallel to the direction of the magnetic field at the location of the particle.

From our numerical analysis, we find that the trajectory followed by a particle depends on the value of the parameter $\alpha = v_\parallel^2/2E$ at the starting location of the particle. If $\alpha = 1$ then the initial velocity of the particle is along the local magnetic field. For $\alpha = 0$ the velocity is perpendicular to the local magnetic field. When the wire is located at the center of the loop, as in Fig. 1, there exists a critical value $\alpha = \alpha_c$ such that for $\alpha < \alpha_c$ the particle bounces between two turning points and it drifts. This corresponds to a trapped particle orbit. For $\alpha \geqslant \alpha_c$, the particle goes around azimuthally in one direction. This corresponds to a passing orbit. The normalized energy is chosen so that the Larmor radius $\rho$ of the particle at the starting location is much smaller than the magnetic field scale length $L_B$ at that point. $\rho = v_\perp/\Omega_c$ where $\Omega_c = qB/m$ with $B$ being the magnitude of the magnetic field at the location of the particle, and $1/L_B = |\nabla B|/B$. In theoretical studies [13], it has been noted that the magnetic moment of the particle $\mu = v_\perp^2/2B$ is an adiabatic invariant provided $\rho/L_B \ll 1$. The evolution of both $\rho/L_B$ and $\mu$ along the trajectory of a particle will be discussed below.

We study the effect of chaotic magnetic field lines on particle orbits by choosing the initial positions of the particles to be in the chaotic region of Fig. 3. The trajectories of particles are followed numerically and we plot a point in the $x-z$ plane every time a particle crosses this plane with its velocity having a positive y component. Figure 4 shows the intersection points for four different particles for $\alpha = 0.9$. The orbits intersect the $x-z$ plane for both $x > 0$ and $x < 0$; the orbit of particles 1 and 4 are as labelled while those of 2 and 3 lie sequentially between them. These are trapped particle orbits for which the turning points are precessing in the azimuthal direction. While it is apparent that the particle trajectories are not chaotic, it is interesting to note that the particles have the same trajectories if we were to assume that the loop-wire system is symmetric as in Fig. 1. In other words, the particles whose orbits are shown in Fig. 4 behave as if the magnetic field lines are not chaotic, even though the field lines are fully chaotic. In Figs. 5 and 6 we plot, respectively, $\mu/\mu_0$ and $\rho/L_B$ along the orbit of particle 2 in Fig. 4. Here $\mu_0$, not to be confused with the free-space permeability, is the value of $\mu$ at the starting point for this particle. Figures 5 and 6 are approximately for one cycle - the curves repeating over subsequent times. From Fig. 5 we note that $\mu$ varies by more than a factor of 2 from its minimum to its maximum value along the orbit. This variation implies that $\mu$ is definitely not an adiabatic invariant. However, the average change in $\mu$ over an entire cycle is zero. Figure 5 shows that $\rho/L_B$ is much less than 1 along the entire cycle. One would expect from theoretical studies [13] that $\mu$ will not vary much along the particle orbit for $\rho/L_B \ll 1$. But that is not the case. It is the average value of $\mu$ over one or more cycles that does not change, i.e., $\mu$ is a constant only in an average sense. The average value of $\mu$ over an integer number of cycles remains zero. So $\mu$ is not an adiabatic invariant as a function of time, but it is essentially invariant over one or more periodic cycles.

Figure 7 displays particle orbits in the $x-z$ plane for $\alpha = 0.989$. Some of these orbits are different from those that would be obtained in the symmetric loop-wire system. This is illustrated in Fig. 8 which is a magnified view of an island structure in Fig. 7. The outermost particle orbit displayed in Fig. 7 is the same as that for a particle moving in a symmetric loop-wire configuration. So a part of phase space is affected by the chaotic magnetic field for this $\alpha$.

Figure 9 shows chaotic particle orbits for two particles with $\alpha = 0.990625$. From numerical simulations we find that this value of $\alpha$ is the critical value for which chaotic particle orbits are present. The orbit indicated by blue dots has no turning points and intersects the $x-z$ plane moving in one azimuthal direction. For both these particles the variation of $\mu$ along the particle orbit is also chaotic while $\rho/L_B \ll 1$.

In Fig. 10 we indicate the intersection points (blue dots) in the $x-z$ plane of a particle orbit. Here $\alpha = 1$ so that the initial particle velocity is aligned along the magnetic field line. Starting at the same initial spatial location as the particle, we plot magnetic field intersection points (red dots) for the chaotic magnetic field in Fig. 3. The number of intersection points of the magnetic field line is the same as those for the particle orbit. The differences in the two sets of points show that, even when the initial velocity is along the magnetic field line, the particle does not follow the magnetic field line. It separates away from the field line, covering a broader range of space and, thus, diffuses across the magnetic field line.

## V. CONCLUSIONS

From Figs. 9 and 10 it is evident that chaotic magnetic field lines can lead to spatial transport of particles. The spatial transport goes hand-in-hand with chaotic motion of the particle since the particle meanders over a wider range in space as compared to the orbits shown in Figs. 4 and 7. The spatial transport is limited to only those particles which are started off with $\alpha$ greater than some critical value $\alpha_c$. As the energy of particles is increased the range of $\alpha$'s for which the particle motion becomes chaotic increases - $\alpha_c$ decreases relative to 1 as the energy increases. For low energy particles the range of $\alpha$'s is limited as $\alpha_c$ approaches 1. This implies that spatial transport across the magnetic field lines is more likely for energetic particles.

In the loop-wire current system, chaotic magnetic fields are generated by small deviations from a symmetric configuration. The magnetic field lines no longer lie on some nice two dimensional surfaces. Thus, it would seem that non-chaotic magnetic field lines require special constraints while chaotic magnetic field lines may be easier to generate.

In the presence of chaotic magnetic field lines, the motion of particles is not necessarily chaotic. There exist a class of particle whose orbits are quite regular and are similar to the orbits that would occur in the presence of a non-chaotic magnetic field. The orbits of only those particles are chaotic whose initial velocities are closely aligned with the direction of the local magnetic field.

The magnetic field line equations and the particle orbit equations are solved numerically using the Gauss and Lobatto Collocation methods [16]. We have used methods of order 4 and 6. These methods are symplectic and possess desirable stability properties as discussed in reference [16]. In the case of particle orbits, the energy is conserved to almost within machine accuracy. The step sizes for the numerical solutions are chosen such that decreasing the step size does not affect the results. This is necessary in order for us to look at changes in the particle orbits as $\alpha$ is varied over small ranges. The results shown in Figs. 7 and 8 required stable algorithms that are accurate.

## VI. ACKNOWLEDGEMENTS

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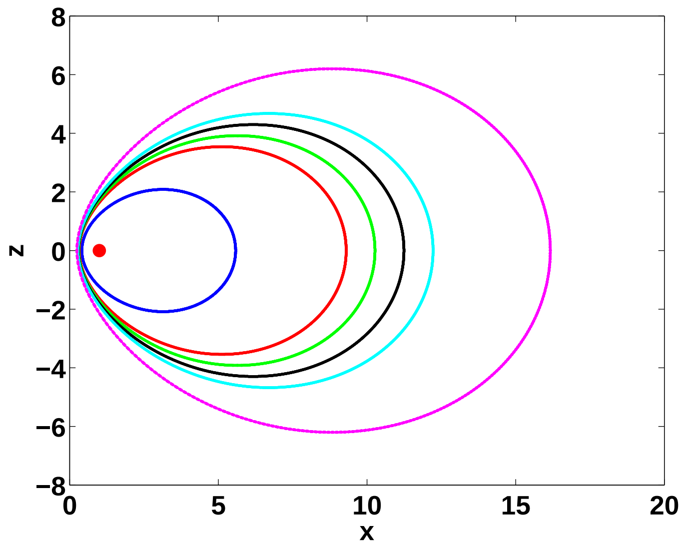


FIG. 1: Poincaré surface-of-section of six magnetic field lines. The dot at $x = 1$ and $z = 1$ marks the intersection of the current loop with this plane and the current wire is along the $z$-axis.

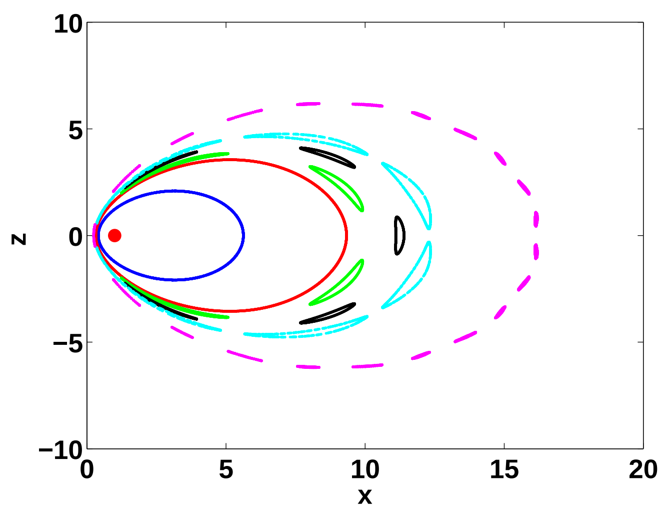


FIG. 2: Surface-of-section of six magnetic field lines when the wire is displaced by a distance $\Delta r = 0.001$ relative to the centre of the loop.

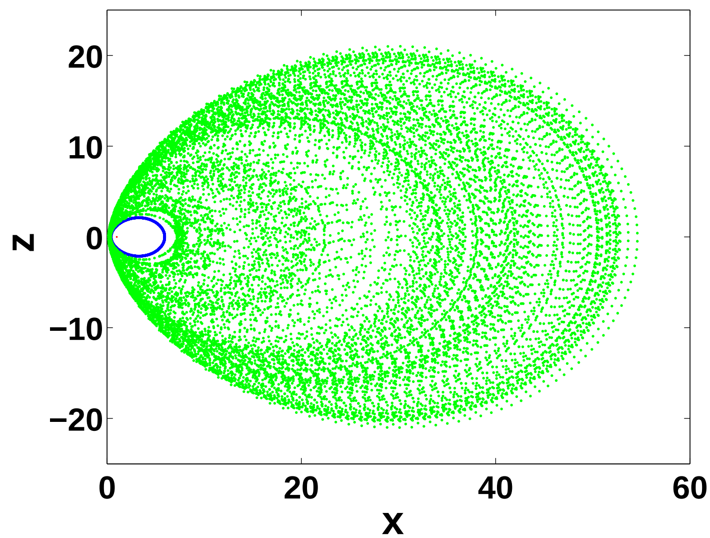


FIG. 3: Surface-of-section of two magnetic field lines with the wire displaced by a distance $\Delta r = 0.01$ relative to the centre of the loop. The inner most curve corresponds to one field line that is not chaotic.

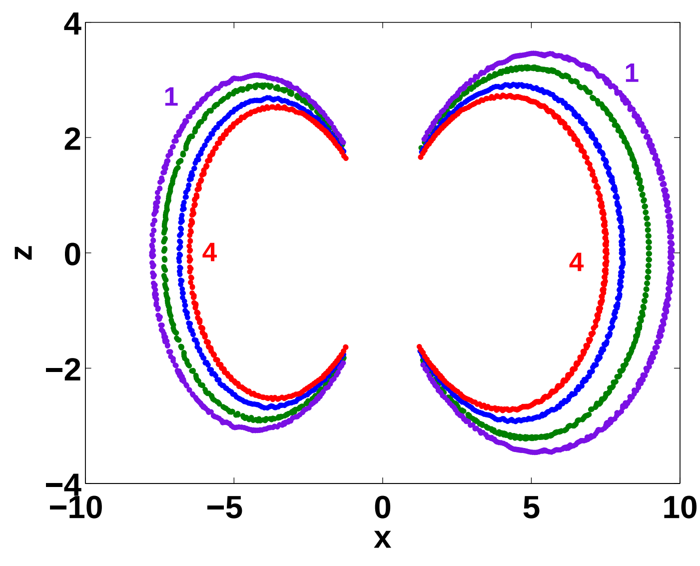


FIG. 4: Intersection points in the $x - z$ plane of the trajectories of four particles moving in the chaotic magnetic field configuration of Fig. 3. The normalized energy is $E = 1.25 \times 10^{-5}$ and $\alpha = 0.9$. The intersection points of particles 1 and 4 are as labelled while 2 and 3 follow sequentially. The four particles are started at different spatial locations which all lie within the region of chaotic magnetic field line shown in Fig. 3.

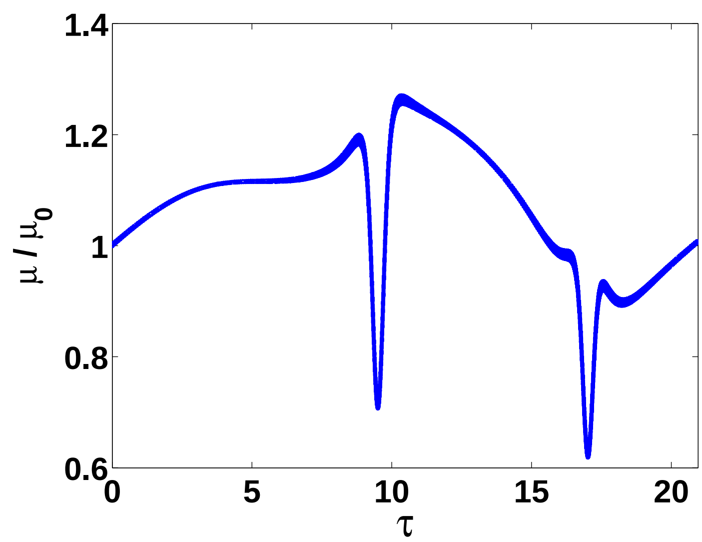


FIG. 5: $\mu/\mu_0$ along the particle orbit for particle 3 in Fig. 4. $\mu$, sometimes referred to as the first adiabatic invariant, is the magnetic moment of the particle. $\mu_0$ is the initial value of $\mu$ at the starting position of the particle.

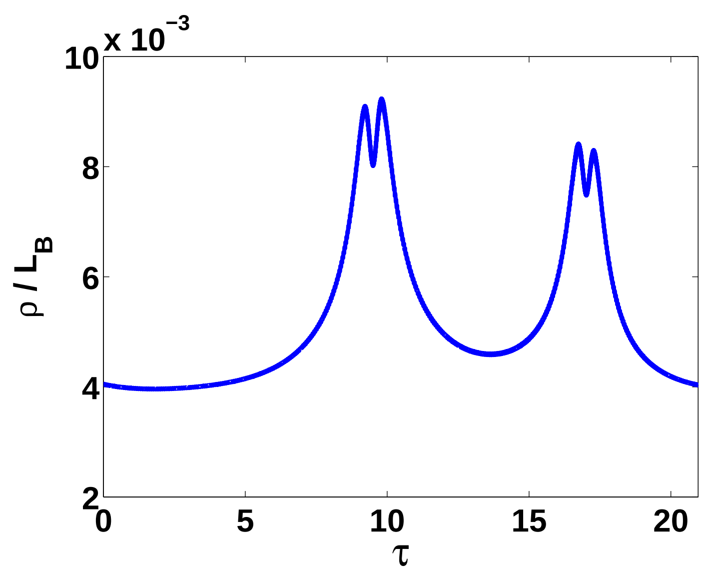


FIG. 6: $\rho/L_B$ along the particle orbit referred to in Fig. 5. $\rho$ is the local Larmor radius of the particle and $L_B$ is the magnetic field scale length at the spatial location of the particle.

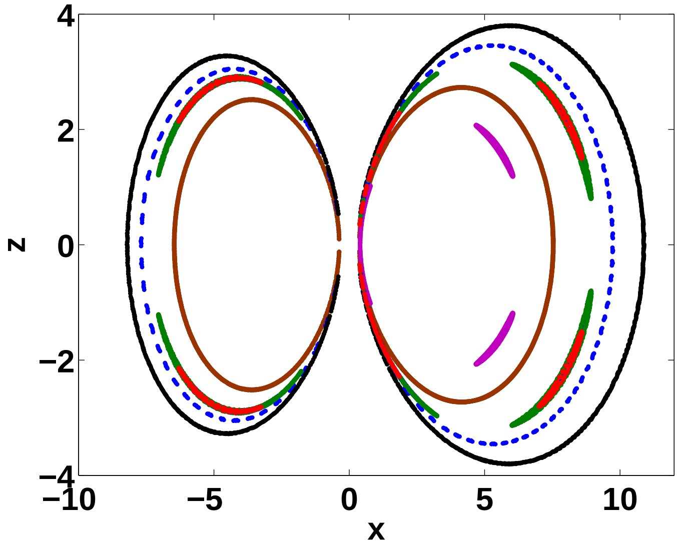


FIG. 7: Intersection points in the x-z plane of various particle orbits for $\alpha = 0.989$.

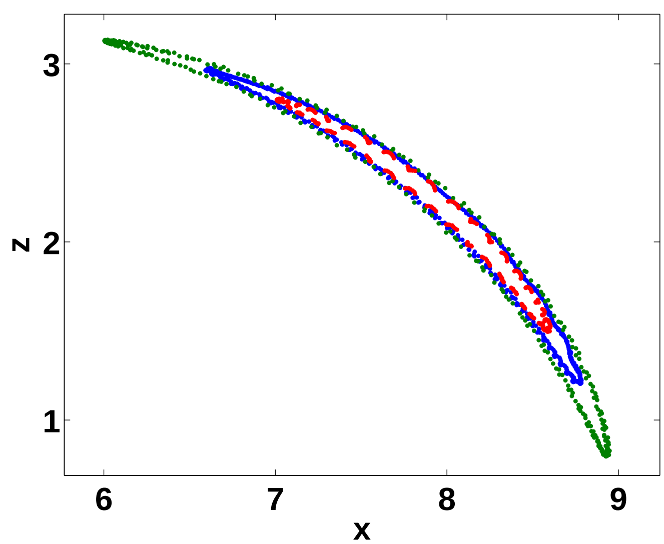


FIG. 8: A magnified view of one of the island structures displayed in Fig. 7.

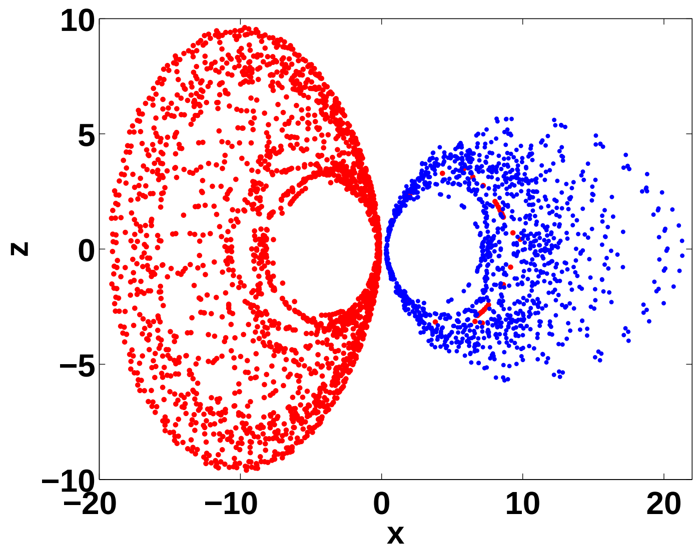


FIG. 9: Intersection points in the $x - z$ plane of two particle orbits (red and blue dots) for $\alpha = 0.990625$. Note the appearance of the red dots in the $x > 0$ region indicating that this particle is partially trapped.

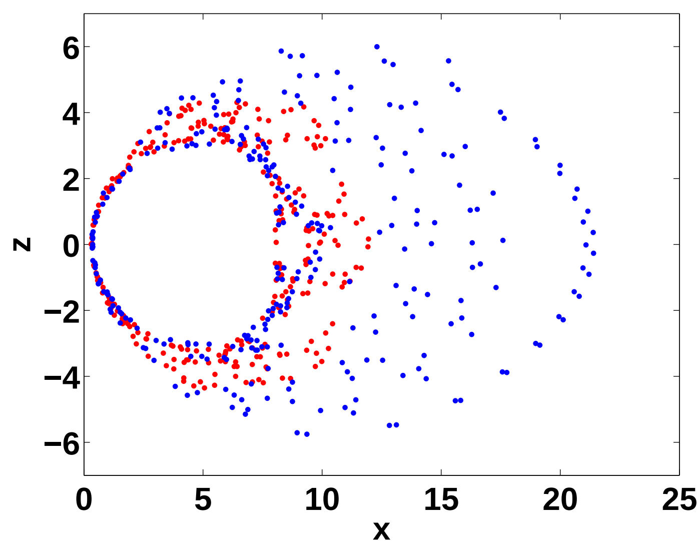


FIG. 10: The blue dots are the intersection points in the $x - z$ plane of a single particle orbit for $\alpha = 1$, corresponding to the initial velocity of the particle being along the magnetic field line. The red dots are the intersection points of the field line. The number of points corresponding to the particle orbit is the same as those for the field line. The spatial structure and extent of the field line is less than that shown in Fig. 3 since we consider only a limited subset of points.