Linear 1-D Analysis of Oscillation Instabilities in Stationary Plasma Thrusters

by
Reid A. Noguchi

Submitted to the Department of Aeronautics and Astronautics in partial fulfillment of the requirements for the degree of Master of Science at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

A one-dimensional, linear perturbation model was formulated to aid in the analysis of the low-frequency (33 kHz), axial oscillation instabilities observed during the operation of Stationary Plasma Thrusters. The first six solutions of this 1st order perturbation model were determined by scanning the complex frequency plane and were addressed in this study. Of these modes, two notable patterns of behavior were found and discussed. The first group of modes had waves which translated through the thruster, with a fundamental harmonic of 49.7 kHz. The second group, having a fundamental harmonic of 74.2 kHz, exhibited a sloshing type of behavior and was found to have several characteristics of a predator-prey type cycle. Although all modes were found to be highly damped, contrary to the existence of oscillation instabilities, it is believed that an ionization-acoustic resonance instability could be excited.

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Chapter 1

Introduction

1.1 Background on SPT’s
Stationary Plasma Thrusters (SPT’s), also called Closed Drift Thrusters, or Hall Thrusters, were developed in Russia in the 1960’s. The typical geometry for an SPT is a metal shell and inner core connected by a metal back-plate which houses the anode and supports the propellant feed system. The inner surface of the shell and outer surface of the core are coated with a thin ceramic thermal/electric insulator and form the annular plasma channel. Located outside the channel is the cathode, which provides electrons for producing the electric field to accelerate the ionized propellant (Xenon), and for neutralizing the ion beam beyond the channel’s exit. Electromagnetic coils are situated around the outside of the shell and provide a primarily radial magnetic field at the channel’s exit. The magnetic circuit is completed through the inner core, back-plate, and shell, which are all made of a ferromagnetic material.

Like most electric propulsion systems (excluding electrothermal systems), an SPT converts electrical energy to kinetic energy by accelerating charged particles with an electric field created by the anode-cathode potential difference. One of the primary characteristics that distinguishes an SPT from other electric propulsion systems is the way in which this electric field is localized at the thruster’s exit. Instead of using a physical cathode grid to accelerate positive ions, the SPT localizes the effect of the electric field by confining electrons at the thruster’s exit with a radial magnetic field. The strength of the magnetic field is designed such that the Larmor radius of the electrons is much smaller than the scale of the SPT’s channel length, yet the curvature of the heavier ions’ path is not significantly affected. This allows the electrons, spiraling around the magnetic field lines, to col-
lide with, and ionize, neutrals (Xenon) in the thruster's channel. Through collisions, the electrons gain mobility across the magnetic field lines and diffuse to the anode. Having been ionized, the charged propellant is accelerated by the axial electric field induced by the confined electrons at the channel exit. Upon exiting the thruster, the Xenon ions are quasi-statically neutralized by electrons from the cathode. Further downstream, the ions recombine with the electrons and neutrals are reformed, completing the energy cycle and by conservation of charge, assuring electrostatic neutrality of the thruster and spacecraft. A schematic illustrating the operation of an SPT is shown below, in Figure 1.1.

![Figure 1.1: Operation of an SPT](image)

Unlike conventional propulsion, where the ejection of mass provides momentum exchange through high particle velocity collisions with a shaped nozzle, the SPT generates propulsion in a different way. At the channel's exit, the crossing of the radial magnetic field lines with the axial electric field lines induces an azimuthal drift (E x B drift) of both the charged propellant and the electrons. Because the Xenon ions have more inertia than
the electrons, their path is not significantly altered by the azimuthal, or Hall drift. The electrons, however, remain in a closed, azimuthal drift and their velocity vector crossed with the radial magnetic field lines produce a force on the magnetic coils. This force translates to the spacecraft and it is the mode of its propulsion.

1.2 Motivation
Since their introduction by Russia, SPT's have been of interest to several U.S. research laboratories and private companies. Because of their high specific impulse (about 1600 sec) and high propellant efficiency (45%-55% for Xenon) SPT's have preferred performance characteristics for orbit raising and station-keeping maneuvers for near-Earth spacecraft. Accordingly, in the past decade, SPT's have become a part of many of the major U. S. satellite manufacturers' next-generation designs. Due to the industry's need for low cost, low mass, reliable systems, certain areas of SPT design and operation must be addressed. Some of these issues are: plume divergence; plume impingement on spacecraft components; electromagnetic interference of the plume; thruster lifetime; and complexity of the Power Processing Unit (PPU);

Although a lot of research is invested in understanding and controlling the behavior of the SPT's plume, the issue of reducing the complexity of the PPU has a direct affect on two major drivers in the commercial industry, cost and mass. The PPU, or Power Processing Unit, is the electronic device which monitors, regulates, and controls the operation of an SPT. For a well behaved electrical system, the PPU would simply be programmed with the control logic of the SPT’s operation. However, the dynamics of SPT operation are laden with oscillations which lead to frequent high amplitude voltage spikes. Such erratic fluctuations in voltage are not only harmful to the spacecraft which carries the SPT, but to the PPU which has to control them. This places a requirement that the PPU must be electrically hardened to survive these fluctuations in addition to damping them out, conse-
sequently, increasing its complexity and cost. Since hardening increases the mass of the PPU, and complexity increases its cost, eliminating these oscillations is a significant issue for satellite manufacturers. Currently, the cost of an SPT's PPU can be several times the cost of the SPT itself, thereby allowing a large margin for improvement in the total cost of the system if the PPU's cost can be reduced.

1.3 Objective
The objective of this study is to formulate a model for the plasma dynamics of an SPT which, not only captures the oscillatory fluctuations observed independently through experimental data and numerical simulation, but also provides a tool for understanding, and possibly controlling, their behavior.
Chapter 2

Literature Review

2.1 Overview of Relevant Literature
Although SPT's have been studied since their introduction in the 1960's, there have been few papers dedicated to formulating an analytical model for the low frequency oscillation instabilities that occur during their operation. Instead, many papers focus primarily on the measurement and observation of these modes and thereafter discuss the analysis and conclusions of their study. Even fewer are those papers in which numerical simulations of SPT operation were developed and implemented, and the results of which were analyzed for compatibility with past experiments and prediction for future experiments. Detailed below, are the various aspects of oscillation behavior which are relevant to this study, and should be understood before proceeding to develop a model to analyze their dynamics.

2.2 Experimental Observations
There have been many papers focused on the measurement, characterization and verification of the oscillation instabilities observed in SPT’s. Although some of these papers focus on higher frequency oscillations, only those which discuss the low frequency, axial oscillations, in the tens of kHz, concern the scope of this paper.

Providing a general overview of SPT oscillations is Choueiri’s paper, which discusses the broad range of frequencies, from 1 kHz - 60 MHz [1]. In this paper, it is suggested that, in the range of the 20 - 100 kHz instabilities, the oscillations might either be driven by gradients in the plasma or through ionization processes. Instabilities in this frequency regime are noted by Choueiri to be often referred to in the Russian literature as “loop”, “circuit”, or “contour” oscillations. Choueiri also comments that it is the oscillations in the 20 - 100 kHz regime which impact the PPU design, and that the higher frequencies do not.
Several other papers have focused on the use of a broad range of instruments which have the capability to observe, measure, and record data essential to characterizing the plasma dynamic behavior, both inside and outside the SPT channel [2]-[5]. From these experiments, several results should be noted for the purpose of this study. With various experiments using optical sensors, electrostatic probes, and CCD cameras, measurements have been taken that verify the existence of low frequency oscillations during the operation of SPT’s. Most of these have been characterized as oscillations in either the azimuthal or axial direction. The azimuthal oscillations are not discussed further in this paper since it is the axial oscillations that are believed to affect the operation of the SPT’s PPU [1], [3].

Although the nature of the axial oscillations have not been definitively explained, they are thought to be induced by ionization type dynamics due to their measured frequency in the tens of kHz [3]-[7].

2.3 Numerical Simulation
In addition to experimental evidence of the existence of these oscillations, as well as measurements of their frequency, there has also been a well documented numerical simulation that leads to the same conclusions. The paper by Fife, Martinez-Sanchez, and Szabo, shows the results of a two-dimensional numerical simulation which also demonstrates the oscillatory behavior of SPT operation [8]. From this simulation, axial oscillations in the tens of kHz have appeared, and correspond to those observed by experimental measurement. Again, the cause of these oscillations are unknown, but a predator-prey type of fluctuation is alluded to in this paper as a possible mode through which these oscillations are produced and sustained. This process, of ion and neutral number densities oscillating 90 degrees out of phase, leads again to the idea that ionization dynamics are crucial to development and propagation of these low frequency oscillations.
2.4 Analysis of Steady State Dynamics

Although experimental observation and numerical simulation have provided many insights into the characteristics of the oscillatory instabilities during SPT operation, an adequate theory to describe the nature of these oscillations has not yet been developed. There has been, however, a few analytical treatments of the steady state behavior of the plasma structure in an SPT channel [9], [10]. In Aledo's and Martinez-Sanchez's paper, a one-dimensional model of SPT behavior has been formulated with a well defined set of governing equations. The primary objective of this model was to retain enough of the system's physics to provide an analytical tool for understanding and predicting the behavior of the plasma in an SPT. One of the secondary objectives of this model was to establish a basis for further study into an analysis of transient behavior.

Because it was focused on steady state plasma characteristics, their formulation does not predict oscillation instabilities. However, for the purpose of this study, which involves a linear perturbation of the steady state variables, Aledo's and Martinez-Sanchez's model contains many aspects directly applicable to the analysis of unsteady behavior. Since this model also includes ionization effects, which appears to be a process that may induce the low frequency oscillations pursued in this study, it provides a good foundation for modeling the transient, oscillatory characteristics, of SPT operation.
Chapter 3

Plasma Dynamics of an SPT

3.1 Generalized Plasma Dynamics

The dynamic behavior of a continuum of particles in three dimensions can be represented by the conservation equations of mass, momentum and energy. Their expressions are given below.

The mass conservation equation is,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \dot{\rho} = m \dot{n}, \tag{3.1}$$

where $\rho$ is the density of the fluid, which is equivalent to the product of the mass, $m$, and number density, $n$. $\mathbf{u}$ is the velocity of the fluid.

The momentum conservation equation,

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = - \nabla \cdot \mathbf{p} + q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \dot{\mathbf{M}}, \tag{3.2}$$

as shown above, includes the contributions of both an electric field, $\mathbf{E}$, and a magnetic field, $\mathbf{B}$, acting on a charge, $q \cdot \mathbf{p}$ is the pressure of the fluid.

The energy conservation equation is shown below,

$$\frac{\partial \left( \rho \frac{3}{2} k T_e \right)}{\partial t} + \nabla \cdot \left( \rho \frac{3}{2} k T_e \mathbf{u} \right) = - \nabla \cdot \mathbf{q} - \mathbf{p} : \nabla \mathbf{u} + \mathbf{E'} \cdot q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \dot{\varepsilon}, \tag{3.3}$$

with $k$ as the Boltzmann constant, $T_e$ as the electron temperature, and $\mathbf{E'}$ as the electric field as seen with respect to the moving fluid.
These equations govern the dynamics of a conducting fluid, or plasma, subjected to both a magnetic field and an electric field [11]. Source terms, denoted as \( f \), are included to account for ionization effects, collisional processes, and other phenomena which affect the conservation equations. \( \dot{n} \), \( \dot{M} \), and \( \dot{\varepsilon} \) are the source terms for the number density, momentum, and energy, respectively. The definitions for these source terms depend on the physical characteristics of the particular system being studied, and are detailed in Section 3.3, "Definition of Source Terms".

In general, each of these three conservation equations can be separated into equations for neutrals, ions, and electrons. The nine resulting equations can then be integrated to determine their dynamic behavior. With several simplifications, detailed in Section 3.2, this system of equations may be reduced to a smaller, and simpler set.

3.2 Simplifications and Assumptions
Since the generalized equations are applicable to any continuum plasma, by constraining the system to the specifics of this analysis, several simplifications can be made without sacrificing accuracy within the scope of this study. Invariance along degrees of freedom, and terms of negligible magnitude, are the two primary avenues for simplification in the dynamics of an SPT.

3.2.1 Variation in One Dimension
In accordance with experimental observations, the dynamics of an SPT which relate to the oscillation instabilities of this study can be focused on by neglecting degrees of variance which do not contribute to them. Since it is believed that azimuthal variations do not significantly contribute to the axial oscillations of this study, the differential equations shown in Section 3.1 need not be integrated over the azimuthal direction [2]. Furthermore, by using the steady state formulation as a basis for this study, it can analogously be defined that all state variables are to be averaged radially. By doing so, variations in the radial
direction are eliminated, and integration along that axis is no longer required.

3.2.2 Terms of Negligible Magnitude

There are four significant simplifications that can be made by eliminating terms in the general dynamic equations that contain factors which are orders of magnitude smaller than the other terms in the equation they belong to. Ion temperature, electron inertia, heavy particle interactions, and the magnetic field effect on ions, are the specific terms which can be neglected for this particular study.

Since ion temperature is relatively small, when compared to the electron temperature, simplifications can be made in the ion-momentum and ion-energy equation if it is neglected. By doing so, because pressure gradients are proportional to temperature, the ion pressure gradient, \( \nabla \cdot p \), can be removed from the ion-momentum equation. Furthermore, since ion temperature is negligible, the ion-energy equation can be omitted from the initial set of nine equations.

With the mean free path of the heavy particles, neutrals and ions, being much longer than the length scale of an SPT channel, their behavior can be considered collisionless [12]. Given this, the heavy particle interactions can be neglected. This simplification removes the necessity for source terms, in all the equations, for momentum and energy losses due to heavy particle collisions. Other collision events, such as electron-neutral interactions, must still remain in the plasma dynamic equations, since ionization must be accounted for. These terms are discussed in detail in Section 3.3, “Definition of Source Terms”. Furthermore, since neutrals are assumed not to interact with other particles, aside from ionization which is accommodated by a source term, the neutral-momentum and neutral-energy equations contain no additional dynamic information, and can be removed from consideration. This reduces the system from eight equations to six.
In the momentum conservation equation for electrons, all terms that contain the electron mass, $m_e$, can be removed since its magnitude is orders of magnitude smaller than other terms in that equation. Although the electron temperature, hence pressure, number density, and velocity are key elements to the plasma dynamics, the electron inertia is of higher order, and for purposes of this study, can be ignored.

The last simplification that can be made by neglecting terms of relatively small magnitude, is to ignore the magnetic field terms in the ion-momentum equation. Because the magnetic field strength is designed to only confine electrons, its ability to affect the trajectories of the much more massive ions is insignificant.

3.2.3 Assumptions
To reduce the complexity of the plasma dynamic system, several assumptions in the model are introduced. These assumptions were also made in the formulation the steady state model, which is used as a basis for this transient analysis [9]. Because of this, there would be no additional accuracy introduced if these assumptions are not made. For the purposes of compatibility, simplification, and verification of the steady state, the set of assumptions detailed below, are included in the formulation.

The first assumption is that the electric and magnetic field terms in the electron-momentum equation contribute in such a way as to negate each other's effects. Reasoning for this assumption comes from the fact that, due to the closed drift behavior of SPT’s, the electrons, in general, move in the azimuthal direction. Although the electrons are attracted to the anode by the electrostatic force, the opposing Lorentz force (produced by the azimuthal movement of the electrons through the radial magnetic field) keeps the electrons from moving in the axial direction (although they may still diffuse to the anode or gain mobility across magnetic field lines through collisions). Since these two forces balance
each other out, the electromagnetic terms in the electron-momentum equation can be ignored.

The next assumption is that wall effects are ignored. Doing so reduces the complexity of the analysis, and is assumed to not affect the behavior of the plasma dynamics which are relevant to this study.

Quasi-neutrality is also assumed in this model. This property fixes the ion and electron number densities to be equal, or $n_i = n_e$, hence eliminating the development of space-charges in the thruster. This is a typical assumption for most plasmas that deal with time scales longer than that of the plasma frequency and length scales bigger than the Debye length. The plasma frequency is a measure of how quickly the plasma can react to an instantaneously developed space-charge. Since the time scale of the low frequency oscillations is well beyond that of the plasma frequency, this assumption is valid.

Finally, it is assumed that there is no electron heat conduction, or that the electrons behave as an adiabatic fluid. With this being the case, the term $\nabla \cdot q$ in the energy conservation equation can be neglected.

### 3.3 Definition of Source Terms

Given the simplifications and assumptions detailed in the previous section, the source terms for mass, momentum, and energy, in the governing differential equations can be defined as discussed below. These source terms account for particle interactions, such as ionization and other collision effects.

#### 3.3.1 Number Density Sources

Without ionization, the law of conservation of mass would preclude any creation or destruction of particles. With it, however, assuming all neutrals are ionized only once, each ionizing collision would simultaneously produce, a Xenon ion and an electron, as
well as destroy a neutral particle. The frequency (per electron) at which these ionization events occur, can be defined as,

$$\nu_{\text{ion}} = \sigma_0 n_n \sqrt{\frac{8kT_e}{\pi m_e}} \left(1 \pm 2\frac{kT_e}{eE_i} \right) e^{\frac{eE_i}{kT_e}},$$

(3.4)

where $\sigma_0$ is the ionization cross section, $n_n$ is the neutral number density, $e$ is the charge of an electron, and $E_i$ is the 1st ionization energy (12.1 eV for Xenon). With this definition, the source terms for ions, electrons, and neutrals, $\dot{n}_i$, $\dot{n}_e$, and $\dot{n}_n$, respectively, can be expressed as

$$\dot{n}_i = n_n \nu_{\text{ion}}$$

(3.5)

$$\dot{n}_e = n_n \nu_{\text{ion}}$$

(3.6)

$$\dot{n}_n = -n_n \nu_{\text{ion}}.$$ (3.7)

Note that the sign for the neutral number density source term is negative since a neutral is destroyed at the frequency at which ions and electrons are produced. These expressions not only satisfy the conservation of mass, but also the conservation of charge, so that no space-charges develop (given that this is not in a sheath).

3.3.2 Momentum Sources

Since the heavy particles, ions and neutrals, are considered collisionless in this model, there is only one source of momentum exchange, electron-neutral interactions. This type of event augments the plasma dynamic equations in two ways. The first way is the momentum exchange between ion and neutral fluids during ionization, and the second is the momentum lost from an electron during an interaction with a heavy particle.

During an ionizing collision, electrons interact with neutrals with the frequency $\nu_{\text{ion}}$ and converts a neutral particle into an ion. Since the average ion has a higher momentum
than the average neutral, this amounts to a source of momentum exchange, whose form is expressed in Equation (3.8) below.

\[ M = -\nu_{ion} m_i (v_i - v_n) \]  

(3.8)

\( m_i \) and \( v_i \) are the ion mass and velocity, respectively, and \( v_n \) is the neutral particle velocity.

When an electron collides with a neutral, the electron experiences a loss of momentum. Simultaneously, the electron jumps to a new magnetic field line, providing mobility across the magnetic field lines which constrain them (spatially at the exit of the thruster’s channel). Electrons may also undergo a Bohm-type diffusion and field-driven mobility toward the anode. This phenomenon is similar to that due to collisions, except that in the “Bohm” case, the scattering events are due to plasma turbulence microfields. With these two processes affecting the momentum of electron-neutral collisions, an effective collision frequency per electron can be expressed in the form,

\[ \frac{m_i \nu_{\text{eff}}}{m_e} = \frac{(eB/m_e)^2}{8 \pi k T_e \sigma_{en} + \alpha_B (eB/m_e)} \]  

(3.9)

\( \sigma_{en} \) is the electron-neutral collision cross section, and \( \alpha_B \) is a coefficient which incorporates the effect of Bohm diffusion, and is given the value \( \frac{1}{16} \).

By substituting the characteristic cyclotron frequency, \( \omega_C \), with

\[ \omega_C = \frac{qB}{m_i \alpha_B} \]  

(3.10)

Equation (3.9) can be rewritten as,
\[
\frac{m_i}{m_e} \frac{\nu_H}{c} = \frac{(\alpha_R \omega_c \frac{m_i}{m_e})^2}{n_n \sigma_{en} \sqrt{\frac{8 \kappa T_e}{\pi m_e} + \alpha_H \omega_c \frac{m_i}{m_e}}},
\]
(3.11)

This is the same formulation as in Ahedo’s and Martinez-Sanchez’s steady state analysis [9].

Looking at Equation (3.9), in the low-density limit, where \( n_n \to 0 \), this reduces to Bohm’s case,

\[
\frac{m_i}{m_e} \frac{\nu_H}{c} = 16eB/m_e,
\]
(3.12)

which yields a diffusivity, \( \frac{kT_e}{16eB} \), and a mobility of \( \frac{1}{16B} \). In the opposite limit, when

\[
\frac{m_i}{m_e} \frac{\nu_H}{c} = \frac{(eB/m_e)^2}{\nu_e},
\]
(3.13)

where,

\[
\nu_e = n_n \sigma_{en} \sqrt{\frac{8 \kappa T_e}{\pi m_e}},
\]
(3.14)

the high Hall-parameter limit of the classical result, \( \nu_e \left[ 1 + \frac{(eB/m_e)^2}{\nu_e^2} \right] \), is attained.

3.3.3 Energy Source
Because of the presence of ionization, the energy conservation equation, Equation (3.3), must include a source term which accounts for the energy added to the particles during this process. If each neutral is ionized only once, the quantity of energy lost per event
would be the atom’s 1st ionization energy, $E_i$. Since the total number of ionization events, per unit time, occur at a rate of $n_e \nu_{ion}$, the expression for the source term in the energy conservation equation is

$$\dot{M} = -n_e \nu_{ion} E_i^\prime.$$  

(3.15)

where $E_i^\prime$ is the effective energy required to produce and ionization event. This effective value is estimated to be $2.5E_i$, again, as formulated in the steady state model [9]. The factor, 2.5, accounts for energy lost by excitation to high-lying electronic levels, with prompt radiative decay. The factor is, in general, a function of $T_e$, with this value being only a rough average.

### 3.4 Simplified Equations

With the simplifications and assumptions detailed in the previous section, the plasma dynamic equations that govern the behavior of the low frequency, axial oscillation instabilities of an SPT, are tabulated below.
<table>
<thead>
<tr>
<th>Plasma Dynamic Equations</th>
<th>Principle of Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(v_e n_e) = n_e v_{ion}$</td>
<td>conservation of ions</td>
</tr>
<tr>
<td></td>
<td>$n_e = $ electron number density</td>
</tr>
<tr>
<td></td>
<td>$v_i = $ ion velocity</td>
</tr>
<tr>
<td></td>
<td>$v_{ion} = $ ionization frequency per electron</td>
</tr>
<tr>
<td>$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(v_e n_e) = n_e v_{ion}$</td>
<td>conservation of electrons</td>
</tr>
<tr>
<td></td>
<td>$v_e = $ electron velocity</td>
</tr>
<tr>
<td>$\frac{\partial n_n}{\partial t} + \frac{\partial}{\partial x}(v_n n_n) = -n_e v_{ion}$</td>
<td>conservation of neutrals</td>
</tr>
<tr>
<td></td>
<td>$n_n = $ neutral number density</td>
</tr>
<tr>
<td></td>
<td>$v_n = $ neutral velocity</td>
</tr>
<tr>
<td>$m_i \left( \frac{\partial v_i}{\partial t} + v_{ion} \frac{\partial v_i}{\partial x} \right) = -q \frac{\partial \phi}{\partial x} - v_{ion} m_i (v_i - v_n)$</td>
<td>conservation of momentum for ions</td>
</tr>
<tr>
<td></td>
<td>$m_i = $ ion mass</td>
</tr>
<tr>
<td></td>
<td>$\phi = $ electric potential</td>
</tr>
<tr>
<td>$0 = q \frac{\partial \phi}{\partial x} - \frac{1}{n_e} \frac{\partial}{\partial x} (n_e kT_e) - v_H m_i n_e$</td>
<td>conservation of momentum for electrons</td>
</tr>
<tr>
<td></td>
<td>$k = $ Boltzman constant</td>
</tr>
<tr>
<td></td>
<td>$T_e = $ electron temperature</td>
</tr>
<tr>
<td></td>
<td>$\frac{m_i}{m_e} v_H = $ effective collision frequency for axial diffusion</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial t} \left( \frac{3}{2} n_e kT_e \right) + \frac{\partial}{\partial x} \left( \frac{5}{2} n_e kT_e v_e \right) = n_e \left( q v_e \frac{\partial \phi}{\partial x} - v_{ion} E_i \right)$</td>
<td>conservation of energy</td>
</tr>
<tr>
<td></td>
<td>$q = $ electric charge</td>
</tr>
<tr>
<td></td>
<td>$E_i = $ effective ionization energy</td>
</tr>
<tr>
<td>$v_{ion} = \sigma_0 n_n \sqrt{\frac{8 k T_e}{\pi m_e} \left( 1 + 2 \frac{k T_e}{e E_i} \right)} \frac{e E_i}{kT_e}$</td>
<td>ionization frequency</td>
</tr>
<tr>
<td></td>
<td>$\sigma_0 = $ ionization cross section</td>
</tr>
<tr>
<td></td>
<td>$m_e = $ electron mass</td>
</tr>
<tr>
<td>$\frac{m_i}{m_e} v_H = \frac{\left( \alpha_B \omega_c \frac{m_i}{m_e} \right)^2}{n_n \sigma_{en} \sqrt{\frac{8 k T_e}{\pi m_e} + \alpha_B^2 \omega_c^2 \frac{m_i}{m_e}}}$</td>
<td>effective collision frequency for axial diffusion</td>
</tr>
<tr>
<td></td>
<td>$\alpha_B = $ Bohm diffusion coefficient</td>
</tr>
<tr>
<td></td>
<td>$\omega_c = $ characteristic cyclotron frequency</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{en} = $ electron-neutral collision cross section</td>
</tr>
</tbody>
</table>

Table 3.1: Simplified Plasma Dynamic Equations
These are the equations for which the formulated model is based upon in Chapter 4. They are a set of six non-linear, coupled, partial differential equations, with two non-linear parameters. In general, there is no closed-form analytical solution to these equations and therefore they must either be solved numerically, or simplified further. The next chapter, "Separation into Steady and Unsteady Parts", deals with deriving a model which still contains enough of the required physics to describe the plasma behavior, but is simple enough to solve with relatively low computational effort.
Chapter 4

Separation into Steady and Unsteady Parts

4.1 Requirements
In general, each state variable is a function of four coordinates, three for space, and one for time, and can be nonlinear with respect to any, or all, of them. For the axisymmetric geometry of an SPT, the three spacial coordinates are the radial, azimuthal, and axial coordinates. When the general form of the state is used in the plasma dynamic equations, given in Section 3.4, "Simplified Equations" on page 27, each partial derivative, in general, will also be nonlinear in both space and time. Integrating this system of nonlinear, partial differential equations is computationally intensive and, although it leads to numerical solutions, it does not provide a convenient tool for the analysis and understanding of the dynamic behavior of the system.

To assist in the analysis of the SPT dynamics, the model must be formulated to retain enough complexity to preserve the behavior of the system, while being simple enough so that generalizations of the system can be made analytically. With too much complexity, as in a numerical simulation, the processes which generate the dynamics are lost and only a numerical result is computed. Too little complexity means not capturing the necessary dynamics in the model, in this case, the low frequency oscillation instabilities.

4.2 Linearization
To achieve a model which is more applicable for analysis and computationally simpler to integrate, each state variable is considered as having a (non-linear) steady state, or time-invariant, behavior, along with small linear perturbations that can vary with time. Since the perturbation part has the property of superposition of solutions, the partial derivatives in the plasma dynamic equations become a set of two systems of equations, the non-linear
steady state equations and the linear perturbation equations, of which there can be several modes. The solutions of each mode, plus the steady solution, can then be superposed to create the total solution for each state.

With the assumption of linearity, each state variable, in general, can be written in the form,

\[ f(\tau, t) = \bar{f}(\tau) + f'(\tau, t) \]  \hspace{1cm} (4.1)

where \( \bar{f}(\tau) \) is the 0th order, or steady state, behavior of the state variable, \( f(\tau, t) \), and \( f'(\tau, t) \) is the 1st order, or linear perturbation, of the steady state behavior. As discussed in Section 3.2.1, “Variation in One Dimension” on page 20, the state variables need only vary in the axial direction, \( x \). Equation (4.1) can now be reduced to

\[ f(x, t) = \bar{f}(x) + f'(x, t) . \]  \hspace{1cm} (4.2)

By definition, the steady state term of the variable is independent of time, and therefore the linear perturbation term contains all the temporal dynamics of the state. However, even though the spacial dynamics have been modeled as having a steady state behavior with an overlaid spacial perturbation, the temporal variation of the perturbation has not been modeled. This is addressed in the following section, “Temporal Variation”.

4.3 Temporal Variation
Since the requirement of the model is to produce a tool for studying oscillatory behavior, the form of the state variable can be defined to confine the model to such modes. The form of the state variables is now defined to be,

\[ f(x, t) = \bar{f}(x) + \sum_{\text{modes, } j} \Re \{ \hat{f}_j(x)e^{i\omega_j t} \} \]  \hspace{1cm} (4.3)
where each term in the summation belongs to a particular mode. As mentioned before, the steady state term, \( \bar{f}_j(x) \), is time invariant, but now each linear perturbation term, \( \tilde{f}_j(x, t) \), is allowed to oscillate (and possibly decay) with complex frequency, \( \omega_j \), and amplitude, \( \tilde{f}_j(x) \). With the spacial and temporal variation of the perturbation separated, this form of the state variable will generate linear, partial differential equations when substituted into the dynamic equations given in Section 3.4, “Simplified Equations” on page 27.

### 4.4 Model Equations

By substituting the state variables, in the form shown above in Equation (4.3), the formulated model equations, can be derived from the plasma dynamic equations given in Section 3.4, “Simplified Equations” on page 27. This process is described in the following section and the resulting equations are listed in Section 4.4.2 and Section 4.4.3.

#### 4.4.1 Deriving the Model Equations

To produce the 0\textsuperscript{th} order and 1\textsuperscript{st} order equations for ion conservation, the state functions for the electron number density, \( n_e \), the ion velocity, \( v_i \), and the ionization frequency per electron, \( v_{ion} \), are substituted into the corresponding plasma dynamic equation listed in Table 3.1, “Simplified Plasma Dynamic Equations,” on page 28. The dynamic equation for ion conservation is given below.

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(v_i n_e) = n_e v_{ion}
\]  

(4.4)

With the forms of the state functions as defined by Equation (4.3), the electron number density, ion velocity, and ionization frequency (per electron), respectively, can be expressed as,
\[ n_e(x, t) = \bar{n}_e(x) + \hat{n}_e(x)e^{i\omega t}, \]  
\[ v_i(x, t) = \bar{v}_i(x) + \hat{v}_i(x)e^{i\omega t}, \]  
\[ v_{ion}(x, t) = \bar{v}_{ion}(x) + \hat{v}_{ion}(x)e^{i\omega t}. \]  

Since the perturbation terms are linear, each mode can be determined independently and later superimposed to form the total solution. For this treatment, the subscript, \( j \), denoting the mode number will be neglected since only one mode will be pursued at a time.

Equation (4.4) can now be written as,

\[ \frac{\partial}{\partial t}(\bar{n}_e + \hat{n}_e e^{i\omega t}) + \frac{\partial}{\partial x}[(\bar{v}_i + \hat{v}_i e^{i\omega t})(\bar{n}_e + \hat{n}_e e^{i\omega t})] = (\bar{n}_e + \hat{n}_e e^{i\omega t})(\bar{v}_{ion} + \hat{v}_{ion} e^{i\omega t}) \]  

(4.8)

Since all variables \( \bar{f} \) and \( \hat{f} \) are functions only of \( x \), their time derivatives are zero. Likewise, all factors \( e^{i\omega t} \) have spatial derivatives that are zero. With these simplifications, and grouping terms of like-power in \( e^{i\omega t} \), Equation (4.8) can be rewritten as,

\[ \frac{\partial}{\partial x}(\bar{v}_i \hat{n}_e) + \left[ i\omega \bar{n}_e + \frac{\partial}{\partial x}(\bar{v}_i \hat{n}_e) \right] e^{i\omega t} + \frac{\partial}{\partial x}(\hat{v}_i \hat{n}_e) e^{2i\omega t} \]  

(4.9)

\[ = \hat{n}_e \bar{v}_{ion} + (\bar{n}_e \hat{v}_{ion} + \hat{n}_e \bar{v}_{ion})e^{i\omega t} + \hat{n}_e \hat{v}_{ion} e^{2i\omega t} \]

Matching the coefficients of the powers of \( e^{i\omega t} \), the 0th order equation becomes,

\[ \frac{\partial}{\partial x}(\bar{v}_i \hat{n}_e) = \bar{v}_i \frac{\partial \hat{n}_e}{\partial x} + \hat{n}_e \frac{\partial \bar{v}_i}{\partial x} = \bar{n}_e \bar{v}_{ion}, \]  

(4.10)

or

\[ \frac{\partial \ln(\bar{n}_e)}{\partial x} + \frac{1}{\bar{v}_i} \frac{\partial \hat{v}_i}{\partial x} = \frac{\bar{v}_{ion}}{\bar{v}_i}. \]  

(4.11)

and the 1st order equation becomes,
\[ i \omega \hat{n}_x + \frac{\partial}{\partial x}(\hat{v}_x \hat{n}_x) + \frac{\partial}{\partial x}(\hat{v}_r \hat{n}_r) = \hat{n}_e \hat{u}_{ton} + \hat{n}_i \hat{u}_{ton}, \text{ or} \]  
(4.12)

\[ i \omega \hat{v}_x + \frac{1}{\hat{v}_i} \hat{v}_e \hat{v}_r \hat{n}_r + \frac{1}{\hat{v}_i} \hat{v}_e \hat{v}_i \hat{n}_e + \frac{1}{\hat{v}_i} \hat{v}_i \hat{v}_r \hat{v}_r + \frac{1}{\hat{v}_i} \hat{v}_i \hat{v}_i \hat{v}_i = \hat{v}_{ton} \hat{v}_x + \hat{v}_{ton} \hat{f}_{ton} \]  
(4.13)

where ratios of perturbation variables to steady state variables, for certain state variables, are grouped for convenience. For this equation, \( \hat{n}_e = \frac{\hat{n}_e}{\hat{n}_i} \), and \( \hat{f}_{ton} = \frac{\hat{v}_{ton}}{\hat{v}_{ton}} \). The ratio \( \frac{\hat{v}_i}{\hat{v}_i} \) is not grouped since \( \hat{v}_i \) passes through zero when the ions transition from travelling toward the inlet to toward the exit. This zero-point would cause a singularity in the ratio, \( \frac{\hat{v}_i}{\hat{v}_i} \), and so it is not incorporated into the equations.

The higher order equation is disregarded for the scope of this 1st order analysis.

After substituting the state variables into the plasma dynamic equations and matching coefficients of like-powers of \( e^{iot} \), there are two separate sets of equations. The first set is the 0th order, or steady state equations, and the second set is the 1st order, linear perturbation equations, which contain terms proportional to \( e^{iot} \). Both sets are coupled, and to be numerically integrated, they need to be either decoupled, or manipulated until a back substitution process is capable of generating each decoupled equation.

Having pursued the latter method, the steady state system of equations have been manipulated so that the partial derivative of potential is decoupled from rest of the system, and can be readily integrated. Since the electron temperature, ion velocity, and electron number density derivatives, contain only the derivative of the potential, and no other derivatives, once the potential derivative is computed, the rest can be computed as well. The same method is used for the perturbation equations, again beginning with the potential derivative.

These two sets of uncoupled equations are shown below in the order in which they must be integrated, and in terms of normalized quantities. These normalization factors are
the same as used in Aledo’s and Martinez-Sanchez’s formulation [9], and can be found in Appendix A.

4.4.2 0th Order, 1-D, Steady State, Plasma Dynamic Equations

When derived through the process described above, the equation for the partial derivative of the potential is,

\[
\frac{\partial \phi}{\partial x} = \frac{1}{\tilde{T}_e - \tilde{\tau}_i \tilde{v}_i^2} \left[ -\tilde{u}_{i\alpha} \tilde{T}_e (2\tilde{v}_i - \tilde{v}_n) + \tilde{u}_{i\alpha} \tilde{v}_i^2 \left( \frac{2\tilde{T}_e}{\tilde{v}_e} + \tilde{T}_e \right) - \tilde{v}_i^2 \tilde{u}_{\alpha\beta} \tilde{v}_e \right].
\]

(4.14)

At a sonic point, when \( \tilde{M}_i = \pm 1 \), where

\[
\tilde{M}_i^2 = \frac{3}{5} \frac{m_i \tilde{v}_i^2}{kT_e}, \quad \text{or}
\]

\[
\tilde{M}_i^2 = \frac{3}{5} \frac{\tilde{v}_i^2}{\tilde{T}_e}.
\]

(4.15)

(4.16)

this potential derivative will diverge and numerical integration will not be possible. If, however, a change of variables is made with an auxiliary variable, \( \tilde{\xi} \), where

\[
\frac{d\tilde{x}}{d\tilde{\xi}} = \tilde{T}_e - \frac{3}{5} \tilde{v}_i^2.
\]

(4.17)

the potential derivative now becomes

\[
\frac{\partial \phi}{\partial \tilde{\xi}} = -\tilde{u}_{i\alpha} \tilde{T}_e (2\tilde{v}_i - \tilde{v}_n) + \tilde{u}_{i\alpha} \tilde{v}_i^2 \left( \frac{2\tilde{T}_e}{\tilde{v}_e} + \tilde{T}_e \right) - \tilde{v}_i^2 \tilde{u}_{\alpha\beta} \tilde{v}_e.
\]

(4.18)
and the singularity at the sonic points are removed. This change of variables, in effect, rescales the axis of integration to accommodate the singularities at the sonic points on the original scale. Since this is a mere remapping of the x-coordinate for the physical dimensions of the channel, it does not have a corresponding perturbation equation (as can be seen in the next section).

The remainder of the system of partial derivatives for the steady state variables is listed below, in terms of the new scale in $\tilde{\xi}$.

\[
\frac{\partial \tilde{T}_e}{\partial \tilde{\xi}} = \frac{1}{v_i^2} \left\{ - (\tilde{W}_e - \tilde{v}_i^2) \frac{\partial \tilde{\phi}}{\partial \tilde{\xi}} - \left[ \tilde{\nu}_{ion} \tilde{T}_e (2 \tilde{v}_i - \tilde{v}_n) + \tilde{v}_i \tilde{\nu}_{He} \tilde{v}_e \frac{d \tilde{\xi}}{d x} \right] \right\}
\]  

\[
\frac{\partial \tilde{v}_i}{\partial \tilde{\xi}} = \frac{1}{v_i^2} \left[ \frac{\partial \tilde{\phi}}{\partial \tilde{\xi}} - \tilde{\nu}_{ion} (\tilde{v}_i - \tilde{v}_n) \frac{d \tilde{\xi}}{d x} \right]
\]  

\[
\frac{\partial \ln(\tilde{n}_e)}{\partial \tilde{\xi}} = \frac{1}{v_i^2} \left[ \frac{\partial \tilde{\phi}}{\partial \tilde{\xi}} + \tilde{\nu}_{ion} (2 \tilde{v}_i - \tilde{v}_n) \frac{d \tilde{\xi}}{d x} \right]
\]  

These equations, in addition to the equation for the potential, are derived from the dynamic equations for ion conservation, electron conservation, momentum conservation, and energy conservation. They can be solved for after determining the derivative of the steady state potential function given in Equation (4.18), above.

From the integral of the difference of the steady state ion-conservation and electron-conservation equations, Equation (4.22) can be derived,

\[
\tilde{v}_e = \tilde{v}_i - \frac{\tilde{\Gamma}_d}{\tilde{n}_e}
\]  

where $\tilde{\Gamma}_d$, the discharge current flux per unit charge, is the constant of the integration.
By taking the sum of the steady state ion-conservation and neutral-conservation equations and integrating, Equation (4.23) can be derived as

\[ \tilde{n}_n = \frac{\tilde{\gamma}_m - \tilde{n}_e \tilde{v}_e}{\tilde{v}_n}, \]  

with the mass flux, \( \tilde{\gamma}_m \), being the constant of integration.

Equation (4.24) and Equation (4.25) follow directly from the substituting the state variables into the corresponding plasma dynamic equations and determining the 0th order relations.

\[ \tilde{\nu}_{ion} = \tilde{\sigma}_0 \tilde{\gamma}_m \frac{8 \tilde{T}_e}{\pi} (1 + 2 \tilde{T}_e) e^{-\frac{1}{2 \tilde{T}_e}} \]  

\[ \tilde{\nu}_H = \frac{\tilde{\omega}^2}{\tilde{n}_e \tilde{\sigma}_{en} \frac{8 \tilde{T}_e}{\pi} + \tilde{\alpha}' \tilde{\omega} \tilde{\theta}} \]  

4.4.3 1st Order, 1-D, Linear Perturbation, Plasma Dynamic Equations

Following the method described in Section 4.4.1, "Deriving the Model Equations" on page 33, the 1st order equations can be derived. As mentioned in the previous section, no perturbation equation is required for rescaling the axial coordinate, \( \tilde{x} \), with \( \tilde{\xi} \). The remainder of the steady state equations have analogous perturbation equations, with a few differences that will be detailed through the course of this section.

From the conservation of ions, electrons, neutrals, ion-momentum, electron-momentum, and energy, the decoupled system of partial differential equations can be determined to be,
\[
\frac{\partial \Phi}{\partial \xi} = \frac{1}{\bar{v}_e} \left[ \frac{1}{\bar{\tau}_e} \left( -\bar{v}_i \frac{\partial \Phi}{\partial \xi} \left[ 2\bar{v}_i \bar{\tau}_e \bar{v}_i - 2\bar{v}_i \bar{\tau}_e \bar{v}_i \right] + \bar{v}_i \left( \bar{\tau}_e \frac{\partial \ln(\bar{n}_e)}{\partial \xi} + \bar{v}_i \frac{\partial \bar{T}_e}{\partial \xi} \right) \bar{v}_i \right] \right) (2\bar{v}_i, -\bar{v}_i, \bar{\tau}_e) \tag{4.26} \]

+ \frac{i\omega}{\bar{v}_e} \bar{v}_i \bar{\tau}_e (\bar{v}_i \bar{n}_e - \bar{v}_i) + \frac{2}{\bar{\tau}_e} \bar{v}_i \bar{\tau}_e (\bar{v}_i \bar{\tau}_e - \bar{v}_i \bar{\tau}_e) + \bar{v}_i \bar{\tau}_e \bar{v}_i \bar{\tau}_e (\bar{v}_i - \bar{v}_i, \bar{\tau}_e) \]

\[
- \bar{u}_{\text{ion}} \bar{v}_i \bar{\tau}_e (2\bar{v}_i, -\bar{v}_i, \bar{\tau}_e) \]

\[
= \frac{2}{\bar{\tau}_e} \bar{v}_i \left[ i\omega \bar{\tau}_e \left( 2\bar{v}_i \bar{n}_e - 3\bar{v}_i \bar{\tau}_e \right) + \frac{2}{\bar{\tau}_e} \bar{v}_i \bar{\tau}_e \bar{v}_i (\bar{v}_e + \bar{\tau}_e - \bar{\tau}_e) + \bar{u}_{\text{ion}} \bar{\tau}_e (\bar{v}_e - \bar{\tau}_e) \right] \]

\[
\frac{\partial \bar{\tau}_e}{\partial \xi} = \frac{1}{\bar{v}_e} \left[ \frac{1}{\bar{\tau}_e} \left( \left( \bar{v}_i + 2\bar{\tau}_e \right) \bar{v}_i - 2\bar{\tau}_e \bar{v}_i \right) \right] \tag{4.27} \]

+ \frac{i\omega}{\bar{v}_e} \bar{v}_i \bar{\tau}_e (\bar{v}_i \bar{n}_e - \bar{v}_i) + \bar{v}_i \bar{\tau}_e \bar{v}_i (\bar{v}_i \bar{\tau}_e - \bar{v}_i \bar{\tau}_e) + \bar{v}_i \bar{\tau}_e \bar{v}_i \bar{\tau}_e (\bar{v}_i - \bar{v}_i, \bar{\tau}_e) \]

\[
- \bar{u}_{\text{ion}} \bar{v}_i \bar{\tau}_e (2\bar{v}_i, -\bar{v}_i, \bar{\tau}_e) \frac{d\bar{\tau}_e}{d\xi} \right) \]

\[
\frac{\partial \bar{v}_i}{\partial \xi} = \left[ \frac{\bar{v}_i}{\bar{v}_e} \left( \bar{v}_i \frac{\partial \bar{v}_i}{\partial \xi} - \bar{v}_i \frac{\partial \bar{\tau}_e}{\partial \xi} + \frac{2}{\bar{\tau}_e} \bar{v}_i \bar{\tau}_e (\bar{v}_i \bar{\tau}_e) \right) \right] \tag{4.29} \]

\[
\frac{\partial \bar{n}_e}{\partial \xi} = \frac{1}{\bar{n}_e} \left[ i\omega \bar{v}_e \bar{n}_e + \bar{n}_e \bar{u}_{\text{ion}} (\bar{n}_e + \bar{n}_e + \bar{\tau}_e) \right] \frac{d\bar{\tau}_e}{d\xi} \tag{4.30} \]

In these equations, as a matter of convenience, ratios of the perturbation variables to their corresponding steady state variables are used. These ratios are defined as, \( \bar{\tau}_e = \frac{\bar{\tau}_e}{\bar{\tau}_e}, \)

\( \bar{n}_e = \frac{\bar{n}_e}{\bar{n}_e}, \) \( \bar{\tau}_e = \frac{\bar{\tau}_e}{\bar{\tau}_e}, \) \( \bar{\tau}_e = \frac{\bar{\tau}_e}{\bar{\tau}_e}, \) \( \bar{\tau}_e = \frac{\bar{\tau}_e}{\bar{\tau}_e}, \) and \( \bar{\tau}_e = \frac{\bar{\tau}_e}{\bar{\tau}_e}. \)
The partial differential equation for the electron velocity perturbation need not be included in the system above since the function itself can be computed by a different method. As was done for the steady state equation for the electron velocity, the perturbation equation can be derived from the difference of the ion-conservation and electron-conservation dynamic equations shown in Table 3.1, “Simplified Plasma Dynamic Equations,” on page 28. By integrating this difference (with the same constant of integration, \( \tilde{\Gamma}_d \)), substituting the state functions, and finding the 1\textsuperscript{st} order relation, the following equation can be derived.

\[
\hat{V}_e = \frac{1}{\hat{n}_e \tilde{v}_e} \left[ \hat{n}_e (\hat{v}_i - \hat{v}_e) \hat{\tilde{\Gamma}}_e + \hat{n}_e \hat{\tilde{v}}_i - \tilde{\Gamma}_d \hat{\tilde{\Gamma}}_d \right] \tag{4.31}
\]

where \( \hat{V}_e = \frac{\hat{v}_e}{\tilde{v}_e} \) and \( \hat{\tilde{\Gamma}}_d = \frac{\tilde{\Gamma}_d}{\tilde{\Gamma}_d} \).

The following expressions, in Equation (4.32) and Equation (4.33) can be found directly through method described in Section 4.4.1, “Deriving the Model Equations” whose derivation is analogous to those for Equation (4.24) and Equation (4.25) in the previous section.

\[
\hat{f}_{\text{ion}} = \hat{n}_e + \left( \frac{1}{2} + \frac{2}{\hat{T}_e} + \frac{1}{\tilde{v}_e} \right) \hat{\tilde{\Gamma}}_e \tag{4.32}
\]

\[
\hat{J}_H = \left[ \frac{1}{1 + \frac{\hat{\omega}_B}{\hat{n}_e \tilde{\omega}_{en} \hat{\tilde{\Gamma}}_d}} \right] \left( -\hat{n}_e - \frac{1}{2} \hat{\tilde{\Gamma}}_e \right) \tag{4.33}
\]
Chapter 5

Boundary Conditions

5.1 0th Order Boundary Conditions

The steady state system is comprised of five differential equations, and hence requires five boundary conditions to determine the constants for their integration. Using the same formulation as Ahedo and Martinez-Sanchez as a basis for this 1st order analysis, a particular choice of parameters for the 0th order boundary conditions is shown below in Table 5.1, "0th Order Boundary Conditions" [9]. The subscript, B, in accord with the paper, denotes the value of the state at the inlet of the chamber (just beyond the anode sheath).

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Physical Interpretation of Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{x}_B = 0$</td>
<td>fixed position of inlet</td>
</tr>
<tr>
<td>$\tilde{T}_{eB} = 0.1$</td>
<td>arbitrary value; see below</td>
</tr>
<tr>
<td>$\tilde{\Phi}_B = 1$</td>
<td>arbitrary value; only potential difference, $\tilde{\Phi}_E - \tilde{\Phi}_B$, affects plasma dynamic behavior</td>
</tr>
<tr>
<td>$\tilde{M}_{iB} = -1$</td>
<td>ions are sonic into anode at interface of space-charge sheath</td>
</tr>
<tr>
<td>$\tilde{\Gamma}_{iB} = -0.01$</td>
<td>arbitrary value; see below</td>
</tr>
</tbody>
</table>

**Table 5.1: 0th Order Boundary Conditions**

When the two conditions, $\tilde{T}_{eB}$ and $\tilde{\Gamma}_{iB}$, are used in the integration process, the length of the channel cannot be specified independently. Instead, it is a function of the values given to these two parameters. Since, in general, the length of the channel has a set value, only one of the parameters is truly arbitrary. However, for this analysis, a specific value of the channel length is not required and the values chosen for the two parameters are those used in the steady state paper [9]. If used for a specific geometry, one parameter can be set.
arbitrarily and the other must be iterated upon to achieve a channel length equal to that of the physical system.

5.2 1st Order Boundary Conditions
Like the steady state system, the perturbation system consists also of five differential equations and requires five boundary conditions to account for their integration constants. Below, in Table 5.2, “1st Order Boundary Conditions”, is the list of these conditions along with the reasoning for their chosen values. As with the 0th order formulation, the subscripts B and E, respectively, denote the values of the state at the chamber’s inlet and exit.

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Physical Interpretation of Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\Gamma}_{mB} = 0 )</td>
<td>fixed mass flow rate, independent of fluctuations (choked injector)</td>
</tr>
<tr>
<td>( \hat{M}_{ib} = 0 )</td>
<td>required condition at sonic point; see Section 5.3, “1st Order Mach Condition at Sonic Points”</td>
</tr>
<tr>
<td>( \hat{r}_{cE} = 0 )</td>
<td>downstream plume matching condition; selected for convenience (a plume model is needed for a better definition)</td>
</tr>
<tr>
<td>( \hat{\phi}_E = \hat{\phi}_B )</td>
<td>fixed potential difference across channel (assumes a large capacitor is in parallel with the thruster)</td>
</tr>
<tr>
<td>( \hat{M}_{iE} = 0 )</td>
<td>required condition at sonic point; see Section 5.3, “1st Order Mach Condition at Sonic Points”</td>
</tr>
</tbody>
</table>

Table 5.2: 1st Order Boundary Conditions

5.3 1st Order Mach Condition at Sonic Points
To determine the special condition to be imposed at the two sonic points at the inlet and exit of the channel, it is necessary to examine the differential equations for the perturbation state variables. The expression for the potential perturbation, Equation (4.26), given
the substitution,

$$\frac{d\tilde{x}}{d\tilde{\xi}} = \tilde{T}_e \frac{3}{5} \tilde{v}_e^2 = \tilde{T}_e \left( 1 - \frac{3}{5} \frac{\tilde{v}_e^2}{\tilde{T}_e} \right) = \tilde{T}_e (1 - \tilde{M}_e^2), \quad (5.1)$$

can be rewritten as

$$\tilde{T}_e (1 - \tilde{M}_e^2) \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} = \frac{1}{\tilde{v}_e} \left[ \frac{1}{\tilde{T}_e (1 - \tilde{M}_e^2)} \tilde{A} + \tilde{B} \right], \quad (5.2)$$

where

$$\tilde{A} = \left[ -\frac{\partial \tilde{\Phi}}{\partial \tilde{\xi}} \left( 2 [\tilde{v}_e^2 - \tilde{T}_e] \tilde{v}_e - \frac{2}{5} \tilde{v}_e^2 \tilde{\xi}_e \right) + \tilde{v}_e \tilde{v}_i \tilde{T}_e \left( \frac{3}{5} \frac{\partial \ln(\tilde{\eta}_e)}{\partial \tilde{\xi}_e} + \frac{\partial \tilde{T}_e}{\partial \tilde{\xi}_e} \right) (2 \tilde{v}_i - \tilde{v}_e \tilde{\xi}_e) \right] \quad (5.3)$$

and

$$\tilde{B} = i \tilde{w}_e \tilde{T}_e (\tilde{v}_i \tilde{\eta}_e - \tilde{v}_e) + \tilde{v}_i^2 \tilde{u}_i \tilde{v}_e (2 \tilde{v}_i - \tilde{v}_e \tilde{\xi}_e) + 2 \tilde{u}_{ion} \tilde{v}_i \tilde{T}_e (\tilde{v}_i - \tilde{v}_e \tilde{\xi}_e) \quad (5.4)$$

It can be seen from Equation (5.2) that the perturbation derivative, $\frac{\partial \tilde{\Phi}}{\partial \tilde{x}}$, will, in general, diverge at any sonic point. It is important at this point to distinguish two cases, depending on whether the 0th order (steady) solution has a choked exit or a supersonic exit with a previous smooth sonic passage. Only the first of these possibilities is considered in detail in this paper, although it must be pointed out that even when the "forward" sonic
point is internal and smooth, the "backwards" sonic point, at the inlet, is still choked, and the special acoustic condition for this case will still apply there.

5.3.1 Perturbation Condition for Singular Point

When the exit (or inlet) is sonic, Equation (4.17) shows that \( \frac{d\bar{x}}{d\bar{\xi}} = 0 \) there, and since the right-hand side of Equation (4.18) for \( \frac{\partial \phi}{\partial \bar{\xi}} \) is not, in general, zero in this case, then \( \frac{\partial \phi}{\partial x} \) will be infinite at the sonic point. The same is then true of other 0\(^{\text{th}}\) order derivatives. The behavior near the sonic point is, \( \bar{\phi} = \bar{\phi}_{\text{sonic}} + a\sqrt{x_{\text{sonic}} - x + \ldots} \), etc., so the singularities are integrable, and the sonic variables themselves, \( \bar{f}_{\text{sonic}} \), are finite. It then follows that this same level of singularity is allowable for the perturbation quantities, \( \bar{f} \), but no higher (if \( \bar{\phi} \) diverged faster than \( \bar{\phi} \) near the sonic point this would violate the linearization assumption \( |\bar{\phi}| \ll \bar{\phi} \)). With reference to Equation (5.2), this means that the coefficient \( \hat{\lambda} \) must be zero at the sonic point, but not necessarily \( \hat{\beta} \).

To determine the condition that meets this requirement, \( \hat{\lambda} \) should be computed for \( \hat{M}_i^2 = 1 \). Equation (5.5) and Equation (5.6) are the 0\(^{\text{th}}\) order state derivatives at a sonic point, in terms of the derivative for the potential perturbation.

\[
\left. \frac{\partial \ln(\tilde{n}_e)}{\partial \bar{\xi}} \right|_{\tilde{M} = \pm 1} = \left. \left( \frac{1}{\tilde{M}^2} \frac{\partial \tilde{\phi}}{\partial \bar{\xi}} \right) \right|_{\tilde{M} = \pm 1} \quad (5.5)
\]

\[
\left. \frac{\partial \tilde{T}_e}{\partial \bar{\xi}} \right|_{\tilde{M} = \pm 1} = \left. \frac{2\tilde{\phi}}{5\tilde{M}^2} \right|_{\tilde{M} = \pm 1} \quad (5.6)
\]

Now, substituting these expressions into \( \hat{\lambda} \) with the Mach perturbation relation,

\[
\frac{\hat{M}}{\tilde{M}} \left|_{\tilde{M} = \pm 1} = \frac{\tilde{v}_i}{\tilde{v}_i} \right| \frac{1}{2} \tilde{x}_e \quad (5.7)
\]

forms the condition
\[ \tilde{\lambda}|_{\hat{M} = \pm 1} = \left[ \frac{6 \cdot 3 \tilde{\phi}}{5 \hat{v}_i^2 \frac{\partial}{\partial \xi} \hat{M}} \right]_{\hat{M} = \pm 1}. \]  \tag{5.8}

Since (from Equation (4.18))

\[ \frac{\partial \tilde{\phi}}{\partial \xi} \bigg|_{\hat{M} = \pm 1} = \left[ -\tilde{\nu}_{i\text{on}} \left( \frac{3 \hat{v}_i^2}{5} \right) (2 \hat{v}_i - \hat{v}_n) + \tilde{\nu}_{i\text{on}} \hat{v}_i^2 \left( \frac{2 \hat{v}_i^2 + \frac{3 \hat{v}_i^2}{5}}{\hat{v}_e} \right) - \hat{v}_i^2 \tilde{\nu}_{i\text{on}} \hat{v}_e \right]_{\hat{M} = \pm 1} \neq 0, \tag{5.9} \]

in general, then for the condition \( \tilde{\lambda}|_{\hat{M} = \pm 1} = 0 \) to be satisfied,

\[ \hat{M}|_{\hat{M} = \pm 1} = 0. \tag{5.10} \]

This condition also appears for all similar acoustic problems. As an illustrative example, the problem of an ideal compressible flow in a convergent duct open to vacuum, also requires that the Mach perturbation be zero at the exit. The details of this analysis can be seen in Appendix B.

These two models, the ion-acoustic and the pure acoustic, lead to the same condition that the perturbation of the Mach number must be zero at a choked exit. Although this condition has been inferred through the mathematical formulation of the models, a physical interpretation can also be drawn. In the case of a choked exit, the steady state derivatives, such as flow-acceleration, are very large (infinite in a 1-D analysis) at the sonic passage. Because of this, when considering the dynamics of a perturbation, its convective rate of change dwarfs its local time derivative (unless the frequency is extremely high), and these \( \frac{\partial \tilde{\phi}}{\partial t} \) terms vanish from the balance equations. The dynamics is then locally quasi-steady, and the Mach number remains at unity despite fluctuations, since this is the steady state sonic condition.
5.3.2 Perturbation Condition for a Smooth Sonic Passage

For a smooth sonic transition, the steady state derivatives do not diverge, as in the case of the choked exit. For this type of sonic transition, the flow accelerates from subsonic to supersonic, smoothly, with no singularities, and this usually occurs at a point internal to the channel, instead of at the exit, although, for geometries similar to an SPT, this internal point occurs near the exit of the channel.

In this case, the derivatives, \( \frac{\partial \tilde{f}}{\partial \tilde{x}} \), remain finite, and hence \( \frac{\partial \Phi}{\partial \xi} \) and all other \( \xi \) derivatives become zero at the sonic point. Therefore, from Equation (5.3), the factor \( \hat{A} \) is automatically zero at \( \tilde{M}_i^2 = 1 \), and \( \frac{\hat{A}}{1 - \tilde{M}_i^2} \) is generally finite. Returning to Equation (5.2), we now impose that perturbation derivatives must remain finite (just as the 0th order derivatives), and therefore, the entire right-hand side must vanish at \( \tilde{M}_i^2 = 1 \), or

\[
\frac{1}{\tilde{T}_\epsilon(1 - \tilde{M}_i^2)} \hat{A} + \hat{B} = 0 \text{ (at } \tilde{M}_i^2 = 1) \tag{5.11}
\]

where \( \hat{A} \) and \( \hat{B} \) are defined in Equation (5.3) and Equation (5.4), respectively.

When this condition is met, the differential equation for the potential perturbation, Equation (5.2), becomes indeterminate. To resolve this indeterminacy, \( \frac{\partial \Phi}{\partial \tilde{x}} \) can be found with the expression below.

\[
\frac{\partial \Phi}{\partial \tilde{x}} = \lim_{M_i \to z \tilde{C}} \frac{\hat{D}}{C}
\]

(5.12)

where

\[
\tilde{C} = \tilde{T}_\epsilon(1 - \tilde{M}_i^2)
\]

(5.13)

\[
\hat{D} = \frac{1}{\tilde{M}_i \tilde{T}_\epsilon(1 - \tilde{M}_i^2)} \hat{A} + \hat{B}
\]

(5.14)
Since $\dot{C}$ at a sonic point is zero, and $\dot{D}$ must be zero to match both sides of Equation (5.2), then Equation (5.12) must be solved using l'Hopital's Rule. By taking the derivative of $\dot{C}$ and $\dot{D}$ individually with respect to $\dot{M}_i$, and solving the limit, the value of $\frac{\partial \rho}{\partial x}$ can be determined at the sonic point. Having this value, the rest of the states can be determined with Equation (4.17) through Equation (4.33). Since this condition must be met for a smooth sonic transition, the numerical integration scheme should begin at this point, and proceed away from it.

At the time of this paper no further research has been done to pursue this solution of the linear perturbation equations. Such a study is beyond the scope of this paper, but may be necessary in order to capture the entire plasma dynamic behavior of the system.
Chapter 6

Integration Scheme

6.1 0\textsuperscript{th} Order Integration

The steady state solution is computed from a set of five differential equations consisting of Equation (4.17) through Equation (4.21). These equations are a set of non-linear, coupled, partial differential equations and must be solved using an appropriate integration scheme. With the five boundary conditions detailed in Section 5.1, “0th Order Boundary Conditions” on page 41, and the first integrals listed as Equation (4.22) and Equation (4.23), this system can be solved using ODE45, an integration package predefined in Matlab 5.2. ODE45 is a medium order, nonlinear, partial differential equation integrator which solves systems given initial boundary conditions. This scheme was used to generate the 0\textsuperscript{th} order solution in Abedo’s and Martinez-Sanchez’s paper [9].

Since ODE45 requires all boundary conditions to be in terms of the variables being integrated, as well as defined at a single point, either the inlet or exit, the set of boundary conditions described in Section 5.1, “0th Order Boundary Conditions” on page 41, must be used to extract the full set of integration variables, $\tilde{x}$, $\tilde{n}$, $\tilde{v}$, $\tilde{T}$, and $\tilde{\phi}$, from the specified inlet conditions $\tilde{x}_B$, $\tilde{T}_eB$, and $\tilde{\phi}_B$, plus $\tilde{M}_{iB}$ and $\tilde{\Gamma}_{iB}$. For this purpose, we use the relations given below.

\[
\tilde{M}_i = \frac{\tilde{v}_i}{\sqrt[5]{\tilde{T}_e}} \\
\tilde{\Gamma}_i = \tilde{n}_\epsilon \tilde{v}_i
\]  

(6.1)  

(6.2)
Now, all five boundary conditions can be expressed at one point, the inlet, in terms of the variables to be integrated. These conditions can be found below in Table 6.1, "0th Order Inlet Conditions".

<table>
<thead>
<tr>
<th>Inlet Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x}_B = 0 )</td>
</tr>
<tr>
<td>( \dot{n}<em>{eB} = \frac{\dot{\Gamma}</em>{mB}}{\dot{v}_{lB}} )</td>
</tr>
<tr>
<td>( \dot{v}<em>{lB} = -\frac{S}{\sqrt{3}} \dot{T}</em>{eB} )</td>
</tr>
<tr>
<td>( \dot{T}_{eB} = 0.1 )</td>
</tr>
<tr>
<td>( \dot{\phi}_B = 1 )</td>
</tr>
</tbody>
</table>

Table 6.1: 0th Order Inlet Conditions

The solution of the system requires the definition of two free parameters whose values are arbitrary for this study, the discharge current density per unit charge, \( \dot{\Gamma}_d \), and the mass flux, \( \dot{\Gamma}_m \). Plots for the 0th order solution of the state variables are shown below, in Figure 6.1, where the ionization rate is defined as

\[
R_{ion} = n_e \nu_{ion}, \text{ and likewise } \dot{R}_{ion} = \dot{n}_e \dot{\nu}_{ion} \tag{6.3}
\]
Figure 6.1: 0th Order State Variables

Notice the narrow ionization layer near the sonic exit, where the electron temperature has a sharp maximum, and the long diffusion layer preceding it. The ion velocity is very low (and mostly negative) in the diffusion layer, and accelerates rapidly in the ionization region.

6.2 1st Order Integration
The 1st order scheme is a system of five linear partial differential equations and five boundary conditions. The system consists of Equation (4.26) through Equation (4.30), and the boundary conditions are as discussed in Section 5.2, “1st Order Boundary Conditions” on page 42. A free parameter, chosen to be the discharge current perturbation, $\dot{y}_d$, can be defined arbitrarily since the linear system’s solution consists of functions with arbitrary
magnitudes and phases, and must be computed with respect to some fixed reference.

For simplicity, a forward Euler integration scheme is used. The iteration relation is shown below in Equation (6.4)

$$\ddot{f}(\xi_{n+1}, \tilde{\omega}) = \dot{f}(\xi_n, \tilde{\omega}) + \frac{\partial}{\partial \xi} \dot{f}(\xi, \tilde{\omega}) \bigg|_{\xi = \xi_n} d\xi$$  

(6.4)

where

$$\dot{\xi}_{n+1} = \dot{\xi}_n + d\xi.$$  

(6.5)

The step size, $d\xi$, is chosen so that the solutions generated remain relatively invariant with small changes in $d\xi$. For an error on the order of a percent, $d\xi = 0.003$ is sufficient. Furthermore each perturbation derivative is a function of the steady state variables, their derivatives, the perturbation variables, and the complex frequency $\tilde{\omega}$, or

$$\frac{\partial}{\partial \xi} \dot{f}(\xi, \tilde{\omega}) \bigg|_{\xi = \xi_n} = F \left[ \dot{f}(\xi_n, \tilde{\omega}), \frac{\partial}{\partial \xi} \dot{f}(\xi, \tilde{\omega}) \bigg|_{\xi = \xi_n}, \dot{f}(\xi_n, \tilde{\omega}) \right].$$  

(6.6)

Since the 1st order solution is a function of the 0th order quantities, at each 1st order integration step there must be a corresponding 0th order step to provide the necessary values. Because of this, although the 0th order system does not need a step size as small as 0.003, this step size must be used to accommodate the 1st order integration. (For an error on the order of a percent, the 0th order integration requires a step size of approximately 0.01.)

For this integration scheme, as with ODE45 from Matlab 5.2, the five boundary conditions must be defined in terms of the integrated variables, from one end, either inlet or exit.
Unlike the 0th order boundary conditions, however, the 1st order conditions are not all defined at the inlet. Instead, there are two conditions at the inlet, and three at the exit. Because the two inlet conditions are not integration variables, they only give relations that can be used to find such conditions. These relations are expressed in the two equations below.

\[
\frac{\tilde{V}_{mB}}{\tilde{V}_{mB}} = \tilde{Y}_{mB} = \frac{\tilde{\eta}_{nB}}{\tilde{\eta}_{nB}^{\prime} \tilde{v}_{iB}} \left( \tilde{\eta}_{eB} + \frac{\tilde{\eta}_{eB}}{\tilde{v}_{iB}} \right) = 0 \quad (6.7)
\]

\[
\frac{\tilde{M}_{iB}}{\tilde{M}_{iB}} = \frac{\tilde{v}_{iB}}{\tilde{v}_{iB}} - \frac{1}{2} \tilde{\tau}_{eB} = 0 \quad (6.8)
\]

With these two relations, in order to integrate the system from the inlet, three additional boundary conditions are needed. The method for solving this issue is detailed in the following section.

6.3 1st Order Boundary Condition Matching

Since the system is linear, these inlet conditions can be given set values for three separate cases, and three solutions will then be generated from the integration scheme. A linear combination of these three solutions will then produce the actual solution of the system. This is represented below in Equation (6.9)

\[
\tilde{f}(\tilde{\xi}, \tilde{\omega}) = c_1 \tilde{f}^{(1)}(\tilde{\xi}, \tilde{\omega}) + c_2 \tilde{f}^{(2)}(\tilde{\xi}, \tilde{\omega}) + c_3 \tilde{f}^{(3)}(\tilde{\xi}, \tilde{\omega}) \quad (6.9)
\]

Having set the values for the three unknown inlet conditions (\(\tilde{\eta}_{nB}, \tilde{\tau}_{eB}, \) and \(\tilde{\phi}_{B}\)), the other two can be determined from Equation (6.7) and Equation (6.8). Now, all boundary conditions are defined at the inlet, in terms of the integration variables. The three sets of inlet conditions, computed from those in Table 6.1, "0th Order Inlet Conditions", are listed below in Table 6.2, "Set of Inlet Conditions for Superposition of Solutions".

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Table 6.2: Set of Inlet Conditions for Superposition of Solutions

With these sets of inlet conditions, the integration is carried through and three solutions are generated. The actual solution to the system, which is some linear combination of the three generated solutions, must satisfy the exit conditions described in Section 5.2, “1st Order Boundary Conditions” on page 42, namely, $\dot{M}_{iE} = 0$, $\dot{v}_{iE} = 0$, and $\dot{\phi}_E = \dot{\phi}_B$. From these, $\dot{v}_{iE} = 0$ as well. This requirement can be expressed as shown in the system below:

$$
\dot{v}_{iE}(\omega) = 0 = c_1 \dot{v}_{iE}^{(1)} + c_2 \dot{v}_{iE}^{(2)} + c_3 \dot{v}_{iE}^{(3)} \tag{6.10}
$$

$$
\dot{\tau}_{iE}(\omega) = 0 = c_1 \dot{\tau}_{iE}^{(1)} + c_2 \dot{\tau}_{iE}^{(2)} + c_3 \dot{\tau}_{iE}^{(3)} \tag{6.11}
$$

$$
\dot{\phi}_E(\omega) = \dot{\phi}_B = c_1 \dot{\phi}_E^{(1)} + c_2 \dot{\phi}_E^{(2)} + c_3 \dot{\phi}_E^{(3)} = c_1 \dot{\phi}_B^{(1)} + c_2 \dot{\phi}_B^{(2)} + c_3 \dot{\phi}_B^{(3)} = c_3 \tag{6.12}
$$

or

$$
M(\omega)c = \begin{bmatrix}
\dot{v}_{iE}^{(1)} & \dot{v}_{iE}^{(2)} & \dot{v}_{iE}^{(3)} \\
\dot{\tau}_{iE}^{(1)} & \dot{\tau}_{iE}^{(2)} & \dot{\tau}_{iE}^{(3)} \\
\dot{\phi}_E^{(1)} & \dot{\phi}_E^{(2)} & (\dot{\phi}_B - 1)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = 0 \tag{6.13}
$$
The matrix $M(\tilde{\omega})$ is a non-linear function of the oscillation frequency, $\tilde{\omega}$, and the vector, $\xi$, contains the coefficients to linearly compose the actual solution of the 1st order state variables, $\tilde{J}(\tilde{\xi}, \tilde{\omega})$. The system $M(\tilde{\omega})_c = 0$ is only satisfied for a valid $\tilde{\omega}$. Determining the set of valid frequencies to satisfy this system is the issue discussed in the next section, "Frequency Determination" on page 55.

### 6.4 Frequency Determination

Since the system $M(\tilde{\omega})_c$ is highly non-linear with $\tilde{\omega}$, there is no simple, analytical way, to determine the value of a frequency which satisfies the system. Due to this difficulty, it is necessary to guess a trial, complex value, for $\tilde{\omega}$ and generate $M(\tilde{\omega})$ through the numerical integration process. Because the system is homogenous, a valid choice of $\tilde{\omega}$ would make the determinant of $M(\tilde{\omega})$ equal to zero and the vector of linear combination coefficients, $\xi$, an eigenvector of $M(\tilde{\omega})$.

Using the expression,

$$MOE = \frac{1}{|M|},$$  \hspace{1cm} (6.14)

as the measure of effectiveness for the choice of $\tilde{\omega}$, the complex $\tilde{\omega}$ plane of possible values for the frequency can be scanned. When the measure of effectiveness is small, it means the chosen frequency results in three simultaneous equations that cannot be linearly combined to match the exit boundary conditions of the system. A large measure of effectiveness, or an infinite value, means that the chosen frequency leads to a valid solution of the 1st order state variables. For these cases, the coefficients, $\xi$, can be computed and used to form the state in accord with Equation (6.9) above.

As with most acoustic or analogous problems, solutions inherently contain an infinite set of harmonics associated with a particular fundamental harmonic. For the purpose of
this study, only the first few harmonics were pursued. Plots and analysis of their computed behavior can be found in the following chapter, “Results and Analysis of Model Solution”.
Chapter 7

Results and Analysis of Model Solution

7.1 Solution Set of Frequencies
Using the integration scheme detailed in the previous section, the model equations from Section 4.4, "Model Equations" on page 33 were solved for values of the frequency which satisfied the boundary conditions. Since an infinite number of solutions exist, only the first few modes were pursued. These are the solutions closest to the origin, where the fundamental harmonics, in general, are located. Figure 7.1 shows the domain of the complex $\omega$ plane scanned for this study. At each $\omega$, (with increments of 0.1 for both the real and imaginary axes) the 1st order system was solved and the measure of effectiveness, defined in Section 6.4, "Frequency Determination", was computed. Each contour on the plot represents a constant value of the computed measure of effectiveness.
**Figure 7.1:** Measure of Effectiveness for Scanned Domain of Complex $\omega$

As can be seen in the contour plot, there are areas where there is a high concentration of relatively large values for the measure of effectiveness. The clear regions above and below these concentrations have values of the measure of effectiveness which are orders of magnitude smaller than those in the concentrated areas, and therefore no contour lines were generated to represent them. It should also be noted that no solutions were found with a negative imaginary part. Solutions of this type were expected because they would diverge with time and would be considered unstable.

Also from Figure 7.1, it can be seen that in each concentrated area, the contour lines tend to focus inward with increasing values for the measure of effectiveness. This implies
convergence of the solutions to those which satisfy the conditions of integration detailed in Chapter 6. To get more accuracy for the value of this frequency, these regions were scanned with a resolution smaller than 0.1. This resolution was reduced until the resulting solutions remained relatively insensitive to small changes in the step size of $\omega$. A step size of 0.001 was determined to be adequate for this study.

Having scanned the domain of the complex $\omega$ plane shown in Figure 7.1, six frequencies were found which satisfy the conditions of integration. These frequencies are shown below in Table 7.1. As the formulation deals with non-dimensional quantities, the frequencies are normalized by the normalization factor, $\omega^*$, where

$$
\omega^* = 2.35 \times 10^3 \left[ \frac{\text{rad}}{\text{s}} \right].
$$

The true values of the frequencies are also shown in the table below.

<table>
<thead>
<tr>
<th>Normalized Frequency</th>
<th>True Frequency [rad/sec]</th>
<th>Real Part of True Frequency [kHz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.981i</td>
<td>231$\times 10^3 i$</td>
<td>0</td>
</tr>
<tr>
<td>1.279 + 0.651i</td>
<td>301$\times 10^3 + 153\times 10^3 i$</td>
<td>47.9</td>
</tr>
<tr>
<td>1.982 + 1.473i</td>
<td>466$\times 10^3 + 346\times 10^3 i$</td>
<td>74.2</td>
</tr>
<tr>
<td>2.717 + 1.103i</td>
<td>638$\times 10^3 + 259\times 10^3 i$</td>
<td>102</td>
</tr>
<tr>
<td>3.250 + 0.935i</td>
<td>764$\times 10^3 + 220\times 10^3 i$</td>
<td>122</td>
</tr>
<tr>
<td>4.332 + 1.706i</td>
<td>1020$\times 10^3 + 401\times 10^3 i$</td>
<td>162</td>
</tr>
</tbody>
</table>

**Table 7.1: Frequencies Satisfying Conditions of Integration**

The solutions to the 1st order system generated with these frequencies are shown and discussed in the following section.
7.2 1\textsuperscript{st} Order State Variables

The results of numerically integrating the 1\textsuperscript{st} order system with each of the six frequencies are shown in Figure 7.2 through Figure 7.13. Each solution is represented by two separate plots, one of magnitude, and one of phase. Two plots are necessary to characterize the behavior of each state variable since they are, in general, complex functions.

Note, that all variables are normalized by those factors described in Appendix A.2, and that all phase angles are in radians. Also, to avoid ambiguity, asterisks are used to indicate the location on the plot of the inlet and exit values.
**Figure 7.2:** Perturbation Magnitudes for $f_r = 0kHz$ ($\bar{\omega} = 0.981i$)

**Figure 7.3:** Perturbation Phases for $f_r = 0kHz$ ($\bar{\omega} = 0.981i$)
Figure 7.4: Perturbation Magnitudes for $f_r = 47.9kHz \ (\tilde{\omega} = 1.279 + 0.651i)$

Figure 7.5: Perturbation Phases for $f_r = 47.9kHz \ (\tilde{\omega} = 1.279 + 0.651i)$
Figure 7.6: Perturbation Magnitudes for $f_r = 74.2kHz$ ($\tilde{\omega} = 1.982 + 1.473i$)

Figure 7.7: Perturbation Phases for $f_r = 74.2kHz$ ($\tilde{\omega} = 1.982 + 1.473i$)
**Figure 7.8:** Perturbation Magnitudes for $f_r = 102 \text{kHz}$ ($\tilde{\omega} = 2.717 + 1.103i$)

**Figure 7.9:** Perturbation Phases for $f_r = 102 \text{kHz}$ ($\tilde{\omega} = 2.717 + 1.103i$)
Figure 7.10: Perturbation Magnitudes for $f_r = 122kHz$ ($\omega = 3.250 + 0.935i$)

Figure 7.11: Perturbation Phases for $f_r = 122kHz$ ($\omega = 3.250 + 0.935i$)
Figure 7.12: Perturbation Magnitudes for $f_r = 162kHz$ ($\tilde{\omega} = 4.332 + 1.706i$)

Figure 7.13: Perturbation Phases for $f_r = 162kHz$ ($\tilde{\omega} = 4.332 + 1.706i$)
The first mode is a stationary mode because the real part of its frequency is zero. Since it has a positive imaginary part, this mode decays with time with no wave movement. The next five modes, however, have wave motion due to the non-zero value of their frequency's real component. As can be expected with wave characteristics, the magnitudes of modes two through six, in general, have increasing curvature with larger real components of frequency. Also, it can be noted that the second, fourth, and fifth modes have similar phase characteristics, whereas the third and sixth modes, although similar to each other, have different phase characteristics from the first group. This grouping of phase characteristics is discussed further in the next section, “Patterns of Behavior of the Various Modes”.

In all the modes, as was noted in with the 0th order variables in Section 6.1, “0th Order Integration”, there are two layers with distinct behavior, an ionization layer near the exit, preceded by a longer diffusion layer. It can also be seen that in the ionization region, perturbation magnitudes are much larger, indicating a higher level of activity.

7.3 Patterns of Behavior of the Various Modes
From the 1st order solutions, determined through the integration process, the perturbation state variables are characterized by two plots, magnitude and phase. Another method by which these solutions can be assessed is by looking at plots of their real and imaginary parts. The real part of the state is the oscillation that can be observed in the laboratory. The imaginary part, in general, contributes to the phase of the oscillation, and deals with the relative behavior of the waves with respect to some reference. The variable which defines the arbitrary, zero-phase, reference is \( \dot{\gamma}_d \), or the perturbation of the discharge current, normalized by its steady state value. Because \( \dot{\gamma}_d \) is a parameter with an arbitrary value of unity, the phase is set to be identically zero.
Since these variables are defined, by Equation (4.3), to oscillate with their corresponding frequency, when propagated though time, their real part will be the linear perturbation of the state. This relation is expressed below, where \( n_\epsilon' \), for example, is the real part of the electron/ion number density perturbation.

\[
n_\epsilon' = \Re\{\hat{n}_\epsilon e^{i\omega t}\}
\]  

(7.2)

From Equation (4.3), it is inherent that all state variables will oscillate with the same frequency in each mode. This being the case, only one state variable need be analyzed for oscillatory behavior. For this particular study, the electron number density perturbation is most relevant. To visualize the dynamics of this state variable, two plots have been produced, one a surface plot, which shows the variation of magnitudes during propagation, and the other, a contour plot, which more clearly illustrates propagation. These plots have been generated for each frequency, and can be seen in Figure 7.14 through Figure 7.25.
Figure 7.14: Propagation of Electron Number Density Perturbation for $f_r = 0\,kHz$
\[ \tilde{\omega} = 0.981i \]

Figure 7.15: Propagation of Electron Number Density Perturbation for $f_r = 0\,kHz$
\[ \tilde{\omega} = 0.981i \]
Figure 7.16: Propagation of Electron Number Density Perturbation for $f_r = 47.9\text{kHz}$

($\tilde{\omega} = 1.279 + 0.651i$)

Figure 7.17: Propagation of Electron Number Density Perturbation for $f_r = 47.9\text{kHz}$

($\tilde{\omega} = 1.279 + 0.651i$)
Figure 7.18: Propagation of Electron Number Density Perturbation for $f_r = 74.2kHz$

($\tilde{\omega} = 1.982 + 1.473i$)

Figure 7.19: Propagation of Electron Number Density Perturbation for $f_r = 74.2kHz$

($\tilde{\omega} = 1.982 + 1.473i$)
Figure 7.20: Propagation of Electron Number Density Perturbation for $f_r = 102 kHz$

($\tilde{\omega} = 2.717 + 1.103i$)

Figure 7.21: Propagation of Electron Number Density Perturbation for $f_r = 102 kHz$

($\tilde{\omega} = 2.717 + 1.103i$)
**Figure 7.22:** Propagation of Electron Number Density Perturbation for $f_r = 122\text{kHz}$ 
$(\tilde{\omega} = 3.250 + 0.935i)$

**Figure 7.23:** Propagation of Electron Number Density Perturbation for $f_r = 122\text{kHz}$ 
$(\tilde{\omega} = 3.250 + 0.935i)$
Figure 7.24: Propagation of Electron Number Density Perturbation for $f_r = 162kHz$

\[ \omega = 4.332 + 1.706i \]

Figure 7.25: Propagation of Electron Number Density Perturbation for $f_r = 162kHz$

\[ \omega = 4.332 + 1.706i \]
By inspection of the surface plots, all solutions are highly damped, and decay within a few periods. As is to be expected with higher order modes, the higher frequency surface plots tend to have more curvature to them.

As can be seen in Figure 7.16, Figure 7.18, Figure 7.20, Figure 7.22, and Figure 7.24, there are two distinct modes of oscillation. In the second, fourth, and fifth surface plots, there is a translational propagation of the waves. This can also be seen in their respective contour plots. The third and sixth plots, show a different type of behavior. Although difficult to see in the surface plots, the contour plots clearly show that these two solutions propagate from each end of the channel and converge inwards. If damping is neglected, these modes tend to form a complex wave structure that resembles a standing wave. A standing wave form can either be thought of as two identical waves, travelling in opposite directions, or a single wave that is stationary in space, and whose amplitude oscillates with time. By regarding the plasma in the latter sense, the state perturbations appear to have a sloshing type of motion during this pattern of behavior. This is discussed further in Section 7.3.2, “Sloshing Modes”.

7.3.1 Translational Modes

From the contour plots, Figure 7.17, Figure 7.21, and Figure 7.23, it can be seen that the waves propagate in near linear fashion. The phase plots of the neutral and electron/ion number density perturbations provides similar information. These plots are shown in Figure 7.29, Figure 7.33, and Figure 7.35.
Figure 7.26: Perturbation Magnitudes in Ionization Layer for $f_r = 0kHz$ ($\tilde{\omega} = 0.981i$)

Figure 7.27: Perturbation Phases in Ionization Layer for $f_r = 0kHz$ ($\tilde{\omega} = 0.981i$)
Figure 7.28: Perturbation Magnitudes in Ionization Layer for $f_r = 47.9\text{kHz}$
($\bar{\omega} = 1.279 + 0.651i$)

Figure 7.29: Perturbation Phases in Ionization Layer for $f_r = 47.9\text{kHz}$ ($\bar{\omega} = 1.279 + 0.651i$)
Figure 7.30: Perturbation Magnitudes in Ionization Layer for $f_r = 74.2\,kHz$ 
($\bar{\omega} = 1.982 + 1.473i$)

Figure 7.31: Perturbation Phases in Ionization Layer for $f_r = 74.2\,kHz$ ($\bar{\omega} = 1.982 + 1.473i$)
Figure 7.32: Perturbation Magnitudes in Ionization Layer for $f_r = 102\,kHz$
($\tilde{\omega} = 2.717 + 1.103i$)

Figure 7.33: Perturbation Phases in Ionization Layer for $f_r = 102\,kHz$ ($\tilde{\omega} = 2.717 + 1.103i$)
Figure 7.34: Perturbation Magnitudes in Ionization Layer for $f_r = 122 kHz$

($\tilde{\omega} = 3.250 + 0.935i$)

Figure 7.35: Perturbation Phases in Ionization Layer for $f_r = 122 kHz$ ($\tilde{\omega} = 3.250 + 0.935i$)
Figure 7.36: Perturbation Magnitudes in Ionization Layer for $f_r = 162 kHz$ ($\tilde{\omega} = 4.332 + 1.706i$)

Figure 7.37: Perturbation Phases in Ionization Layer for $f_r = 162 kHz$ ($\tilde{\omega} = 4.332 + 1.706i$)
From the phase plots, it appears that, in the ionization region of the first translational mode, Figure 7.29, the neutral and electron/ion number densities have similar phases, about 20 degrees apart. Similarly, for the higher order modes, shown in Figure 7.33 and Figure 7.35, the phase differences are approximately, 3 and 5 degrees, respectively, or almost exactly in phase.

The ionization zone is defined by the region where the rapid increase of the ionization rate is located, and can be seen in Figure 7.28. This follows the physical picture that the perturbation of both, ions and neutrals, travel together with the wave, in an acoustic or perhaps convective fashion, and exit the thruster. This leads to the notion that the fluctuations in the ionization region is closely related to the fluctuations of the ion and neutral number densities, and not those in electron temperature. This idea more directly concerns the behavior of the sloshing modes, and is detailed in the next section.

7.3.2 Sloshing Modes

To visualize the sloshing behavior of the third and sixth modes, the perturbation quantity for the ion velocity,

\[ v'_i = \Re \{ \hat{v}_i e^{i\omega t} \} \] (7.3)

is plotted in the same manner that \( n'_e \) was plotted to visualize the behavior of translational modes.
Figure 7.38: Propagation of Ion Velocity Perturbation for $f_r = 0kHz$ ($\omega = 0.981i$)

Figure 7.39: Propagation of Ion Velocity Perturbation for $f_r = 0kHz$ ($\omega = 0.981i$)
Figure 7.40: Propagation of Ion Velocity Perturbation for $f_r = 47.9 kHz$
($\omega = 1.279 + 0.651i$)

Figure 7.41: Propagation of Ion Velocity Perturbation for $f_r = 47.9 kHz$
($\omega = 1.279 + 0.651i$)
**Figure 7.42:** Propagation of Ion Velocity Perturbation for $f_r = 74.2 kHz$

($\tilde{\omega} = 1.982 + 1.473i$)

**Figure 7.43:** Propagation of Ion Velocity Perturbation for $f_r = 74.2 kHz$

($\tilde{\omega} = 1.982 + 1.473i$)
**Figure 7.46:** Propagation of Ion Velocity Perturbation for $f_r = 122 kHz$

($\hat{\omega} = 3.250 + 0.935i$)

**Figure 7.47:** Propagation of Ion Velocity Perturbation for $f_r = 122 kHz$

($\hat{\omega} = 3.250 + 0.935i$)
Figure 7.46: Propagation of Ion Velocity Perturbation for $f_r = 122\, kHz$
($\tilde{\omega} = 3.250 + 0.935i$)

Figure 7.47: Propagation of Ion Velocity Perturbation for $f_r = 122\, kHz$
($\tilde{\omega} = 3.250 + 0.935i$)
Figure 7.48: Propagation of Ion Velocity Perturbation for $f_r = 162kHz$
($\omega = 4.332 + 1.706i$)

Figure 7.49: Propagation of Ion Velocity Perturbation for $f_r = 162kHz$
($\omega = 4.332 + 1.706i$)
Figure 7.43 and Figure 7.49 show the dynamics of the sloshing modes. As can be seen by the contour plots, lobes of ion velocity perturbation seem to propagate toward the center of the thruster’s channel. In Figure 7.43 the two lobes at the bottom of the plot show the velocity perturbations of the ions. Since the left lobe has a positive velocity (toward the right), and the right lobe has a negative velocity (toward the left) the sloshing effect can be readily seen. The second of the two sloshing modes, shown in Figure 7.49, contains similar features. Because this is a higher order mode, instead of two lobes of opposing velocities, there are four lobes with alternating velocities. Beginning from the left, the first and third of the lobes have a positive velocity, and the second and the fourth lobes have a negative velocity. This can be visualized as two sets of converging lobes instead of only one in the case of the lower order sloshing mode.

Similar traits can be seen in the surface plots in Figure 7.42 and Figure 7.48. In addition to these three dimensional plots, in a bit more subtle way, the plots of the 1st order solutions, Figure 7.2 through Figure 7.13, show the counter-flow action of the sloshing mode. Instead of the value of the perturbation’s real part propagating backwards, from exit to inlet, the reversal of the wave propagation can be seen through the phase change of the perturbation solution.

As can be seen in Figure 7.5, the phase plot for \( \tilde{v}_i \) increases almost linearly with the location in the thruster. Since this behavior is for the translational mode, this increasing phase can be regarded as a forward movement of the wave when the state is propagated in time. When comparing the behavior of the sloshing mode, Figure 7.7, the phase of \( \tilde{v}_i \) begins to increase at the inlet, peaks near the center of the channel, and finally decreases toward the exit. Although this is not an indication that the electron number density is higher in the center of the channel, it does show the relative motion of the waves with the propagation of time. For instance, at the inlet, the phases for both the translational and
sloshing modes increase, indicating similar motion. However, near the exit, the phase of
the translational mode continues to increase, while that of the sloshing mode flips, and
starts to decrease. This is an indication of an upstream wave from the exit, and a sloshing
behavior when regarding the entire channel at once.

Figure 7.50 shows that this particular standing wave (third mode) has one nodal point
in mid-channel, where $n_r$ does not fluctuate. The same is true of the sixth mode, while the
fifth has a quasi-node, with small, but finite amplitude oscillations. The plot of the first
mode shows its stationary behavior. Its wave form does not move in space with respect to
time, and instead, just decays away.
Figure 7.50: Propagation of Electron Number Density Perturbation

Translational modes do not have nodes since the waves sweep from inlet to outlet, with finite amplitude.
7.3.3 Predator-Prey Behavior of Sloshing Modes

In addition to the sloshing behavior of modes three and six, upon closer examination of their phase plots (in the ionization layer), there seems to be some similarity to the predator-prey phenomenon alluded to in the paper by Fife, Martinez-Sanchez, and Szabo [8]. Unlike with the translational mode where the neutral and electron number perturbations are in phase, in the ionization region of the first sloshing mode, shown in Figure 7.31, their phase difference is about 60 degrees. For the higher order mode, shown in Figure 7.37, the difference in phase is about 90 degrees. When the electron and neutral number densities are out of phase by 90 degrees, they exhibit a behavior analogous to a predator-prey cycle. During this type of oscillation, when the neutral number density is high, and the electron density is low, there is an affinity for more neutrals to be ionized, thus increasing the ionization rate. After undergoing a high ionization rate, the number of neutrals decreases and more electrons/ions are produced. When the number of electrons become too high, and the number of neutrals low, the ionization rate decreases. Since neutrals are constantly being introduced at the inlet, their number density begins to increase again, thus completing the cycle. This type of behavior is also referred to as a feast-famine type of cycle.

With phase differences near 90 degrees, it appears that the sloshing modes of oscillation exhibit a predator-prey type of cycle. Further support for this type of cycle is derived from the fact that the ionization rate perturbation is dominated by the perturbations of the neutral and electron number densities, and not those of the temperature. This can be shown through the definition of the recombination rate, as shown in Equation (6.3), and its perturbation,

\[ \tilde{K}_{\text{ion}} = (n e \tilde{V}_{\text{ion}}) = \tilde{n}_e \tilde{V}_{\text{ion}} \left( \frac{\tilde{n}_e}{n_e} + \frac{\tilde{V}_{\text{ion}}}{V_{\text{ion}}} \right) = \tilde{K}_{\text{ion}} \left( \frac{\tilde{n}_e}{n_e} + \frac{\tilde{V}_{\text{ion}}}{V_{\text{ion}}} \right), \]

(7.4)
which can be rewritten, using Equation (4.32), as

$$\frac{\dot{R}_{\text{ion}}}{R_{\text{ion}}} = \frac{\dot{n}_e}{n_e} + \frac{\dot{v}_{\text{ion}}}{v_{\text{ion}}} = \dot{n}_e + \frac{\dot{v}_{\text{ion}}}{v_{\text{ion}}} + \left(1 + \frac{2}{1 + 2\frac{1}{T_e}}\right)\frac{\dot{T}_e}{T_e}. \tag{7.5}$$

Looking at the magnitude plots for the sloshing modes (74.2 kHz and 162 kHz) in the ionization layer (Figure 7.30 and Figure 7.36, respectively), as well as the ionization plots for the 0th order variables found below in Figure 7.51, the relative magnitudes of the three terms in Equation (7.5) can be computed. The results of these computations are tabulated in Table 7.2.

![Figure 7.51: 0th Order State Variables in the Ionization Layer](image)

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Table 7.2: Relative Magnitudes of Terms in Ionization Rate Perturbation

<table>
<thead>
<tr>
<th>Sloshing Mode</th>
<th>$\tilde{n}_x \tilde{n}_e$</th>
<th>$\tilde{n}_n \tilde{n}_e$</th>
<th>$\tilde{\dot{T}}_x \tilde{T}_e$</th>
<th>$\left( \frac{1}{2} + \frac{2}{\tilde{T}_x} + \frac{1}{\tilde{T}_e} \right) \tilde{\dot{T}}_x \tilde{T}_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>74.2 kHz</td>
<td>0.3</td>
<td>1</td>
<td>0.07</td>
<td>0.1</td>
</tr>
<tr>
<td>162 kHz</td>
<td>0.4</td>
<td>1</td>
<td>0.07</td>
<td>0.1</td>
</tr>
</tbody>
</table>

As can be seen through the computations, the magnitude of the perturbation temperature is smaller than those for the electron/ion and neutral number densities. This confirms the fact that the fluctuations in the ionization rate are dominated by changes in the electron/ion and neutral number density, supporting the notion of the predator-prey cycle.

In the paper by Fife, Martinez-Sanchez, and Szabo [8], the ionization box model which pointed to the predator-prey cycle, led to the relation

$$\omega_r = \xi(T_e) \sqrt{\tilde{n}_e \tilde{n}_n}$$  \hspace{1cm} (7.6)

where $\omega_r$ is the real component of the oscillation frequency, $n_e$ is the electron/ion number density, $n_n$ is the neutral number density, and $\xi(T_e)$ is defined as

$$\xi(T_e)n_e n_n = R_{\text{ion}} = n_e \nu_{\text{ion}}.$$  \hspace{1cm} (7.7)

Rewriting Equation (7.6) in terms of the ionization rate, $R_{\text{ion}}$, and converting to normalized variables, the expression for the predator-prey frequency now becomes

$$\bar{\omega}_r = \frac{\tilde{R}_{\text{ion}}}{\sqrt{\tilde{n}_e \tilde{n}_n}}.$$  \hspace{1cm} (7.8)
This expression can be used to compare the predator-prey model with the sloshing modes found in this study. The predator-prey frequency can be computed from taking averages (in the ionization layer) of the 0th order state variables shown in Figure 7.51. Table 7.3 shows the results of this calculation.

<table>
<thead>
<tr>
<th></th>
<th>Real Part of Normalized Frequency</th>
<th>Real Part of True Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Sloshing Mode</td>
<td>1.982</td>
<td>74.2 kHz</td>
</tr>
<tr>
<td>2nd Sloshing Mode</td>
<td>4.332</td>
<td>162 kHz</td>
</tr>
<tr>
<td>Predator-Prey Model</td>
<td>1.25</td>
<td>46.8 kHz</td>
</tr>
</tbody>
</table>

Table 7.3: Comparison of 1st Order Modes to Predator-Prey Model

As can be seen from the calculated values, the predator-prey frequency is considerably lower than the either of the two sloshing modes. Although this is does not support the existence of predator-prey type of behavior, the fact that neutral and electron number density perturbations are nearly 90 degrees out of phase, and that they contribute more to perturbations in the ionization rate than the electron temperature perturbations do, tends to indicate that this type of feast-famine cycle may still be a plausible theory.

7.4 Two Oscillator Model

In Section 7.2, “1st Order State Variables”, it is noted that there are two distinct regions of behavior, a long diffusion layer, followed by a thin ionization layer at the channel exit. Because the dynamics in these two regions seem to have notably different characteristics, it may be illustrative to construct and analyze a simple model of two oscillators that are coupled at the diffusion-ionization interface. For this purpose, the diffusion layer is modeled as an acoustic channel with a closed end at the inlet, and the ionization fluctuation is modeled by a mass-spring oscillator at the exit.
Modeled as a one-dimensional, constant cross-section channel containing an ideal, adiabatic fluid the acoustic behavior of the diffusion layer is governed by the following equations.

<table>
<thead>
<tr>
<th>Governing 1-D Equation</th>
<th>Principle of Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(\rho u A) = 0 )</td>
<td>conservation of mass</td>
</tr>
<tr>
<td>( A ) = area of channel</td>
<td></td>
</tr>
<tr>
<td>( \rho = ) density of fluid</td>
<td></td>
</tr>
<tr>
<td>( u = ) velocity of fluid</td>
<td></td>
</tr>
<tr>
<td>( \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} = 0 )</td>
<td>conservation of momentum, convective form</td>
</tr>
<tr>
<td>( p = ) pressure of fluid</td>
<td></td>
</tr>
<tr>
<td>( \frac{p}{\rho^\gamma} = \text{constant} )</td>
<td>adiabatic condition; no heat transfer</td>
</tr>
<tr>
<td>( \gamma = ) ratio of specific heats of fluid</td>
<td></td>
</tr>
<tr>
<td>( p = \rho RT )</td>
<td>ideal gas law</td>
</tr>
<tr>
<td>( R = ) universal gas constant</td>
<td></td>
</tr>
<tr>
<td>( T = ) temperature of fluid</td>
<td></td>
</tr>
<tr>
<td>( c = \sqrt{\gamma RT} )</td>
<td>definition of speed of sound</td>
</tr>
</tbody>
</table>

Table 7.4: Governing Equations for Diffusion Layer of Two Oscillator Model

Assuming solutions of the form,

\[
f(x, t) = \bar{f}(x) + \sum_{j \in \text{modes}} f_j(x, t) = \bar{f}(x) + \sum_{j \in \text{modes}} \Re \{ \hat{f}_j(x) e^{i \omega_j t} \},
\] (7.9)

where each state variable, \( f(x, t) \), is the sum of a steady state solution, \( \bar{f}(x) \), and linear combination of perturbations, \( f_j(x, t) \), which consist of a spatially varying amplitude, \( \hat{f}_j(x) \), and a time varying oscillation with frequency, \( \omega_j \). Since perturbations are linear, each mode can be solved for separately, then combined together, and summed with the steady state function to form the total solution. Because of this, only one frequency, \( \omega \), will be carried through in this analysis.
In addition to this assumption, two other simplifications are made. The first is that the steady state velocity, \( \bar{u} \), is equal to zero. As can be seen in Figure 6.1, the 0\(^{th}\) order solution of the ion velocity has a near-zero magnitude in the diffusion layer and can be approximated as zero (in the ionization layer, however, this assumption cannot be made). The second simplification is that the cross sectional area, \( A \), is constant throughout the length of the channel and does not vary with time. This results from the constant-area geometry of an SPT channel.

With these assumptions, the perturbation equations can be formed by substituting the state variables in the acoustic equations and separating the 0\(^{th}\) and 1\(^{st}\) order terms. This method follows that which was used for the plasma dynamic equations in this study, and is detailed in Chapter 4. The resulting equations are detailed below.

The steady state equation for mass conservation does not provide any new information, and is not included in this list. Equation (7.10) and Equation (7.11) are derived from the momentum conservation equation and the adiabatic condition, respectively.

\[
\frac{\partial \bar{p}}{\partial x} = 0 \quad (7.10)
\]

\[
\frac{\partial \bar{\rho}}{\partial x} = 0 \quad (7.11)
\]

By combining the ideal gas law and the definition of the speed of sound, Equation (7.12) can be derived.

\[
\bar{p} = \frac{\bar{c}^2}{\gamma} \bar{\rho} \quad (7.12)
\]

Substituting these relations into the 1\(^{st}\) order solutions, the perturbation equations can be derived to be as follows:
\[
\frac{\partial p'}{\partial t} + \frac{\partial \rho'}{\partial x} = 0 \quad (7.13)
\]

\[
\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \quad (7.14)
\]

\[
p' = \tilde{c}^2 \rho' \quad (7.15)
\]

Combining these equations results in the wave equation, as can be expected from acoustic dynamics. Although there is a wave equation for each perturbation variable \((p', u', \text{and } p')\), only the velocity equation will be needed for this analysis. It is expressed below.

\[
\frac{\partial^2 u'}{\partial t^2} = \tilde{c}^2 \frac{\partial^2 u'}{\partial x^2} \quad (7.16)
\]

Since the perturbation of the velocity is defined to be an oscillatory function of time, in the form, \(u' = \hat{u} e^{iat}\), the wave equation can be rewritten in the form,

\[
\frac{\partial^2 \hat{u}}{\partial x^2} + \left(\frac{\omega}{\tilde{c}}\right)^2 \hat{u} = 0. \quad (7.17)
\]

This is a simple harmonic oscillator with the general solution,

\[
\hat{u} = C \sin \left(\frac{\omega}{\tilde{c}} x + \varphi\right), \quad (7.18)
\]

where \(C\) and \(\varphi\) are arbitrary constants of integration and are determined by the boundary conditions of the system. For this particular problem, since the diffusion layer is modeled to be an acoustic channel with a closed end (at \(x = 0\)), then the boundary condition
there is that the velocity perturbation is zero. With this constraint, Equation (7.18) can be simplified to be

\[ \dot{u} = C \sin \left( \frac{\omega}{c} x \right) \]  

(7.19)

and therefore,

\[ u' = C \sin \left( \frac{\omega}{c} x \right) e^{i \omega t} \]  

(7.20)

The other arbitrary constant, \( C \), will be resolved when the ionization layer (at the channel exit) is taken account of. Since the ionization layer is modeled as a mass-spring oscillator, the boundary condition at the exit \( (x = L) \) of the acoustic channel is the equilibrium of forces between the perturbation pressure force and the mass-spring perturbation forces. This is expressed in the equation below.

\[ m \frac{\partial^2 \chi'}{\partial t^2} + k \chi' = -p' A \text{ at } (x = L) \]  

(7.21)

\( m \) and \( k \), respectively, are the mass and spring constant of the mass-spring oscillator. \( \chi' \) and \( p' \) are defined in terms of the fluid velocity, \( u' \). These relations are expressed below in Equation (7.22) and Equation (7.23).

\[ \frac{\partial \chi'}{\partial t} = u' \]  

(7.22)

\[ \frac{\partial p'}{\partial x} = -\bar{\rho} \frac{\partial u'}{\partial t} \]  

(7.23)
By substituting the solution of the velocity perturbation from Equation (7.20) into Equation (7.22) and Equation (7.23), Equation (7.21) can be expressed as

\[
\left(1 - \frac{\lambda^2}{v^2}\right) v \tan v = \mu, \tag{7.24}
\]

where all variables are non-dimensional, and defined as

\[
\lambda = \sqrt{\frac{k/m}{c/L}}, \tag{7.25}
\]

\[
\nu = \frac{\omega}{c/L}, \tag{7.26}
\]

\[
\mu = \frac{BAL}{m} \tag{7.27}
\]

A plot of the solutions to Equation (7.24) is shown below, in Figure 7.52. In this representation, the right-hand side of the equation is plotted against the normalized acoustic frequency, \(\nu\), with the normalized mass-spring frequency, \(\lambda\), as a parameter. The intersection of this curve with the left-hand side of the equation, indicates the frequencies, \(\nu\), which satisfy the complete equation. Since \(\mu\) is the ratio of the mass of the fluid in the acoustic channel to the mass of the mass-spring oscillator, it is a positive number. For this graph, it is assumed, arbitrarily, to be unity.
Figure 7.52: Solution of Frequencies for Two Oscillator Model

As can be seen in this plot, the higher frequency solutions ($v > \pi$) are not significantly affected by the value of the parameter, $\lambda$. This being the case, only those solutions which vary with $\lambda$, or those for $v < \pi$, will be addressed further. It should be noted, though, that due to the periodicity of the tangent function in Equation (7.24), similar characteristics develop if $\lambda$ is offset by a multiple of the period, $\pi$. In that case, the behavior of the plots will also be offset by the same amount.

From Figure 7.52, the normalized frequencies of the mass-spring oscillation, $\lambda$, tend to be lower than those for the acoustic oscillation when it is less than the first acoustic resonance at $v = \frac{\pi}{2}$. For the chosen values of the normalized mass-spring frequency, $\lambda = 0.1$ and 1, the normalized acoustic frequencies were determined to be 0.8 and 1.2, respectively. Furthermore, when $\lambda > \frac{\pi}{2}$, it tends to be higher than the corresponding normalized acoustic frequency, $v$. In the case of $\lambda = 2$, $v = 1.8$. The more interesting case is the solu-
tion when \( \lambda = \frac{\pi}{2} \). At this normalized mass-spring frequency, the corresponding solution of the acoustic frequency disappears, and in this case, the first acoustic mode vanishes. Due to the periodicity of the function, if \( \lambda \) were set to be \( \frac{3\pi}{2} \), the second acoustic mode would vanish instead. This implies that a mass-spring oscillator could be designed to remove one of the system's acoustic modes by matching their frequencies.
Chapter 8

Conclusions

From the formulated, one-dimensional, linear perturbation model, several modes of oscillation were found through numerical integration of the model equations. Although these modes were computed to be of higher frequency than the commonly observed 30 kHz instability, the two fundamental harmonics, at 47.9 kHz and 74.2 kHz, are of the same order, and fall within the bounds (tens of kHz) of previous experimental and numerical studies [3]-[6]. Of these computed modes, two types of oscillation patterns have been found. The first, is a translational mode, where perturbations travel from the inlet to exit. This, in effect would produce the pulsed ejection of ions which has been observed experimentally [3]. The second of these two modal patterns exhibits a sloshing behavior, where ion-velocity waves appear to converge inward from the inlet and exit, simultaneously, and form a standing wave. This backward flow may be the reflection of a wave at the exit where ionization effects occur, making this mode an ionization-type oscillation [3]-[7]. Nonetheless, because of the presence of a counter-flow, a predator-prey type of cycle seems to be established, where electron and neutral number density perturbations fluctuate 90 degrees out of phase. Despite this connection, however, a comparison of the predator-prey model frequency [8] and the frequencies of the sloshing modes do not show close agreement.

Because there appear to be two distinct regions of behavior in the channel, a diffusion layer and an ionization layer, a simplified two-oscillator model was formulated and analyzed. This model, consisting of an acoustic channel with one end closed and the other open and connected to a second oscillator, resulted in the formation of a hybrid mode of
oscillation. When the frequency of the oscillator is not equal to one of the resonance modes of the acoustic channel, it produces a mode of oscillation at a frequency similar to its own. This frequency lies between that of the oscillator and the nearest resonating acoustic frequency, thus creating a hybrid mode. When the oscillator's frequency exactly matches one of the resonance modes, however, they both vanish as solutions with no creation of a hybrid mode. If damping were included in this model (as all physical systems contain damping), an exact match of the frequency would not be necessary to significantly eliminate a mode. Instead only a close match may be required for a particular system. As a consequence of this analysis, it may be possible to eliminate an unwanted mode of fluctuation by generating an oscillatory disturbance at that, or near that, frequency.

This behavior is also seen with other acoustic and mechanical systems. Hemholtz resonators (or acoustic liners), for instance, are used to remove unwanted noise from acoustic systems. Another example of a modal damper is the use of a pendulum on a rotating shaft. This can convert energy from the resonance modes of a rotating, eccentric shaft into movement of a mechanical device which can dissipate that energy. It should be noted, however, that if the energy dissipation rate is lower than the energy input rate, then the mechanical damper (i.e. pendulum) may resonate at its natural frequency at unwanted amplitudes.

A possible way to study this effect with the plasma dynamic model would be to, instead of altering exit conditions to stimulate changes in the ionization rate, the length of the acoustic chamber could be altered to change the frequencies at which the system would resonate. Recalling that setting the values of the two 0th order boundary conditions, $\tilde{r}_{cB}$ and $\tilde{r}_{iB}$ (from Section 5.1), fixes the channel length, by varying one (or both) of them, it would be possible to alter the channel length, and therefore change the acoustic resonance frequencies. If the channel length were to be iterated upon until the resonance
modes matched the ionization fluctuations (determined through the integration of the 1st order system), this would be analogous to the ionization fluctuations passing through the resonating acoustic modes. This analysis remains open for future work.

Another key result of this study, is that all modes were found to be highly damped and propagate for little longer than a period. This being the case, there is no mechanism for an unstable oscillation to develop or be sustained. A possible mechanism, for the occurrence of such of oscillatory instabilities in a system with only damped modes, is if there are external oscillatory disturbances driven with frequencies that pass through a resonating mode. These disturbances must be driven oscillations which provide energy to excite the damped modes of the system into instability. In such a situation, as the frequency of the disturbance nears the damped frequency of a mode, the mode will be excited and begin to diverge. When the disturbance passes beyond the damped frequency, the mode will revert to its normal behavior, and decay rapidly as a function of time. This produces a ringing effect which has been observed in experimental work by Fife and Martinez-Sanchez [13].

If fluctuation in the ionization layer is considered to be such a disturbance whose frequency changes depending on the local conditions of the state, and the fixed-length channel acts as a resonating acoustic chamber, it may be possible that the driving frequency of the fluctuations passes through harmonics of the channel and produces a resonance which manifests itself as this ringing phenomenon. In this case, although the modes found in the 1st order analysis are damped, an unstable oscillation occurrence may be produced.

Other mechanisms which may produce this phenomenon might be wall effects, sheath effects, or other quantities ignored in this formulation. One study has suggested that aging effects may contribute to the development of oscillations [6]. In particular, changes in the condition of the channel surface, like erosion due to ion sputtering or redepositing of
ablited insulator material, may introduce wall effects that are not captured by this formulation.

During lifetime tests of SPT operation, oscillatory instabilities tend to develop at two stages in a thruster’s life. Initially, when the thruster is devoid of insulator deposits, it operates under nominal (stable) conditions. When ion sputtering causes insulator material to erode and then redeposit in the thruster’s channel, the oscillatory instabilities that have been documented in the literature arise. Upon further operation, as the insulator near the channel exit begins to erode and expose new, and clean, insulator surface, the oscillatory instabilities subside, and thruster operation returns nearly, but not quite, to the nominal condition. When erosion of the insulator near the channel exit is complete, and the metallic surface of the inner core is exposed to the plasma, the oscillator instabilities appear again.

This behavior is an indication that wall effects could be the source of energy for these instabilities. Since ionization rate is sensitive to the electron temperature (see Equation (4.24)), and both, deposits of insulator material, and exposed metal from the inner core, lead to locally high electron temperatures, it may be the case that during these stages of thruster operation, the ionization fluctuations pass through the resonating acoustic modes of the channel.

Furthermore, it may be beneficial to incorporate wall effects, such as surface geometry, secondary electron emission, and heat loss into the model. This may either require a full, two-dimensional model, with variation in the axial and radial direction, or the addition of source terms in the one-dimensional formulation which account for the effects of wall conditions. These mechanisms may provide the necessary source of energy for the damped modes of the system to become unstable. At the time of this paper, the formulated model does not include any such energy sources and therefore appears unable to capture
any oscillatory instabilities. Incorporating the response of the plasma dynamic system to excitations may be necessary to account for this.

Also, as discussed in Section 5.3, "1st Order Mach Condition at Sonic Points", only the solutions of the 1st order equations, for the case of a choked exit, were pursued in this study. Subsequent work should also consider solutions to the case of a smooth, internal sonic passage. Such an analysis may be necessary to capture the full plasma dynamic behavior of the system.
Appendix A

Definition of Terms

A.1 Nomenclature
The following is a list of the terms used in the formulation of the steady state and perturbation models. To aid numerics, all terms used in the integration are non-dimensionalized in the manner prescribed in Appendix A.2. Non-dimensional terms are denoted either with tildas, \( \tilde{\cdot} \), or hats, \( \hat{\cdot} \). Note that the perturbation ratios in Appendix A.1.4 are already non-dimensional by their definition and do not use the normalization factors.

A.1.1 Constants and Parameters
\[ \xi = \text{variable to regularize equations by removing an order of singularity} \]
\[ \omega = \text{oscillation frequency of linear perturbations} \]
\[ v_n = \text{neutral particle velocity} \]
\[ E_i = \text{ionization energy} \]
\[ E'_i = 2.5E_i = \text{effective energy loss during ionization} \]
\[ \sigma_0 = \text{ionization cross section} \]
\[ \omega_c = \text{cyclotron frequency} \]
\[ \sigma_{en} = \text{electron-neutral collision cross section} \]
\[ \alpha_B = \text{effective Bohm diffusion coefficient} \]

A.1.2 Steady State Terms
\[ x = \text{axial coordinate in channel} \]
\[ n_n = \text{neutral number density} \]
\[ n_e = \text{electron number density} \]
\[ v_i = \text{ion velocity} \]
\[ v_e = \text{electron velocity} \]
\[ T_e = \text{electron temperature} \]
\[ \phi = \text{electric potential} \]
\[ v_{ion} = \text{ionization frequency} \]
\[ v_H = \text{axial diffusion collision frequency} \]
\[ \Gamma_d = n_e \cdot v_{i} - n_e v_e = \text{discharge current flux per unit charge} \]
\[ \Gamma_m = n_n v_n + n_e v_i = \text{mass flux} \]
\[ M_i = \text{ion Mach number} \]

### A.1.3 Perturbation Terms

\[ n_n = \text{neutral number density perturbation} \]
\[ n_e = \text{electron number density perturbation} \]
\[ v_i = \text{ion velocity perturbation} \]
\[ v_e = \text{electron velocity perturbation} \]
\[ T_e = \text{electron temperature perturbation} \]
\[ \phi = \text{electric potential perturbation} \]
\[ v_{ion} = \text{ionization frequency perturbation} \]
\[ v_H = \text{axial diffusion collision frequency perturbation} \]
\[ \Gamma_d = \text{discharge current flux per unit charge perturbation} \]
\[ \Gamma_m = \text{mass flux per unit ion mass perturbation} \]
\[ M_i = \text{ion Mach number perturbation} \]

### A.1.4 Perturbation Ratios

These ratios are defined for the purpose of simplifying the model equations. They are the 1\textsuperscript{st} order state variables, normalized by the 0\textsuperscript{th} order state variables.

\[ \hat{n}_n = \frac{\dot{n}_n}{n_n} \]
\[ \hat{n}_e = \frac{\dot{n}_e}{n_e} \]
\[ \dot{V}_e = \frac{v_e}{v_e} \]
\[ \dot{\xi}_e = \frac{\dot{v}_e}{\xi_e} \]
\[ \dot{f}_{ion} = \frac{\dot{v}_{ion}}{v_{ion}} \]
\[ \dot{f}_H = \frac{\dot{v}_H}{v_H} \]
\[ \dot{f}_d = \frac{\dot{v}_d}{v_d} \]

**A.2 Normalization**

The normalization factors for all steady state and perturbation variables are as those used in Ahedo’s and Martinez-Sanchez’s steady state formulation [9]. Only the perturbation ratios, listed in Appendix A.1.4, are not normalized by these expressions. Instead, the perturbation ratios are the 1st order state variables, normalized by their corresponding 0th order state variables.

Each factor below, normalizes the variables which have identical labels. In general,

\[ \bar{F}_i = \frac{\bar{F}_i}{F^*} \quad \text{or} \quad \dot{F}_i = \frac{\dot{F}_i}{F^*}. \quad (A.1) \]

The normalization factors are as listed below.

\[ T^*_e \equiv \frac{E_i}{k} \quad (A.2) \]

\[ x^* = l^* = \text{arbitrarily fixed length} \quad (A.3) \]

\[ \xi^* = \frac{l^*}{E_i} \quad (A.4) \]

\[ \nu^* = \frac{E_i}{\kappa m_i} \quad (A.5) \]

\[ \nu^* = \omega^* = \frac{1}{l^* \kappa m_i} \quad (A.6) \]
\[ E_{\text{ion}}^* = E_i \]  \hspace{1cm} (A.7)

\[ \phi^* = \frac{E_i}{q} \]  \hspace{1cm} (A.8)

\[ n^* = \Gamma_m \sqrt{\frac{m_i}{E_i}} \]  \hspace{1cm} (A.9)

\[ \alpha^* = \left( L^* \sqrt{\frac{m_i}{E_i}} \right)^{\frac{1}{2}} \]  \hspace{1cm} (A.10)
Appendix B

1st Order Boundary Condition at a Sonic Exit

B.1 Acoustic Model
The equations that govern 1-D compressible flow in a channel, with a varying cross sectional area, and no heat transfer, are given below.

<table>
<thead>
<tr>
<th>Governing 1-D Equation</th>
<th>Principle of Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u A) = 0$</td>
<td>conservation of mass</td>
</tr>
<tr>
<td>$A = \text{area of channel}$</td>
<td></td>
</tr>
<tr>
<td>$\rho = \text{density of fluid}$</td>
<td></td>
</tr>
<tr>
<td>$u = \text{velocity of fluid}$</td>
<td></td>
</tr>
<tr>
<td>$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} = 0$</td>
<td>conservation of momentum, convective form</td>
</tr>
<tr>
<td>$p = \text{pressure of fluid}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{p}{\rho^\gamma} = \text{constant}$</td>
<td>adiabatic condition; no heat transfer</td>
</tr>
<tr>
<td>$\gamma = \text{ratio of specific heats of fluid}$</td>
<td></td>
</tr>
<tr>
<td>$p = \rho RT$</td>
<td>ideal gas law</td>
</tr>
<tr>
<td>$R = \text{universal gas constant}$</td>
<td></td>
</tr>
<tr>
<td>$T = \text{temperature of fluid}$</td>
<td></td>
</tr>
<tr>
<td>$M = \frac{u}{\sqrt{\gamma RT}}$</td>
<td>definition of Mach number</td>
</tr>
</tbody>
</table>

Table B.1.1: Governing 1-D Acoustic Equations

B.2 Model Equations
For the form of the state variables being

$$f(x, t) = \tilde{f}(x) + \mathcal{R} e\{ \tilde{f}(x)e^{\imath \omega t} \},$$

(B.1)

denotes the steady state, or 0th order equations can be derived to be as shown in Equation (B.2) through Equation (B.5), and the linear perturbation, or 1st order equations, as in Equation (B.6) through Equation (B.10).
\[
\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\alpha \bar{M}^2}{1 - \bar{M}^2} \quad (B.2)
\]

\[
\frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial \bar{x}} = -\frac{\alpha}{1 - \bar{M}^2} \quad (B.3)
\]

\[
\frac{1}{\bar{T}} \frac{\partial \bar{T}}{\partial \bar{x}} = \frac{\alpha(\gamma - 1)\bar{M}^2}{1 - \bar{M}^2} \quad (B.4)
\]

\[
\frac{1}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial \bar{x}} = \frac{\alpha \bar{M}^2}{1 - \bar{M}^2} \quad (B.5)
\]

\[
\left(\frac{1 - \bar{M}^2}{\bar{M}^2}\right) \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{T}}{\bar{\rho}}\right) = \left(-\frac{i\omega}{\bar{u}} + \frac{2\alpha}{1 - \bar{M}^2}\right)\frac{\bar{M}}{\bar{M}} + \left(\frac{\gamma - 3}{2}\right)\frac{\bar{\rho}}{\bar{\rho}} \quad (B.6)
\]

\[
\left(\frac{1 - \bar{M}^2}{\bar{M}^2}\right) \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{u}}{\bar{\rho}}\right) = \left(\frac{i\omega}{\bar{u}} - \frac{2\alpha}{1 - \bar{M}^2}\right)\frac{\bar{M}}{\bar{M}} - \left(\frac{\gamma - 1}{2}\right)\frac{\bar{\rho}}{\bar{\rho}} + \left(\frac{\gamma - 3}{2}\right)\frac{\bar{\rho}}{\bar{\rho}} \quad (B.7)
\]

\[
\left(\frac{1 - \bar{M}^2}{\bar{M}^2}\right) \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{T}}{\bar{\rho}}\right) = \left[(\gamma - 1)\left(-\frac{i\omega}{\bar{u}} + \frac{2\alpha}{1 - \bar{M}^2}\right)\frac{\bar{M}}{\bar{M}} + \left(\frac{\gamma - 1}{2}\right)\frac{\bar{\rho}}{\bar{\rho}} \right. \quad (B.8)
\]

\[
\left(\frac{1 - \bar{M}^2}{\bar{M}^2}\right) \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{M}}{\bar{\rho}}\right) = \left[\gamma\left(-\frac{i\omega}{\bar{u}} + \frac{2\alpha}{1 - \bar{M}^2}\right)\frac{\bar{M}}{\bar{M}} + \left[\gamma\left(\frac{\gamma - 3}{2}\right)\right]\frac{\bar{\rho}}{\bar{\rho}} \right. \quad (B.9)
\]

\[
\left(\frac{1 - \bar{M}^2}{\bar{M}^2}\right) \frac{\partial}{\partial \bar{x}} \left(\frac{\bar{M}}{\bar{\rho}}\right) = \left[\left(\gamma + 1\right)\frac{i\omega}{\bar{u}} - \frac{\alpha(\gamma + 1)}{1 - \bar{M}^2}\right]\frac{\bar{M}}{\bar{M}} - \left[\left(\frac{\gamma - 1}{2}\right)^2 - \frac{1}{\bar{M}^2}\right]\frac{\bar{\rho}}{\bar{\rho}} \quad (B.10)
\]

where, since the geometry of the channel is fixed with respect to time, or \(A(x, t) = \bar{A}(x)\),

\[
\alpha = \frac{1}{\bar{A}} \frac{\partial \bar{A}}{\partial \bar{x}}. \quad (B.11)
\]

From the 0th order equations, Equation (B.2) through Equation (B.5), the basic flow gradients are singular at \(\bar{M} = 1\), and behave as \(\frac{1}{1 - \bar{M}^2}\) in the vicinity of this point (exit). From the 1st order equations, it can be seen that at a sonic point, or when \(\bar{M} = \pm 1\), the term, \(\frac{1}{1 - \bar{M}^2}\), will diverge, and lead to perturbation derivatives of order \(\left(\frac{1}{1 - \bar{M}^2}\right)^2\), greater
than that of the basic derivatives. To avoid this singular point in the case of a choked exit, the coefficient of this term must be identically zero. This coefficient, is always proportional to $\alpha \dot{M}$, and since $\alpha$, in general, is not identically zero, in order to resolve this singularity,

$$\dot{M} \bigg|_{\dot{M} = \pm 1} = 0$$

(B.12)
Appendix C

Matlab Code for Numerical Integration

C.1 Main Program

```matlab
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% NOGUCHI_THESIS.M %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% %
% WRITTEN BY: REID NOGUCHI %
% LAST MODIFIED: 14 APRIL 1999 %
% PURPOSE: LINEAR 1-D ANALYSIS OF OSCILLATION INSTABILITIES %
% IN STATIONARY PLASMA THRUSTERS %
% ACKNOWLEDGMENTS: EDUARDO AHEDO %
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% NOGUCHI_THESIS.M %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all;

% initialization values normalization of variables
ei=1; % ionization energy
flm=1; % mass flow
sigma0=1; % cross-section for ionization * sqmim
% constants and parameters
sqmim=490.4; % sqrt of mass ratio for Xe
sigmamn=7.5.*sigma0; % cross section for elastic collisions
alfab=1/16; % Bohm diffusion parameter
vn=.13; % velocity of neutrals
alfab=alfab.*sqmim;
eip=2.5*ei; % effective energy loss per ionization
omegac=1; % ion frequency / alfab
omega_B=alfab.*omegac; % hybrid frequency: qB/sqrt(m_i*m_e)
% boundary conditions
% 0th order
fld=1.1; % discharge current
fli_B=-.01; % ion flow at B
te_B=.1; % temeperature at B
pot_B=1; % potential at B (arbitrary)
x_B=0;
% related conditions at point B
% 0th order
vi_B=sqrt(5/3.*te_B);
```
ne_B=fli_B./vi_B;
nn_B=(flm-ne_B.*vi_B)./vn;
ve_B=vi_B-fld./ne_B;
y_0_B=[x_B, log(ne_B), vi_B, te_B, pot_B]';
% boundary conditions
% 0th order integration
% options=odeset('events','on', 'RelTol', 1e-4);
dx_i_0=0.003;
[xi,y_0,xie,sole,esub]=ode45('noguchi_unsteady_steady_fdif', [0:dx_i_0:1000], y_0_B, options, ei, flm, omega_B, fld, vn, ei, sigma0, sigmaen, alfabc);
% 0th order integration
% 0th order solution
x=y_0(:,1)';
ne=exp(y_0(:,2))';
vi=y_0(:,3)';
te=y_0(:,4)';
pot=y_0(:,5)';
fl=ne.*vi;
fln=flm-fl;
flc=flf-fl;
ve=flc./ne;
mach=vi./sqrt(5./3.*te);
nn=(flm-fl)/vn;
nui=nn.*sigma0.*sqrt(8.*te./pi).*(1+2.*te./ei).*exp(-ei./te);
nue=nn.*sigmaen.*sqrt(8.*te./pi)+alfabc.*omega_B;
nuh=omega_B.^2./nue;
% 0th order solutions
% 1st order boundary conditions
% 1st order
gamd_hat=1;
etan_B_1=1;
taoe_B_1=0;
pot_hat_B_1=0;
etan_B_2=0;
taoe_B_2=1;
pot_hat_B_2=0;
etan_B_3=0;
taoe_B_3=0;
pot_hat_B_3=1;

% related conditions at point B
% 1st order
vi_hat_B_1=.5.*vi_B.*taoe_B_1;
vi_hat_B_2=.5.*vi_B.*taoe_B_2;
vi_hat_B_3=.5.*vi_B.*taoe_B_3;
etae_B_1=(vi_hat_B_1/vi_B)-(nn_B.*vn./(ne_B.*ve_B)).*etan_B_1;
etae_B_2=(vi_hat_B_2/vi_B)-(nn_B.*vn./(ne_B.*ve_B)).*etan_B_2;
etae_B_3=(vi_hat_B_3/vi_B)-(nn_B.*vn./(ne_B.*ve_B)).*etan_B_3;

% solution at boundary point B
y_B=etan_B_1. etae_B_1. vi_hat_B_1. taoe_B_1. pot_hat_B_1.etan_B_2. etae_B_2. vi_hat_B_2. taoe_B_2. pot_hat_B_2.etan_B_3. etae_B_3. vi_hat_B_3. taoe_B_3. pot_hat_B_3];

% boundary conditions
%
% 0th order differential equation solutions
dy_0=zeros(5,length(ne));
dy_0(1:.)=te.*vi.^2;
dy_0(5:.)=te.*((te.^2./vi.^2).*nui.*((4.*eip.*te+1).*vi.^2./ve+vi.^2.*nui.*ve./te);
dy_0(4:.)=(te./vi.^2-1).*dy_0(5:.)-dy_0(1:.).*te.*(nui./vi).*((2-vn./vi)+nui.*ve);
dy_0(3:.)=-(1./vi).*dy_0(5:.)-nui.*(1-vn./vi).*dy_0(1:);
dy_0(2:.)=1./vi.*dy_0(5:.)+nui.*(2-vn./vi).*dy_0(1:)./vi;

dw_real=0.1;
dw_imag=0.1;
real_index=1;
for w_real=[0:dw_real:0],
  for w_real=[1.279:dw_real:1.279],
  for w_real=[1.982:dw_real:1.982],
  for w_real=[2.717:dw_real:2.717],
  for w_real=[3.250:dw_real:3.250],
  for w_real=[4.332:dw_real:4.332],

imag_index=1;
for w_imag=[0.981:dw_imag:0.981],
  for w_imag=[0.651:dw_imag:0.651],
  for w_imag=[1.473:dw_imag:1.473],
  for w_imag=[1.103:dw_imag:1.103],
  for w_imag=[0.935:dw_imag:0.935],
  for w_imag=[1.706:dw_imag:1.706],

w=w_real+i.*w_imag;
y=zeros(15,1);
mag_factor=1;
dxi=dxi_0*mag_factor;
y=y_B;
dy=zeros(15,1);

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N=length(ne);
stop_index=fix((N-1)/mag_factor+1);
set_points(1)=1;
for index=2:stop_index
    set_points(index)=set_points(index-1)+mag_factor;
end;
if mod(N,mag_factor)>0
    set_points(stop_index+1)=N;
dx_i_last=(N-set_points(index)).*dx_i_0;
else
    dx_i_last=0;
end;
N_set=length(set_points);

for index=1:(N_set-1),

index_0=set_points(index);

x_set(index)=x(index_0);
ne_set(index)=ne(index_0);
vi_set(index)=vi(index_0);
t_e_set(index)=te(index_0);
pot_set(index)=pot(index_0);
fl_i_set(index)=fli(index_0);
fl_n_set(index)=fli(index_0);
fl_e_set(index)=fle(index_0);
ve_set(index)=ve(index_0);
mach_set(index)=mach(index_0);
n_n_set(index)=nn(index_0);
nui_set(index)=nui(index_0);
nue_set(index)=nue(index_0);
nuh_set(index)=nuh(index_0);

%% 1st order variables, case 1 (etan_B=1, taoe_B=0, pot_hat_B=0)
etan_1=y(1,index);
etae_1=y(2,index);
vi_hat_1=y(3,index);
taoe_1=y(4,index);
pot_hat_1=y(5,index);

%% 1st order variables, case 2 (etan_B=0, taoe_B=1, pot_hat_B=0)
etan_2=y(6,index);
etae_2=y(7,index);
vi_hat_2=y(8,index);
taoe_2=y(9,index);
pot_hat_2=y(10,index);
% 1st order variables, case 3 (etan_B=0, taoe_B=0, pot_hat_B=1)
etan_3=y(11,index);
etae_3=y(12,index);
vi_hat_3=y(13,index);
taoe_3=y(14,index);
pot_hat_3=y(15,index);
%
% calculation of parameters

Ve_1=(1./ne(index_0).*ve(index_0)).*(ne(index_0).*vi(index_0)-
    ve(index_0)).*etae_1+ne(index_0).*vi_hat_1-fld.*gamd_hat;

Ve_2=(1./ne(index_0).*ve(index_0)).*(ne(index_0).*vi(index_0)-
    ve(index_0)).*etae_2+ne(index_0).*vi_hat_2-fld.*gamd_hat;

Ve_3=(1./ne(index_0).*ve(index_0)).*(ne(index_0).*vi(index_0)-
    ve(index_0)).*etae_3+ne(index_0).*vi_hat_3-fld.*gamd_hat;

fion_1=etan_1+(0.5+2.*(1./te(index_0)+2)+1./te(index_0)).*taoe_1;
fion_2=etan_2+(0.5+2.*(1./te(index_0)+2)+1./te(index_0)).*taoe_2;
fion_3=etan_3+(0.5+2.*(1./te(index_0)+2)+1./te(index_0)).*taoe_3;

fh_1=(1./(1+alfabp.*sqrt(pi./(8.*te(index_0)))).*omega_B./(nn(index_0).*sigmaen)).*(-etan_1-
    0.5.*taoe_1);

fh_2=(1./((1+alfabp.*sqrt(pi./(8.*te(index_0)))).*omega_B./(nn(index_0).*sigmaen))).*(-etan_2-
    0.5.*taoe_2);

fh_3=(1./((1+alfabp.*sqrt(pi./(8.*te(index_0)))).*omega_B./(nn(index_0).*sigmaen))).*(-etan_3-
    0.5.*taoe_3);
%
% set of differential equations

dy(15)=(1./vi(index_0)).*((1./dy_0(1,index_0)).*(dy_0(5,index_0).*vi(index_0).*
    te(index_0))).*vi_hat_3-
    .4.*vi(index_0).^3.taoe_3+vi(index_0).^2.*(te(index_0).^2+dy_0(2,index_0)+dy_0(4,index_0)).*(2.*vi_hat_3-vi(index_0)).*etae_3-
    vi_hat_3+nui(index_0).^2.*ve(index_0).*vi(index_0).*ve(index_0).^2.*2.*vi_hat_3-vi(index_0).*Ve_3-
    vi(index_0).*fh_3+2.*nui(index_0).*vi(index_0).*te(index_0).*vi(index_0).*Ve_3-vi(index_0).*nui(index_0).*ve(index_0).*vi(index_0).*fe(index_0).*
    nui(index_0).*ve(index_0).*vi(index_0).*ve(index_0).*ve(index_0).*ve(index_0).*Ve_3-taoe_3+4.*nui(index_0).*eip.*(Ve_3-taoe_3-
    fion_3)+nui(index_0).*te(index_0).*Ve_3-fion_3));
\[ \begin{align*}
_1\cdot \text{vi}(\text{index}_0) \cdot \text{taoe}_1 \rangle + \text{i} \cdot \text{w} \cdot \text{vi}(\text{index}_0) \cdot \text{te}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot \text{etae}_1 \cdot \\
\text{vi}_\text{hat}_1 \rangle + \text{nui}(\text{index}_0) \cdot \text{ve}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot \text{ve}(\text{index}_0) \cdot ^{2} \cdot (2 \cdot \text{vi}_\text{hat}_1 \cdot \text{vi}(\text{index}_0) \cdot \text{Ve}_\text{hat}_1 \cdot \\
\text{vi}(\text{index}_0) \cdot \text{fh}_1 \rangle + 2 \cdot \text{nui}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot \text{te}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot \text{fion}_1 \rangle \\
\text{nui}(\text{index}_0) \cdot \text{ve}(\text{index}_0) \rangle \cdot (i \cdot \text{w} \cdot \text{te}(\text{index}_0) \cdot \text{eip}(\text{Ve}_1 \cdot \text{taoe}_1 \cdot \\
fion_1 \rangle + \text{nui}(\text{index}_0) \cdot \text{te}(\text{index}_0) \cdot (\text{Ve}_1 \cdot \text{fion}_1 \rangle) \}
\]

\[
\text{dy}(4) = (1 / \text{vi}(\text{index}_0) \cdot ^{3} \cdot (1 / \text{te}(\text{index}_0)) \cdot ((\text{vi}(\text{index}_0) \cdot ^{2} \cdot \text{te}(\text{index}_0)) \cdot \text{vi}(\text{index}_0) \cdot \text{dy}(5) \cdot \\
\text{dy}(0) \cdot (5 / \text{index}_0) \cdot \text{vi}_\text{hat}_1 \rangle + \text{vi}(\text{index}_0) \cdot \text{ve}(\text{index}_0) \cdot ^{2} \cdot (\text{te}(\text{index}_0) \cdot \text{dy}(0) / \text{index}_0) + \text{dy}(0) / (4 \cdot \text{index}_0)) \cdot (2 \cdot \text{vi}_\text{hat}_1 \cdot \\
\text{vi}(\text{index}_0) \cdot \text{taoe}_1 \rangle + (i \cdot \text{w} \cdot \text{vi}(\text{index}_0) \cdot \text{te}(\text{index}_0) \cdot \text{etae}_1 \cdot \\
\text{vi}_\text{hat}_1 \rangle + \text{nui}(\text{index}_0) \cdot \text{ve}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot ^{2} \cdot (2 \cdot \text{vi}_\text{hat}_1 \cdot \text{vi}(\text{index}_0) \cdot \text{Ve}_1 \cdot \\
\text{vi}(\text{index}_0) \cdot \text{fh}_1 \rangle + 2 \cdot \text{nui}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot \text{te}(\text{index}_0) \cdot \text{vi}(\text{index}_0) \cdot \text{fion}_1 \rangle \\
\text{nui}(\text{index}_0) \rangle \cdot \text{ve}(\text{index}_0) \rangle \cdot (2 \cdot \text{vi}_\text{hat}_1 \cdot \text{vi}(\text{index}_0) \cdot \text{fion}_1 \rangle) \cdot \text{dy}(0) / (1 \cdot \text{index}_0) \}
\]

\[
\text{dy}(3) = (1 / \text{vi}(\text{index}_0) \cdot ^{2} \cdot (\text{vi}(\text{index}_0) \cdot \text{dy}(5) \cdot \\
\text{dy}(0) / (3 \cdot \text{index}_0) \cdot \text{vi}_\text{hat}_1 \rangle \\
\text{vi}(\text{index}_0) \rangle \cdot \text{dy}(5) + 2 \cdot \text{dy}(0) / (5 \cdot \text{index}_0) \cdot \text{vi}_\text{hat}_1 \rangle + (i \cdot \text{w} \cdot \text{vi}(\text{index}_0) + \text{nui}(\text{index}_0) \rangle \\
\text{vi}(\text{index}_0) + 2 \cdot \text{vn}) \rangle \cdot \text{vi}_\text{hat}_1 \rangle + \text{nui}(\text{index}_0) \rangle \cdot \text{vi}(\text{index}_0) \rangle \cdot \text{vi}(\text{index}_0) - \text{vn} \rangle \cdot \text{fion}_1 \rangle \cdot \text{dy}(0) / (1 \cdot \text{index}_0) \}
\]

\[
\text{dy}(2) = (1 / \text{vi}(\text{index}_0) \cdot ^{3} \cdot (- \text{vi}(\text{index}_0) \cdot ^{2} \cdot \text{dy}(3) + \text{vi}(\text{index}_0) \cdot \text{dy}(0) / (3 \cdot \text{index}_0) + \text{vi}_\text{hat}_1 \rangle + \\
i \cdot \text{w} \cdot \text{vi}(\text{index}_0) \cdot ^{2} \cdot \text{etae}_1 \cdot \\
\text{nui}(\text{index}_0) \rangle \cdot \text{vi}(\text{index}_0) \rangle \cdot \text{vi}_\text{hat}_1 \rangle + \text{nui}(\text{index}_0) \rangle \cdot \text{vi}(\text{index}_0) \rangle \cdot ^{2} \cdot \text{fion}_1 \rangle \cdot \text{dy}(0) / (1 \cdot \text{index}_0) \}
\]

\[
\text{dy}(1) = (-1 / \text{mn}(\text{index}_0) \rangle \cdot \text{vn}) \rangle \cdot (i \cdot \text{w} \cdot \text{vn} \cdot \text{etan}_1 + \text{ne}(\text{index}_0) \rangle \cdot \text{nui}(\text{index}_0) \rangle \cdot (\text{etae}_1 + \text{fion}_1 \cdot \text{etan}_1) \rangle \cdot \text{dy}(0) / (1 \cdot \text{index}_0) \}
\]

set of differential equations

if index==N_set-1 & dxi_last>0
    y(:,index+1)=y(:,index)+dy.*dxi_last;
else
    y(:,index+1)=y(:,index)+dy.*dxi;
end;
mach_set(index)=mach(index_0);
nn_set(index)=nn(index_0);
nui_set(index)=nui(index_0);
nue_set(index)=nue(index_0);
nuh_set(index)=nuh(index_0);

---

% 1st order integration %

% 1st order solution
etan_1=y(1,:)';
etae_1=y(2,:)';
vi_hat_1=y(3,:)';
Vi_1=vi_hat_1/vi_set';
Ve_1=((1/(ne_set.*ve_set)).*(ne_set.*(vi_set-ve_set).*etae_1'+ne_set.*vi_set.*Vi_1'-fld.*gamd_hat'));
taoe_1=y(4,:)';
pot_hat_1=y(5,:)';
P_1=pot_hat_1/pot_set';
fion_1=etan_1+(0.5+2./(1./te_set'+2)+1./te_set').*taoe_1;
fh_1=(1/(1+alfabp.*sqrt(pi/(8.*te_set'))).*omega_B./(nn_set'.*sigmaen)).*(-etan_1-0.5.*taoe_1);

etan_2=y(6,:)';
etae_2=y(7,:)';
vi_hat_2=y(8,:)';
Vi_2=vi_hat_2/vi_set';
Ve_2=((1/(ne_set.*ve_set)).*(ne_set.*(vi_set-ve_set).*etae_2'+ne_set.*vi_set.*Vi_2'-fld.*gamd_hat'));
taoe_2=y(9,:)';
pot_hat_2=y(10,:)';
P_2=pot_hat_2/pot_set';
fion_2=etan_2+(0.5+2./(1./te_set'+2)+1./te_set').*taoe_2;
fh_2=(1/(1+alfabp.*sqrt(pi/(8.*te_set'))).*omega_B./(nn_set'.*sigmaen)).*(-etan_2-0.5.*taoe_2);

etan_3=y(11,:)';
etae_3=y(12,:)';
vi_hat_3=y(13,:)';
Vi_3=vi_hat_3/vi_set';
Ve_3=((1/(ne_set.*ve_set)).*(ne_set.*(vi_set-ve_set).*etae_3'+ne_set.*vi_set.*Vi_3'-fld.*gamd_hat'));
taoe_3=y(14,:)';
pot_hat_3=y(15,:)';
P_3=pot_hat_3/pot_set';
fion_3=etan_3+(0.5+2./(1./te_set'+2)+1./te_set').*taoe_3;
fh_3=(1/(1+alfabp.*sqrt(pi/(8.*te_set'))).*omega_B./(nn_set'.*sigmaen)).*(-etan_3-0.5.*taoe_3);

% linear superposition of solutions %
exit_len=length(etan_1);
M=[vi_hat_1(exit_len) vi_hat_2(exit_len) vi_hat_3(exit_len)];
\( \text{taoe}_1(\text{exit}_\text{len}) \text{taoe}_2(\text{exit}_\text{len}) \text{taoe}_3(\text{exit}_\text{len}); \)
\( \text{pot}_\text{hat}_1(\text{exit}_\text{len}) \text{pot}_\text{hat}_2(\text{exit}_\text{len}) \text{pot}_\text{hat}_3(\text{exit}_\text{len})-1); \)

\( M2=\{\text{vi}_\text{hat}_1(\text{exit}_\text{len}) \text{vi}_\text{hat}_2(\text{exit}_\text{len}) \text{vi}_\text{hat}_3(\text{exit}_\text{len}); \)
\( \text{taoe}_1(\text{exit}_\text{len}) \text{taoe}_2(\text{exit}_\text{len}) \text{taoe}_3(\text{exit}_\text{len})\}; \)

\( \text{lin}_\text{coeff}=\text{null}(M2); \)
\( \text{c1}=\text{lin}_\text{coeff}(1,1); \)
\( \text{c2}=\text{lin}_\text{coeff}(2,1); \)
\( \text{c3}=\text{lin}_\text{coeff}(3,1); \)

\( A = \{\text{M}(1,2) \text{M}(1,3); \text{M}(2,2) \text{M}(2,3)\}; \)
\( B = \{-\text{M}(1,1); -\text{M}(2,1)\}; \)
\( C = \text{inv}(A)*B; \)
\( \text{D} = [1; \text{C}]; \)
\( \text{d}=\text{D}/\text{norm}(\text{D}); \)

\( \text{etan}=\text{c1}.*\text{etan}_1+\text{c2}.*\text{etan}_2+\text{c3}.*\text{etan}_3; \)
\( \text{etae}=\text{c1}.*\text{etae}_1+\text{c2}.*\text{etae}_2+\text{c3}.*\text{etae}_3; \)
\( \text{vi}_\text{hat}=\text{c1}.*\text{vi}_\text{hat}_1+\text{c2}.*\text{vi}_\text{hat}_2+\text{c3}.*\text{vi}_\text{hat}_3; \)
\( \text{Ve}=\text{c1}.*\text{Ve}_1+\text{c2}.*\text{Ve}_2+\text{c3}.*\text{Ve}_3; \)
\( \text{taoe}=\text{c1}.*\text{taoe}_1+\text{c2}.*\text{taoe}_2+\text{c3}.*\text{taoe}_3; \)
\( \text{pot}_\text{hat}=\text{c1}.*\text{pot}_\text{hat}_1+\text{c2}.*\text{pot}_\text{hat}_2+\text{c3}.*\text{pot}_\text{hat}_3; \)
\( \text{fion}=\text{c1}.*\text{fion}_1+\text{c2}.*\text{fion}_2+\text{c3}.*\text{fion}_3; \)
\( \text{fh}=\text{c1}.*\text{fh}_1+\text{c2}.*\text{fh}_2+\text{c3}.*\text{fh}_3; \)

%%%%%%% linear superposition of solutions %%%% %%%%%

\( \text{real}_\text{omega} = \text{real}_\text{index}, 1 = \text{w}_\text{real}; \)
\( \text{imag}_\text{omega} = \text{imag}_\text{index}, 1 = \text{w}_\text{imag}; \)
\( \text{moe} = \text{imag}_\text{index}, \text{real}_\text{index} = 1/\text{abs} \text{det}(\text{M}) \)
\( \text{moe}_\text{pot} = \text{imag}_\text{index}, \text{real}_\text{index} = 1/\text{abs} \text{pot}_\text{hat}(1)-\text{pot}_\text{hat}(\text{exit}_\text{len}); \)
\( \text{figure}(2); \)
\( \text{clf}; \)
\( \text{contour} = \text{real}_\text{omega}, \text{imag}_\text{omega}, \text{moe}; \)
\( \text{figure}(5); \)
\( \text{clf}; \)
\( \text{contour} = \text{real}_\text{omega}, \text{imag}_\text{omega}, \text{moe}_\text{pot}; \)

\( \text{w} \)

\( \text{imag}_\text{index} = \text{imag}_\text{index} + 1; \)
\( \text{end}; \)
\( \text{real}_\text{index} = \text{real}_\text{index} + 1; \)
\( \text{end}; \)
C.2 0th Order Differential Equations Used in ODE45

function [out1,out2,out3]=dy(xi,y,flag,eif|eifm,omega_B,fld,vn,eip,sigma0,sigmaen,alfabp)

% state variables
x=y(1);
lnne=y(2);
vi=y(3);
te=y(4);
pot=y(5);
ne=exp(lnne);

% parameters
% ion flux
fli=ne.*vi;

% electron velocity
vex=vi-fld./ne;

% neutral number density
nn=(eifm-fli)./vn;

% ionization frequency
nui = nn.*sigma0.*sqrt((8.*te./pi).*(1+2.*te./ci)).*exp(-ci./te);

% diffusion frequency
nue = nn.*sigmaen.*sqrt((8.*te./pi)+alfabp.*omega_B);
nuh = omega_B.^2./nue;

% differential equations to solve
dy = zeros(5,1);

dy(1) = te-0.6.*vi.^2;
dy(5) = -(nui.*(2.*vi-vn).*nui.*(4.*eip./te+1).*vi.^2./vex+vi.^2.*nuh.*vex./te);
dy(4) = -(te./vi.^2-1).*dy(5)-dy(1).*te.*(nui./vi).*((2-vn./vi)+nuh.*vex);
dy(3) = -(1./vi).*dy(5)-nui.*(1-vn./vi).*dy(1);
dy(2) = (1./vi.^2).*dy(5)+nui.*(2-vn./vi).*dy(1)./vi;

% condition to end integration
if nargin < 3 | isempty(flag)
    out1 = dy;
else
    switch(flag)
    case 'events'
        mach = vi/sqrt(te/0.6);
        out1 = [dy(1).*te-0.02; ne-.01; mach-5];
        len = length(out1);
        out2 = ones(len,1);
        out3 = zeros(len,1);
        end
end
References


