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## ABSTRACT

Neoclassical transport theory is developed in a Lagrangian rather than the usual Eulerian formulation. We show that an underlying asymmetry exists in the neoclassical pinch and bootstrap effects and demonstrate the physical basis of the Onsager symmetry relationship in the pinch-bootstrap duality. It is suggested that low frequency turbulence can destroy the bootstrap current at levels too low to effect the Ware Pinch.

Neoclassical transport theory[1] has long predicted the existence of a "bootstrap current" driven by the radial density gradient and a pinch effect driven by the toroidal electrical field. Both effects have substantial practical implications for tokamak reactors. Neither, however, has been confirmed by experiment. Inward plasma flow has been clearly, if indirectly, observed[2] although not quantitatively explained by the neoclassical pinch.[3] The bootstrap current[6] has not been seen even in experiments where it should have been easily detected.[4] The pinch effect is often explained as a purely kinematic, collisionless inward flow of trapped particles. The underlying mechanism of the bootstrap current is somewhat obscure, although it clearly is a collision driven phenomenon involving circulating particles.[7] These two phenomena display an Onsager symmetry relation and have equal coefficients. Why effects involving different classes of particles should be Onsager conjugate has never been satisfactorily explained.

This letter considers the questions of Onsager symmetry and the non-observance of the bootstrap current in light of a recently developed Lagrangian formulation of neoclassical theory.[8] This formulation has the virtue of being a direct expression of the elementary kinematics and collision processes, allowing the various flows to be identified at the microscopic level. We show the *net* kinematic contributions to both pinch and bootstrap current to be small and symmetric. However, the individual kinematic processes involved in the pinch are large and there is an underlying asymmetry of collisionless processes. Collisional processes involving circulating particles drive both the pinch (contrary to the accepted models) and bootstrap effects. These collisional processes are microscopically inverse and clearly Onsager conjugate. However, the bootstrap current is a result of scattering by particles in a very narrow layer near the trapped particle boundary. We argue that the nature of this layer is such that the bootstrap current, but not the Ware pinch, is easily destroyed by low level turbulent scattering.

The physical basis of the Lagrangian formulation is that in the lowest collisionality (banana) regime, the distribution function; f, relaxes in a sequence of well-ordered time scales. The fastest is the orbital time scale, on which f relaxes to a function of constants of motion (or actions) alone. The actions $J_1$, $J_2$, $J_3$ are chosen to be respectively the magnetic moment, parallel invariant and bounce averaged poloidal flux. These actions then scatter under the influence of collisions to relax f to a local Maxwellian on the collisional time scale. The radial action action gradients relax to produce transport on the (longer) diffusion time scale.

The Lagrangian formulation follows this hierarchy of relaxation processes, first expressing the kinetic equation in action angle variables, and averaging over the orbital time scale (or equivalently, the angle variables). The result is a kinetic equation in terms of the actions alone,

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \underline{J}} \cdot \frac{q}{2\pi} \underline{a}(\underline{J}) V_T f = C^J(f), \tag{1}$$

where $V_T$ is the toroidal voltage, with coefficient, $\underline{a}(\underline{J}) = d^3\theta \frac{1}{mR} \underline{e}_\zeta \cdot \nabla_{\underline{v}} \underline{J}$, and the collision operator is (for the Lorentz model)

$$C^J(f) = \frac{\partial}{\partial \underline{J}} \cdot \underline{\underline{D}} \cdot \frac{\partial}{\partial \underline{J}} = \frac{\partial}{\partial \underline{J}} \cdot \int d^3\theta \nu (\nabla_{\underline{v}} \underline{J})^T \cdot (v^2 \underline{\underline{I}} - \underline{v}\underline{v}) \cdot (_{\underline{v}} \underline{J}) \cdot \frac{\partial}{\partial \underline{J}} f. \tag{2}$$

For evaluating <u>neoclassical</u> fluxes in an axisymmetric system, the angle averages are operationally bounce averages, $d^3\theta \rightarrow \omega_2 ds/u$. Equation (2) gives the bounce averaged effect of collisions, (i.e., velocity scattering at fixed spatial position) on the actions. The three actions, $J_1$, $J_2$, $J_3$ have well ordered associated frequencies, $\omega_1 >> \omega_2 >> \omega_3$, so that the energy is principally a function of $J_1$ and $J_2$, the "velocity" variables, while $J_3$ is a "radial" parameter. More precisely, $J_3 = \frac{2\pi q}{c} \psi + \Delta J_3$, where $\omega_2 \oint \frac{ds}{u} \Delta J_3 = 0$. Note that $C^J$ contains terms of the form $\frac{\partial}{\partial J_3} D_{33} \frac{\partial}{\partial J_3} f$, which give, explicitly, radial diffusion due to action scattering under collisions. There are also collisional cross processes of the form, $\frac{\partial}{\partial J_2} D_{23} \frac{\partial}{\partial J_3}$ and $\frac{\partial}{\partial J_3} D_{32} \frac{\partial}{\partial J_2} f$, which are ultimately responsible for the pinch and bootstrap effects. Transport equations are obtained directly by the reduced moments over $J_1$ and $J_2$. One can, in fact, show that the density per unit $J_3$, $n_3 \equiv \int dJ_1 dJ_2 f$, is equal to the flux surface average spatial density times the specific volume, $dV/dJ_3 = \frac{c}{2\pi q} dV/d\psi$. In Equation (1), the $J_3$ derivatives are smaller than the $J_1$, $J_2$ derivatives by a factor $\rho_p/a$, the small parameter that, for tokamaks, measures the relative slowness of radial scattering to velocity scattering. The transport coefficients can be obtained by a straight forward expansion of (1) in powers of $\rho_p/a$, using a maximal ordering where $V_T \sim \rho_p/a$, $\frac{\partial}{\partial t} \sim (\rho_p/a)^2$.

The leading order consequence of Equation (1) is $C_o^J(f_o) = 0$, where $C_o^J$ is the "velocity" scattering part of the full operator. This operator has a local H-theorem (when like-particle collisions are included) so that $f_o$ is a local Maxwellian of the form $f_o = N(J_3) \exp(-H(\underline{J})/T(J_3))$.

The first order equation is

$$C_o(f_1) = \frac{q}{2\pi} V_T \underline{a}_\perp \cdot \frac{\partial}{\partial \underline{J}_\perp} f_o - \left(\frac{\partial}{\partial \underline{J}_\perp} \cdot \underline{\underline{D}} \cdot \underline{e}_3\right) \frac{\partial f_o}{\partial J_3}, \qquad (3)$$

where $\frac{\partial}{\partial \underline{J}_\perp} = \underline{e}_1 \frac{\partial}{\partial J_1} + \underline{e}_2 \frac{\partial}{\partial J_2}$, is effectively a velocity gradient. It is thus the electric field and the collisional cross processes that drive perturbations $f_1$. Only the circulating particles are effected and $f_1 = 0$ in trapped space.

With $f_1$ known, the particle and energy moments of Eq. (1)

are determined to second order (since $C_o(f_2)$ is annihilated) and provide the transport equations. These are formalized by defining generalized forces, $A_1 = d\ell nn/dJ_3$, $A_2 = d\ell nT/dJ_3$, $A_3 = V_T/T$ and their respective fluxes of particles, heat and charge, $F_1 = \Gamma$, $F_2 = q$, $F_3 = I_T$, related by $F_i = \sum_i T_{ij} A_j$, with the transport matrix, $T_{ij}$, exhibiting Onsager symmetry, $T_{ij} = T_{ji}$. Defining coefficients, $\alpha_i$, in (3) such that $C_o(f_1) = \sum_i \alpha_i f_o A_i$, $f_1$ can be expressed as a sum over the thermodynamic forces, $f_1 = \sum_i g_i A_i$, with individual responses determined by $C_o(g_i) = \alpha_i f_o$. These relations are useful for writing the transport equations in a compact form and for proving Onsager symmetry.

For example, the particle moment of Eq. (1), can be written, to second order, as

$$0 = \frac{\partial n_3}{\partial t} + \frac{\partial}{\partial J_4} \left[\int d^2 J_\perp \left(\frac{q}{2\pi} a_3 f_o V_T - \alpha_1 f_1 - D_{33} \frac{\partial f_o}{\partial J_3}\right)\right] \qquad (4)$$

The particle flux transport coefficients, $T_{ij}$, can be inferred from (4). Specifically, the pinch coefficient, $T_{13}$, $(g,h) \equiv \int d^2\underline{J}_\perp gh$, is $T_{13} = (\frac{qT}{2\pi} a_3, f_o) - (\alpha_1, g_3) \equiv T_{13}^e + T_{13}^i$. The decomposition into explicit (superscript e) and implicit (superscript i) is a key feature of this formulation. The fluxes that do not require the calculation of $f_1$ are termed <u>explicit</u>, and represent processes depicted in explicit form in the original kinetic equation (1). Thus the first flux term in (4), proportional to $V_T$ is an explicit, purely kinematic, radial flow, independent of collisions. The Ware effect appears in this term. The implicit or indirect fluxes result from $f_1$.

To calculate the toroidal current, one starts from

$$J_T = \int d^3J d^3\theta\, \delta\, (x - x(\underline{J},\underline{\theta}))\; q\underline{e}_\zeta \cdot v(J,\underline{\theta})\; f(\underline{J}) .$$

This contains the Pfirsch-Schluter current in addition to the parts, constant on a flux surface, determined from transport theory. We now weight $J_T$ by $1/2\pi R$, flux surface average and multiply by $dV/dJ_3$, to give an angular current per unit $J_3$,

$$I_T = \int d^3J\omega_2 \oint \frac{ds}{u} \frac{c}{2\pi q} \left(\psi - \frac{c}{2\pi q} (J_3 - \Delta J_3)\right) \frac{q}{2\pi R}\, \underline{e}_\zeta \cdot \underline{V} f$$

$$= \int d^2J_\perp \omega_2 \oint \frac{ds}{u}\; \frac{q}{2\pi R}\, \underline{e}_\zeta \cdot \underline{v}\, \left[f_1 (\underline{J}_\perp, \frac{2\pi q}{e}\psi) + \Delta J_3 \frac{\partial f_o}{\partial J_3}\right] + \ldots \tag{5}$$

This expression is correct to order $\rho_p/a$. Here again we have implicit and explicit parts. The explicit piece is a toroidal current associated with the departure of the orbits from the average flux surfaces analogous to the perpendicular diamagnetic flow associated with the departure of gyro-orbits from the guiding center. Transport coefficients may be inferred from Eq. (6), the bootstrap coefficients being $T^e_{31}$ and $T^i_{31} = -(\alpha_3, g_1)$.

Constructing the rest of the transport matrix, one gets results for the net coefficients, $T_{ij}$, identical to the conventional Eulerian theory, and having Onsager symmetry.[8] In the Lagrangian formulation, however, there are two symmetry theorems, one for the implicit and another for the explicit piece. The former follows immediately from the self-adjointness of $C_o$ viz. $T^i_{ij} = (\alpha_i, g_j) = (C_o(g_i), g_j) = (g_i, C_o(g_i)) = (\alpha_j, g_i) = T^i_{ji}$. Explicit symmetry can be demonstrated term by term.[8] Recall that only the circulating particles contribute to the implicit fluxes. Trapped particle contributions are all explicit.

We now come to describing the elementary processes involved in producing the pinch and bootstrap effects. It is useful to define the dimensionless transport coefficients $I_{13}$ and $I_{31}$ according to $T_{13} = \frac{3}{4} qTn_3 I_{13}$, $T_{31} = \frac{3}{4} qTn_3 I_{31}$. In the limit of small $\varepsilon = a/R$, the overall coefficients[1] are $I_{13}$ and $I_{31} \hat{=} 1.38\sqrt{2\varepsilon}$.

In the conventional explanation, the pinch is associated predominantly with the kinematic flow of trapped particles, $I^{e,tr}_{13}$ (the Ware effect).

Indeed, one finds, $I_{13}^{e,tr} = \frac{8}{3\pi}\sqrt{2\varepsilon}$, which is 62% of the full coefficient. The circulating particle contributions $I_{13}^{e,cir}$ and $I_{13}^{i,cir}$ are of opposite sign and tend to cancel. The difficulty with this interpretation is that the process conjugate to the Ware effect, the trapped particle banana current, $I_{31}^{e,tr}$ is negligible (of order $\varepsilon^{3/2}$) in the small $\varepsilon$ limit. Also, $I_{31}^{e,cir} = \sigma(\varepsilon^{3/2})$, so that $I_{31}^{e}$ is negligible. The bootstrap current is all implicit. It is a result of the collisional cross process, and entirely a consequence of circulating particles. This picture does not give the physical basis of Onsager symmetry.

To clarify the symmetry first note that the explicit symmetry $I_{31}^{e} = I_{13}^{e}$, implies that to order $\varepsilon^{3/2}$ $I_{13}^{e} = 0$, and, therefore, that the explicit circulating particle flow is actually radially <u>out</u>, at a rate which cancels entirely the Ware pinch! Thus, one does not have symmetry of the elementary processes represented by $I_{13}^{e,tr}$ and $I_{31}^{e,tr}$ (or $I_{13}^{e,cir}$ and $I_{31}^{e,cir}$) but only of the net coefficients $I_{13}^{e}$ and $I_{31}^{e}$. Nonetheless, $I_{13}^{e} \hat{=} 0$, and this leads to an alternate interpretation of the pinch as a *collisional* process involving circulating particles.

The question is now reduced to understanding the implicit flows. Recall that these arise from the cross processes in the collision operator, reflecting correlations in the scattering process between jumps in radius and jumps in velocity.

Figure (1a) compares representative orbits for different types of particles. Consider two particles, initially well circulating, moving in opposite directions along the magnetic field. Both have orbits lying very nearly on the flux surface $\psi$. As these particles scatter toward trapped space, under the influence of collisions, the $\sigma = -1$ particle scatters to an orbit whose average surface is shifted inward relative to $\psi$ while the $\sigma = +1$ particle scatters out. This is the origin of the correlation between velocity and radial scattering. A radial flow will result whenever the perturbed distribution has unequal fractions of $\sigma = -1$ and $\sigma = +1$ particles, or in other words, carries a current. The current driven by the toroidal electric field has an excess of $\sigma = -1$ particles and drives an inward radial flow. This process, due entirely to circulating particles, causes the pinch effect.

Now invert the process just described. That is, take two marginally circulating particles, oppositely directed along the magnetic field and scatter them back toward the well circulating state. The $\sigma = -1\,(+1)$ particles start on an inner (outer) average surface and end up on $\psi$. With a normal density gradient the result will be more $\sigma = -1$ than $\sigma = +1$ particles on the final $\psi$ surface, and thus an electric current. This is the mechanism behind the bootstrap current. It is the precise microscopic inverse to the process accounting for the pinch effect. The true Onsager symmetry is obvious.

A detailed calculation shows that all the circulating particle contributions, $T_{13}^{e,cir}$, $T_{13}^{i,cir}$, and $T_{31}^{i,cir}$ arise from a very narrow layer near the trapped particle boundary. In terms of the pitch angle variable $\lambda$ (such that $0 \leq \lambda \leq 1-\varepsilon$ is circulating, $1-\varepsilon \leq \lambda \leq 1+\varepsilon$ is trapped) the layer is much thinner than $\varepsilon$. A layer width, $\Delta\hat{\lambda}$, on the order of several percent of $\varepsilon$, accounts for 80% of the effect. In contrast, the Ware effect $T_{13}^{e,tr}$ has equal contributions from all the trapped particles. Since the explicit bootstrap $T_{31}^{e}$ is uniformly small for all $\lambda$, the underlying asymmetry in the explicit processes can be brought out by small modifications of the boundary layer particle dynamics. This can cause a breakdown of the symmetry when turbulence is present.

Whereas neoclassical theory can be viewed as a collisional scattering from one global collisionless orbit to another, in a turbulent medium the collisionless orbits are quite different. In particular, the orbit projections for the boundary layer particles look like the turbulent smear of Fig. 1b. The displacement of the orbit relative to $\psi$ depends not on the $\nabla B$ drifts (and parallel velocity sign) but on the turbulent radial flows which are often larger. Within the layer a particle cannot retain its memory of a trapped or circulating status, but only the mixture of these properties. The turbulent orbit has an average surface, $\psi$, and this does not jump in a *correlated* way depending on $\sigma$. Therefore, collisional scattering of these orbits will not generate the cross processes and

$T^i_{13}$ and $T^i_{31}$ are eliminated. By a related argument, $T^{e,cir}_{13}$ is eliminated. The Ware effect, $T^{e,tr}_{13}$, is more robust. It arises from a secular accumulation of $\nabla B$ drifts due to poloidal rotation of the banana tips, is independent of radial memory along the orbit, and survives the turbulence. We end up with $T_{13} \hat{=} T^{e,tr}_{13}$ , $T_{31} \hat{=} 0$. In short, the circulating particle effects are sensitive to the detailed structure of the boundary layer. At realistic turbulence levels they will be eliminated, leaving the Ware effect and broken symmetry.

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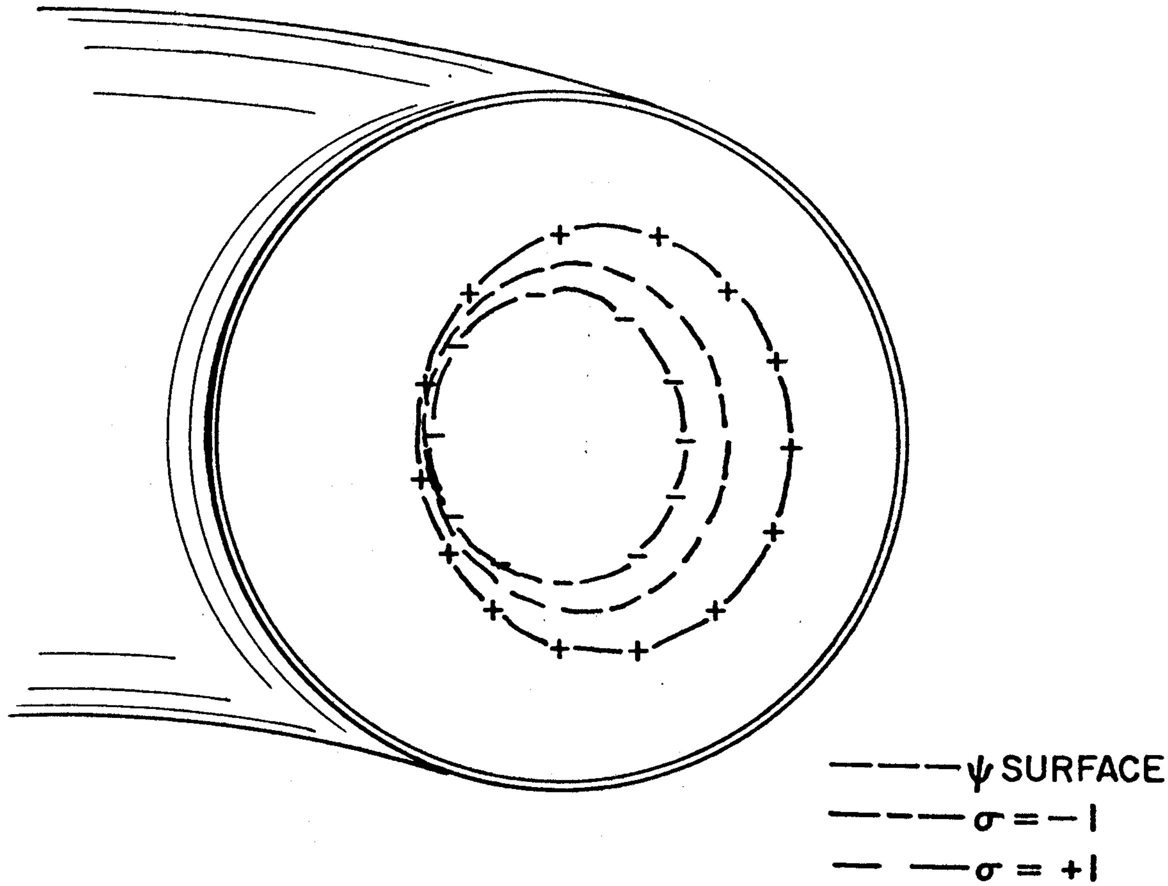
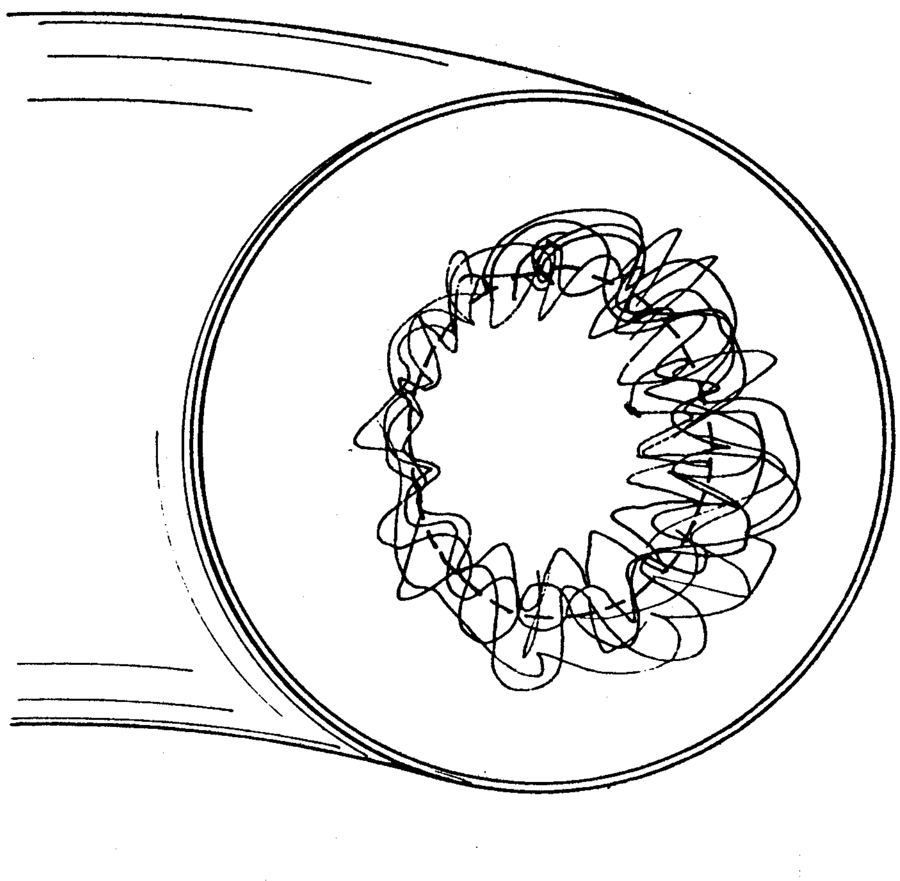
Fig. 1a. Representative orbit projections for circulating particles in quiescant tokamak. Well circulating particle orbits follow the $\psi$ surface very closely. Marginally circulating particles have very distorted orbits, resembling the inner (outer) half of a trapped particle banana orbit for the $\sigma = -1$ direction of parallel velocity. Average surfaces, $\langle\psi\rangle$, for marginally circulating particles are displaced inward from $\psi$ for $\sigma = -1$, and outward for $\sigma = +1$.

Fig. 1b. Boundary layer circulating particle orbits in the presence of turbulence. The orbits are not well defined, but correspond to a mixture of properties in a local region of phase space.




1a



1b