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Analytical Treatment of Linearized Self-Consistent  
Theory of a Gyromonotron with a Nonfixed Structure

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### Abstract

The linearized self-consistent theory of gyrotron is analytically treated for a self-excited oscillator with nonfixed structure. The self-consistent RF axial field profile consists of four waves, namely the forward and backward travelling waves of the waveguide, and two electron cyclotron waves. According to the magnetic field value (or the detuning parameter), the electron cyclotron wave couples strongly with the forward wave, or weakly with backward wave of the waveguide mode. Also, the starting current and the axial field profile are significantly different for the forward versus the backward interaction.

## 1. Introduction

In most analyses of gyrotrontrons, the RF axial field profile has been assumed to be a fixed profile which is independent of the electron beam current, including Gaussian profile (Nusinovich and Erm 1972, Gaponov et al. 1981), sinusoidal profile (Chu et al. 1980, Kre'scher and Temkin 1981), and the cold cavity profile (Fliflet et al. 1982, Saito et al. 1986). The performance of gyrotrontrons has been calculated for these RF axial field profiles. However, the RF axial field profile is modified by the interaction between electrons and RF field and a self-consistent treatment is required. The effect of the electron beam and the RF field interaction on gyrotrontron operation has been considered previously (Bratman et al. 1973, Flyagin et al. 1977, Charbit et al. 1981, Fliflet et al. 1982). The high frequency electron current appears in the wave equation as a source. The wave equation and the electron equation of motion are simultaneously solved. The previous works, however, were numerical treatments because of the complexity of the cavity structure.

Saito and coworkers (Saito et al. 1986) proposed and demonstrated experimentally a gyrotrontron with a minimum Q cavity which has very low reflection in the output taper, in order to reduce the field magnitude in the cavity. They found that the self-consistent field profile of the gyrotrontron is significantly modified by the electron beam.

As an extreme case of such gyrotrontrons with a minimum Q cavity, we have modelled the gyrotrontron as a semi-infinite uniform waveguide with a stipulated interaction length (Bratman et al. 1973). The static magnetic field is assumed to be uniform in the interaction region. The interaction of the gyrotrontron is bounded by the abrupt drop of the static magnetic

field. We performed an analytical treatment of the linearized self-consistent field for this model, which is shown in Fig.1. This analysis shows that the forward and backward travelling waves of the waveguide, and the two electron cyclotron waves, form the self-consistent RF field profile. Also, the interaction of the electron cyclotron wave with the forward or backward travelling wave impacts significantly the gyromonotron performance.

## 2. Theoretical Model

We consider an annular beam of magnetized weakly relativistic electrons interacting with the  $TE_{mp}$  mode field. The wave equation for RF axial field profile  $f$  is given by (Bratman et al. 1973, Fliflet et al. 1982, Tran et al. 1986):

$$d^2f/d\zeta^2 + k^2f = -I\langle P \rangle, \quad (1)$$

where  $\zeta$  is the normalized axial coordinate defined by

$$\zeta = (\beta_{\perp}^2/2\beta_{\parallel})(\omega/c)z \quad (2)$$

Here  $z$  denotes the axial coordinate;  $\beta_{\perp}, \beta_{\parallel}$  are the  $\beta$  values of perpendicular and parallel component of electron velocity divided by  $c$ ;  $\omega$  is the oscillating frequency, and  $c$  is the velocity of light. In Eq.(1),  $k$  is the normalized axial wavenumber defined by

$$k = (2\beta_{\parallel}/\beta_{\perp})(1-\omega_{\text{cut}}^2/\omega^2)^{1/2}, \quad (3)$$

where  $\omega_{\text{cut}}$  is the cutoff frequency of the waveguide. In the right hand side of Eq.(1),  $I$  is the normalized electron beam current

$$I = (16/\pi)(eI_A/\epsilon_0 m \gamma c^3) \beta_{\perp}^{2(n-4)} \beta_{\parallel} (n^n/2^n n!)^2 \times J_{m \pm n}^2(k_{\perp} R_e) / [(v_{mp}^2 - m^2) J_m^2(v_{mp})], \quad (4)$$

where  $I_A$  is the beam current, and  $n$  is the harmonic number. Also,  $v_{mp}$  is the  $p$ th nonzero root of  $J_m'$ , and  $R_e$  is the electron beam radius. The transverse wavenumber  $k_{\perp}$  is defined by  $v_{mp}/R$ , where  $R$  is the waveguide radius. In Eq.(1),  $P$  is the complex electron momentum and satisfies the following equation of motion (Bratman et al. 1973, Fliflet et al. 1982,

Tran et al. 1986).

$$dP/d\zeta + i(\Delta - 1 + w)P = -inw^{n-1}f, \quad (5)$$

where  $w = |P|^{2/n}$ , and  $\Delta$  is the detuning parameter defined by

$$\Delta = (2/\beta_{\perp}^2)(1 - n\omega_{co}/\omega) \quad (6)$$

In Eq.(6),  $\omega_{co}$  is the relativistic electron cyclotron frequency. The symbol  $\langle \rangle$  defines an ensemble average over the electrons at  $\zeta$ .

We linearize Eq.(5) and get the following expression (Tran et al. 1986), assuming that the electrons are monoenergetic and have a uniform phase angle distribution at the initial position  $\zeta=0$ .

$$P = -inE*f - E*(E*f), \quad (7)$$

where \* implies the convolution integral defined by

$$E*f = \int_0^{\zeta} E(\zeta - \zeta')f(\zeta')d\zeta' \quad (8)$$

The transfer function E is

$$E(\zeta) = \exp(-i\Delta\zeta) \quad (9)$$

The electric field of the RF field  $f(\zeta)$  changes the electron momentum at a position  $\zeta$ . This change propagates through the electron cyclotron motion, and E corresponds to the transfer function. The first term of Eq.(7) describes the above-mentioned cumulative effect in terms of the convolution integral. The RF field changes the electron energy  $\gamma$ , and the cyclotron frequency. This change induces the electron bunching in the phase angle distribution. The double convolution integral in Eq.(7) expresses the bunching effect. Now, we have the closed form of

the self-consistent wave equation in the form of integrodifferential equation, substituting Eq.(7) into Eq.(1).

Figure 1 is our model of the gyromonotron. The region  $\zeta < 0$  is below cutoff and gives the boundary condition as

$$f(0) = 0 \quad (10)$$

The interaction is terminated at  $\zeta = \mu$  by an abrupt drop of the static field. The radiation must satisfy the boundary condition at the output section

$$f'(0) = -ikf(\mu), \quad (11)$$

which means the radiation is forward-travelling-wave-like locally at  $\zeta = \mu$ . This boundary condition is widely applied to the gyromonotron with a cavity structure (Bratman et al. 1973, Fliflet et al. 1982).

We introduce the Laplace transformation defined by

$$F(s) = \int_0^{\infty} \exp(-s\zeta) f(\zeta) d\zeta, \quad (12)$$

where  $s$  is the complex wavenumber. Under the Laplace transformation, Eq.(7) becomes products of two (or three) Laplace-transformed quantities by using the convolution theorem. Laplace-transformed equations of Eqs.(1) and (7) give

$$F(s) = \alpha_0 (s + i\Delta)^2 / D(s), \quad (13)$$

where

$$D(s) = (s^2 + k^2)(s + i\Delta)^2 - I(1 - n\Delta + ins) \quad (14)$$

The input boundary condition Eq.(10) is naturally taken into account in the Laplace transformation, and  $\alpha_0 = f'(0)$ . The wave equation Eqs.(1), combined with (7) is a homogeneous equation and has a free constant  $\alpha_0$ .

The RF field amplitude in the coordinate space can be easily obtained by the inverse Laplace transformation given by

$$\begin{aligned}
 f(\zeta) &= (2\pi i)^{-1} \int_{-i\infty+a}^{i\infty+a} \exp(s\zeta) F(s) ds \\
 &= \sum_{j=1}^4 \alpha_j \exp(s_j \zeta),
 \end{aligned} \tag{15}$$

where  $s_j$  is the roots of  $D(s)=0$ , and  $\alpha_j$  is the residue of  $F(s)$  with respect to  $s_j$  defined by

$$\alpha_j = \lim_{s \rightarrow s_j} (s - s_j) F(s) \tag{16}$$

The boundary condition at the output section Eq.(11) gives the following condition

$$\sum_{j=1}^4 \exp(s_j \mu) \alpha_j (s_j + ik) = 0 \tag{17}$$

This complex equation (namely two real equations) determines the eigenvalues of the current  $I$  and the wavenumber  $k$  (or the frequency  $\omega$ ). Physically, the eigenvalues  $I$  and  $\omega$  are the starting current and the frequency of the gyrotron oscillation.

We recall the boundary condition at the initial position Eq.(10), which has already been taken into account in the Laplace transformation. Eqs.(10), and (15) gives another form of the boundary condition.

$$\sum_{j=1}^4 \alpha_j = 0 \tag{18}$$

Equations (17) and (18) imply that the waves travelling in  $\zeta$  and  $-\zeta$



direction are necessary to satisfy the boundary conditions.

### 3. Results and Discussions

We begin with the analysis of the dispersion equation (14). The natural modes of the dispersion equation without the beam current is the forward ( $s = -ik$ ) and the backward ( $s = ik$ ) travelling wave of the waveguide, and the two electron cyclotron waves ( $s = -i\Delta$ ) which are degenerate. The dispersion equation (14) is numerically solved including the beam current. Two situations of the poles are found. In one case, all four roots are pure imaginary and oscillating waves. In another case, the electron cyclotron waves have complex roots (exponential growth and decay waves in the coordinate space), and the waveguide modes are oscillatory. Equicontour curves of the spatial growth rate of the electron cyclotron wave are shown in Fig.2, in the plane of  $k$  and  $\Delta$  for the fixed current  $I = 1 \times 10^{-3}$  and  $n = 1$ . The hatched area corresponds to the oscillatory waves ( $\text{Re}(s_j) = 0$ ). The dispersion relation of the electrons is  $\omega = \omega_{co} + c\beta_{\parallel}k_{\parallel}$ , where  $k_{\parallel}$  is the axial wavenumber. The synchronous condition with the waveguide modes is  $\Delta = \pm k$ , where  $+$ ( $-$ ) sign corresponds to the forward (backward) travelling wave of the waveguide. Figure 2 depicts that the spatial growth rate is enhanced at  $\Delta = \pm k$  due to the strong coupling between the electron cyclotron wave and the waveguide modes.

Figure 3 shows an example of the RF profile for  $n = 1$ ,  $\mu = 15$ ,  $\Delta = -0.2, 0, 0.2$ , and  $0.4$ . The field amplitude and the phase are defined by

$$f(\zeta) = |f(\zeta)| \exp(-i\psi(\zeta)) \quad (19)$$

The starting current  $I$  and the eigenvalue of  $k$  are plotted in Fig.4 as a function of  $\Delta$  for  $n = 1$ ,  $\mu = 15$ . Also, we plot the synchronous condition

$k = \pm \Delta$  between the electron cyclotron wave and the waveguide modes. Figures 5, and 6 show the loci of the corresponding poles  $s_j$  and amplitude  $\alpha_j$  of four waves in the complex planes. The amplitudes are normalized such that the maximum value of  $|f(\zeta)|$  is one. The  $\Delta$  values are marked next to the loci. In Figs.5 and 6,  $\bullet$  and  $\times$  correspond to the forward and backward waveguide modes, and two cyclotron modes, respectively.

In the region  $\Delta > 0.35$ ,  $k < \Delta$  is satisfied and the four waves are purely oscillating as Fig.2 implies. Although the cyclotron waves propagate forward, they can not synchronize with the forward travelling wave of the waveguide because their phase velocity is much faster than that of the waveguide modes. The self-consistent field profile is formed by the interference of the four pure oscillating waves. A high starting current is required to maintain the self-consistent field, because the cyclotron wave can not grow exponentially in the axial direction. Also, the starting current increases rapidly as the  $\Delta$  increases from the synchronous condition with the forward travelling wave of the waveguide. Figure 3 shows an example of the field profile for  $\Delta=0.4$ . The peak of the field is close to the output section and the field is pushed forward.

In the region  $0.0 < \Delta < 0.35$ , the electron cyclotron waves propagate forward and one of them becomes exponential-growth wave in the axial direction ( $\text{Re}(s_j) > 0$ ) mainly due to the coupling with the forward travelling wave of the waveguide. Especially, at  $\Delta=0.2$ , the synchronous condition  $k=\Delta$  between the cyclotron wave and the forward travelling wave is satisfied ( Fig.3 ) and we have the maximum growth rate and the minimum starting current. The field profile for  $\Delta=0.2$  with the minimum starting current is plotted in Fig.3. The peak of the field retreats slightly from the output section. Note that the backward travelling wave is

necessary to satisfy the boundary condition even for the case with the strong interaction with the forward travelling wave, as Fig.5 shows.

In the region  $\Delta < 0.0$ , the cyclotron waves propagate backward ( $\text{Im}(s_j) > 0$ , Fig.4) and couple mainly with the backward travelling wave of the waveguide. However,  $k > -\Delta$  is always satisfied in Fig.4 and there is not strong coupling with the backward travelling wave, unlike the strong coupling with the forward travelling wave at  $\Delta = 0.2$ . Instead, the weak coupling with the backward travelling wave is maintained and one of the electron cyclotron waves is an exponential-growth wave. Therefore, the starting current increases just gradually as the detuning parameter decreases from the optimum value, unlike  $\Delta > 0.35$  region. An example of the field is plotted in Fig.4 for  $\Delta = 0.0$ . The axial field profile is pushed back away from the output section due to the interaction with the backward travelling wave of the waveguide. This means that the interaction with the backward wave increases the stored energy of the gyromonotron working space, for the fixed output power or the fixed field magnitude at the output section. This phenomenon was observed in the self-consistent analysis of a gyromonotron with an open resonator (Saito et al. 1986). As  $\Delta$  decreases further, the field is subject to the transition from the fundamental axial mode ( $q=1$ ) to the second axial mode ( $q=2$ ), where  $q$  is the axial mode number. Figure 3 shows the field profile for  $\Delta = -0.2$ , and there are two spatial peaks. This transition that occurs as  $\Delta$  decreases is a consequence of the retreat of the field due to the enhancement of the interaction with the backward wave. According to gyrotron linear theory (Kreischer and Temkin 1981) in which the cavity axial profile is assumed as a fixed sinusoidal function, the higher axial modes are excited at higher magnetic fields (or small  $\Delta$ )

through the interaction with a backward travelling wave. This result qualitatively agrees with the self-consistent theory for a nonfixed structure gyrotron presented in this paper.

#### 4. Conclusion

The linearized self-consistent theory of a gyromonotron has been analytically treated for a self-excited oscillator with a nonfixed structure. The self-consistent RF axial field profile consists of the forward and back travelling waves of the waveguide, and two electron cyclotron waves, all of which are modified by the interaction between the RF field and the electron beam. The interaction between the cyclotron wave and the waveguide modes varies depending on the value of  $\Delta$ . Consequently, the starting current and the axial field profile are significantly affected.

The minimum starting current occurs at  $\Delta_{\text{opt}}$ , where the synchronous condition of the coupling between the cyclotron wave and the forward wave is roughly satisfied and the spatial growth rate of the cyclotron wave is maximum. For higher  $\Delta$  value, the interaction with the forward wave is terminated and the cyclotron wave becomes a pure oscillating wave. In this region, the starting current increases very rapidly as  $\Delta$  increases. In the region  $\Delta < 0$ , the weak interaction with the backward wave is dominant instead of the forward wave. The axial field profile is pushed backward due to the enhanced interaction with the backward wave. Therefore, stored energy increases for a fixed output power. As  $\Delta$  decreases further, the self-consistent field is subject to the transition from the fundamental axial mode to the higher axial modes.

Although this present model of gyromonotron is simple, it is useful in understanding the physics of the gyromonotron, especially one with a minimum Q cavity.

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### Figure Captions

- Fig.1: Model of the gyrotron with a nonfixed structure.
- Fig.2: Equicontour curves of spatial growth rate  $\text{Re}(s_j)$  in  $k$ - $\Delta$  plane, for  $n=1$  and  $I=1 \times 10^{-3}$ .
- Fig.3: Axial RF field profile for  $n=1$  and  $\mu=15$ , varying  $\Delta$ .
- Fig.4: Starting current  $I$  and wavenumber  $k$  as a function of  $\Delta$  for  $n=1$  and  $\mu=15$ . Synchronous condition  $k=\Delta$  between cyclotron wave and waveguide modes are plotted by broken lines.
- Fig.5: Loci of poles  $s_j$  for the waveguide modes ( $\bullet$ ) and cyclotron modes ( $\times$ ), varying  $\Delta$  value. Values of  $\Delta$  are marked next to the loci. Same conditions as Fig.4.
- Fig.6: Loci of amplitudes  $\alpha_j$  in a complex plane, varying  $\Delta$  value. Same conditions as Fig.4.

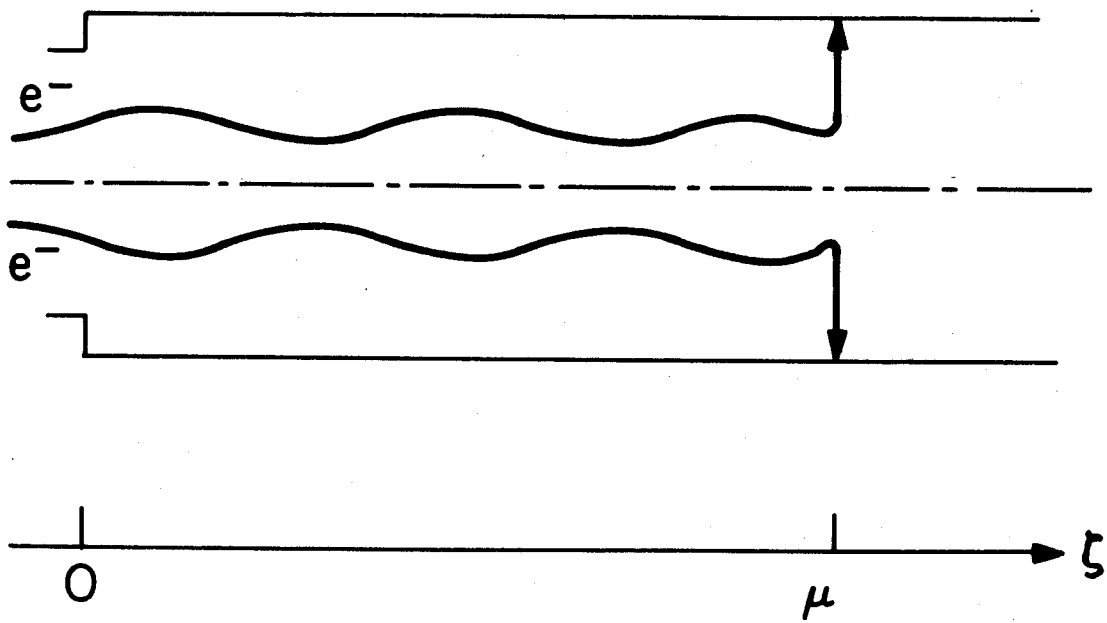


Figure 1

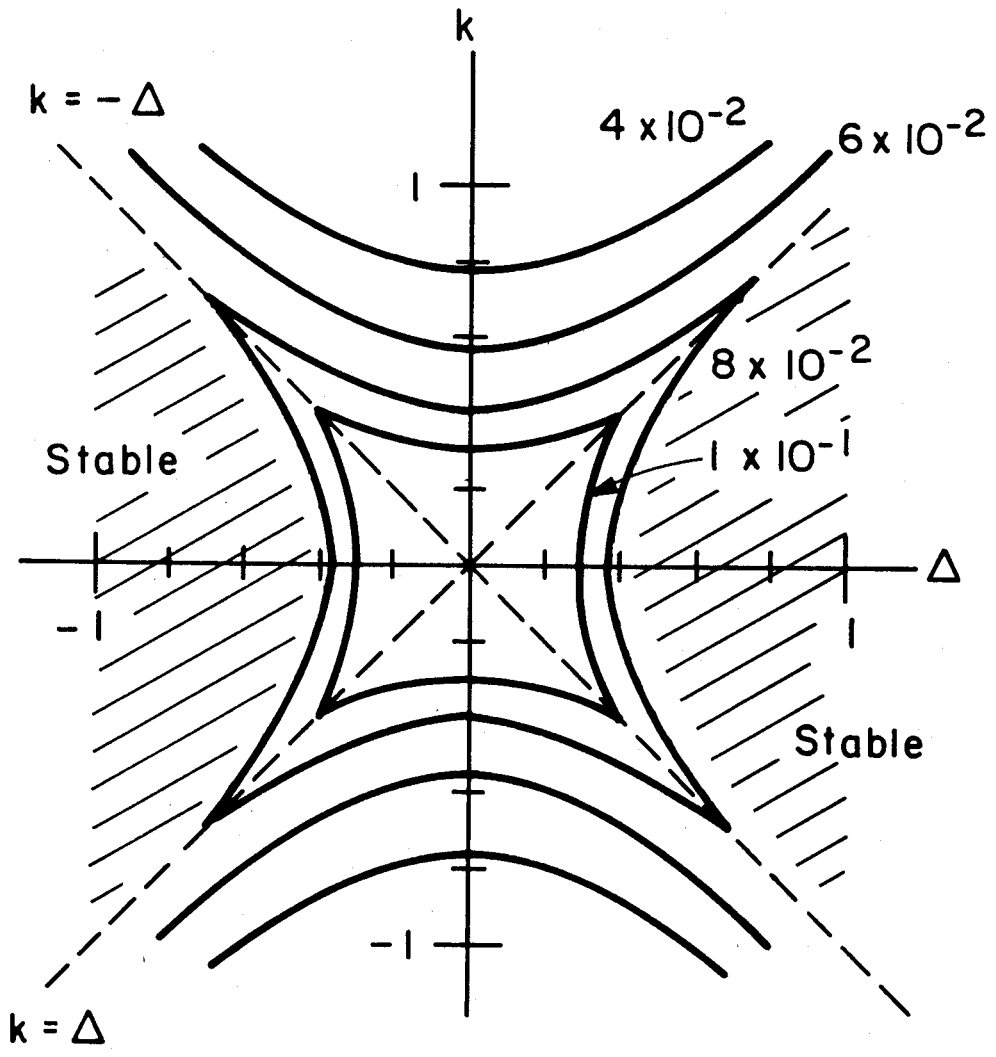


Figure 2

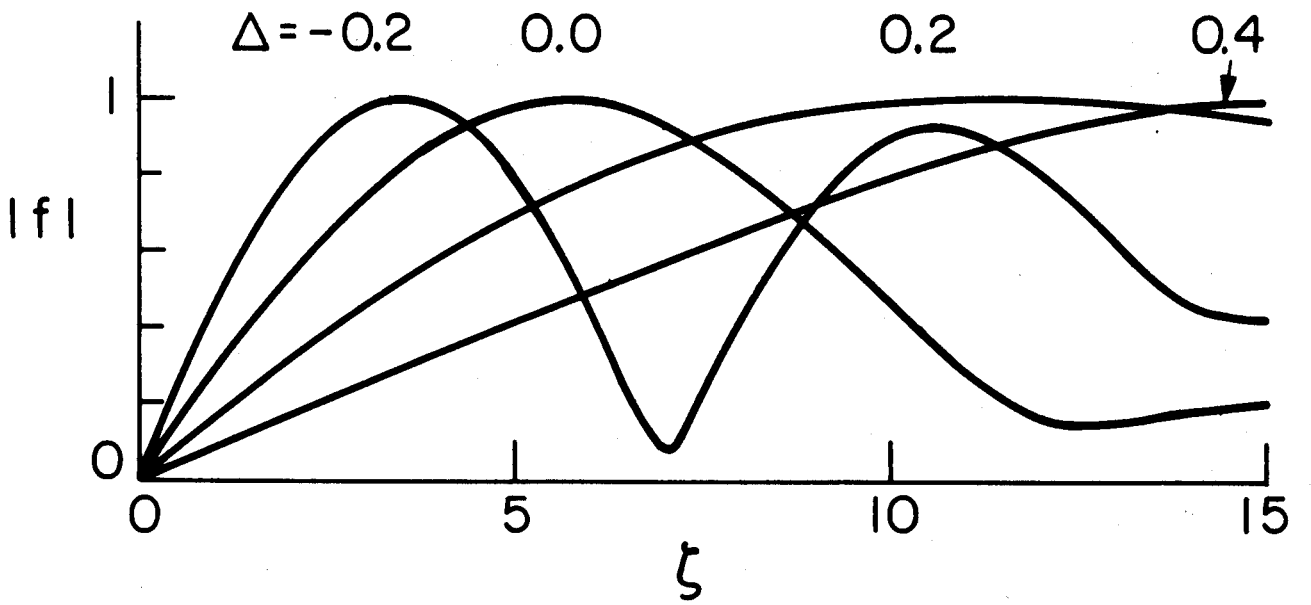


Figure 3

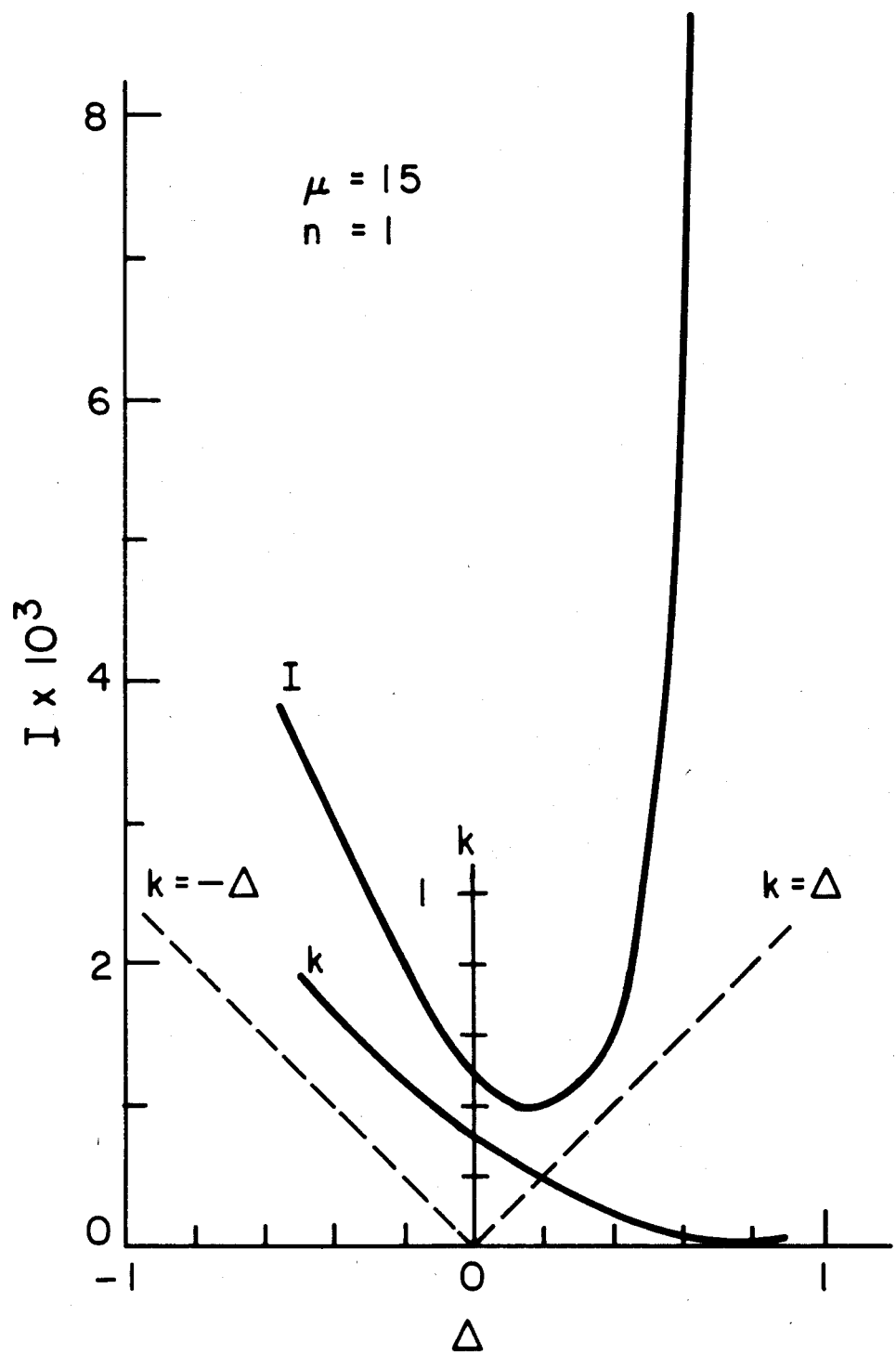


Figure 4

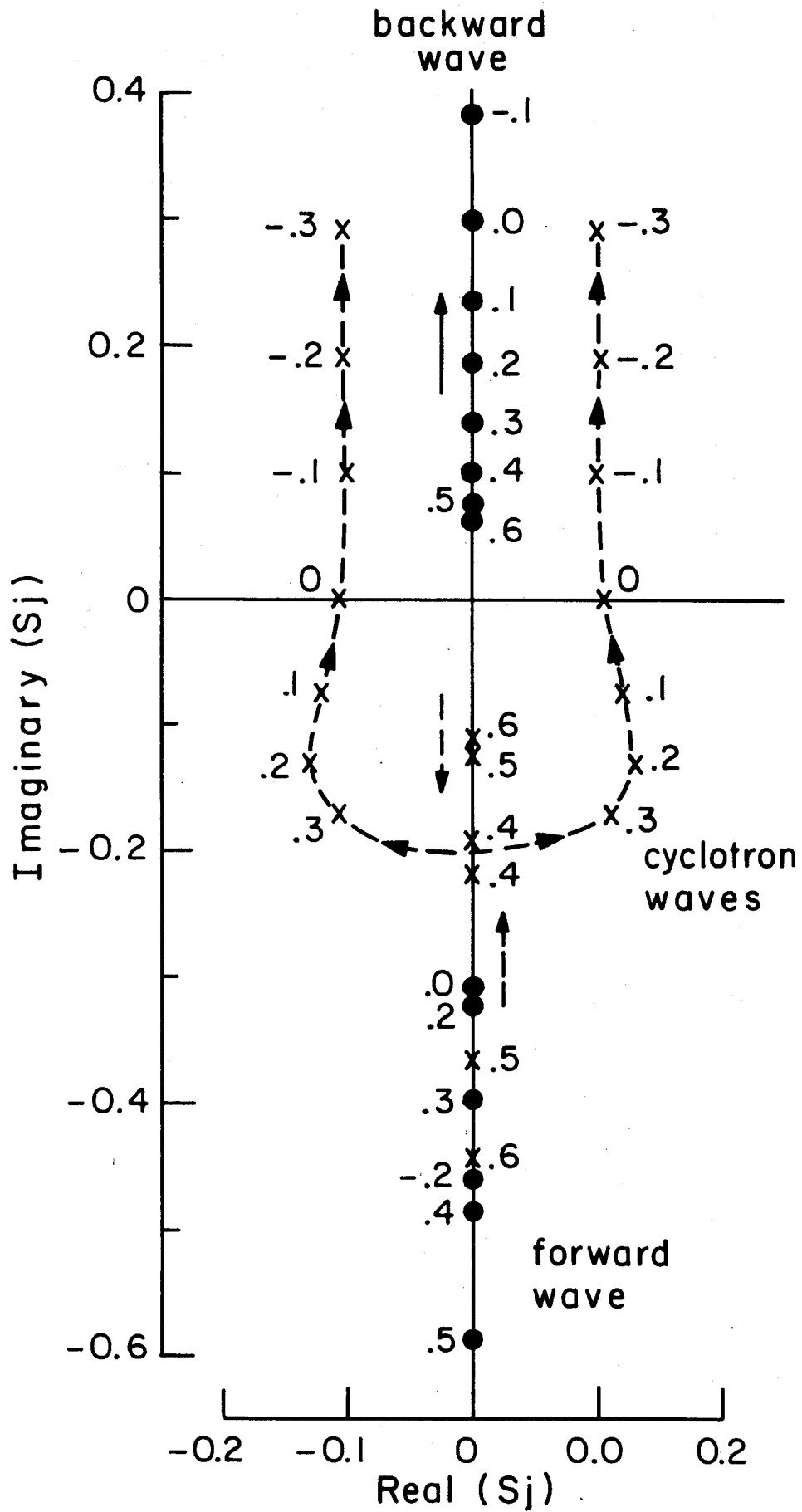


Figure 5

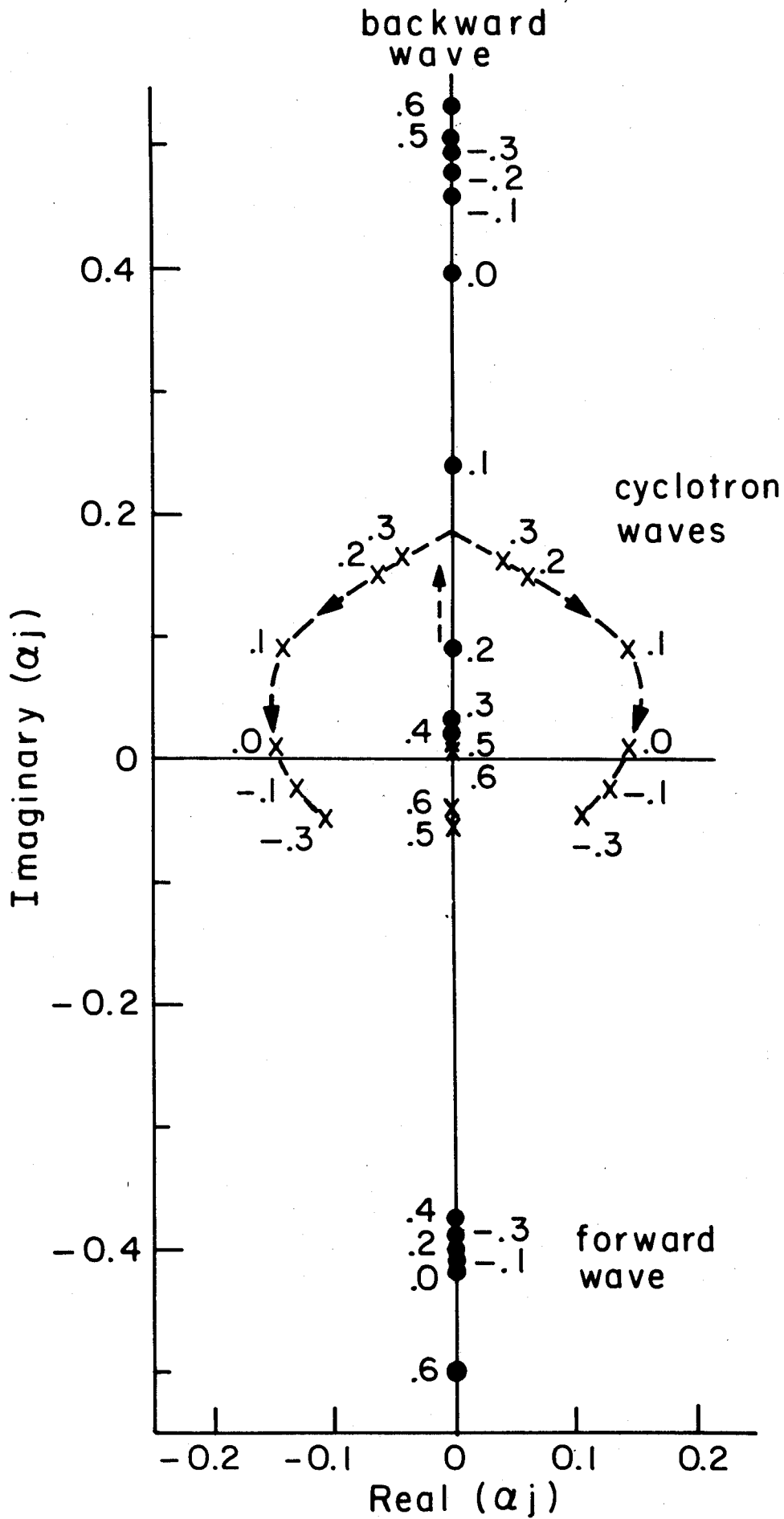


Figure 6