JA/83-5

NONLINEAR TRAVELLING-WAVE EQUILIBRIA FOR FREE ELECTRON LASER APPLICATIONS

Barton Lane and Ronald C. Davidson

Plasma Fusion Center

Massachusetts Institute of Technology

Cambridge, Massachusetts 02139

NONLINEAR TRAVELLING-WAVE EQUILIBRIA

FOR FREE ELECTRON LASER APPLICATIONS

Barton Lane and Ronald C. Davidson

Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

ABSTRACT

The class of large-amplitude travelling wave solutions to the nonlinear Vlasov-Maxwell equations are investigated in which the wave pattern is stationary in a frame of reference moving with the pondermotive phase velocity $v_p = \omega/(k + k_0)$. Here, $\lambda_0 = 2\pi/k_0$ is the wavelength of the transverse helical wiggler field, and (ω, k) are the frequency and wavenumber of the saturated radiation field which is assumed to be monochromatic and circularly polarized. The conservation of (average) density, momentum and energy are imposed as additional exact constraint equations that connect the final (saturated) and initial states of the combined electron beam-radiation field-wiggler field system. These constraint equations reduce the generality of the nonlinear equilibrium BGK solutions, and allow estimates to be made of the saturated field amplitude in terms of initial properties of the beam-wiggler system. As a simple example that is analytically tractable, we consider the case where the initial distribution $F_0(\gamma)$ and the saturated untrapped distribution $F_u(\gamma')$ are prescribed by rectangular distribution functions centered around axial velocity $v_z = \omega/(k + k_0)$, assuming a moderate field amplitude with $b_T = e\delta\hat{B}_T/mc^2k < 1$ and small fractional energy spread in the beam electrons. For a tenuous beam with $\omega \simeq kc$ and $k \simeq (1 + v_p/c)\,\gamma_p^2\,k_0$, where $\gamma_p = (1 - v_p^2/c^2)^{-1/2}$, it is found that the saturated amplitude of the radiation field is given approximately by

$$\delta\hat{B}_T = \frac{\Delta_L}{10(1 + b_w^2)^{1/2}}\;\frac{\omega_p^2}{c^2k_0^2}\left(1 + \frac{v_p}{c}\right)\hat{B}_w$$

where $b_w = e\hat{B}_w/mc^2k_0$, $\Delta_L mc^2$ is the characteristic half-width energy spread in the laborabory frame, and $\omega_p^2 = 4\pi n_b\, e^2/m$ is the nonrelativistic plasma frequency-squared.

1. INTRODUCTION AND SUMMARY

There have been several theoretical[1-5] and experimental[6-8] investigations of the free electron laser (FEL) which generates coherent electromagnetic radiation using an intense relativistic electron beam as an energy source. For beam propagation through a transverse helical wiggler field, there have been many theoretical estimates (e.g., Refs. 1-5) of the gain (growth rate) during the linear phase of instability. Few calculations[9-14], however, have addressed the nonlinear development and saturation of the instability. Particularly important for FEL applications is the development of a self-consistent theoretical model that estimates the saturated amplitude of the radiation field (and hence the overall efficiency of radiation generation) in terms of properties of the electron beam and the wiggler field.

In the present article, we investigate the class of large-amplitude travelling wave solutions to the nonlinear Vlasov-Maxwell equations in which the wave pattern is stationary in a frame of reference moving with the ponderomotive phase velocity [Eq. (15)]

$$v_p = \frac{\omega}{k+k_0}$$

Here, $\lambda_0 = 2\pi/k_0$ is the wavelength of the helical wiggler field [Eq. (1)], and (ω,k) are the frequency and wavenumber of the saturated radiation field. That is, in the final saturated state, the electron beam, helical wiggler field and radiation field are assumed to co-exist in a quasi-steady equilibrium, and the corresponding solutions to the Vlasov-Maxwell equations are determined self-consistently (Secs. 2-4). A very important feature of the present analysis is that the conservation of (average) density, momentum and energy are incorporated as additional exact constraint equations that connect the final (saturated) and initial states of the combined electron beam-radiation field-wiggler field system. These constraint equations reduce the generality of the nonlinear equilibrium solution, and allow estimates to be made of the saturated field amplitude (for example) in terms of initial properties of the beam-wiggler system.

As a general remark, the existence and properties of stationary, travelling-wave solutions to the nonlinear Vlasov-Poisson equations[15,16] have been extensively studied for electrostatic perturbations in the nonrelativistic regime. These solutions are referred to as nonlinear BGK (Bernstein-Greene-Kruskal) waves. In the relativistic, electromagnetic analysis presented here, we will make use of many of the techniques developed in the electrostatic case[15,16].

To briefly summarize specific assumptions, we consider a tenuous relativistic electron beam propagating in the z-direction through a transverse helical wiggler field with $\hat{B}_w$ = const. [Eq. (1)]. For simplicity, perpendicular spatial variations are omitted in the analysis ($\partial/\partial x = 0 = \partial/\partial y$), and the beam density and current are assumed to be sufficiently low that equilibrium electric and magnetic self fields are negligibly small. Moreover, consistent with the tenuous beam approximation, space-charge perturbations are neglected (Compton regime with $\delta\phi \simeq 0$). We examine the class of exact solutions to the fully nonlinear Vlasov-Maxwell equations of the form [Eqs. (19) and (31)]

$$f_b(z,\underset{\sim}{p},t) = n_b\delta(P_x)\delta(P_y)F(z-v_pt,\ p_z),$$

where n_b = const. is the density, P_x and P_y are the exact transverse canonical momenta [Eq. (3)], and the radiation field $(\delta\hat{B}_T,\omega,k)$ is prescribed by the constant-amplitude, circularly polarized waveform in Eq. (10).

Transforming to the pondermotive frame variables (z', p_z', t') defined in Eq. (26), it is found that the most general stationary solution $(\partial/\partial t' = 0)$ to the nonlinear Vlasov equation is given by [Eq. (38)]

$$F_<(\gamma') = F_>(\gamma') \equiv \frac{1}{2} F_T(\gamma')$$

for the trapped particles with $[1 + b_w^2 + b_T^2 - 2b_T b_w \cos k'z']^{1/2} < \gamma' < [1 + (b_w + b_T)^2]^{1/2}$, and by [Eq. (40)]

$$F_u(\gamma') = F_<(\gamma') + F_<(\gamma')$$

for the untrapped particles with $\gamma' > [1 + (b_w + b_T)^2]^{1/2}$. Here $H' = \gamma' mc^2$ is the particle energy in the pondermotive frame and k', b_T and b_w are defined by $k' = (k + k_0)/\gamma_p$, $b_T = e\delta\hat{B}_T/mc^2 k$ and $b_w = e\hat{B}_w/mc^2 k_0$, where $\gamma_p = (1 - v_p^2/c^2)^{-1/2}$, c is the speed of light in vacuo and m is the electron rest mass. Moreover, $F_>(F_<)$ refer to forward (backward) moving particles in the pondermotive frame with $p_z' > 0$ $(p_z' < 0)$. To summarize, with regard to Maxwell's equations, the final nonlinear BGK equilibrium equation that relates ω, k, b_T, b_w, $F_T(\gamma')$ and $F_u(\gamma')$ is given by Eq. (41), supplemented by the consistency conditions (42) and (46) which involve the "initial" distribution function $F_0(\gamma)$. As in the electrostatic BGK analysis[16], it is found from Eq. (41) that a specification of the untrapped distribution function $F_u(\gamma')$ is sufficient information to reconstruct in detail the trapped-particle distribution function $F_T(\gamma')$ [Eq. (48)]

$$F_T(\gamma') = \frac{2}{\pi} \int_{\gamma_+'}^{\infty} d\gamma'' \, F_u(\gamma'') \, \frac{\gamma'}{(\gamma''^2 - \gamma'^2)} \, \frac{(\gamma_+'^2 - \gamma'^2)^{1/2}}{(\gamma''^2 - \gamma_+'^2)^{1/2}} \, ,$$

where $\gamma_+' \equiv [1 + (b_w + b_T)^2]^{1/2}$.

To the extent summarized in the previous paragraph (corresponding to Secs. 2-4.A), since ω, k, $\delta\hat{B}_T$ and $F_u(\gamma')$ are yet unspecified and undetermined quantities, it is important to recognize the tremendous generality (and ambiguity) of the final saturated BGK state and its relationship to the "initial" (t=0) state in the absence of radiation field ($\delta A_x = 0 = \delta A_y$). To reduce the generality inherent in such a standard BGK analysis, in the remainder of Sec. 4 we make use of the exact conservation relations for (average) density, momentum and energy to provide three additional constraint equations that relate ω, k, $\delta\hat{B}_T$ and $F_u(\gamma')$ to the initial distribution function $F_0(\gamma)$ and the wiggler field(B_w,k_0). The conservation equations [Eqs. (55)-(57)] or alternate forms thereof [e.g., Eqs. (69), (71) and (74)] together with Eqs. (42), (46) and (48) then form the final results of this paper and can be used to investigate the properties of saturated FEL states for a broad class of initial distributions $F_0(\gamma)$ and final untrapped equilibria $F_u(\gamma')$.

As a simple example that is analytically tractable, in Sec. 5 we consider the case where $F_0(\gamma)$ [Eq. (82)] and $F_u(\gamma')$ [Eq. (83)] are prescribed by rectangular distribution functions centered around $v_z = \omega/(k + k_0)$, assuming $b_T b_w << 1$ and small fractional energy spread in the beam electrons. For a tenuous beam with $\omega \simeq kc$ and $k \simeq (1 + v_p/c)\gamma_p^2 k_0$, it is found that the saturated field is given approximately by [Eq. (95)]

$$\delta\hat{B}_T = \frac{\Delta_L}{10(1+b_w^2)^{1/2}} \frac{\omega_p^2}{c^2k_0^2} \left(1 + \frac{v_p}{c}\right) \hat{B}_w,$$

where $b_w = e\hat{B}_w/mc^2k_0$, $\Delta_L mc^2$ is the characteristic half-width energy spread in the laboratory frame, and $\omega_p^2 = 4\pi n_b e^2/m$ is the nonrelativistic plasma frequency-squared. Moreover, making use of Eq. (48), the trapped-particle distribution function $F_T(\gamma')$ is given by Eq. (98) for the choice of untrapped distribution function $F_u(\gamma')$ in Eq. (83). Equation (95) can of course be used to estimate the radiated power P_{RAD} [Eq. (96)] as well as the efficiency η of radiation generation [Eq. (97)] for the model choice of distribution functions in Eqs. (82) and (83).

One surprising conclusion in the present analysis is discussed at the end of Sec. 4.C. When the conservation equations are imposed as additional constraints relating the BGK solutions to the initial state, it is found that for $\delta\hat{B}_T \neq 0$ solutions to exist it is necessary that the initial beam distribution function $F_0(\gamma)$ have some degree of energy spread. That is, in the present analysis, an initially cold beam will not lead to acceptable final BGK states with $\delta\hat{B}_T \neq 0$. While at first this conclusion may seem surprising, we hasten to point out that the present analysis is restricted to a very narrow class of BGK equilibria in which it is assumed <u>a priori</u> that the saturated state corresponds for a monochromatic, circularly polarized radiation field. An alternate way to state the conclusion is that a constant-amplitude state in which the waveform is purely monochromatic and circularly polarized <u>is not accessible</u> from initial conditions with zero beam energy spread. Of course, from a practical point of view, the beam emittance in laboratory experiments is small but nonetheless finite.

The organization of this paper is the following. In Secs. 2 and 3, we outline the basic assumptions and equations that describe the self-consistent BGK equilibrium solutions for a relativistic electron beam propagating in combined helical wiggler and circularly polarized monochromatic radiation fields. The solution for the trapped-particle distribution function $F_T(\gamma')$ is derived in Sec. 4 in terms of the untrapped distribution $F_u(\gamma')$, and the conservation equations for average density, axial momentum and energy are expressed in a form useful for subsequent applications. In Sec. 5 we consider a specific example of a saturated BGK equilibrium in which the initial [Eq. (82)] and final untrapped [Eq. (83)] distribution functions have a simple rectangular form that permits straightforward analytic estimates of the trapped-particle distribution function [Eq. (98)] and the saturated radiation field amplitude $\delta\hat{B}_T$ [Eq. (95)].

2. THEORETICAL MODEL AND ASSUMPTIONS

The present analysis assumes a relativistic electron beam with uniform cross section propagating in the z direction. The beam density and current are assumed to be sufficiently small that the influence of equilibrium self electric and self magnetic fields on particle trajectories and stability behavior can be neglected, i.e., $\underset{\sim}{E}_s^0 = 0 = \underset{\sim}{B}_s^0$. Moreover, the electron beam propagates through an equilibrium helical wiggler magnetic field described by

$$\underset{\sim}{B}^0(\underset{\sim}{x}) = -\hat{B}_w \cos k_0 z \; \hat{\underset{\sim}{e}}_x + \hat{B}_w \sin k_0 z \; \hat{\underset{\sim}{e}}_y \ , \tag{1}$$

where $\hat{B}_w$ = const is the field amplitude, $\lambda_0 = 2\pi/k_0$ is the wiggler amplitude, and $\hat{\underset{\sim}{e}}_x$ and $\hat{\underset{\sim}{e}}_y$ are unit vectors in the plane perpendicular to the propagation direction. Strictly speaking, the approximate form of the wiggler field given in Eq. (1) is valid only near the magnetic axis, $k_0^2(x^2 + y^2) \ll 1$, which is the region considered in the present analysis.

Perturbations are considered in which the spatial variations are one-dimensional with $\partial/\partial x = 0 = \partial/\partial y$, and $\partial/\partial z$ generally non-zero. It is also assumed that the electron beam is sufficiently tenuous that the Compton-regime approximation is valid with negligibly small longitudinal fields ($\delta E_z = -\partial\,\delta\phi/\partial z = 0$). The transverse electromagnetic wave fields, $\delta\underset{\sim}{E}_T(\underset{\sim}{x}, t)$ and $\delta\underset{\sim}{B}_T(\underset{\sim}{x}, t)$, can be expressed in terms of the vector potential $\delta\underset{\sim}{A}(\underset{\sim}{x}, t)$ as

$$\delta\underset{\sim}{E}_T = -\frac{1}{c}\frac{\partial}{\partial t}\,\delta\underset{\sim}{A} \ , \qquad \delta\underset{\sim}{B}_T = \nabla \times \delta\underset{\sim}{A} \ , \tag{2}$$

where $\delta\underset{\sim}{A}(\underset{\sim}{x}, t) = \delta A_x(z, t)\,\hat{\underset{\sim}{e}}_x + \delta A_y(z, t)\,\hat{\underset{\sim}{e}}_y$. In the present geometry, there are two exact single-particle invariants in the combined wiggler and radiation fields. These are the canonical momenta, P_x and P_y, transverse to the beam propagation direction, i.e.[3],

$$P_x = p_x - \frac{e}{c} A_x^0(z) - \frac{e}{c}\delta A_x(z, t) = \text{const} ,$$

$$P_y = p_y - \frac{e}{c} A_y^0(z) - \frac{e}{c}\delta A_y(z, t) = \text{const} , \tag{3}$$

where $A_x^0(z) = -(\hat{B}_w/k_0)\cos k_0 z$ and $A_y^0(z) = (B_w/k_0)\sin k_0 z$ are the components of vector potential for the equilibrium wiggler field in Eq. (1). In Eq. (3), $-e$ is the electron charge, c is the speed of light in vacuo, and $p_x = \gamma m v_x$ and $p_y = \gamma m v_y$ are the transverse mechanical momenta.

For present purposes, we examine the class of exact solutions to the fully nonlinear Vlasov equation of the form[3,4]

$$f_b(z, \underset{\sim}{p}, t) = n_b\delta(P_x)\delta(P_y)F(z, p_z, t) , \tag{4}$$

where n_b = const is the density, and P_x and P_y are the exact invariants defined in Eq. (3). From Eq. (4), note that the effective transverse motion of the beam electrons is "cold". Substituting Eq. (4) into the Vlasov equation for $f_b(z, p_z, t)$ gives

$$\left\{\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} - \frac{\partial}{\partial z} H(z, p_z, t) \frac{\partial}{\partial p_z}\right\} F(z, p_z, t) = 0 \tag{5}$$

for the evolution of $F(z, p_z, t)$. Here, $H(z, p_z, t) = \gamma mc^2$ is the particle energy for $P_x = 0 = P_y$, where γ is defined by

$$\gamma = \gamma(z, p_z, t) \equiv \left[1 + \frac{p_z^2c^2}{m^2c^4} + \frac{e^2}{m^2c^4}(A_x^0 + \delta A_x)^2 + \frac{e^2}{m^2c^4}(A_y^0 + \delta A_y)^2\right]^{1/2} . \tag{6}$$

Moreover, $v_z = \partial H/\partial p_z = p_z/\gamma m$ is the axial electron velocity in Eq. (4).

Substituting Eq. (4) into the Maxwell equations for $\delta A_x(z,t)$ and $\delta A_y(z,t)$, the equations describing the fully nonlinear evolution of the vector potential perturbations can be expressed as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\delta A_x = -\frac{4\pi n_b e^2}{mc^2}\left[(A_x^0 + \delta A_x)\int\frac{dp_z}{\gamma}F - A_x^0\int\frac{dp_z}{\gamma_0}F_0\right] , \quad (7)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\delta A_y = -\frac{4\pi n_b e^2}{mc^2}\left[(A_y^0 + \delta A_y)\int\frac{dp_z}{\gamma}F - A_y^0\int\frac{dp_z}{\gamma_0}F_0\right] , \quad (8)$$

where $\gamma = \gamma(z,p_z,t)$ is defined in Eq. (6), $F_0 = F_0(p_z)$ is the unperturbed distribution function in the absence of radiation field ($\delta A_x = 0 = \delta A_y$), $F = F(z,p_z,t)$ solves the nonlinear Vlasov Eq. (5), and γ_0 is defined by

$$\gamma_0 = \gamma_0(p_z) \equiv \left[1 + \frac{p_z^2}{m^2c^2} + \frac{e^2\hat{B}_w^2}{m^2c^4k_0^2}\right]^{1/2} . \quad (9)$$

Note that $\gamma_0 mc^2$ is the particle energy for $\delta A_x = 0 = \delta A_y$ [Eq. (6)] and that use has been made of $(e^2/m^2c^4)(A_x^{02} + A_y^{02}) = e^2\hat{B}_w^2/m^2c^4k_0^2 = \text{const.}$ in obtaining Eq. (9). Equations (7) and (8) are exact within the context of the neglect of equilibrium self field effects and the approximate expression for the helical wiggler field given in Eq. (1). No assumption has been made that the radiation or wiggler field amplitudes are small.

3. MONOCHROMATIC TRAVELLING WAVE BGK EQUILIBRIA

A. Basic Equations

We now specialize to the case where the transverse electromagnetic fields are prescribed by the constant-amplitude circularly polarized waveform

$$\delta \underset{\sim}{B}_T(\underset{\sim}{x},t) = -\delta\hat{B}_T \cos(kz-\omega t)\underset{\sim}{\hat{e}}_x - \delta\hat{B}_T \sin(kz-\omega t)\underset{\sim}{\hat{e}}_y \quad , \tag{10}$$

where the wave magnetic field satisfies $\delta \underset{\sim}{B}_T = \nabla \times \delta\underset{\sim}{A}$ with

$$\delta\underset{\sim}{A}(\underset{\sim}{x},t) = \delta A_x(z,t)\underset{\sim}{\hat{e}}_x + \delta A_y(z,t)\underset{\sim}{\hat{e}}_y$$

$$= \frac{\delta\hat{B}_T}{k}\cos(kz-\omega t)\underset{\sim}{\hat{e}}_x + \frac{\delta\hat{B}_T}{k}\sin(kz-\omega t)\underset{\sim}{\hat{e}}_y \quad , \tag{11}$$

and the wave electric field is given by $\delta\underset{\sim}{E}_T = -(1/c)(\partial/\partial t)\delta\underset{\sim}{A}$. In Eqs. (10) and (11), the amplitude $\delta\hat{B}_T$ = const, and ω and k are the wave frequency and wavenumber, respectively. For $\delta\underset{\sim}{A}$ given by Eq. (11), it is straightforward to show that

$$(A_x^0 + \delta A_x)^2 + (A_y^0 + \delta A_y)^2 = \hat{B}_w^2/k_0^2 + \delta\hat{B}_T^2/k^2$$

$$- 2(\hat{B}_w/k_0)(\delta\hat{B}_T/k)\cos[(k+k_0)z - \omega t] \quad , \tag{12}$$

Therefore, the pondermotive force in Eq. (5) can be expressed as

$$F_z = - mc^2 \frac{\partial\gamma}{\partial z} = \frac{e^2\hat{B}_w \ \delta\hat{B}_T}{k_0 k\gamma mc^2} \frac{\partial}{\partial z} \cos \xi \quad , \tag{13}$$

where the axial coordinate ξ is defined by

$$\begin{aligned} \xi &= (k+k_0)z - \omega t \\ &= (k+k_0)(z - v_p t) \quad . \end{aligned} \tag{14}$$

In Eq. (14)

$$v_p = \frac{\omega}{k+k_0} \tag{15}$$

is the effective phase velocity of the pondermotive potential, i.e., the phase velocity of the beat wave produced by the combined wiggler and transverse electromagnetic wave fields. Moreover, from Eqs. (6) and (12), γ can be expressed as

$$\gamma = \left[1 + \frac{p_z^2}{m^2c^2} + b_w^2 + b_T^2 - 2b_w b_T \cos \xi\right]^{1/2} \quad , \tag{16}$$

where the abbreviated notation

$$b_w = \frac{e\hat{B}_w}{mc^2 k_0} \quad , \qquad b_T = \frac{e \ \delta\hat{B}_T}{mc^2 k} \tag{17}$$

has been introduced in Eq. (16). For future reference, making use of $p_z = \gamma m v_z$, it can be shown from Eq. (16) that v_z can be expressed in terms of γ and $\xi = (k+k_0)z - \omega t$ as

$$\frac{v_z^2}{c^2} = \frac{p_z^2}{\gamma^2 m^2 c^2} = 1 - \frac{1}{\gamma^2} \left[1 + b_w^2 + b_T^2 - 2b_w b_T \cos \xi\right] \quad , \tag{18}$$

where γ is defined in Eq. (16).

In the time-honored manner, having specified the precise waveform in Eqs. (10) and (11), we now make use of the nonlinear Vlasov-Maxwell equations (5), (7) and (8), to determine the corresponding self-consistent BGK equilibrium distribution

$$F = F\,(\xi, p_z) \tag{19}$$

where the explicit dependence on z and t occurs only through the combination $\xi = (k+k_0)z-\omega t \;= (k+k_0)(z-v_p t)$.

Substituting Eq. (19) into Eq. (5) gives

$$\left\{(v_z - v_p)\frac{\partial}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial}{\partial p_z}\right\} F(\xi, p_z) = 0 \quad , \tag{20}$$

where $v_p = \omega/(k+k_0)$, $H = \gamma mc^2$ is defined in Eq. (16), and use has been made of

$$\frac{\partial}{\partial t} = -\, v_p \frac{\partial}{\partial z} \quad , \tag{21}$$

which follows from $\xi = (k+k_0)z-\omega t$. Moreover, substituting Eqs. (11) and (19) into the Maxwell equations (7) and (8) and rearranging terms readily gives

$$\left[k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pb}^2}{c^2}\int \frac{dp_z}{\gamma} F(\xi, p_z)\right] (A_x^0 + \delta A_x) = \left[k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pb}^2}{c^2}\int \frac{dp_z}{\gamma_0} F_0(p_z)\right] A_x^0,$$

$$\left[k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pb}^2}{c^2}\int \frac{dp_z}{\gamma} F(\xi, p_z)\right] (A_y^0 + \delta A_y) = \left[k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pb}^2}{c^2}\int \frac{dp_z}{\gamma_o} F_0(p_z)\right] A_y^0, \tag{23}$$

where $A_x^0 = -(\hat{B}_w/k_0)\cos k_0 z$, $A_y^0 = (\hat{B}_w/k_0)\sin k_0 z$, δA_x and δA_y are defined in Eq. (11), $\omega_{pb}^2 = 4\pi n_b e^2/m$ is the nonrelativistic plasma frequency-squared, $\gamma(\xi,p_z)$ is defined in Eq. (16), and γ_0 is defined in Eq. (9). Given the sinusoidal dependence of A_x^0 and A_y^0 on $(k_0 z)$ and the sinusoidal dependence of δA_x and δA_y on $(kz-\omega t)$, it can be shown (Appendix A) that the square-bracket coefficients in Eq. (23) must separately be equal to zero, i.e.,

$$\omega^2 = c^2k^2 + \omega_{pb}^2\int\frac{dp_z}{\gamma_0(p_z)}F_0(p_z) \quad , \tag{24}$$

and

$$\omega^2 = c^2k^2 + \omega_{pb}^2\int\frac{dp_z}{\gamma(\xi,p_z)}F(\xi,p_z) \quad , \tag{25}$$

where $F(\xi,p_z)$ solves the Vlasov equation (20). In Eq. (24), $F_0(p_z)$ is the "initial" distribution function in the absence of radiation fields ($\delta A_x = 0 = \delta A_y$), and it is assumed that $F_0(p_z)$ is a known (specified) quantity with normalization $\int_{-\infty}^{\infty} dp_z\, F_0(p_z) = 1$. Note that Eq. (24) effectively plays the role of a "dispersion relation" relating the oscillation frequency ω and wavenumber k. On the other hand, Eq. (25) together with Eq. (20) play the role of constraint equations that determine the beam distribution function $F(\xi,p_z)$ given that the electromagnetic field has the circularly polarized waveform prescribed by Eqs. (10) and (11). Analogous to the electrostatic BGK case[16], we will find (Sec. 4) that if the distribution of untrapped beam electrons is specified, then Eqs. (20) and (25) can be used to calculate the corresponding self-consistent distribution of trapped beam electrons.

B. Lorentz Transformation to Pondermotive Frame

For purpose of investigating the nonlinear BGK solutions to the Vlasov equation (20) and the Maxwell equations (23)-(25), it is convenient to transform to the pondermotive frame of reference moving with the beat wave phase velocity $v_p = \omega/(k+k_0)$ relative to the laboratory frame. For present purposes, it is assumed that $k>0$ and $k_0>0$. Therefore, for a tenuous beam and $\omega \simeq kc$, it follows that $v_p<c$. We make the Lorentz transformation from laboratory frame variables (z,p_z,t) to pondermotive frame variables (z', p_z', t'), where

$$\begin{aligned} z' &= \gamma_p \, (z-v_p t) \quad , \\ p_z' &= \gamma_p \, (p_z - v_p \, H/c^2) = \gamma_p \gamma m \, (v_z - v_p) \quad , \\ t' &= \gamma_p \, (t - v_p z/c^2) \quad , \end{aligned} \tag{26}$$

and

$$\gamma_p = (1-v_p^2/c^2)^{-1/2} \tag{27}$$

Here, $v_z' = p_z'/\gamma' m = \partial H'/\partial p_z'$ is the axial velocity in the pondermotive frame, and $H' = \gamma' \, mc^2$ is the energy, where

$$\gamma' = \left[1 + \frac{p_z'^2}{m^2 c^2} + b^2(z') \right]^{1/2} \tag{28}$$

In Eq. (28),

$$b^2(z') = b_w^2 + b_T^2 - 2b_w b_T \cos k'z' \quad , \tag{29}$$

where k' is defined by

$$k' = (k+k_0)/\gamma_p \quad , \tag{30}$$

and b_w and b_T are the normalized magnetic field amplitudes defined in Eq. (17).

In primed variables, the nonlinear Vlasov equation (20) in the pondermotive frame can be expressed as

$$\left\{\frac{\partial H'}{\partial p_z'}\frac{\partial}{\partial z'} - \frac{\partial H'}{\partial z'}\frac{\partial}{\partial p_z'}\right\} F(z', p_z') = 0 \quad , \tag{31}$$

where $\partial H'/\partial p_z' = v_z' = p_z'/\gamma' m$, and $\gamma' = H'/mc^2$ is defined in Eq. (28).

It is clear that $F = F(\gamma')$ solves exactly the nonlinear Vlasov equation (31) in the pondermotive frame. Indeed, the most general solution to Eq. (31) can be expressed as

$$F = F_>(\gamma')\,\Theta(p_z') + F_<(\gamma')\,\Theta(-p_z') \quad , \tag{32}$$

where $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} +1 & , x \geq 0 \ , \\ 0 & , x < 0 \ . \end{cases} \tag{33}$$

The two functions, $F_>$ and $F_<$, correspond to the distribution functions for electrons with positive and negative momentum, $p_z' = \pm mc\,[\gamma'^2 - 1 - b^2(z')]^{1/2}$, in the pondermotive frame. From Eq. (29), the quantity $b^2(z')$ oscillates (spatially) between the maximum value

$$b_+^2 = (b_w + b_T)^2 = \left(\frac{e\hat{B}_w}{mc^2 k_0} + \frac{e\delta\hat{B}_T}{mc^2 k}\right)^2 \quad , \tag{34}$$

and the minimum value

$$b_-^2 = (b_w - b_T)^2 = \left(\frac{e\hat{B}_w}{mc^2 k_0} - \frac{e\delta\hat{B}_T}{mc^2 k}\right)^2 \quad . \tag{35}$$

Moreover, from Eqs. (28), (29) and (32), it is clear that there are two classes of particles: untrapped particles for which

$$\gamma' > (1 + b_+^2)^{1/2} \equiv \gamma_+' \quad , \tag{36}$$

and the trapped particles for which γ' lies in the range

$$\gamma_-'(z') \equiv [1+b^2(z')]^{1/2} < \gamma' < (1+b_+^2) \equiv \gamma_+' \quad . \tag{37}$$

When Eq. (37) is satisfied, the particle motion is periodic in the beat wave potential. When Eq. (36) is satisfied, the particle motion is also affected by the beat wave potential, but the direction of motion does not change, i.e., p_z' does not change sign. For the trapped particles, the density of particles with negative and positive momentum in the pondermotive frame must be identical. We therefore impose

$$F_<(\gamma') = F_>(\gamma') \equiv \frac{1}{2} F_T(\gamma') \quad , \text{ for}$$

$$\gamma_-'(z') < \gamma' < \gamma_+' \quad . \tag{38}$$

For the untrapped particles with $\gamma' > (1+b_+^2)^{1/2}$, however, $F_>(\gamma')$ and $F_<(\gamma')$ can be specified independently.

With regard to Eq. (25), which relates wave frequency ω, wavenumber k, the BGK equilibrium distribution $F(\gamma')$, and the normalized wave amplitude $b_T = e\delta\hat{B}_T/mc^2k$, we first note that the operator $c^{-2}\partial^2/\partial t^2 - \partial^2/\partial z^2$ is a Lorentz invariant, so that Eq. (25) remains valid when Eqs. (7) and (8) are transformed to the pondermotive frame. Moreover, making use of Eq. (26), it is found that $v_z' = dz'/dt' = (v_z - v_p) / (1 - v_p v_z/c^2)$ and $\gamma' = \gamma_p \gamma(1 - v_p v_z/c^2)$. After some straightforward algebra, it then follows that $dp_z'/\gamma' = (\gamma_p/\gamma')\,(1 - v_p v_z/c^2)dp_z = dp_z/\gamma$, and Eq. (25) can be expressed as

$$\omega^2 = c^2k^2 + \omega_{pb}'^{\,2}\int_0^\infty \frac{dp_z'}{\gamma'}\,[F_>(\gamma') + F_<(\gamma')] \quad , \tag{39}$$

where use has been made of $\gamma(-p_z') = \gamma'(p_z')$. In Eq. (39), the quantities $\omega_{pb}'^{\,2} \equiv 4\pi n_b' e^2/m$ and n_b' are the plasma frequency-squared and the beam density, respectively, in the pondermotive frame. For the untrapped electrons, it is useful to define the total distribution function

$$F_u(\gamma') = F_<(\gamma') + F_>(\gamma') \ , \text{ for } \gamma' > \gamma_+' \ . \tag{40}$$

Making use of $dp_z'/\gamma' = m^2c^2 d\gamma'/p_z' = mc\,d\gamma'/[\gamma'^2 - 1 - b^2(z')]^{1/2}$, and dividing the integral into untrapped and trapped particle contributions, Eq. (39) can be expressed as

$$\omega^2 = c^2k^2 + \omega_{pb}'^{\,2}\,mc\int_{\gamma_-'(z')}^{\gamma_+'} \frac{d\gamma'\ F_T(\gamma')}{[\gamma'^2 - 1 - b^2(z')]^{1/2}} + \omega_{pb}'^{\,2}\,mc\int_{\gamma_+'}^{\infty} \frac{d\gamma'\ F_u(\gamma')}{[\gamma'^2 - 1 - b^2(z')]^{1/2}} \tag{41}$$

where $b^2(z')$ is defined in Eq. (29), γ_+' is defined in Eq. (36), and $\gamma_-'(z')$ is defined in Eq. (37).

To summarize, Eq. (41) is the final nonlinear BGK equilibrium equation that relates wave frequency ω, wavenumber k, normalized wave amplitude $b_T = e\delta\hat{B}_T/mc^2k$, normalized wiggler amplitude $b_w = e\hat{B}_w/mc^2k_0$, and the trapped and untrapped electron distributions $F_T(\gamma')$ and $F_u(\gamma')$. Equation (41) of course must be supplemented by the constraint equation (24), which plays the role of a dispersion equation relating ω, k, $b_w = eB_w/mc^2k_0$, and the "unperturbed" or "initial" distribution function F_0 in the absence of radiation field ($\delta A_x = 0 = \delta A_y$). We define the initial distribution function in the pondermotive frame to be $F_0'(\gamma') = F_{0>}'(\gamma') + F_{0<}'(\gamma')$ where $F_0'(\gamma')$ is normalized to unity in the pondermotive frame, i.e., $F_0'(p_z') = (n_b/n_b')$ $F_0[p_z(p_z')]$, and p_z is given in terms of p_z' by Eq (26). With this normalization, since no beam particles are trapped by the wiggler field in the unperturbed state, Eq. (24) can be expressed as

$$\frac{(\omega^2 - c^2k^2)}{\omega_{pb}'^2} = mc \int_{\gamma_W}^{\infty} \frac{d\gamma\ F_0'(\gamma)}{(\gamma^2-1-b_W^2)^{1/2}} \quad , \tag{42}$$

where $\gamma_W \equiv (1+b_W^2)^{1/2}$.

Solutions to Eqs. (41) and (42) will be examined in detail in Sec. 4, where it is shown that a specification of the unperturbed distribution function $F_0(\gamma)$ and the untrapped particle distribution $F_u(\gamma')$ in the presence of the wave field is sufficient information to reconstruct the detailed form of the trapped particle distribution $F_T(\gamma')$. For future reference, it is convenient to rewrite Eq. (41) in the equivalent form

$$mc \int_{\gamma_-'(z')}^{\gamma_+'} \frac{d\gamma'\ F_T(\gamma')}{[\gamma'^2 - \gamma_-'^2(z')]^{1/2}} = G[\gamma_-'(z')] \quad , \tag{43}$$

where $\gamma_-'(z') = [1 + b^2(z')]^{1/2} = (1 + b_T^2 + b_w^2 - 2b_Tb_w \cos k'z')^{1/2}$, and G is defined by

$$G[\gamma_-'(z')] = \frac{(\omega^2-c^2k^2)}{\omega_{pb}'^2} - mc \int_{\gamma_+'}^{\infty} \frac{d\gamma'\ F_u(\gamma')}{[\gamma'^2 - \gamma_-'^2(z')]^{1/2}} \quad . \tag{44}$$

Equation (43) is an inhomogeneous integral equation for the trapped particle distribution $F_T(\gamma')$. The quantity $G[\gamma_-'(z')]$ is determined by specifying $F_u(\gamma')$ subject to the consistency condition [see Eq. (43)]

$$G(\gamma_+') = 0 \quad . \tag{45}$$

Making use of Eqs. (42) and (44), and eliminating $(\omega^2 - c^2k^2)/\omega'^2_{pb}$, the consistency condition (45) can be expressed as

$$\int_{\gamma'_+}^{\infty} \frac{d\gamma' \; F_u(\gamma')}{(\gamma'^2 - \gamma'^2_+)^{1/2}} = \int_{\gamma_w}^{\infty} \frac{d\gamma \; F'_0(\gamma)}{(\gamma^2 - \gamma_w^2)^{1/2}} \quad , \tag{46}$$

where $\gamma_w = (1 + b_w^2)^{1/2}$, and $\gamma'_+ = [1 + (b_w + b_T)]^{1/2}$.

4. DETERMINATION OF TRAPPED PARTICLE DISTRIBUTION FUNCTION

A. Solution to Integral Equation

Assuming that the untrapped distribution function $F_u(\gamma')$ is specified subject to the consistency condition (46), the integral equation (43) can be inverted to determine the trapped particle distribution function $F_T(\gamma')$ in terms of the known quantity G. Paralleling the electrostatic case, we find[16]

$$mc\ F_T(\gamma') = \frac{2}{\pi}\int_{\gamma'}^{\gamma'_+} d\gamma''\ \frac{\gamma'}{(\gamma''^2 - \gamma'^2)^{1/2}} \left[-\frac{d}{d\gamma''}\ G(\gamma'')\right] \quad , \tag{47}$$

which can be verified by direct substitution of Eq. (47) into Eq. (43). Substituting Eq. (44) for $G(\gamma'')$ into Eq. (47) and interchanging the order of integration, we find that the distribution of trapped electrons is given by

$$F_T(\gamma') = \frac{2}{\pi}\int_{\gamma'_+}^{\infty} d\gamma''\ F_u(\gamma'')\ \frac{\gamma'}{(\gamma''^2 - \gamma'^2)}\ \frac{(\gamma'^2_+ - \gamma'^2)^{1/2}}{(\gamma''^2 - \gamma'^2_+)^{1/2}} \quad , \tag{48}$$

where $\gamma'_+ = [1 + (b_w+b_T)^2]^{1/2}$, and $F_u(\gamma'')$ is related to the unperturbed distribution function $F_0(\gamma)$ by the consistency condition (46).

In Sec. 5, we consider a specific example in which the functional form of the untrapped distribution $F_u(\gamma'')$ is specified and Eq. (48) is solved to determine the corresponding distribution of trapped electrons $F_T(\gamma')$ that is consistent with Eq. (48) and the monochromatic, circularly polarized waveform assumed in Eqs. (10) and (11).

B. Conservation Relations

The BGK analysis in Secs. 3 and 4.A is "standard" in the sense that no attempt has been made to connect the initial (unperturbed) state and the final (saturated) BGK state. That is, the formalism contains no information on the detailed time development of the system or accessibility of the final BGK equilibrium state. In this section, for future reference, we make use of the fact that the fully nonlinear Vlasov-Maxwell equations (5) - (7) possess three exact global conservation relations, corresponding to conservation of (average) particle density, total axial momentum and total energy. These conservation relations constitute additional constraint equations that connect the initial unperturbed state characterized by the distribution function $F_0(p_z)$ and wiggler magnetic field $\underset{\sim}{B}^0(\underset{\sim}{x})$ in Eq. (1) to the final BGK state characterized by the distribution function in Eq. (32) and the combined equilibrium wiggler field and circularly polarized wave field in Eqs. (1) and (10). These additional constraint equations of course have the effect of further specifying the details of the final state, i.e., imposing further restrictions on ω, k, $\delta\hat{B}_T$, $F_u(\gamma')$, and $F_T(\gamma')$.

From the fully nonlinear Vlasov-Maxwell equations (5) - (8), it can be shown that the following quantities are exactly conserved (independent of time t as the system evolves): average density,

$$n_b \int_L \frac{dz}{L} \int_{-\infty}^{\infty} dp_z \; F(z,p_z,t) = \text{const}, \tag{49}$$

total average momentum,

$$\int_L \frac{dz}{L} \left\{ n_b \int_{-\infty}^{\infty} dp_z \; p_z \; F(z,p_z,t) + \frac{1}{4\pi c} \left(\delta\underset{\sim}{E}_T \times \underset{\sim}{B}_w + \delta\underset{\sim}{E}_T \times \delta\underset{\sim}{B}_T \right)_z \right\} = \text{const} \quad , \tag{50}$$

total average energy

$$\int_L \frac{dz}{L} \left\{ n_b \int_{-\infty}^{\infty} dp_z \left[m^2c^4 + c^2p_z{}^2 + e^2(A_x^0 + \delta A_x)^2 + e^2(A_y^0 + \delta A_y)^2 \right]^{1/2} F(z,p_z,t) \right.$$

$$\left. + \frac{1}{8\pi} \left[\delta \underset{\sim}{E}_T^2 + (\underset{\sim}{B}_w + \delta \underset{\sim}{B}_T)^2 \right] \right\} = \text{const.} \tag{51}$$

In Eqs. (49) - (51), $\delta \underset{\sim}{B}_T = \nabla \times \delta \underset{\sim}{A}$ and $\delta \underset{\sim}{E}_T = -(1/c)(\partial/\partial t)\delta \underset{\sim}{A}$. Moreover, $F(z,p_z,t)$, $\delta A_x(z,t)$ and $\delta A_y(z,t)$ evolve according to Eqs. (5) - (8), and $\int_L dz/L$ constitutes a spatial average over some basic periodicity length L.

For future reference, it is convenient to introduce the wave frequency and wavenumber, $\hat{\omega}$ and $\hat{k}$, in the pondermotive frame, i.e.,

$$\hat{\omega} = \gamma_p\ (\omega - kv_p)\quad ,$$

$$\hat{k} = \gamma_p\ (k - \omega v_p/c^2)\ , \tag{52}$$

where $\gamma_p = (1 - v_p^2/c^2)^{-1/2}$ and $v_p = \omega/(k+k_0)$ is the beat wave velocity. We also define the form function $\sigma(\theta')$ that occurs in the spatial average of the final BGK state,

$$\sigma(\theta') = \frac{2b_T b_w}{[1 + (b_T + b_w)^2]}\ (1 + \cos\theta')\quad , \tag{53}$$

where $b_w = e\hat{B}_w/mc^2k_0$ and $b_T = e\delta\hat{B}_T/mc^2k$. After some straightforward but tedious algebra (Appendix B), the three conservation equations (49) - (51) can be used to derive simple relations connecting the unperturbed state and the final BGK state. The algebraic manipulation of Eqs. (49) - (51) involves transformation of the integrands to the pondermotive frame, and spatial average over the basic periodicity wavelength $L = \lambda' = 2\pi/k'$, where

$k' = (k+k_0)/\gamma_p$. Making use of the normalization condition for the unperturbed state,

$$1 = \int_{-\infty}^{\infty} dp_z' \; F_0' \; (p_z') = mc \int_{\gamma_w}^{\infty} \frac{d\gamma \; \gamma F_0'(\gamma)}{(\gamma^2-1-b_w^2)^{1/2}} \quad , \tag{54}$$

where $F_0'(\gamma) \equiv F_{0>}'(\gamma) + F_{0<}'(\gamma)$, we find (Appendix B):

Conservation of Average Density:

$$n_b' - n_b' \int_0^{2\pi} \frac{d\theta'}{2\pi} \int_{\gamma_+'}^{\infty} d\gamma' \; \frac{mc \; F_u(\gamma')}{(\gamma'^2 - \gamma_+'^2)^{1/2}}$$

$$\times \left\{ \gamma' - \gamma_+' \; \sigma(\theta') \int_0^{2\pi} \frac{d\theta}{2\pi} \; \frac{\cos^2\theta}{\left[\gamma'/\gamma_+' + (1-\sigma(\theta') \; \cos^2\theta)^{1/2}\right]} \right\} = 0, \tag{55}$$

Conservation of Axial Momentum:

$$n_b' \; mc \int_{\gamma_w}^{\infty} d\gamma \; \gamma \; mc \; [F_{0>}(\gamma) - F_{0<}(\gamma)] - n_b' \; mc \int_{\gamma_+'}^{\infty} d\gamma'\gamma' \; mc$$

$$\times [F_{u>}(\gamma') - F_{u<}(\gamma')] = \frac{\hat{\delta B}_T^2}{4\pi} \frac{\hat{\omega}\hat{k}}{c^2k^2} \quad , \tag{56}$$

Conservation of Energy:

$$n_b' \; mc^2 \int_{\gamma_w}^{\infty} d\gamma \; \frac{\gamma^2 \; mc \; F_0'(\gamma)}{(\gamma^2-1-b_w^2)^{1/2}} - n_b' \; mc^2 \int_0^{2\pi} \frac{d\theta'}{2\pi} \int_{\gamma_+'}^{\infty} \frac{d\gamma' \; mc \; F_u \; (\gamma')}{(\gamma'^2-\gamma_+'^2)^{1/2}}$$

$$\times \left(\gamma'^2 - \frac{1}{2} \gamma_+'^2 \, \sigma(\theta') \right)$$

$$= \frac{\delta \hat{B}_T^2}{8\pi} \frac{(\hat{\omega}^2 + c^2\hat{k}^2)}{c^2 k^2} \quad . \tag{57}$$

In Eqs. (55) - (57), $\theta' = k'z'$, $\gamma_+' \equiv [1 + (b_w+b_T)^2]^{1/2}$, $\gamma_w \equiv (1+b_w^2)^{1/2}$, and $F_u(\gamma') \equiv F_{u>}(\gamma') + F_{u<}(\gamma')$. Moreover, $F_{u>}$ ($F_{u<}$) refers to that portion of the untrapped distribution function with $p_z' > 0$ ($p_z' < 0$) [Eq. (32)]. The quantity n_b', the beam density in the pondermotive frame, can be expressed in terms of the laboratory frame beam density (n_b) and the initial laboratory distribution function $F_0(p_z)$ as

$$n_b' = n_b \int dp_z' \, F_0 \, [p_z \, (p_z')], \tag{58}$$

where p_z is expressed in terms of p_z' according to Eq (26).

Of course, Eqs. (55) - (57) must be supplemented by the constraint equation (46) and Eq. (42). The main point is the following. The three conservation equations (55) - (57) have the effect of further specifying the details of the final BGK state, i.e., imposing additional restrictions on ω, k, $\delta\hat{B}_T$, $F_u(\gamma')$ and $F_T(\gamma')$ that self-consistently relate the final and initial states.

C. Reduction of Contraint and Conservation Equations

In this section we reduce the conservation equations (55) - (58) together with the constraint equation (46) and the dispersion relation (42) to a single equation by eliminating the wave frequency $\hat{\omega}$

and wavenumber $\hat{k}$.

Denoting the left-hand side of Eq. (57) by $n_b' mc^2(\Delta\varepsilon)_b$ and the left-hand side of Eq. (56) by $n_b' mc(\Delta p)_b$, where $(\Delta\varepsilon)_b$ and $(\Delta p)_b$ are dimensionless quantities, we first note that the conservation equations for energy [Eq. (57)] and momentum [Eq. (56)] can be expressed as

$$(\Delta\varepsilon)_b = \frac{1}{2}\frac{(\hat{\omega}^2 + \hat{k}^2c^2)}{c^2k^2}\frac{\omega_{cT}^2}{\omega_{pb}'^2} \tag{59}$$

$$(\Delta p)_b = \frac{\hat{\omega}\hat{k}c}{c^2k^2}\frac{\omega_{cT}^2}{\omega_{pb}'^2} \tag{60}$$

and the constraint equation (42) can be expressed as

$$\hat{\omega}^2 - \hat{k}^2c^2 = \omega_{pb}'^2\alpha\ , \tag{61}$$

where α is the dimensionless factor defined by

$$\alpha \equiv \int_{\gamma_w}^{\infty} d\gamma\ \frac{F_0'(\gamma)}{(\gamma^2 - \gamma_w^2)^{1/2}}\ , \tag{62}$$

and ω_{cT} and ω_{pb}' are defined by

$$\omega_{cT} = \frac{e\delta\hat{B}_T}{mc} \equiv ckb_T\ , \tag{63}$$

and

$$\omega_{pb}'^2 = \frac{4\pi n_b' e^2}{m}\ . \tag{64}$$

The quantities $\hat{\omega}$ and $\hat{k}$ can be eliminated from Eq. (59) - (61) to give

$$(\Delta\varepsilon)_b^2 - (\Delta p)_b^2 = \frac{1}{4}\frac{\omega_{cT}^2}{c^2k^2}\alpha^2\ . \tag{65}$$

We now introduce the renormalized untrapped distribution function $\bar{F}_u(\gamma')$ defined by

$$\bar{F}_u(\gamma') = \frac{mc}{\alpha} F_u(\gamma) \ . \tag{66}$$

For notational simplicity the integral operator

$$K\{A\} = \int_0^{2\pi} \frac{d\xi}{2\pi} \int_1^{\infty} db \, \frac{\bar{F}_u(b\gamma'_+)}{(b^2 - 1)^{1/2}} A(b,\xi) \tag{67}$$

is introduced, where $\xi \equiv k'z'$ and $b \equiv \gamma'/\gamma'_+$. The constraint equations (46) and (42) then become

$$K\{1\} = 1. \tag{68}$$

Making use of Eqs. (67) and (68), the combined constraint equation (65) can now be expressed as

$$(\Delta\hat{\varepsilon})_b^2 - (\Delta\hat{p})_b^2 = \frac{1}{4}\left(\frac{e\delta\hat{B}_T}{\gamma'_+ mc}\right)^4 \frac{1}{c^4 k^4} \ , \tag{69}$$

where $(\Delta\hat{\varepsilon})_b$ and $(\Delta\hat{p})_b$ are defined by

$$(\Delta\hat{\varepsilon})_b \equiv \left(\frac{1}{\gamma'_+\alpha}\right)^2 \beta - K\left\{b^2 - \frac{\sigma}{2}\right\} \ ,$$

$$(\Delta\hat{p})_b \equiv \left(\frac{1}{\gamma'_+\alpha}\right) \int_{-\infty}^{+\infty} dp'_z \, \frac{p'_z c}{mc^2 \gamma'_+} F'_0 - \int_0^{2\pi} \frac{d\xi}{2\pi} \int_1^{\infty} db \; b[\bar{F}_{u>}(b\,\gamma'_+) - \bar{F}_{u<}(b\gamma'_+)] \ , \tag{70}$$

and the factor $(\gamma'_+\alpha)^{-1}$ is given by the particle number conservation equation [Eq. (55)]

$$\left(\frac{1}{\gamma'_+\alpha}\right) = K\left\{b - \sigma \, \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \, \frac{\cos^2\theta}{b + (1 - \sigma\cos^2\theta)^{1/2}}\right\} \ , \tag{71}$$

and α is related to the initial distribution function by Eq. (62).

In Eq. (70), the quantity β is defined by

$$\beta = \int_{\gamma_w}^{\infty} d\gamma \, \frac{\gamma^2 mcF_0'(\gamma)}{(\gamma^2 - \gamma_w^2)^{1/2}} \int_{\gamma_w}^{\infty} d\gamma \, \frac{mcF_0'(\gamma)}{(\gamma^2 - \gamma_w^2)^{1/2}} \, , \tag{72}$$

and F_0', the initial distribution function in the pondermotive frame is normalized to unity, i.e.,

$$\int_{\gamma_w}^{\infty} d\gamma \, \frac{\gamma mcF_0'(\gamma)}{(\gamma^2 - \gamma_w^2)^{1/2}} = 1 \, . \tag{73}$$

Note that if $F_0'(\gamma)$ is a delta function in energy, i.e., $F_0'(\gamma) = [(\bar{\gamma}^2 - \gamma_w^2)^{1/2}/(\bar{\gamma}mc)]\delta(\gamma-\bar{\gamma})$, then $\beta = 1$ follows from Eq. (72).

It is informative to explicitly substitute the particle number conservation equation (71) into the expression for the normalized particle energy loss $(\Delta\hat{\varepsilon})_b$ given by Eq. (70). After some straightforward algebraic manipulation, we obtain

$$\begin{aligned}
(\Delta\hat{\varepsilon})_b = & \left(\beta K\{b\}^2 - K\{b^2\}\right) \\
& - \left[\beta K\{b\} K\left\{\frac{\sigma}{2b}\right\} - K\left\{\frac{\sigma}{2}\right\}\right] \\
& -\beta \left[K\{b\} K\left\{\frac{1}{b}\,\sigma^2\,\frac{2}{\pi}\int_0^{\pi/2} d\theta \, \frac{\cos^4\theta}{[b+(1-\sigma\cos^2\theta)^{1/2}]^2}\right\}\right. \\
& \left. -K\left\{\frac{1}{b}\,\frac{2}{\pi}\,\sigma b\int_0^{\pi/2} d\theta \, \frac{\cos^2\theta}{b+(1-\sigma\cos^2\theta)^{1/2}}\right\}^2\right] \\
& -\beta K\{b\} K\left\{\frac{\sigma(b^2-1)}{b}\,\frac{2}{\pi}\int_0^{\pi/2} d\theta \, \frac{\cos^2\theta}{[b+(1-\sigma\cos^2\theta)^{1/2}]^2}\right\} .
\end{aligned} \tag{74}$$

From Eq. (69), for physically acceptable solutions, $(\Delta\hat{\varepsilon})_b$ must be positive. By use of the Schwartz inequality, however, it can be shown that the expressions within the square brackets in Eq. (74) are positive. Thus, for physically acceptable solutions to exist, it is necessary for the first term in curved brackets on the right-hand side of Eq. (74) to be positive. This leads to the somewhat surprising conclusion that the quantity β defined in Eq. (72) must be larger than unity. That is, an initially cold beam equilibrium will not lead to acceptable final BGK states with $\delta B_T \neq 0$.

By examining the expression for β in Eq. (72), it is clear that β will exceed unity for initial distribution functions with sufficiently broad energy spread $\Delta\gamma_0$. Moreover, it is observed experimentally that the energy spread $\Delta\gamma_f$ in the final distribution function exceeds that in the initial distribution function. In Sec. 5, we will consider a model distribution function with finite initial energy width and obtain an estimate for the amplitude $\delta\hat{B}_T$ of the saturated radiation field.

D. General Properties of Trapped Particle BGK Solutions

In this section, we discuss the general properties of the trapped particle BGK solutions given in Eq. (48) and consider a specific example in which a shifted Gaussian is chosen for the untrapped particle distribution function. In Eq. (48), the trapped particle distribution function $F_T(\gamma')$ is expressed in terms of an integral over the untrapped distribution function $F_u(\gamma')$. In analyzing the properties of $F_T(\gamma')$, it is convenient to change variables of integration in Eq. (48). Defining

$$x' = \frac{(\gamma''^2 - \gamma_+'^2)^{1/2}}{(\gamma_+'^2 - \gamma'^2)^{1/2}} , \tag{74}$$

and

$$a' = \left(1 - \frac{\gamma'^2}{\gamma_+'^2}\right)^{1/2} , \tag{75}$$

Eq. (48) can be expressed as

$$F_T(\gamma') = \frac{2}{\pi}\frac{\gamma'}{\gamma_+'}\int_0^\infty \frac{dx' F_u\left\{\gamma_+'[1+(a'x')^2]^{1/2}\right\}}{[1+(a'x')^2]^{1/2}(1+x'^2)} . \tag{76}$$

Since the integrand in Eq. (76) is positive definite, bounded, and vanishes as $x' \to \infty$, we conclude that the trapped distribution function is also positive and bounded for untrapped distribution functions which are normalizable. At the boundary between the trapped and untrapped particles, the trapped distribution function $F_T(\gamma')$ joins continuously to the untrapped distribution function $F_u(\gamma')$. This follows directly by setting $\gamma' = \gamma_+'$ and $a' = 0$ in Eq. (76), which yields

$$F_T(\gamma_+') = F_u(\gamma_+'). \tag{77}$$

If we differentiate Eq. (48) with respect to γ', integrate by parts, and simplify the integrand, we obtain an expression for the derivative of $F_T(\gamma')$ in terms of an integral over the derivative of the untrapped particle distribution function, i.e.,

$$\frac{dF_T(\gamma')}{d\gamma'} = -\frac{1}{(\gamma_+'^2 - \gamma'^2)^{1/2}}\frac{2}{\pi}\int_{\gamma_+'}^\infty d\gamma'' \frac{dF_u(\gamma'')}{d\gamma''}\frac{\gamma''(\gamma''^2 - \gamma_+'^2)^{1/2}}{(\gamma''^2 - \gamma'^2)} . \tag{78}$$

If $dF_u(\gamma'')/d\gamma''$ is everywhere negative and bounded, the integral is finite, but the factor $-(\gamma_+'^2 - \gamma'^2)^{-1/2}$ will force the derivative of $F_T(\gamma')$ to be positive and infinite as $\gamma' \to \gamma_+'$. Thus the derivative of $F_T(\gamma')$ is not necessarily continuous with the derivative of $F_u(\gamma')$ at the boundary between trapped and untrapped particles. Of course, the continuity of the derivative of the distribution function across this boundary could be imposed as an additional constraint on the functional form of the untrapped distribution function.

As a concrete example, we consider the case where the untrapped distribution function is a shifted Gaussian,

$$F_u^{\pm}(\gamma') = A\exp\{-[(\gamma'^2 - \gamma_+'^2)^{1/2} - \gamma_s^{\pm}]^2/\Delta_{\pm}^2\} \quad . \tag{79}$$

Here, $F_u^+(\gamma')$ $[F_u^-(\gamma')]$ is the untrapped distribution function for positive (negative) velocities in the pondermotive frame, and γ_s^+ and γ_s^- are the respective shifts for the two distribution functions. The constant A is determined by the consistency condition (46) in terms of the initial distribution function $F_0'(\gamma)$. Substituting Eq. (79) into Eq. (76) gives

$$F_T(\gamma') = \frac{\gamma'}{\gamma_+'}\frac{2}{\pi}\int_0^{\infty} dx' \, \frac{\{\exp[-(a'x'-\hat{\gamma}_s^+)^2/\hat{\Delta}_+^2] + \exp[-(a'x'-\hat{\gamma}_s^-)^2/\hat{\Delta}_-^2]\}}{[1+(a'x')^2]^{1/2}(1+x'^2)} \quad , \tag{80}$$

where $a' \equiv [1 - (\gamma'/\gamma_+')^2]^{1/2}$, $\hat{\gamma}_s^{\pm} = \gamma_s^{\pm}/\gamma_+'$ and $\hat{\Delta}\pm = \Delta\pm/\gamma_+'$. The numerical evaluation of the integral in Eq. (80) is straightforward except in the boundary region as $\gamma' \to \gamma_+'$, where the exponential decays on a length scale of order $\hat{\Delta}/a' >> 1$, and where the factor $(1+x'^2)^{-1}$ decays on a length scale of order 1. By exploiting these separate length scales we can, however, obtain an asymptotic expression for the integral in

Eq. (80). It can be shown that the functional form for the trapped distribution function $F_T(\gamma')$ near the boundary of the trapped and untrapped particles ($\gamma' \simeq \gamma'_+$) is given by

$$F_T(\gamma' \to \gamma'_+) \simeq \text{const.} - \frac{2}{\pi}\frac{\gamma'}{\gamma'_+}\, 2\left(\frac{\hat{\gamma}_s^+}{\hat{\Delta}_+^2} + \frac{\hat{\gamma}_s^-}{\hat{\Delta}_-^2}\right) a'\ln a' + \text{const.} \times a' \ . \tag{81}$$

For purpose of illustration, the shape of distribution function is shown in Fig. 1 assuming that $\hat{\gamma}_s^+ = 0.1$ and $\hat{\Delta}_+ = 0.05$.

5. EXAMPLE OF SATURATED BGK EQUILIBRIUM STATE

A. Constraint Equations for a Model Distribution Function

In order to illustrate the nature of the constraints imposed by the conservation equations, we consider a simple model for the initial distribution function and the final untrapped distribution function. As an example, the initial distribution function in the pondermotive frame is assumed to be given by

$$F_0'(\gamma') = A[\Theta(p_z')\Theta(\Delta_R + \gamma_w - \gamma') + \Theta(-p_z')\Theta(\Delta_L + \gamma_w - \gamma')] \ , \tag{82}$$

where $\gamma_w = (1+b_w^2)^{1/2}$, and an energy skew proportional to $\Delta_R - \Delta_L \equiv \delta$ is allowed in Eq. (82). Moreover, the constant A is determined from the normalization condition in Eq. (73). For simplicity, it is assumed that the skew is small with $\delta << (\Delta_R+\Delta_L)/2 \equiv \Delta_0$.

Similarly, it is assumed that the renormalized untrapped distribution function in the final BGK state is given by

$$\bar{F}_u(\gamma') = B\Theta(\Delta' + \gamma_+' - \gamma')[\Theta(p_z') + \Theta(-p_z')] \ , \tag{83}$$

where $\gamma_+' = [1+(b_w+b_T)^2]^{1/2}$, the quantity $\bar{F}_u(\gamma')$ is taken to be symmetric about $p_z' = 0$ in the pondermotive frame, and the constant B is determined by imposing the consistency condition in Eq. (68). Finally, it is assumed that the amplitude of the radiation field is small, specifically, $\hat{\sigma} \equiv 2b_T b_w/[1+(b_w+b_T)^2] << 1$, where $b_T = e\delta\hat{B}_T/mc^2k$ and $b_w = e\hat{B}_w/mc^2k_0$ [Eq. (53)].

We first impose the particle conservation constraint [Eq. (71)] in order to determine a relation between the characteristic energy widths of the initial and final distribution functions. To lowest order in $\hat{\sigma}$, Eq. (71) gives

$$\frac{1}{\gamma'_+\alpha} = K\{b\} - \frac{\hat{\sigma}}{2} K\left\{\frac{1}{b+1}\right\} \quad , \tag{84}$$

where $b=\gamma'/\gamma'_+$, and α is defined in Eq. (62). Substituting Eq. (82) into Eq. (62) and Eq. (83) into the operator expression (67) for K{ }, and assuming that $\Delta_0, \Delta' << 1$, we evaluate the integrals in Eq. (84) to first order in Δ_0 and Δ'. This gives the approximate result

$$\frac{\gamma_w}{\gamma'_+}\left[1 + \frac{1}{3}\left(\frac{\Delta_0}{\gamma_w}\right)\right] = 1 + \frac{1}{3}\left(\frac{\Delta'}{\gamma'_+}\right) - \frac{1}{4}\hat{\sigma} \; , \tag{85}$$

or to lowest order in b_T,

$$\left(\frac{\Delta_0}{\gamma_w} - \frac{\Delta'}{\gamma'_+}\right) = \frac{3}{2}\,\frac{b_T b_w}{[1+(b_w+b_T)^2]} \; , \tag{86}$$

where use has been made of $\gamma_w = (1+b_w^2)^{1/2}$, $\gamma'_+ = [1+(b_T+b_w)^2]^{1/2}$ and $\hat{\sigma} = 2b_T b_w/[1+(b_w+b_T)^2]$.

We now make use of Eq. (74) to evaluate $(\Delta\hat{\varepsilon})_b$, which is proportional to the energy lost by the particles to the radiation field. The first term in Eq. (70) can be rewritten using the definition of β [Eq. (72)], the normalization condition on $F'_0(\gamma)$ [Eq. (73)] and the definition of the integral operator K{ } in Eq. (67). This gives

$$\beta K\{b^2\} - K\{b^2\} = \frac{K_0\{b^2\}K\{b\}^2 - K\{b^2\}K_0\{b\}^2}{K_0\{b\}^2} \; , \tag{87}$$

where

$$K_0\{A(b)\} = \left[\int_1^\infty db\, \frac{F'_0(b\gamma_0)A(b)}{(b^2-1)^{1/2}}\right] \times \left[\int_1^\infty \frac{db F'_0(b\gamma_0)}{(b^2-1)^{1/2}}\right]^{-1} \; . \tag{88}$$

Substituting the explicit expressions for $F'_0(\gamma)$ and $F_u(\gamma')$ into the integral operators K{ } and K_0\{ \}, expanding in powers of Δ_0 and Δ', and neglecting terms proportional to δ yields the approximate result

$$\beta K\{b\}^2 - K\{b^2\} = \frac{4}{45}\left[\left(\frac{\Delta_0}{\gamma_w}\right)^2 - \left(\frac{\Delta'}{\gamma'_+}\right)^2\right]$$
$$= \frac{4}{45}\left(\frac{\Delta_0}{\gamma_w} - \frac{\Delta'}{\gamma'_+}\right)\left(\frac{\Delta_0}{\gamma_w} + \frac{\Delta'}{\gamma'_+}\right) . \tag{89}$$

Making use of Eq. (86), it follows that Eq. (89) can be expressed as

$$\beta K\{b\}^2 - K\{b^2\} = \frac{2}{15}\left(\frac{\Delta_0}{\gamma_w} + \frac{\Delta'}{\gamma'_+}\right) \frac{b_T b_w}{[1+(b_T+b_w)^2]} . \tag{90}$$

The largest remaining term in Eq. (74) is the final term, which can be approximated by

$$\beta K\{b\}K\left\{\sigma\,\frac{(b^2-1)}{b}\,\frac{2}{\pi}\int_0^{\pi/2} d\theta\,\frac{\cos^2\theta}{[b+(1-\cos^2\theta)^{1/2}]^2}\right\}$$
$$= \frac{1}{12}\,\hat{\sigma}\,\frac{\Delta'}{\gamma'_+} . \tag{91}$$

The second term in square brackets in Eq. (74) is proportional to $\hat{\sigma}\Delta'^2$ and the third term in square brackets is proportional to $\hat{\sigma}^2\Delta$. We therefore neglect these terms to lowest order. Combining Eq. (90) and Eq. (91) then gives the approximate result

$$(\Delta\hat{\varepsilon})_b = \left[\frac{1}{10}\left(\frac{\Delta_0}{\gamma_w}\right)\right] \frac{b_T b_w}{[1+(b_w+b_T)^2]} , \tag{92}$$

where $\gamma_w = (1+b_w^2)^{1/2}$, $b_T = e\hat{\delta}B_T/mc^2k$ and $b_w = e\hat{B}_w/mc^2k_0$.

An alternate expression can be obtained for $(\Delta\hat{\varepsilon})_b$ directly from Eqs. (57) and (59) assuming that $\hat{\omega} = \gamma_p(\omega - kv_p)$ and $\hat{k} = \gamma_p(k - \omega v_p/c)$ are known quantities. Equation (59) can be expressed in the equivalent form

$$(\Delta\hat{\varepsilon})_b = \frac{1}{2\alpha\gamma_+'^2}\left(\frac{e\delta\hat{B}_T}{kmc^2}\right)^2 \frac{(\hat{\omega}^2+c^2\hat{k}^2)\gamma_p}{\omega_p^2} , \tag{93}$$

where $\gamma'_+ = [1+(b_w+b_T)^2]^{1/2}$, $\omega_p^2 = 4\pi n_b e^2/m$, $\gamma_p = (1-v_p^2/c^2)^{-1/2}$, and α is defined in Eq. (62). In the limit of a tenuous beam, we now approximate $\omega \simeq kc$ so that $\hat{\omega} = \gamma_p(1-v_p/c)kc$ and $\hat{k} = \gamma_p(1-v_p/c)k$. Combining Eqs. (92) and (93) then gives

$$\delta\hat{B}_T = \frac{\Delta_0}{10\gamma_w} \frac{\omega_p^2}{\gamma_p c^2 \hat{k}^2} \frac{k}{k_0} \hat{B}_w ,$$

which reduces to

$$\delta\hat{B}_T = \frac{\Delta_0}{10\gamma_w} \frac{\omega_p^2}{\gamma_p c^2 k_0^2} \frac{k_0}{k} \gamma_p^2 (1+v_p/c)^2 \hat{B}_w , \tag{94}$$

where $\gamma_w \equiv (1+b_w^2)^{1/2}$ and use has been made of $\hat{k}^2 = \gamma_p^2(1-v_p/c)^2k^2 = k^2/[\gamma_p^2(1+v_p/c)^2]$ and $\gamma_p = (1-v_p^2/c^2)^{-1/2}$. In the tenuous beam limit, if we further assume that the simultaneous resonance conditions $\omega - kV_b = k_0V_b$ and $\omega = kc$ are satisfied, where $V_b \simeq \omega/(k+k_0) = v_p$ is the axial velocity of the electron beam, then the wavenumber k of the radiation field is given approximately by $k = k_0/(1-v_p/c) = \gamma_p^2(1+v_p/c)k_0$. Using this value of k in Eq. (94) then gives as the estimate for $\delta\hat{B}_T$,

$$\delta\hat{B}_T = \frac{\Delta_L}{10\,(1+b_w^2)^{1/2}} \frac{\omega_p^2}{c^2k_0^2} \left(1 + \frac{v_p}{c}\right) \hat{B}_w , \tag{95}$$

where $\Delta_L mc^2 \equiv (\Delta_0 mc^2)/\gamma_p$ is the half-width energy spread in the laboratory frame, $\omega_p^2 = 4\pi n_b e^2/m$ is the nonrelativistic plasma frequency-squared, and $b_w = e\hat{B}_w/mc^2k_0$ is the normalized wiggler amplitude.

In obtaining the estimate for $\delta\hat{B}_T$ in Eq. (95), it is important to note that we have chosen to specify an approximate value for ω ($\simeq kc$) in the tenuous beam limit rather than directly impose the momentum conservation constraint in Eqs. (56) and (60). Within the context of the assumptions that $\hat{\sigma}$, Δ_0, $\Delta' \ll 1$, it can be shown from the momentum conservation equation that the skew δ which is required to supply momentum to the radiation field is much smaller than Δ_0 or Δ'. The skew δ therefore does not enter into the energy conservation equations that we have used to estimate the amplitude $\delta\hat{B}_T$ of the radiation field.

Keeping in mind the simple rectangular models for the initial [Eq. (82)] and final untrapped [Eq. (83)] distribution functions, and the assumption of a tenuous beam with $\omega \simeq kc$, the expression for the field amplitude $\delta\hat{B}_T$ in Eq. (95) provides a very important estimate of the saturated level of FEL radiation expressed in terms of initial beam parameters (Δ_L, n_b, and $V_b \simeq v_p$) and properties of the wiggler field ($\hat{B}_w$ and k_0).

To conclude this section, it is of considerable practical interest to make use of Eq. (95) to estimate the efficiency η of conversion of beam energy to radiation energy. Defining η as the ratio of average electromagnetic field energy in the saturated state to beam kinetic energy, we find

$$\begin{aligned}\eta &= \left(\frac{\delta\hat{B}_T^2}{8\pi} + \frac{\delta\hat{E}_T^2}{8\pi}\right) / [Fn_b(\gamma_p-1)mc^2] \\ &= \left(\frac{\Delta_L(1+v_p/c)}{10(1+b_w^2)^{1/2}}\,\frac{\omega_p^2}{c^2k_0^2}\right)^2 \frac{\hat{B}_w^2}{4\pi Fn_b(\gamma_p-1)mc^2} .\end{aligned} \tag{97}$$

In Eq. (97), we have introduced a phenomenological geometric filling factor F which is related to the ratio of electron beam cross-sectional area to the effective cross-sectional area of the radiation field ($F = R_b^2/R_w^2$ for the simple model assumed in the previous paragraph).

B. Characteristics of the Trapped-Particle Equilibrium for Model Distribution Function

It is informative to examine the characteristics of the trapped-particle equilibrium assuming the untrapped distribution function given in Eq. (83). Making use of Eq. (48), we obtain

$$\bar{F}_T(\gamma') = B\,\frac{1}{\pi}\left\{\arcsin\left[\frac{\gamma'}{\gamma'_+} - \frac{\gamma_+'^2 - \gamma'^2}{\gamma'_+(\gamma'_+ + \Delta' - \gamma')}\right] + \arcsin\left[\frac{\gamma'}{\gamma'_+} + \frac{\gamma_+'^2 - \gamma'^2}{\gamma'_+(\gamma'_+ + \Delta' - \gamma')}\right]\right\}. \tag{98}$$

Note that $\bar{F}_T(\gamma'_+) = B = \bar{F}_u(\gamma'_+)$ for $\gamma' = \gamma'_+$. From Eq. (78), the derivative of $\bar{F}_T(\gamma'_+)$ is given by

$$\frac{d\bar{F}_T}{d\gamma'} = \frac{1}{(\gamma_+'^2 - \gamma'^2)^{1/2}}\,\frac{2}{\pi}\left\{\frac{(\gamma'_+ + \Delta')[(\gamma'_+ + \Delta')^2 - \gamma_+'^2]^{1/2}}{[(\gamma'_+ + \Delta')^2 - \gamma'^2]^{1/2}}\right\}. \tag{99}$$

Evidently, as $\gamma' \to \gamma'_+$,

$$\frac{d\bar{F}_T(\gamma')}{d\gamma'} \to \frac{1}{(\gamma_+'^2 - \gamma'^2)^{1/2}}\,\frac{2}{\pi}\,(\gamma'_+ + \Delta')\ , \tag{100}$$

and $\bar{F}_T(\gamma')$ has an infinite slope at the boundary between the trapped and untrapped particles. As before, we find that the derivative of the trapped-particle distribution function is discontinuous at the boundary for the choice of $\bar{F}_u(\gamma')$ in Eq. (83).

6. CONCLUSIONS

In the present article, we have investigated the class of large-amplitude travelling wave solutions to the nonlinear Vlasov-Maxwell equations in which the wave pattern is stationary in a frame of reference moving with the pondermotive phase velocity $v_p = \omega/(k+k_0)$. Here, $\lambda_0 = 2\pi/k_0$ is the wavelength of the helical wiggler field and (ω,k) are the frequency and wavenumber of the saturated radiation field. That is, in the final saturated state, the electron beam, helical wiggler field and radiation field co-exist in a quasi-steady equilibrium, and the corresponding solutions to the Vlasov-Maxwell equations have been determined self-consistently (Secs. 2-4). A very important feature of the present analysis is that the conservation of (average) density, momentum and energy are incorporated as additional exact constraint equations that connect the final (saturated) and initial states of the combined electron beam-radiation field-wiggler field system. These constraint equations reduce the generality of the nonlinear equilibrium solutions, and allow estimates to be made of the saturated field amplitude (for example) in terms of initial properties of the beam-wiggler system.

As a simple example that is analytically tractable, in Sec. 5 we considered the case where the initial beam distribution $F_0(\gamma)$ [Eq. (82)] and the final untrapped equilibrium $F_u(\gamma')$ [Eq. (83)] are prescribed by rectangular distribution functions centered around $v_z = \omega/(k+k_0)$, assuming $b_T b_w << 1$ and small fractional energy spread in the beam electrons. For a tenuous beam with $\omega \simeq kc$ and $k \simeq (1 + v_p/c)\ \gamma_p^2\ k_0$, it is found that the saturated amplitude of the radiation field is given approximately by

[Eq. (95)]

$$\delta\hat{B}_T = \frac{\Delta_L}{10(1 + b_w^2)^{1/2}} \frac{\omega_p^2}{c^2k_0^2} \left(1 + \frac{v_p}{c}\right) \hat{B}_w,$$

where $b_w = e\hat{B}_w/mc^2k_0$, $\Delta_L mc^2$ is the characteristic half-width energy spread in the laboratory frame, and $\omega_p^2 = 4\pi n_b e^2/m$ is the nonrelativistic plasma frequency-squared. Moreover, making use of Eq. (48), the trapped-particle distribution function $F_T(\gamma')$ is given by Eq. (98) for the choice of distribution function in Eq. (83). Equation (95) can of course be used to estimate the radiated power P_{RAD} [Eq. (96)] as well as the efficiency η of radiation generation [Eq. (97)] as for the model choice of distribution functions in Eqs. (82) and (83).

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the benefit of useful discussion with Wayne A. McMullin. This research was supported in part by the Office of Naval Research and in part by the Air Force Aeronautical Systems Division. The research of one of the authors (R.C. Davidson) was supported in part by ONR Contract N00014-79-C-0555 with Science Applications Inc., Boulder, Colorado.

REFERENCES

1. N.M Kroll and W.A. McMullin, Phys. Rev. A17, 300 (1978).
2. T. Kwan and J.M. Dawson, Phys. Fluids 22, 1089 (1979).
3. R.C. Davidson and H.S. Uhm, Phys. Fluids 23, 2076 (1980).
4. P. Sprangle and R.A. Smith, Phys. Rev. A21, 293 (1980).
5. I.B. Bernstein and J.L. Hirshfield, Phys. 20A, 1661 (1979).
6. D.A.G. Deacon, L.R. Elias, J.M.M. Madey, G.J. Ramian, H.A. Schwettman and T.I. Smith, Phys. Rev. Lett. 38, 897 (1977).
7. J.N. Eckstein, J.M.J. Madey, K. Robinson, T.I. Smith, S. Benson, D. Deacon, R. Taber and A. Gaupp, "Additional Experimental Results from the Stanford Three Micron Free Electron Laser", Stanford High Energy Physics Laboratory Report #894, May (1981).
8. D.B. McDermott, T.C. Marshall, and S.P. Schlesinger, R.K. Parker, V.L. Granatstein, Phys. Rev. Lett. 41, 1368 (1978).
9. R.C. Davidson and W.A. McMullin, Phys. Rev. A25, July (1982).
10. N.M. Kroll, P.L. Morton, and M.N. Rosenbluth, IEEE J. Quant. Electron. QE-17, 1436 (1981).
11. W.H. Louisell, J.F. Lam, D.A. Copeland, and W.B. Colson, Phys. Rev. A 19, 288 (1979).
12. F.A. Hopf, P. Meystre, M.O. Scully, and W.H. Louisell, Phys. Rev. Lett. 37, 1342 (1976).
13. T. Taguchi, K. Mima, and T. Mochizuki, Phys. Rev. Lett. 46, 824 (1981).
14. P. Sprangle, C.M. Tang, and W.M. Manheimer, Phys. Rev. A 21, 302 (1980).
15. I.B. Bernstein, J.M. Green and M.D. Kruskal, Phys. Rev. 108, 546 (1957).
16. R.C. Davidson, Methods in Nonlinear Plasma Theory (Academic Press, New York, 1972), pp 72-80.

FIGURE CAPTIONS

Fig. 1 Numerical plot of trapped-particle [Eq.(80)] and untrapped particle [Eq.(79)] distribution functions for $p_z' > 0$, $\hat{\gamma}_s^{+} = \hat{\gamma}_s^{-} = 0.1$, $\hat{\Delta}_{+} = \hat{\Delta}_{-} = 0.05$, $b_T = 0.01$, $b_w = 1.0$, and $\gamma_{MIN}/\gamma_{+}' = [1 + (b_w - b_T)^2]^{1/2}/\gamma_{+}' = 0.99$.

Fig. 2 Schematics of (a) initial distribution function [Eq.(82)], and (b) final saturated state [Eqs.(83) and (98)], for the model BGK equilibrium discussed in Sec. 5. Here, $\gamma_w = (1 + b_w^2)^{1/2}$, $\gamma_{+}' = [1 + (b_w + b_T)^2]^{1/2}$ and $\gamma_{MIN} = [1 + (b_w - b_T)^2]^{1/2}$ where $b_w > 0$ and $b_T > 0$ are assumed.

APPENDIX A

DERIVATION OF CONSTRAINT EQUATIONS (24) AND (25)

The two equations in Eq. (23) can be combined to give the single complex equation

$$D(\xi)\left\{-\frac{\hat{B}_w}{k_0}\exp(-i\,k_0\,z)+\frac{\delta\hat{B}_T}{k}\exp\left[i\,(kz-\omega t)\right]\right\}$$

$$=D_0\left[-\frac{\hat{B}_w}{k_0}\exp(-i\,k_0\,z)\right]\quad, \tag{A.1}$$

where $\xi = (k+k_0)z-\omega t$, and $D(\xi)$ and D_0 are defined by

$$D(\xi)\equiv k^2-\frac{\omega^2}{c^2}+\frac{\omega_{pb}^2}{c^2}\int\frac{dp_z}{\gamma}F(\xi,\,p_z)\quad, \tag{A.2}$$

$$D_0\equiv k^2-\frac{\omega^2}{c^2}+\frac{\omega_{pb}^2}{c^2}\int\frac{dp_z}{\gamma_0}F_0\,(p_z)\quad. \tag{A.3}$$

The real (imaginary) parts of Eq. (A.1) give the x(y) components of Eq. (23). Multiplying Eq. (A.1) by $\exp(ik_0z)$ gives

$$D(\xi)\left\{-\frac{\hat{B}_w}{k_0}+\frac{\delta\hat{B}_T}{k}\exp\left[i(k+k_0)z-\omega t\right]\right\}=-D_0\frac{\hat{B}_w}{k_0}\quad. \tag{A.4}$$

Since $D(\xi)$ is real, the imaginary part of Eq. (A.3) gives

$$D(\xi)\,\frac{\delta\hat{B}_T}{k}\sin\xi=0\quad. \tag{A.5}$$

Assuming that $D(\xi)$ is continuous, Eq. (A.4) implies that $D(\xi) = 0$ [Eq. (25)]. The real part of Eq. (A.3) then gives $D_0 = 0$ [Eq. (24)].

APPENDIX B

EVALUATION OF CONSERVATION EQUATIONS IN THE PONDERMOTIVE FRAME

The three conservation equations, Eq.'s (49), (50), and (51), are most easily evaluated in the pondermotive frame. In this frame, the conservation of average density can be expressed as

$$n_b' = n_b' \int_0^{\lambda'} \frac{dz'}{\lambda'} \int_{-\infty}^{+\infty} dp_z' \left[F_u(\gamma') + F_T(\gamma')\right], \tag{B.1}$$

where $\lambda' = 2\pi/k'$ is the periodicity length in the pondermotive frame. Substituting the expression for the trapped distribution function given by Eq. (48) into the right-hand side of Eq. (B.1), changing variables from p_z' to γ' [Eq. (28)], and interchanging the order of integration gives

$$\begin{aligned} n_b' = n_b' \int_0^{\lambda'} \frac{dz'}{\lambda'} \Bigg\{ \int_{\gamma_+'}^{\infty} d\gamma'' \, \frac{mcF_u(\gamma'')}{(\gamma''^2 - \gamma_+'^2)^{1/2}} \\ \times \left[\frac{2}{\pi} \int_{\gamma_-'(z')}^{\gamma_+'} d\gamma' \, \frac{\gamma'^2(\gamma_+'^2 - \gamma'^2)^{1/2}}{[\gamma'^2 - \gamma_-'^2(z)]^{1/2} \, (\gamma''^2 - \gamma'^2)} \right. \\ \left. + \gamma'' \left(\frac{\gamma''^2 - \gamma_+'^2}{\gamma''^2 - \gamma_-'^2(z')} \right)^{1/2} \right] \Bigg\} \end{aligned} \tag{B.2}$$

The first term in square brackets can be rewritten using the change of variables

$$\sin\theta = \left(\frac{\gamma'^2 - \gamma_-'^2(z')}{\gamma_+'^2 - \gamma_-'^2(z')} \right)^{1/2} . \tag{B.3}$$

The second term in the square brackets can be expressed as an integral and combined with the first term using the identity,

$$\left(\frac{\gamma''^2 - \gamma_+'^2}{\gamma''^2 - \gamma_-'^2(z')}\right)^{1/2} = (\gamma''^2 - \gamma_+'^2)\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \frac{1}{[\gamma''^2 - \gamma_+'^2 + \gamma_+'^2\,\sigma(\theta')\cos^2\theta]} \tag{B.4}$$

where $\theta' = k'z'$, and

$$\sigma(\theta') \equiv \frac{\gamma_+'^2 - \gamma_-'^2(z')}{\gamma'^2_+} = \frac{2b_T b_w}{[1 + (b_T + b_w)^2]} (1 + \cos\theta') \quad . \tag{B.5}$$

Rearranging the terms in the resulting expression gives Eq. (55).

The equation for conservation of energy can be written in the pondermotive frame as

$$n_b' mc^2 \int_0^{\lambda'} \frac{dz'}{\lambda'}\left[\int_{\gamma_w}^{\infty} d\gamma' \frac{\gamma'^2 mcF_0'(\gamma')}{(\gamma'^2 - 1 - b_w^2)^{1/2}} + \frac{E_w'^2 + B_w'^2}{8\pi}\right]$$

$$= n_b' mc^2 \int_0^{\lambda'} \frac{dz'}{\lambda'}\left[\int_{\gamma_-'(z')}^{\gamma_+'} d\gamma' \frac{\gamma'^2 mcF_T(\gamma')}{[\gamma'^2 - \gamma_-'^2(z')]^{1/2}} + \int_{\gamma_+'}^{\infty} d\gamma' \frac{\gamma'^2 mcF_u(\gamma')}{[\gamma'^2 - \gamma_-'^2(z')]^{1/2}}\right]$$

$$+ \int_0^{\lambda'} \frac{dz'}{\lambda'}\left[\frac{(E_w' + \delta E')^2}{8\pi} + \frac{(B_w' + \delta B')^2}{8\pi}\right] \quad . \tag{B.6}$$

Note that the particle and electromagnetic energies have been redefined to have their usual form in the pondermotive frame. The first two terms on the right-hand side of Eq. (B.6) represent the particle energy in the final saturated state. Following similar steps as in the reduction of Eq. (B.1), these terms can be simplified and combined to give the second term in Eq. (57).

In order to reduce the electromagnetic energy density terms, we express the electric and magnetic fields in terms of the vector potential $\underset{\sim}{A}'(z',t') = \underset{\sim}{A}'_0(z',t') + \delta\underset{\sim}{A}'(z',t')$ obtained by Lorentz transforming the laboratory vector potentials, Eq. (3) and Eq. (11), to the pondermotive frame using Eq. (26). For illustration, consider the contribution to the energy density from the electric field

$$\underset{\sim}{E}'^2 = \left(-\frac{1}{c}\frac{\partial \underset{\sim}{A}'_0}{\partial t'}\right)^2 + \left(-\frac{1}{c}\frac{\partial \delta\underset{\sim}{A}'}{\partial t'}\right)^2 + 2\left(\frac{1}{c}\frac{\partial \underset{\sim}{A}'_0}{\partial t'} \cdot \frac{\partial \delta\underset{\sim}{A}'}{\partial t'}\right) \quad . \qquad \text{(B.7)}$$

The first term is associated with the wiggler electric field energy and cancels from both sides of Eq. (C.6). The second term in Eq. (.7) reduces to

$$\left(-\frac{1}{c}\frac{\partial \delta\underset{\sim}{A}'}{\partial t'}\right)^2 = \gamma_p^2 \frac{(\omega - kv_p)^2}{c^2} \frac{\delta\hat{B}_T^2}{k^2} \equiv \frac{\hat{\omega}^2}{k^2c^2} \delta\hat{B}_T^2 \quad . \qquad \text{(B.8)}$$

The sinusoidal dependences in the third term of Eq. (B.7) beat together to give a term proportional to cos $k'z'$ which vanishes in the subsequent average over z'. The magnetic contribution to the total field energy can be evaluated in a similar fashion.

The conservation of axial momentum, Eq. (56), involves integrals over only the untrapped particle distribution function since in the pondermotive frame the trapped particles carry no net momentum. The reduction of the electromagnetic momentum is similar to the evaluation of the electromagnetic energy outlined above and leads directly to the right-hand side of Eq. (56).

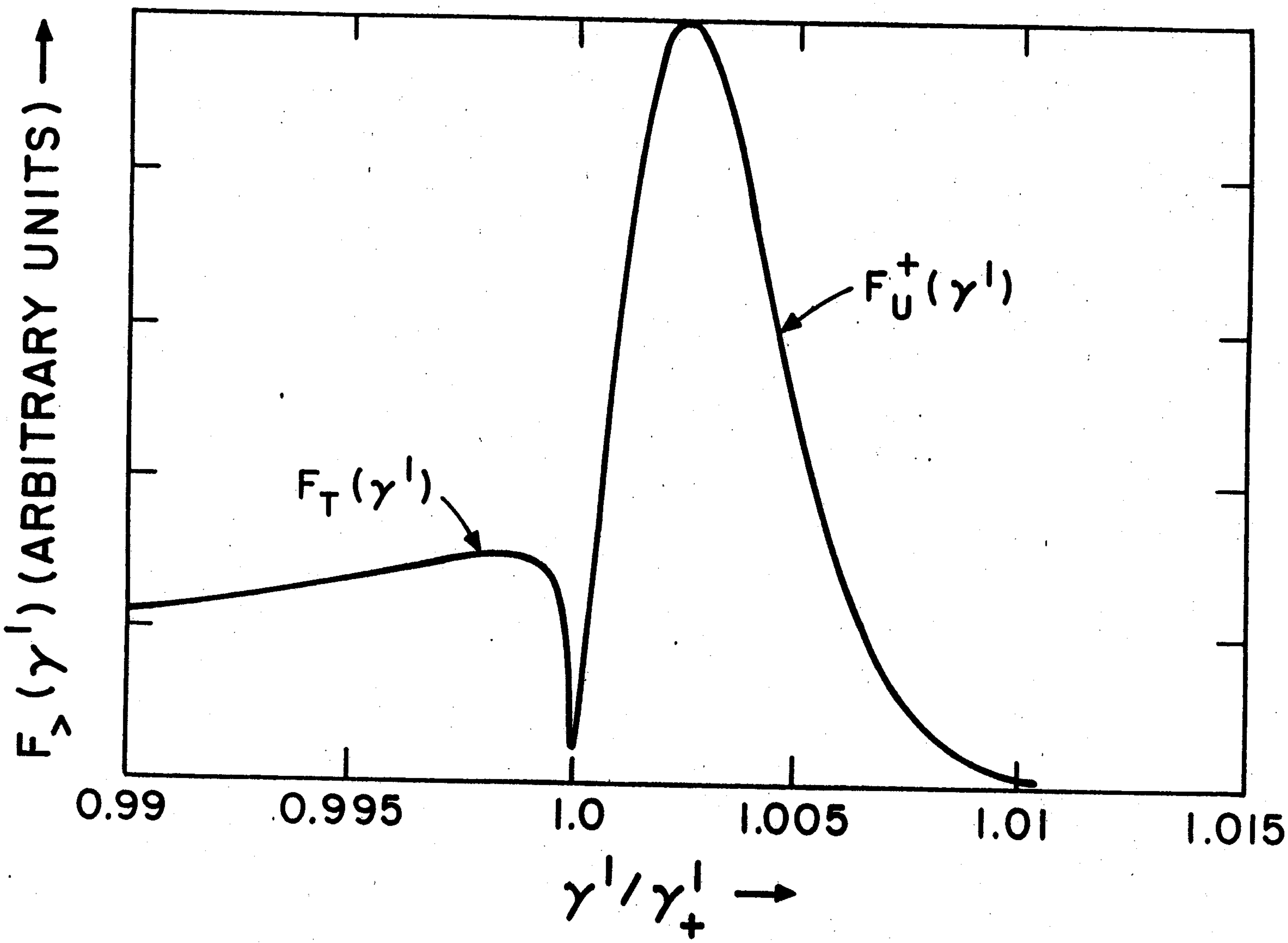
$F_>(\gamma')$ (ARBITRARY UNITS) →
$F_T(\gamma')$
$F_U^+(\gamma')$
0.99
0.995
1.0
1.005
1.01
1.015
γ'/γ'_+ →

Figure 1

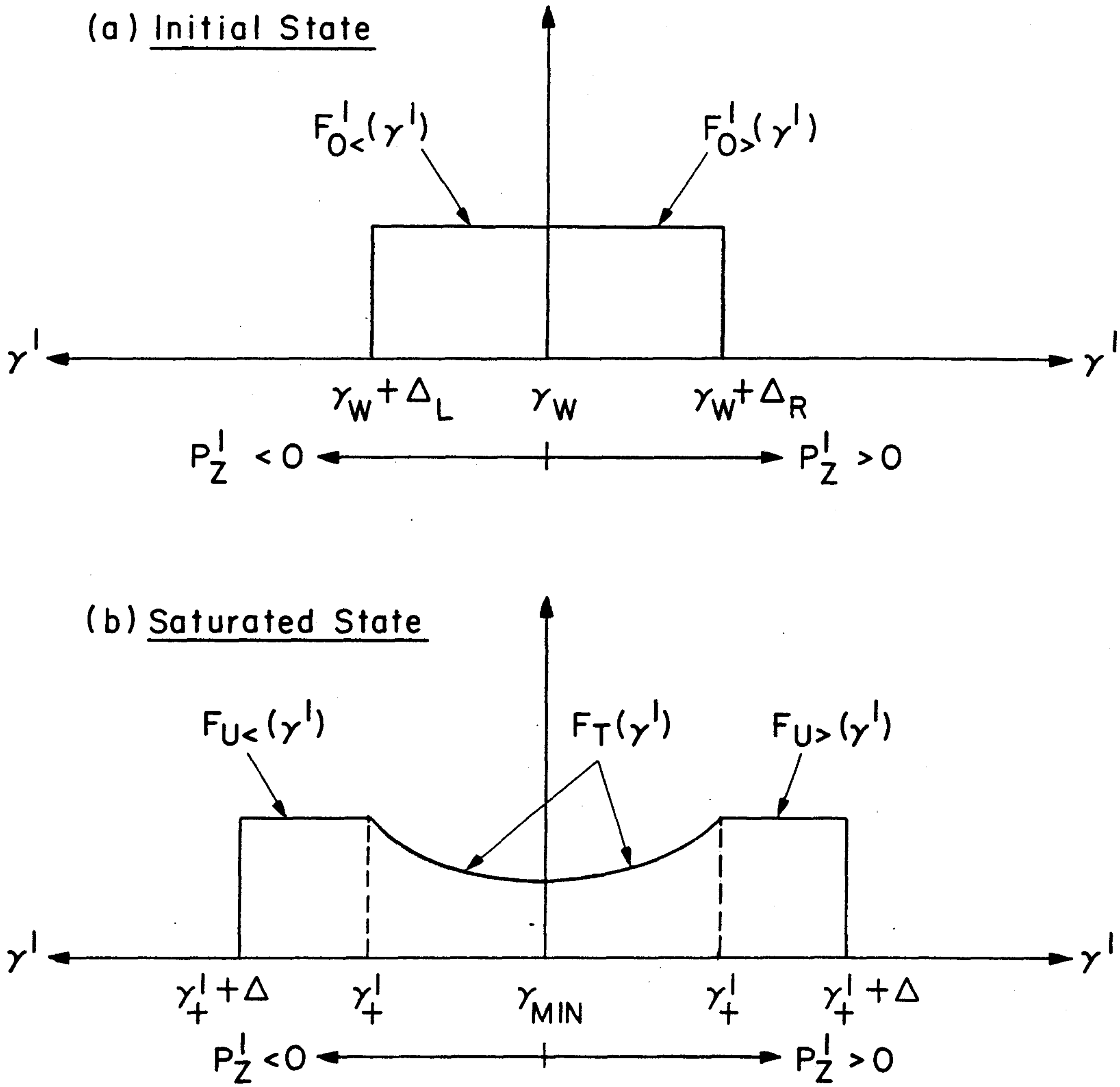
(a) Initial State
$F'_{0<}(\gamma')$
$F'_{0>}(\gamma')$
γ'
γ'
$\gamma_W + \Delta_L$
γ_W
$\gamma_W + \Delta_R$
$P'_Z < 0$
$P'_Z > 0$
(b) Saturated State
$F_{U<}(\gamma')$
$F_T(\gamma')$
$F_{U>}(\gamma')$
γ'
γ'
$\gamma'_+ + \Delta$
γ'_+
γ_{MIN}
γ'_+
$\gamma'_+ + \Delta$
$P'_Z < 0$
$P'_Z > 0$

Figure 2