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# MULTIVARIABLE ROOT LOCI ON THE REAL AXIS

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## ABSTRACT

Some methods for determining the number of branches of multivariable root loci which are located on the real axis at a given point are obtained by using frequency domain methods. An equation for the number of branches is given for the general case, and simpler results for the special cases when the transfer function G(s) has size 2 x 2, and when G(s) is symmetric, are also presented.

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## 1. Introduction

It is generally very difficult to plot root loci precisely for finite gains. Exact analytical expressions for the various branches are usually difficult or impossible to obtain, and attempts to construct the locus by actually plotting the closed-loop poles for various values of the scalar gain k tend to be onerous at best. These difficulties hold even in the single-input-single-output (SISO) case; they are considerably greater in the multivariable case.

There is, however, one part of the root locus that can be plotted easily: the portion that lies on the real axis. The form of the locus on the real axis is of course known exactly, and, in addition, the number of branches of the root locus on the real axis can change only at a finite number of points. Thus a relatively small amount of work may yield an exact plot of a sizable portion of the root locus, and in some cases all of it (see Example 2 below). The knowledge of the asymptotes and of the angles of arrival and departure is often sufficient to sketch the rest of the locus.

In the SISO case the rule for the location of root loci on the real axis is very simple (see, e.g., D'Azzo and Houpis 1975): for positive gains, there is a single root-locus branch at a point s on the real axis if and only if there is an odd number of real poles and zeros located to the right of s. The simplicity of this rule is due to the fact that only one branch of the root locus can lie on the real axis at any given point. However, in the multivariable case, several branches can lie on the real axis at a given point. Thus the problem is not only one of determining whether a branch is present, but also one of determining how many branches, if any, are present. Moreover, since multivariable root loci are branches of an algebraic function (see Postlethwaite and MacFarlane 1979), their behavior is much more unusual than that of SISO root loci. In particular, a branch lying on the real axis can turn around at a branch point and double back on itself. This behavior will be exhibited in Example 2 below; for more details see Yagle (1981).

The root locus problem that will be considered here is the standard one, where we assume that G(s) is a proper, rational transfer function matrix of size mxm. In addition, we assume that G(s) has full rank (det G(s) $\not\equiv 0$). Then, the root locus problem consists of plotting the evolution of the system closed-loop poles as k varies for a negative output feedback matrix $K = -kI_m$, $0<k<\infty$ (the same gain multiples all channels). The closed-loop poles are given by

$$\det(I + kG(s))=0 \tag{1}$$

or equivalently if $g=-k^{-1}$, by

$$\det\,(gI - G(s))=0. \tag{2}$$

We note first that, unlike in the SISO case, the knowledge of the pole and zero locations alone is not sufficient for determining the number of loci on the real axis. The following example makes this clear.

Example 1. Consider the root loci of

$$G_1(s) = \begin{bmatrix} \frac{s+1}{s+2} & 0 \\ 0 & \frac{s-2}{s-1} \end{bmatrix} \quad \text{and } G_2(s) = \begin{bmatrix} \frac{s-2}{s+2} & 0 \\ 0 & \frac{s+1}{s-1} \end{bmatrix}.$$

Since each of these represents two decoupled SISO systems, we may immediately plot the root loci as shown in Figure 1. Note that although $G_1(s)$ and $G_2(s)$ have the same poles and zeros, the number of loci on the real axis between -1 and 1 is different.

Despite this difficulty, some equations for the number of branches of the root locus on the real axis at any given point may be found. Also, these equations are not too complicated to be useful. We consider first the case when m=2, then the general case, and finally the case when G(s) is symmetric.

## 2. The case of two input-two output systems

When G(s) has size 2 x 2, the following theorem provides a step-by-step procedure for determining the number of branches of the root locus on the real axis

**Theorem 1.** If m=2, define

$$\Delta(s)=(\mathrm{tr}G(s))^2 - 4 \det G(s) \tag{3}$$

where s is real. Then, we have

(i) If $\det G(s)<0$, exactly one branch lies on the real axis at s

(ii) If $\det G(s)>0$, two or zero branches lie on the real axis at s:

- (a) If $\Delta(s)<0$, zero branches lie on the real axis at s;
- (b) If $\Delta(s)>0$ and $\mathrm{tr}\, G(s)>0$, zero branches lie on the real axis at s;
- (c) If $\Delta(s)>0$ and $\mathrm{tr}\, G(s)<0$, exactly two branches lie on the real axis at s.

*Proof* Note that the closed-loop poles are given by

$$\det(gI-G(s)) = g^2 - \mathrm{tr}\, G(s)\, g + \det G(s) = 0 \tag{4}$$

and that the root loci are obtained by letting g vary from $-\infty$ to 0. This means that the number of branches at a point $s_o$ on the real axis is equal to the number of negative real roots of (4) with $s=s_o$. Since the roots of (4) are given by

$$g = \frac{1}{2}\,(\mathrm{tr}G(s) \pm \Delta(s)^{\frac{1}{2}}) \tag{5}$$

we need only to find how many negative real values of g we get for various values of $\Delta(s)$, tr $G(s)$ and det $G(s)$. If $\Delta(s)<0$ the two values of g are complex, and there are no branches on the real axis at s. But if det $G(s)<0$, then $\Delta(s)>0$ and

$$|\mathrm{tr}\ G(s)| < ((\mathrm{tr}\ G(s))^2 - 4\ \det G(s))^{\frac{1}{2}} \tag{6}$$

and the two values of g are real and of opposite sign. Hence there is exactly one branch on the real axis at s. The other rules follow similarly. ■

The following comments illustrate the main features of this theorem:

1. The number of branches on the real axis changes by one whenever det $G(s)$ changes sign. This makes sense since branches start at poles and end at zeros and since det $G(s)$ changes sign at poles and zeros of odd order.
2. The number of branches on the real axis may change by two when $\Delta(s)$ changes sign. This behavior is related to the existence of branch points (cf. Postlethwaite and MacFarlane 1979) for the algebraic function g(s) defined by (4). The branch points are the points s where (4) has multiple roots, and they are given by $\Delta(s)=0$. At such points $\Delta(s)$ changes sign and a branch of the root locus turns around (see Example 2), so that the number of branches on the real axis changes by two.

3. The number of branches on the real axis will occasionally change by two at points where trG(s) changes sign. This happens when there is a double pole or zero with both branches departing or arriving on the same side. For example, consider

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix} .$$

Clearly there will be two branches both departing from the pole at -1 in the positive direction, and it may be seen that

$$trG(s)= \frac{2s-3}{s+1}$$

changes sign at -1.

The following example which is taken from Postlethwaite and MacFarlane (1979) illustrates the implementation of Theorem 1.

Example 2 Let

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

Then, one has

$$detG(s) = \frac{1}{(s+1)(s+2)}$$

so that det G(s)<0 for -2<s<-1, and consequently there is one branch on the real axis for -2<s<-1. We also have

$$trG(s) = \frac{2s-3}{(s+1)(s+2)}$$

and

$$\Delta(s) = \frac{1-24s}{(s+1)^2(s+2)^2} .$$

This shows that $\Delta(s)<0$ for $s>1/24$ and therefore there are no branches on the real axis for $s>1/24$. Finally, if we consider the values of s not already discussed, we have tr $G(s)<0$ for $s>-1$ or $s<-2$, so that there are two branches on the real axis everywhere else.

The root locus branches on the real axis are plotted in Figure 2. Note how one branch turns around at the branch point at $s=1/24$. Note also that since there are two poles, no zeros and two asymptotes on the real axis, this is the complete root locus.

## 3. The general case

The general case when $m>2$ is more complicated than the case when $m=2$. However, provided that we assume that the poles and zeros of $G(s)$ on the real axis are simple, the number of branches on the real axis can be determined easily by evaluating only a few quantities.

Definition: Let the Smith-McMillan form of $G(s)$ be $\text{diag}\left(\frac{n_i(s)}{d_i(s)}\right)$ and let p be a pole of order n of $G(s)$. Then, the pole p is simple if $(s-p)^n | d_1(s)$ and if $d_i(p) \neq 0$ for $i>1$. Similarly, z is a simple zero of order n of $G(s)$ if $(s-z)^n | n_m(s)$ and if $n_i(z) \neq 0$ for $i<m$.

A characterization of simple poles and zeros that will be useful in the following is given by:

Lemma 1. Let p be a pole of order n of $G(s)$. Then, p is a simple pole if and only if the Laurent series expansion

$$G(s) = \frac{G_{-n}}{(s-p)^n} + \ldots + \frac{G_{-1}}{s-p} + G_o + \ldots \tag{7}$$

at s=p is such that

$$\text{rank } G_{-n} = 1 \quad . \tag{8}$$

Furthermore, if $G_{-n}$ has a simple null structure, i.e., if the rank of $G_{-n}$ is equal to its number of non-zero eigenvalues, the condition (8) is equivalent to

$$\text{tr } G_{-n} = (s-p)^n \text{ tr } G(s)\Big|_{s=p} \neq 0. \tag{9}$$

Similarly, if z is a zero of order n of G(s), z is simple if and only if the Laurent series expansion

$$G^{-1}(s) = \frac{H_{-n}}{(s-z)^n} + \ldots + \frac{H_{-1}}{s-z} + H_o + \ldots \tag{10}$$

at s=z is such that

$$\text{rank } H_{-n} = 1. \tag{11}$$

Also, if $H_{-n}$ has a simple null structure, (11) is equivalent to

$$\text{tr } H_{-n} = (s-z)^n \text{ tr } G^{-1}(s)\Big|_{s=z} \neq 0 \ . \tag{12}$$

Proof: see Appendix.

In the following, in addition to assuming that the poles and zeros of G(s) on the real axis are simple, we will also assume that the leading coefficient matrices in the Laurent series expansions of G(s) and $G^{-1}(s)$ at these points have simple null structure, so that (9) and (12) will be assumed to hold throughout. Note that the simple null structure assumption plays an important role in the results of Kouvaritakis and Shaked (1976), Owens (1978), and Sastry and Desoer (1980). This property was also shown by Byrnes and Stevens (1981) to be generic, so that there is little loss of generality in assuming it holds.

Finally, we will assume that there exists no single point loci on the real axis (see Postlethwaite and MacFarlane 1979 for a description of such points). The significance of these assumptions appears more clearly if we note that:

Lemma 2. Let p be a simple pole of order n of G(s) such that (9) is satisfied. Assume also that p is not a single point locus. Then the n branches of the root locus leaving p form a single Butterworth pattern with angles of departure

$$\Theta_{depart} = \frac{1}{n} (\mathrm{Arg}[- \mathrm{tr}\ G_{-n}] + k\ 360°) \quad , \quad k=0,1,\ ...,\ n-1, \qquad (13)$$

Similarly, if z is a simple zero of order n of G(s) such that (12) holds, and such that z is not a single point locus, the n branches of the root locus arriving at z form a single Butterworth pattern with angles of arrival

$$\Theta_{arrival} = \frac{1}{n} (\mathrm{Arg}[\mathrm{tr}\ H_{-n}] + k360°) \quad , \quad k=0,1,...n-1. \qquad (14)$$

Proof: see Appendix.

Another result that will be needed in our derivation of the main theorem deals with the description of the effect of branch points on multivariable root loci. We recall that if

$$\Phi(g,s) = A_m(s) \det \big(gI - G(s)\big) \tag{15}$$
$$= A_m(s)\, g^m + A_{m-1}(s)\, g^{m-1} + \dots + A_o(s)$$

is the polynomial obtained by multiplying det $\big(gI - G(s)\big)$ by the pole polynomial $A_m(s)$ of $G(s)$, then $s_o$ is a branch point (see Yagle 1981) if

$$\left.\frac{ds}{dg}\right|_{s_o} = 0 \quad , \tag{16}$$

or, equivalently, if $(g_o,s_o)$ is a common solution of

$$\Phi(g,s)=0 \text{ and } \frac{\partial\Phi}{\partial g}(g,s)=0 \quad . \tag{17}$$

But (16) implies that $s_o$ is a stationary point of the root locus — a point where a branch turns around and doubles back on itself. We show now that it is possible to determine on which side of a branch point a branch of the root locus will approach, reach the branch point, and turn around.

Lemma 3. Given a branch point $s_o$ on the real axis, the root locus will approach it, turn around, and depart from it on the left side (respectively on the right side) if

$$\operatorname{sgn}\left(\frac{\partial^2\Phi}{\partial g^2} \cdot \left.\frac{\partial\Phi}{\partial s}\right|_{s=s_o}\right) = 1 \qquad \text{(respectively } -1\text{)} \ . \tag{18}$$

Proof: see Appendix.

Given these preliminary results, we can now prove the main theorem

Theorem 2. The number N of branches of the root locus at a point $s_o$ on the real axis is given by

$$N = \sum_{\substack{\text{poles } p_i \text{ of}\\ \text{odd order to}\\ \text{right of } s_o}} \operatorname{sgn}\left( (s-p_i)^{n_i} \operatorname{tr} G(s)\Big|_{s=p_i} \right) + \begin{pmatrix}\text{number of}\\ \text{asymptotes}\\ \text{at } +\infty\end{pmatrix}$$

$$+ \sum_{\substack{\text{zeros } z_i \text{ of}\\ \text{odd order to}\\ \text{right of } s_o}} \operatorname{sgn}\left( (s-z_i)^{n_i} \operatorname{tr} G^{-1}(s)\Big|_{s=z_i} \right) \tag{19}$$

$$+2 \sum_{\substack{\text{branch points } b_i\\ \text{to right of } s_o}} \operatorname{sgn}\left( \frac{\partial^2 \Phi}{\partial g^2} \cdot \frac{\partial \Phi}{\partial s}\Big|_{s=b_i} \right)$$

where

(i) the $n_i$ are the orders of the poles and zeros;

(ii) the summations are taken over the poles and zeros of odd order, and branch points, on the real axis to the right of $s_o$;

(iii) it is assumed that the poles and zeros on the real axis are simple and satisfy the simple null structure assumption, and that there are no single point loci on the real axis.

Note that in order to apply Theorem 2 it is only necessary to evaluate the sign of a quantity at each pole and zero of odd order and each branch point on the real axis. Once this has been done the number of branches on the real axis may be determined immediately for all points.

Proof: We use a conservation of loci argument: each branch must start somewhere, end somewhere, and be continuous in between. We claim first that if there are only first-order poles and zeros on the real axis, then

$$N = \begin{pmatrix}\text{number of poles to right of } s_o \\ \text{with a branch departing at } 180^\circ\end{pmatrix} - \begin{pmatrix}\text{number of zeros to right of } s_o \\ \text{with a branch arriving at } 180^\circ\end{pmatrix} + \begin{pmatrix}\text{number of zeros to right of } s_o \\ \text{with a branch arriving at } 0^\circ\end{pmatrix} - \begin{pmatrix}\text{number of poles to right of } s_o \\ \text{with a branch departing at } 0^\circ\end{pmatrix} \quad (20)$$

This is easy to see, since the first two terms give the number of branches moving in the negative real direction and the last two terms give the number of branches moving in the positive real direction at $s_o$.

We now extend this to higher-order poles and zeros that are simple. Recall that the loci departing from or arriving at a simple pole or zero do so in a single Butterworth pattern. By symmetry, it is clear that a simple pole or zero of <u>even</u> order can have no effect on the number of branches on the real axis, while one of <u>odd</u> order must have exactly <u>one</u> branch departing or arriving at either $0^\circ$ or $180^\circ$. The angle may be determined by using Lemma 2, and since all quantities are real, we may use sgn instead of Arg in (13) and (14).

It should be evident that break-in and break-out points have no effect* on the number of branches on the real axis, while asymptotes on the real axis at $+\infty$ should be added in (consider them as infinite zeros).

Finally, we must introduce branch points since we have seen that at these points a branch can turn around and double back on itself. The side from which a branch approaches a branch point $b_i$, turns around, and departs

*It is shown in Yagle (1981), p.76, that the branches approaching and leaving a break point are evenly distributed over $360^\circ$ and are interleaved (they alternate).

from it is given by Lemma 3. Depending on whether the locus is to the left or right side of $b_i$, we should <u>add</u> or <u>subtract</u> two to the number of loci as $s_o$ moves to the left of $b_i$. By using Lemma 3, this yields the final term in (19). ■

The following corollary is interesting, primarily because it is the closest we can come to generalizing the SISO rule for loci on the real axis to the multivariable case. It may also be used as a check when applying Theorem 2, and may even provide sufficient information by itself for some applications.

Corollary 1. Assume that there are no asymptotes on the real axis at $+\infty$, and that the assumptions of Theorem 2 are valid. Then, counting multiplicities, at least one branch (in fact, an odd number of branches) of the root locus lies on the real axis at a given point $s_o$ if there is an odd number of poles and zeros to the right of $s_o$.

Remark: If there is an even number of poles and zeros to the right of $s_o$, then there is an even number of branches on the real axis at $s_o$. Unfortunately, zero is an even number.

Proof: Note that by making obvious substitutions, (20) can be written as $N = x_1 - x_2 + x_3 - x_4$. The total number of poles and zeros to the right of $s_o$, counting multiplicities, is $x_1 + x_2 + x_3 + x_4$ and it is clear that N will be odd if and only if this quantity is odd, guaranteeing at least one branch on the real axis at $s_o$. Recalling that break points have no effect on the number of branches on the real axis, and that branch points can only change the number of branches by an even number, completes the proof. ■

To illustrate Theorem 2, we consider the following example:

Example 3. Let

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} s^2 + 2s - 3 & s^2 + 3s & s+3 \\ -6s - 18 & s^2 + s - 6 & s^2 + 7s + 12 \\ 0 & 0 & s^2 + 3s + 2 \end{bmatrix}$$

There are first order poles at s = −1, −2, −3; no finite zeros; and a branch point at s=1/24. It is easy to see that all the eigenvalues of G(s) as $s \to +\infty$ are positive, so that there are no asymptotes at $+\infty$. We have

$$(s+1)\,\mathrm{tr}\, G(s)\Big|_{s=-1} = -5$$

$$(s+2)\,\mathrm{tr}\, G(s)\Big|_{s=-2} = 7$$

$$(s+3)\,\mathrm{tr}\, G(s)\Big|_{s=-3} = 1$$

and the corresponding signs of these quantities are respectively −1, 1 and 1. Also, one gets

$$\Phi(g,s) = (s+1)(s+2)(s+3)\, g^3 - (3s^2 + 6s - 7)g^2 + 3sg - 1.$$

The gain at the branch point $s_o = 1/24$ is $g_o = -24/35$, and

$$\mathrm{sgn}\left(\frac{\partial^2\Phi}{\partial g^2} \cdot \frac{\partial \Phi}{\partial s}\Big|_{s_o,g_o}\right) = \mathrm{sgn}\left(\left[6(s+1)(s+2)(s+3)\, g - (6s^2 + 12s - 14)\right] \cdot \left[(3s^2 + 12s + 11)\, g^3 - (6s+6)\, g^2 + 3g\right]\Big|_{s_o,g_o}\right)$$

where, by inspection, it is clear that both terms being multiplied are negative, so that the entire quantity is positive. Using (19), we find that

$N = 0$ for $1/24 < s$

$N = 2$ for $-1 < s < 1/24$

$N = -1 + 2 = 1$ for $-2 < s < -1$

$N = -1 + 1 + 2 = 2$ for $-3 < s < -2$

$N = -1 + 1 + 1 + 2 = 3$ for $s < -3$ .

The corresponding root locus is plotted in Figure 3. Note that once again the entire root locus is on the real axis.

4. The case of symmetric G(s)

In this section we consider the case when G(s) is symmetric. Since G(s) is symmetric for reciprocal networks, this case does have some practical applications. The motivation for considering this class of systems is that in this case the results of the previous section simplify considerably, and no assumptions are required on the poles and zeros of G(s). Since our final result depends on matrices obtained from G(s) by several transformations, we will first construct these transformations, and then state the results as a theorem at the end of this section.

We observe from (2) that the number of branches of the root locus at a point s on the real axis is the number of negative real eigenvalues of G(s). However, if G(s) is symmetric then all its eigenvalues are real, and we need only to determine how many of them are positive and how many of them are negative. To do so, we will use the signature of G(s).

Definition: Let M be a nonsingular real symmetric matrix, and define

$m_+$ = number of positive eigenvalues of M

$m_-$ = number of negative eigenvalues of M.

Then, the signature $\sigma(M)$ of M is defined as

$$\sigma(M) = m_+ - m_- \tag{21}$$

Remark Since M is nonsingular, we have $m_+ + m_- = m$ where m is the size of M. Therefore, we may determine $m_+$ and $m_-$ from $\sigma(M)$.

An important property of the signature of a matrix is that it is invariant under congruency transformations. Thus, if L is a nonsingular real matrix and if

$$P = L\,M\,L' , \tag{22}$$

then $\sigma(P) = \sigma(M)$.

Now, consider a left matrix fraction description

$$G(s) = D^{-1}(s)\,N(s) \tag{23}$$

where D(s) and N(s) are left coprime polynomial matrices. The poles and zeros of G(s) are, respectively, the zeros of det D(s) and det N(s). Since the product of the eigenvalues $g_i(s)$ of G(s) is given by

$$\prod_{i=1}^{m} g_i(s) = \det G(s) = \frac{\det N(s)}{\det D(s)} \tag{24}$$

the eigenvalues $g_i(s)$ can only change sign at the real poles and zeros of G(s).

For all points on the real axis that are not poles of G(s),

$$D(s)G(s)D'(s) = N(s)D'(s) \triangleq P(s) \tag{25}$$

is a congruency transformation of G(s), so that

$$\sigma(G(s))=\sigma(P(s)). \tag{26}$$

Since: (i) the number of loci on the real axis at s is the number of negative real eigenvalues of G(s); (ii) the number of negative real eigenvalues of G(s) may be determined from $\sigma(G(s))=\sigma(P(s))$; and (iii) $\sigma(P(s)) = \sigma(G(s))$ can only change at a pole or zero $s_o$ of G(s), we now investigate how $\sigma(P(s))$ changes near a real pole or zero $s_o$.

Near such a point, P(s) can be expanded as

$$P(s) = P_o + P_1\ (s-s_o) + \ldots + P_d(s-s_o)^d \tag{27}$$

where $P_o$ is singular, and where the matrices $P_i$ are real and symmetric. Then, we note that

<u>Lemma 4.</u> If $x = s-s_o$, there exists a sequence of congruency transformations that transforms P(s) into

$$Q(s) = \begin{bmatrix} Q_o + 0(x) & & & 0 \\ & Q_1 x + 0(x^2) & & \\ & & \ddots & \\ 0 & & & Q_k x^k \end{bmatrix} + 0(x^{k+1}) \tag{28}$$

where the matrices $Q_i$ , i=1 ... k, are real, nonsingular and symmetric.

<u>Proof:</u> The proof is similar to one that appears in Bitmead and Anderson (1977) and Owens (1978). Since $P_o$ is singular, there exists a real nonsingular matrix $T_o$ such that

$$T_o'\ P_o\ T_o = \begin{bmatrix} Q_o & 0 \\ 0 & 0 \end{bmatrix}$$

where $Q_o$ is real, symmetric and nonsingular. Then we introduce

$$\hat{P}(s) = T_o'\ P(s) T_o = \begin{bmatrix} Q_o & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} A_i & B_i \\ B_i' & C_i \end{bmatrix} x^i$$

where $\hat{P}(s)$ is congruent to P(s). The matrix $B_1$ can be eliminated by using another congruency transformation. To do so, we define

$$V_1 = Q_o^{-1} B_1$$

and

$$P^{(1)}(s) = \begin{bmatrix} I & 0 \\ -V_1' x & I \end{bmatrix} \hat{P}(s) \begin{bmatrix} I & -V_1 x \\ 0 & I \end{bmatrix}$$

and we note that $P^{(1)}(s)$ can be written as

$$P^{(1)}(s) = \begin{bmatrix} Q_o & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & C_1^{(1)} \end{bmatrix} x + 0\,(x^2) \qquad (29)$$

where $C_1^{(1)}$ is real and symmetric. If $C_1^{(1)}$ has full rank, (29) has the desired form given in (28), and the result is proved.

If $C_1^{(1)}$ does not have full rank, the previous procedure may be repeated with $C_1^{(1)}$ taking the place of $P_o$. This means that there exists a real non-singular matrix $T_1$ such that

$$T_1' C_1^{(1)} T_1 = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $Q_1$ is real, symmetric and nonsingular. Now define the congruency transformation

$$\hat{P}^{(1)}(s) = \begin{bmatrix} I & 0 \\ 0 & T_1' \end{bmatrix} P^{(1)}(s) \begin{bmatrix} I & 0 \\ 0 & T_1 \end{bmatrix}$$

and write

$$\hat{P}^{(1)}(s) = \left[\begin{array}{c|c} G_o & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|cc} A_1 & & 0 \\ \hline 0 & G_1 & 0 \\ & 0 & 0 \end{array}\right] x + \sum_{i=2}^{d_1} \left[\begin{array}{c|c} A_i^{(1)} & B_i^{(1)} \\ \hline B_i^{(1)'} & C_i^{(1)} \end{array}\right] x^i .$$

The matrix $B_2^{(1)}$ can be eliminated by using a congruency transformation of the type

$$P^{(2)}(s) = T^{(1)'}(s)\hat{P}^{(1)}(s)T^{(1)}(s)$$

where

$$T^{(1)}(s) = \left[\begin{array}{c|c} I & (-V_1^{(1)}x \\ & -V_2^{(1)}x^2) \\ \hline 0 & I \end{array}\right] ;$$

and the off-diagonal blocks of $A_2^{(1)}$ can be eliminated in the same way. Then, depending on whether the resulting $C_2^{(2)}$ has full rank or not, the construction (28) terminates, or we need to run the previous procedure another time. Note however that in the end we obtain a polynomial matrix of the form given in (28). ■

We may now investigate how $\sigma(P(s))$ changes near $s_o$. Since Q(s) was obtained from P(s) by a sequence of congruency transformations, we have $\sigma(Q(s)) = \sigma(P(s))$. In the vicinity of $s_o$, Q(s) can be approximated by

$$Q(s) = \text{diag}(Q_i\, x^i) \qquad (30)$$

and the eigenvalues of Q(s) are approximately the eigenvalues of $Q_o$, $Q_1x$, ... and $Q_kx^k$.

Now consider what happens to the signs of the eigenvalues of $Q_i x^i$ if x changes sign from positive to negative. If i is even the signs do not change; but if i is odd, all the positive eigenvalues become negative, and vice-versa. Then $\sigma(Q_i x^i)$ changes sign, so that the change in $\sigma(Q_i x^i)$ is $-2\sigma(Q_i)$. (Note that if x is positive, $\sigma(Q_i x^i)=\sigma(Q_i)$.) It follows immediately that the change in $\sigma(Q(s))$ is

$$\Delta\sigma(Q(s)) = -2\sum_{i\ \mathrm{odd}} \sigma(Q_i) \tag{31}$$

Since we have

$$m_-(Q(s)) = \frac{1}{2}\,(m - \sigma(Q(s))) \tag{32}$$

where $m_-(Q(s))$ is the number of negative eigenvalues of Q(s) and where m is the size of Q(s), the change in the number of negative real eigenvalues is

$$\Delta m_-(Q(s)) = -\frac{1}{2}\,\Delta\sigma(Q(s)) = \sum_{i\ \mathrm{odd}} \sigma(Q_i)\ . \tag{33}$$

Now let s vary along the real axis from $+\infty$ to $-\infty$, and assume that $G(+\infty)$ is positive definite (this is equivalent to assuming that there are no asymptotes on the real axis at $+\infty$) . For each pole or zero $s_j$ on the real axis, we can compute a set of matrices $Q_i^{(j)}$ by using Lemma 4. Then, recalling that $\sigma(Q(s)) = \sigma(P(s)) = \sigma(G(s))$ and that the number of branches on the real axis at s is the number of negative real eigenvalues of G(s), we obtain:

<u>Theorem 3.</u> Assume that there are no asymptotes on the real axis at $+\infty$, and that G(s) is symmetric. For each pole and zero $s_j$ on the real axis compute the matrices $Q_i^{(j)}$, using the procedure of Lemma 4. Then the number N of

branches on the real axis at s is given by

$$N = \sum_{\substack{\text{all poles and} \\ \text{zeros } s_j \text{ to} \\ \text{right of } s}} \sum_{i \text{ odd}} \sigma(Q_i^{(j)}) \ . \tag{34}$$

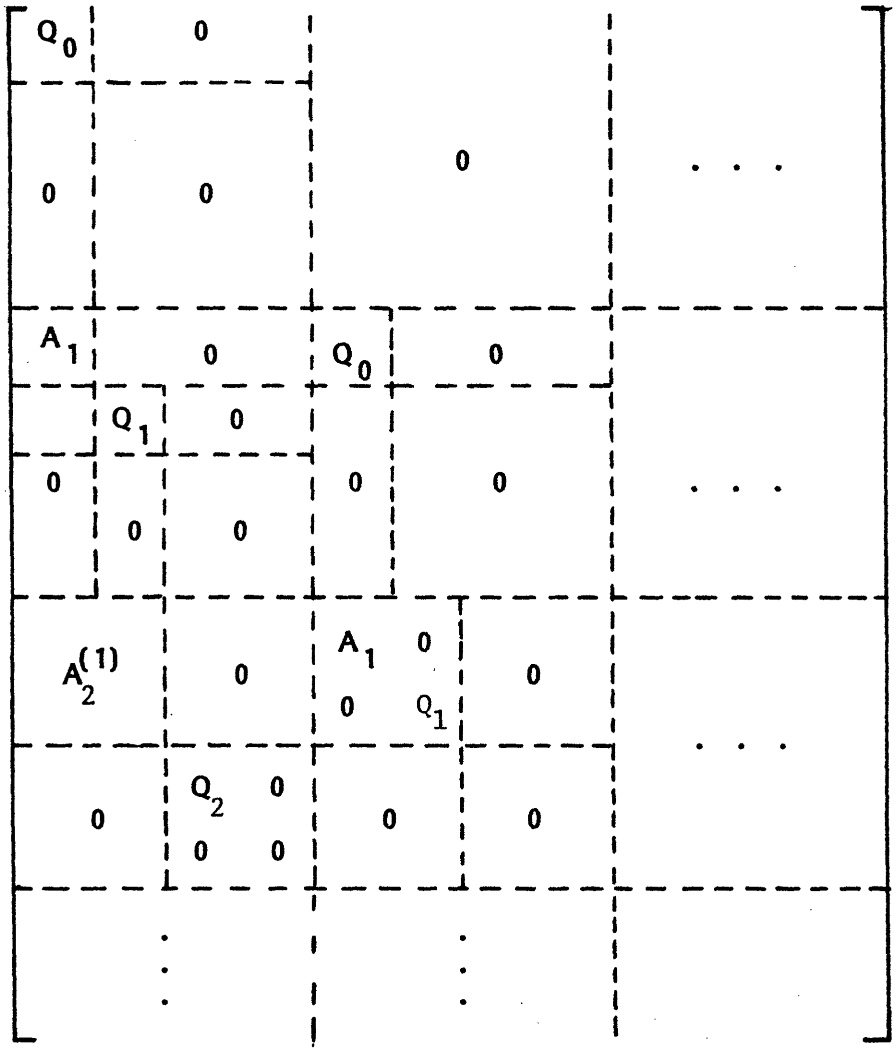
There is an interesting observation that may be made on the procedure for generating the matrices $Q_i$. Consider the set of Toeplitz matrices

$$T_i = \begin{bmatrix} P_o & & 0 \\ P_1 & \ddots P_o & \\ P_{i-1} & P_1 & P_o \end{bmatrix}, \quad i=1,2 \ldots , d+1 \ . \tag{35}$$

It was shown by VanDooren, Dewilde and Vandewalle (1979), and by Kailath and Verghese (1981) that the zero structure of P(s) at $s_o$ can be obtained by computing the ranks of the matrices $T_i$. If $r_i$ denotes the number of McMillan indices $\{k_j\}$ of P(s) at $s_o$ which are equal to i, we have

$$\text{rank} \ \ T_i = ir_0 + (i-1)r_1 + \ldots + r_{i-1} \ . \tag{36}$$

However, it may also be shown that the congruency transformations used to generate the matrices $Q_i$ may be applied to the $T_i$, yielding matrices of the form



so that we have

$$\text{rank } Q_i = r_i \,, \quad i=0,1, \ldots d \,. \tag{37}$$

Therefore, the zero structure of P(s) at $s_o$ is determined by the <u>ranks</u> of the $Q_i$, while the number of branches of the root locus at $s_o$ is determined by the <u>signatures</u> of the $Q_i$.

## 5. Conclusion

The problem of determining the number of branches of multivariable root loci on the real axis at any point has been studied for three different cases. The case of two-input-two-output systems was found to be relatively simple, requiring only the solution of three polynomial inequalities. The general case was found to be much more difficult due to the possible presence of branch points and the possibility of several branches on the real axis at the same point. Nevertheless, an equation was exhibited that required only the evaluation of the sign of a quantity at each pole and zero of odd order, and branch point, on the real axis. Finally the case when G(s) is symmetric was found to be solvable by computing the signatures of certain matrices formed by congruency transformations of G(s). More work needs to be done in finding other special cases that admit to simple solutions, and in finding ways of simplifying the general equation (19).

## Appendix

### Proof of Lemma 1

According to the definition given in section 3, a pole of order n is simple if its McMillan indices $[k_1, k_2, \dots k_m]$ are such that $k_1 = -n$ and $0 \leq k_2 \dots \leq k_n$ (note that G(s) may have some zeros at p). Then, the characterization of the McMillan indices given in VanDooren, Dewilde and Vandewalle (1979) implies that

$$\text{rank } G_{-n} = 1 \; . \tag{A.1}$$

Conversely if (A.1) holds we must have $k_1 = -n$, and since the total polar order at p is n, the other structure indices $k_i$ must be $\geq 0$ so that p is simple.

If in addition we assume that $G_{-n}$ has simple null structure, (A.1) implies that $G_{-n}$ must have exactly one nonzero eigenvalue, so that

$$\text{tr } G_{-n} \neq 0 \; . \tag{A.2}$$

To show the converse, we note that if (A.2) holds, then rank $G_{-n} \geq 1$; and since the total polar order at p is only n, we must have (A.1).

### Proof of Lemma 2

If s belongs to the root locus, we have

$$\det\left(I+k\, G(s)\right) = 1+k \text{ tr } G(s) + \dots + k^m \det G(s) = 0 \tag{A.3}$$

for some k real and positive. By multiplying (A.3) by the pole polynomial $A_m(s)$ one gets

$$A_m(s) - k\, A_{m-1}(s) + \dots + (-k)^m\, A_o(s) = 0 \quad , \tag{A.4}$$

where we note that, since p is a pole of order n of G(s),

$$A_m(s) = (s-p)^n \tilde{A}_m(s)$$

with $\tilde{A}_m(p) \neq 0$. Using a Newton diagram (see Yagle 1981), it may be shown that the branches of the root locus leaving p can be approximated by

$$s-p \simeq ck^{1/n} \quad , \tag{A.5}$$

and by substitution in (A.4) one finds

$$(c^n\tilde{A}_m(p) - A_{m-1}(p))k + 0\ (k^2) = 0 \ .$$

Neglecting the higher-order terms in k gives

$$c^n = \frac{A_{m-1}(p)}{\tilde{A}_m(p)} = -(s-p)^n \operatorname{tr} G(s)\Big|_{s=p} = -\operatorname{tr} G_{-n} \ , \tag{A.6}$$

and since by assumption $\operatorname{tr} G_{-n} \neq 0$, it is clear from (A.6) that the branches of the root locus leave p in a single Butterworth pattern with angles of departure

$$\operatorname{Arg} c = \frac{1}{n} [\operatorname{Arg}[-\operatorname{tr} G_{-n}] + k\ 360°] \tag{A.7}$$

with k=0,1, ... n-1 .

For the angles of arrival at a simple zero z, make the substitution h=1/k in (A.4). Again using a Newton diagram (see Yagle 1981), it may be shown that the branches of the root locus arriving at z can be approximated by

$$s-z \simeq bh^{1/n} \tag{A.8}$$

and substitution in (A.4) as before yields

$$b^n = (s-z)^n \left.\frac{A_1(s)}{A_o(s)}\right|_{s=z} = (s-z)^n \frac{1}{\det G(s)} \left.\left(\sum \begin{matrix}\text{principal minors of}\\ \text{order m-1 of } G(s)\end{matrix}\right)\right|_{s=z}$$

$$= (s-z)^n \operatorname{tr} G^{-1}(s)\Big|_{s=z} . \tag{A.9}$$

The last equality follows from the familiar equation

$$G^{-1}(s) = \frac{\operatorname{adj} G(s)}{\det G(s)} .$$

The rest of the argument parallels the one given above for the angles of departure.

Proof of Lemma 3

In the vicinity of a branch point $s_o$, define $\delta s = s-s_o$. Then for a small perturbation $\delta g$ in g, write the Taylor series expansion

$$\delta s = \left.\frac{ds}{dg}\right|_{s_o} \delta g + \frac{1}{2} \left.\frac{d^2 s}{dg^2}\right|_{s_o} (\delta g)^2 + \ldots \quad . \tag{A.10}$$

The first term is zero, so that we have

$$\operatorname{sgn} \delta s = \operatorname{sgn}\left(\left.\frac{d^2 s}{dg^2}\right|_{s_o}\right) . \tag{A.11}$$

This means that regardless of the sign of $\delta g$ (i.e. regardless of whether the locus arrives at or departs from $s_o$) the sign of $\delta s$ is the same. Thus, s is always on the same side of $s_o$.

By using the identity $\Phi(g,s) \equiv 0$, we find that

$$\frac{ds}{dg} = - \frac{\partial\Phi}{\partial g} \Big/ \frac{\partial\Phi}{\partial s} \tag{A.12}$$

and by differentiating (A.12) with respect to g, we get

$$\frac{d^2 s}{dg^2} = -\left(\frac{\partial\Phi}{\partial s} \frac{d}{dg}\left(\frac{\partial\Phi}{\partial g}\right) - \frac{\partial\Phi}{\partial g} \frac{d}{dg}\left(\frac{\partial\Phi}{\partial s}\right)\right) \Big/ \left(\frac{\partial\Phi}{\partial s}\right)^2 . \tag{A.13}$$

Then, if we note that at a branch point $(g_o,s_o)$ one has $\frac{\partial\Phi}{\partial g}(g_o,s_o) = 0$, we obtain

$$\left.\frac{d^2s}{dg^2}\right|_{s_o} = -\left.\frac{\partial^2\Phi}{\partial g^2} \bigg/ \frac{\partial\Phi}{\partial s}\right|_{s = s_o} \tag{A.14}$$

so that

$$\text{sgn}\ \delta s = -\ \text{sgn}\left(\left.\frac{\partial^2\Phi}{\partial g^2}\cdot\frac{\partial\Phi}{\partial s}\right|_{s = s_o}\right) . \tag{A.15}$$

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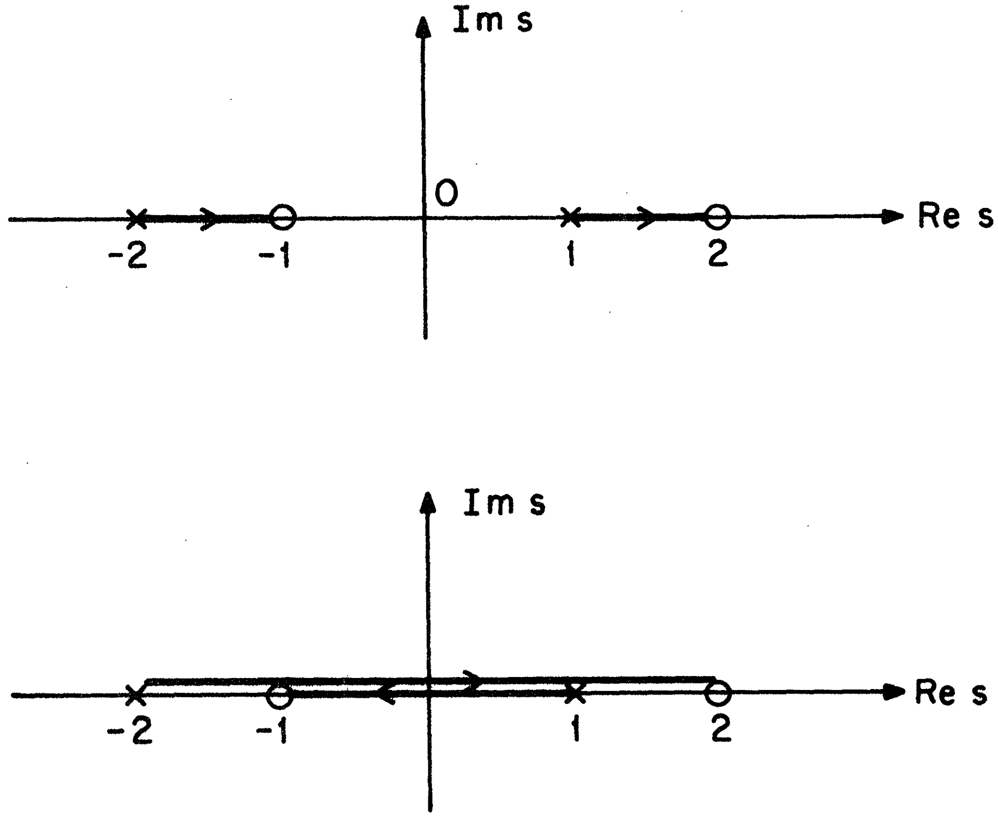


Figure 1. ROOT LOCI FOR EXAMPLE 1.

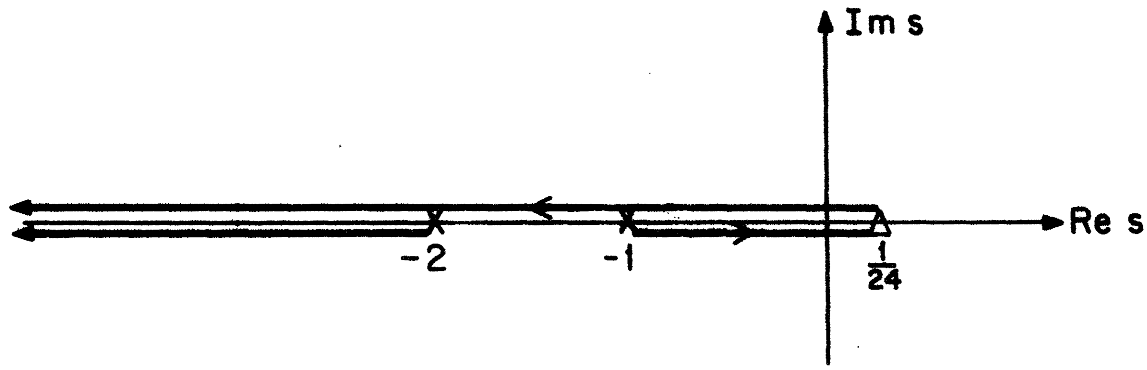


Figure 2. ROOT LOCUS FOR EXAMPLE 2.

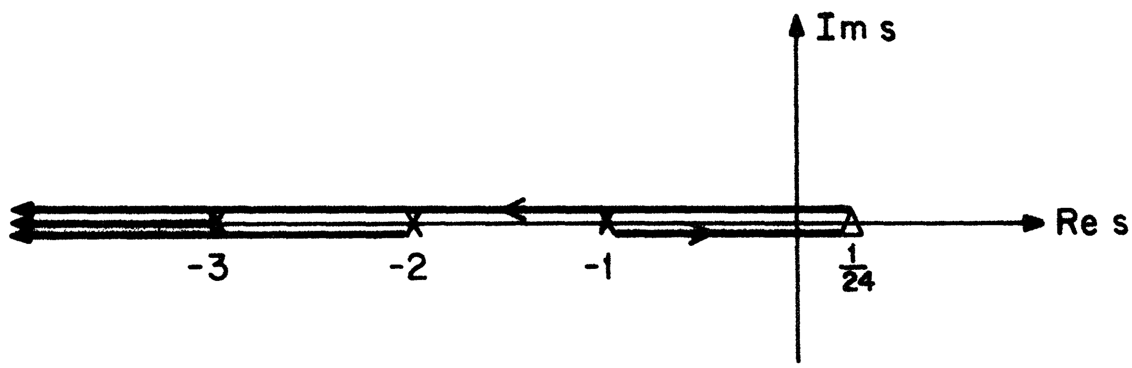


Figure 3. ROOT LOCUS FOR EXAMPLE 3.