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On the Theoretical Foundations of the Tokamak $\beta aB/I$ Limit

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Abstract

It is shown that for tokamak equilibria with vanishing current density at the plasma edge, the ratio $\beta aB/I$ has an upper bound which itself depends only on the safety factor at the magnetic axis and the geometry of the plasma boundary. This result follows solely from tokamak equilibrium constraints.

One of the most important issues in the ideal magnetohydrodynamic theory of tokamak plasmas is the determination of the maximum ratio β of plasma kinetic pressure to magnetic pressure that can be achieved under conditions of MHD equilibrium and stability. In the last few years it has become well established [1-4] that, under a wide variety of circumstances, the maximum β compatible with ideal MHD stability obeys a law of the form

$$\beta_{max} = CI/aB \;, \tag{1}$$

where I is the plasma current, a its minor radius, B is the magnetic field and C is a proportionality factor that depends on the geometrical characteristics of the plasma cross section [5]. This scaling applies for values of I below the current limit set by the external kink instability (usually $q_a > 2$ where q_a is the edge safety factor), namely for the range of plasma parameters where ideal MHD stability limits are set by pressure driven modes.

The expression (1) for the tokamak beta limit is of very general validity, as it holds for both low [1,2] and high [1,3,5] toroidal mode numbers with only some variation in the numerical value of C, and agrees with the available experimental evidence [4]. However, despite its generality and simplicity, this scaling has been obtained only empirically as the result of fits to either experimental data or numerical simulations: no satisfactory demonstration based on first principles is known to this date. Another intriguing feature is the linear relation between β_{max} and the normalized current I/aB: essentially β is inversely proportional to the square of the toroidal field (or the edge safety factor) whereas I/aB is inversely proportional to the toroidal field (or q_a), so that a quadratic relation between β_{max} and I/aB might have been expected.

In this work we show that for a fixed plasma boundary geometry and a fixed safety factor at the magnetic axis q_o, the ratio $\beta aB/I$ must have an upper bound for tokamak equilibria whose toroidal current density vanishes (and tends to zero smoothly) at the plasma edge. The existence of this upper bound is due solely to equilibrium constraints which is highly suggestive of the universal character of the $\beta_{max} = CI/aB$ scaling, where only the numerical value of C would be determined by the stability constraints.

As pointed out earlier, the proportionality factor C between β_{max} and I/aB depends on the geometry of the plasma boundary. We shall not address such dependence in the present work, but rather shall carry out the analysis for a fixed boundary shape. On the other hand, we shall also carry out the analysis for a fixed value of q_o, and consider C to be a function of q_o as well. (One may later optimize C with respect to q_o as done in Ref. 2). Retaining the dependence of C on q_o is useful because this helps correct the mismatch in powers of B (or q) between β and I/aB, provided C scales like q_o^{-1}. Of course it remains to be explained why C should scale like q_o^{-1} and not like q_a^{-1} (or $q_a^{-x} q_o^{x-1}$). Our work shows that $q_o \beta aB/I$ is bounded from above, but no such bound can be proven for $q_a^x q_o^{1-x} \beta aB/I$ with $x > 0$.

Let us begin by defining a dimensionless characterization of fixed boundary tokamak equilibria. All dimensioned equilibrium quantities can be "scaled out" by using the trivial invariance of the Grad-Shafranov equation under changes of units. Thus, up to trivial scalings, a tokamak equilibrium solution is completely specified by a dimensionless representation Γ of its boundary and two dimensionless flux functions. Usually the geometry of the plasma boundary is defined by a set of dimensionless parameters, $\Gamma \equiv (A, \kappa, \delta \ldots)$, representing the aspect ratio, elongation, triangularity, etc. These shall be held constant throughout our analysis. We find it convenient to split the representation of the two independent equilibrium flux functions into two normalized functions that carry only "profile shape" information, plus two dimensionless parameters that define the proper scale of the relevant flux functions. We choose as our independent, pure profiles a normalized average current density $\hat{\jmath}(\hat{\psi})$ and a normalized pressure $\hat{p}(\hat{\psi})$ defined by $\hat{\jmath} \equiv j_{av}(\psi)/j_{av}(\psi_o)$, $\hat{p} \equiv p(\psi)/p(\psi_o)$ and $\hat{\psi} \equiv (\psi - \psi_o)/(\psi_a - \psi_o)$; here $j_{av}(\psi)$ is some magnetic surface average of the plasma current density, $2\pi\psi$ is the poloidal flux and the subscripts "a" and "o" refer to the plasma edge and the magnetic axis respectively. By definition $\hat{\jmath}$ and $\hat{p}$ satisfy $\hat{\jmath}(0) = \hat{p}(0) = 1$. Besides we consider only equilibria with vanishing pressure and current density at the plasma edge, hence $\hat{p}(1) = d\hat{p}(1)/d\hat{\psi} = \hat{\jmath}(1) = 0$. The dimensionless parameters that set the proper scale of the current density and the pressure are chosen to be the safety factor on axis q_o (to be held constant throughout our analysis) and the beta-like $\beta_* \equiv \beta q_a^2/\epsilon$ whose precise definition is given below. Therefore we consider $\Gamma, \hat{\jmath}(\hat{\psi}), \hat{p}(\hat{\psi})$, q_o and β_* as the independent input variables that define an equilibrium. All other dimensionless equilibrium quantities (i.e., q_a or β_p) are derived from these.

Next we give the definitions of several equilibrium functions and parameters. We define a "radial" flux function $r(\psi)$ by $V(\psi) \equiv 2\pi^2 R_o r^2(\psi)$, where $V(\psi)$ is the volume enclosed by the flux surface and R_o is the radius of the magnetic axis. The plasma minor radius a is defined to be equal to the value of r at the plasma edge, and our inverse aspect ratio parameter ϵ is defined as $\epsilon \equiv a/R_o = (V_a/2\pi^2 R_o^3)^{1/2}$. Notice that ϵ is not a purely geometrical quantity since R_o depends on the details of the tokamak equilbirum. However ϵ is bounded by two purely geometrical parameters, $\epsilon_- \leq \epsilon \leq \epsilon_+$, ϵ_+ and ϵ_- being obtained respectively by using the minimum and maximum values of R around the plasma boundary instead of R_o in the above definition. The plasma beta and poloidal beta parameters are defined as

$$\beta \equiv 2B_o^{-2} V_a^{-1} \int_o^{V_a} p \, dV \ , \tag{2}$$

$$\beta_p \equiv 4I^{-2} R_o^{-1} \int_o^{V_a} p \, dV \ , \tag{3}$$

where B_o is the vacuum field at the magnetic axis $[R_o B_o = (RB_t)(\psi_a)]$, V_a is the total plasma volume and I is the total plasma current. Then,

$$\frac{\beta a B_o}{I} = \frac{1}{2\pi}(\beta\beta_p)^{1/2} = \frac{\beta_*^{1/2}(\epsilon\beta_p)^{1/2}}{2\pi q_a} \ . \tag{4}$$

We can now proceed to optimize the ratio $\beta a B_o/I$ at constant Γ and q_o. We do this in two stages. In the first one we hold β_* fixed and maximize $\beta a B_o/I$ with respect to variations of the normalized profiles $\hat{\jmath}(\psi)$ and $\hat{p}(\hat{\psi})$, subject to the condition of ideal MHD stability plus the constraints $\hat{\jmath}(0) = \hat{p}(0) = 1$, $\hat{\jmath}(1) = \hat{p}(1) = \hat{p}'(1) = 0$. The result (if it exists) is a function $(\beta a B_o/I)_{opt} = F(\Gamma, q_o, \beta_*)$. In the second stage we let β_* vary and obtain the maximum of $F(\Gamma, q_o, \beta_*)$ with respect to β_*.

The existence of the function $F(\Gamma, q_o, \beta_*)$ for any finite β_* is guaranteed by the equilibrium limit on $\epsilon\beta_p$. This implies that $\epsilon\beta_p$ cannot exceed a certain value $(\epsilon\beta_p)_{lim}$ of the order of one, that depends on the geometrical characteristics of the plasma and is determined by the formation of a separatrix at the plasma boundary. To our knowledge, no general proof of this assertion exists, but the overwhelming evidence supports it. Thus in the particular equilbrium solution of Haas [6], the relation between β_* and $\epsilon\beta_p$ is $\beta_* = \epsilon\beta_p/(1-\epsilon^2\beta_p^2)$, with the separatrix touching the plasma edge at $\epsilon\beta_p = 1$. In all numerical solutions of the Grad-Shafranov equation (see e.g. Ref. 7) $\epsilon\beta_p$ saturates about some value of order unity. A recent analytic result [8] for large aspect ratio circular tokamaks with general profiles also shows the limit on $\epsilon\beta_p$. Here we show numerically the $\epsilon\beta_p$ limit for a sequence of tokamak equilibria satisfying our constraints of fixed q_o and vanishing edge current density. For the sake of simplicity and to facilitate comparison with analytic models, we assume a large aspect ratio ($A = 10$) circular plasma boundary. The flux surface averaged current density used to specify our equilibria is the ohmic current $j_{av} = j_{oh}(\psi) \equiv \langle \mathbf{j}\cdot\mathbf{B}\rangle / \langle R_o \nabla\phi\cdot\mathbf{B}\rangle$, where $\langle\ldots\rangle$ stands for the conventional flux surface average. We generate a sequence of equilibria characterized by the profiles $p(\psi) = p_o(1-\hat{\psi}^2)^2$ and $j_{oh}(\psi) = j_o(1-\hat{\psi})$, where j_o is adjusted to keep $q_o = 1.00$ and p_o is increased to increase the beta parameters. We find that $\epsilon\beta_p$ cannot exceed a value of 2 at which point q_a tends to infinity and beyond which the equilbrium code fails to converge. The relation between $\epsilon\beta_p$ and β_* is shown in Fig. 1. Once we accept the existence of the equilibrium limit on $\epsilon\beta_p$, it is immediate to derive an upper bound for $\beta a B_o/I$ at fixed β_*, from which the existence of $F(\Gamma, q_o, \beta_*) = (\beta a B_o/I)_{opt}$ follows. To this end we only need to use the tokamak condition $q_a \geq q_o$ and write:

$$\frac{\beta a B_o}{I} = \frac{\beta_*^{1/2}(\epsilon\beta_p)^{1/2}}{2\pi q_a} \leq \frac{\beta_*^{1/2}(\epsilon\beta_p)_{lim}^{1/2}}{2\pi q_o} \, . \tag{5}$$

The second part of our argument consists of showing that, as β_* varies, $F(\Gamma, q_o, \beta_*)$ also has an upper bound. For finite β_* this is an obvious consequence of the above bound (5). However $\beta_* = \beta q_a^2/\epsilon$ can grow arbitrarily large. Thus as we approach the conventional tokamak equilibrium limit at fixed plasma current β/ϵ is limited but q_a tends to infinity, whereas in a flux conserving sequence q_a is fixed but β/ϵ is unlimited. In any case if, as assumed from the start, we restrict ourselves to equilibria with vanishing current density at the plasma edge, we can prove that, as β_* tends to infinity, q_a/q_o also tends to infinity at least as fast as $\beta_*^{1/2}$ as a consequence of the zero edge current density constraint. Therefore

$$\frac{\beta a B_o}{I} = \frac{\beta_*^{1/2}}{q_a/q_o} \frac{(\epsilon\beta_p)^{1/2}}{2\pi q_o} \tag{6}$$

is also bounded from above as $\beta_* \to \infty$. To show that, for vanishing edge current density, $(q_a/q_o)(\beta_* \to \infty) \gtrsim \beta_*^{1/2}$, we introduce the flux function

$$g(r) \equiv -\frac{2R_o^3 q^2(r)}{r^3 (RB_t)^2(r)} \int_o^r r'^2 \frac{dp}{dr'} dr' , \tag{7}$$

with the properties $g(0) = 0$ and $g(a) = \beta_*$. We also define $s(r) \equiv (r/q)\; dq/dr$ and $\nu(r) \equiv (r/g)\; dg/dr$. Now, the condition that the toroidal current density be equal to zero around the plasma boundary implies

$$\frac{dp(a)}{dr} = \frac{d(RB_t)(a)}{dr} = 0 , \tag{8}$$

hence

$$\nu(a) = 2s(a) - 3 . \tag{9}$$

The ratio q_a/q_o can be expressed as

$$\frac{q_a}{q_o} = \exp\left(\int_o^a \frac{1}{q}\frac{dq}{dr} dr\right) \tag{10}$$

or, changing variables from r to $g(r)$,

$$\frac{q_a}{q_o} = \exp\left(\int_o^{\beta_*} \frac{s}{\nu}\frac{dg}{g}\right) . \tag{11}$$

In the limit $\beta_* \to \infty$ the integral in Eq. (11) diverges logarithmically because $(s/\nu)(g = \beta_*) = s_a/(2s_a - 3) \geq 1/2$. No such divergence occurs at the lower integration limit because $(s/\nu)(g = 0) = 0$. Therefore for $\beta_* \to \infty$, the main contribution to that integral comes from the values of g near its upper limit, β_*. In this region we approximate s/ν by its limit value:

$$\frac{s}{\nu} \simeq \frac{s_a}{\nu_a} = \frac{s_a}{2s_a - 3} , \tag{12}$$

(assuming that the plasma current density tends to zero smoothly as we approach the plasma edge, $g \to \beta_*$). Thus we obtain

$$\frac{q_a}{q_o} \underset{\beta_* \to \infty}{\sim} \beta_*^{\frac{s_a}{2s_a-3}} \geq \beta_*^{1/2} . \tag{13}$$

Notice that the bound $(q_a/q_o)(\beta_* \to \infty) \gtrsim \beta_*^{1/2}$ is the least needed to prove the boundedness of $\beta a B_o/I$, so that $q_a^x \beta a B_o/I$ with $x > 0$ need not be bounded. Moreover, it only guarantees that the function $F(\Gamma, q_o, \beta_*)$ is bounded (not that F tends to zero) as $\beta_* \to \infty$. Therefore the maximum of F with respect to β_*, $F_{max} \equiv C(\Gamma, q_o) = (\beta a B_o/I)_{max}$ may be rather broad. This would explain an approximate linear scaling $\beta_{max} \sim CI/aB_o$ if one were to maximize β at constant I/aB_o.

These predictions are verified in our numerical example. Figures 2 and 3 display the dependence of q_a and $\beta a B_o/I$ on β_* for our equlibrium sequence. They show clearly that q_a is proportional to $\beta_*^{1/2}$ as $\beta_* \to \infty$, and that $\beta a B_o/I$ is bounded. Its maximum value is $C = 0.086$, or in terms of the more familiar $C_T = 10^8 \beta (aB_o/I)_{MKS} = 40\pi C = 10.9$. Figure 4 shows the $q(\psi)$ profile for the highest $\epsilon\beta_p$ equilibrium obtained ($\epsilon\beta_p = 2$), right before the equilibrium limit. It shows that the largest increase in q occurs near the plasma edge, which confirms the validity of our approximation (13). The flux and toroidal current density contours for this highest $\epsilon\beta_p$ equilibrium are displayed in Fig. 5.

As regards a further optimization of $C(\Gamma, q_o)$ with respect to q_o we have

$$C(\Gamma, q_o) = \frac{1}{2\pi q_o} \left[\frac{\beta_*^{1/2} (\epsilon\beta_p)^{1/2}}{q_a/q_o} \right]_{max} \tag{14}$$

where $\left[\beta_*^{1/2}(\epsilon\beta_p)^{1/2} q_o/q_a\right]_{max}$ is itself a function of Γ and q_o. If the latter dependence is weak, we can expect that, due to the more fundamental factor of q_o^{-1}, $C(\Gamma, q_o)$ is a decreasing function of q_o at large q_o. At the opposite end, arbitrarily low values of q_o are forbidden by the Mercier stability criterion. Therefore the existence of an optimum $\beta a B_o/I$ when q_o is also varied should be expected. In order to test the dependence of C on q_o, we generate a second sequence of numerical equilibria with the same characteristics as the one discussed before, except that now q_o is fixed at 2.00. The maximum equilibrium value of $\beta a B_o/I$ is found to be $C_T = 5.5$, to be compared with $C_T = 10.9$ for $q_o = 1.00$.

A point worth mentioning is that since the limit $\beta a B_o / I \leq C$ results only from equilibrium constraints, it should apply equally to plasmas in the first and second stability regimes. The numerical value of C may of course be different for these two regimes, but the ratio $\beta a B_o / I$ cannot be arbitrarily large even in the second stability regime.

Finally we stress that in order to obtain our bound on $\beta a B_o / I$ we need that the tokamak equilibrium profiles be "sufficiently mild". Specifically, the toroidal current density should tend to zero smoothly near the plasma edge and q_a should be greater than q_o or better yet $q(r)$ and $g(r)$ should be monotonically increasing functions of r. Thus the $\beta a B_o / I$ limit might be avoided by considering equilibria with more exotic though physically possible profiles. Examples of these are flux conserving equilibria which violate the zero edge current density condition, equilibria with sharp current gradients at the plasma edge such as in H-mode plasmas, and equilibria with hollow profiles.

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Figure Captions

Fig. 1 Variation of $\epsilon\beta_p$ with β_* for a sequence of circular tokamak equilibria characterized by the profiles $p = p_o(1 - \hat{\psi}^2)^2$ and $j_{oh} = j_o(1 - \hat{\psi})$, and constant $q_o = 1.00$.

Fig. 2 Dependence of q_a on β_* for our $q_o = 1$, zero edge current density equilibrium sequence.

Fig. 3 Dependence of $\beta a B_o / I$ on β_* for our $q_o = 1$, zero edge current density equilibrium sequence.

Fig. 4 Inverse rotational transform profile at the highest β_* equilibrium computed, right before the equilibrium limit.

Fig. 5 Flux and toroidal current density contours for our highest β_* equilibrium. The shift of the magnetic axis equals 2/3 of the minor radius.

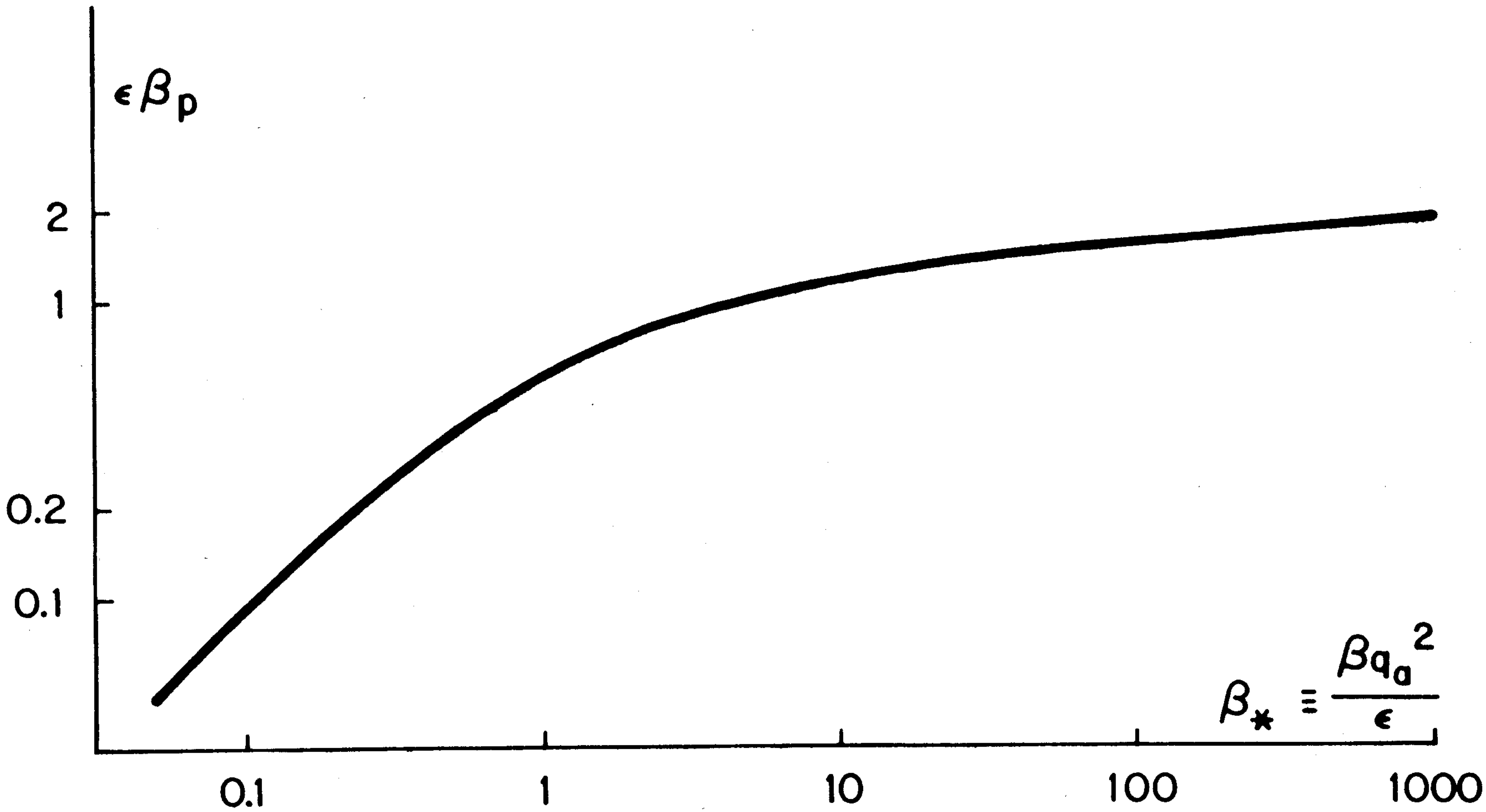
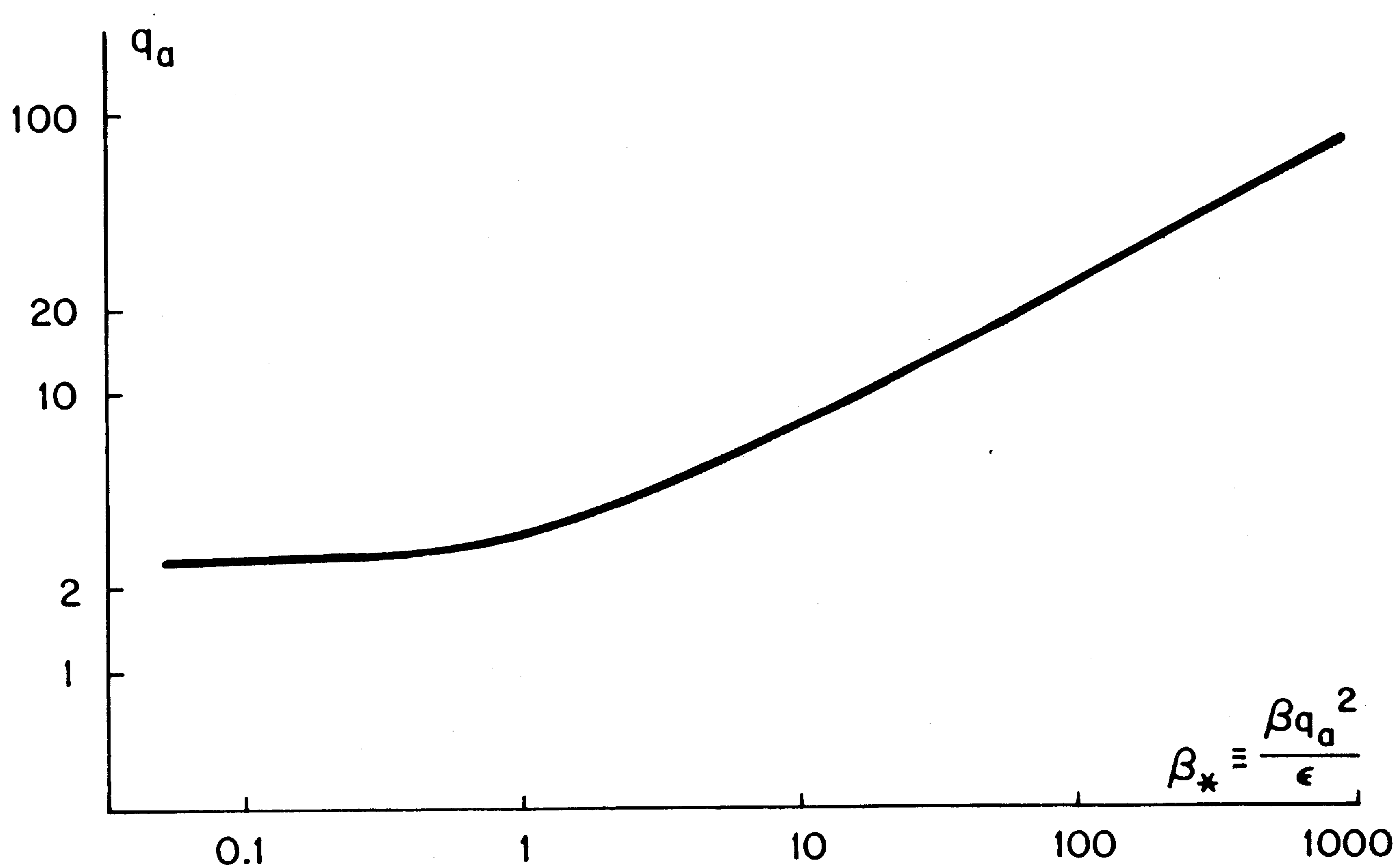
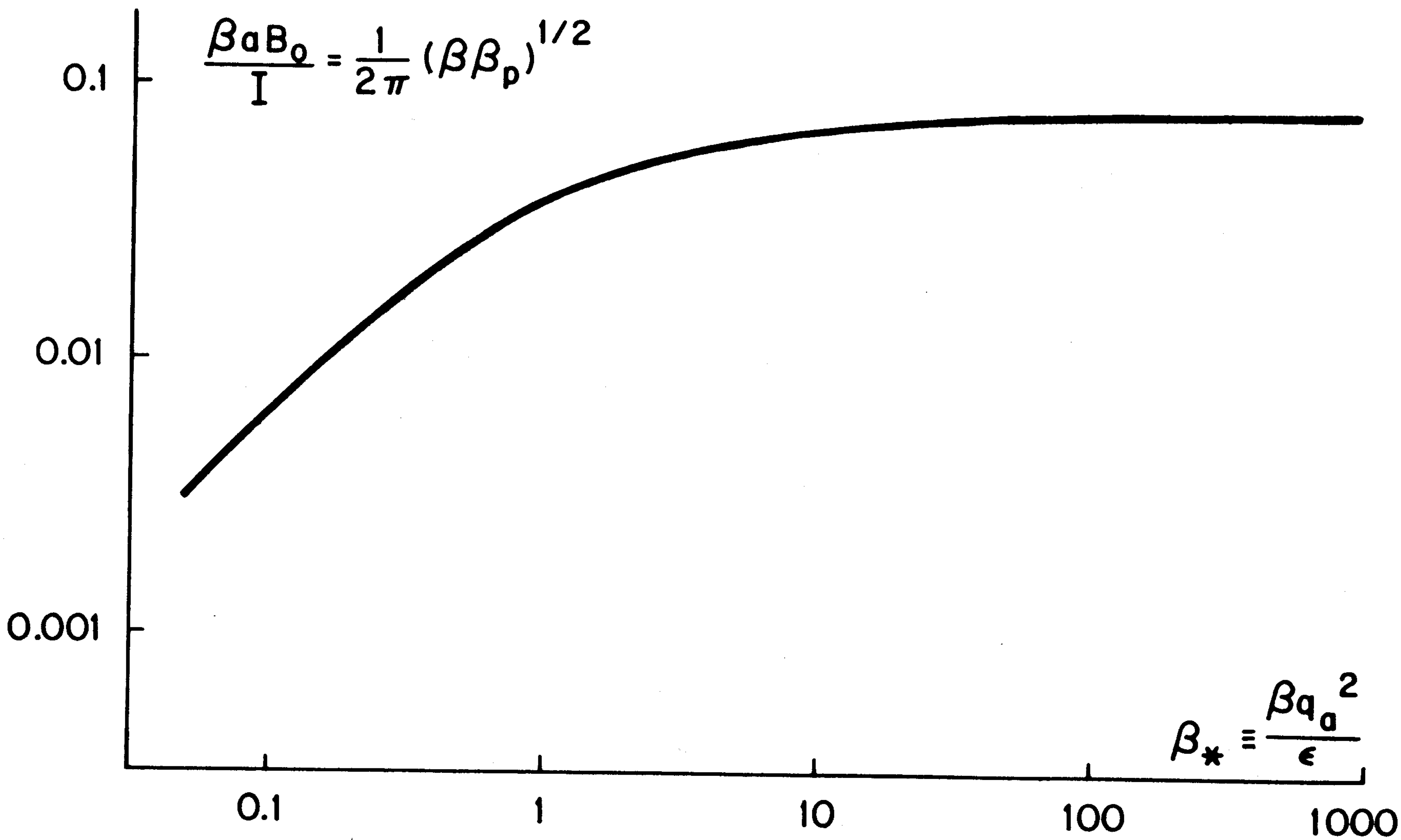

$\epsilon\beta_p$
2
1
0.2
0.1
0.1
1
10
100
1000
$\beta_* \equiv \frac{\beta q_a^2}{\epsilon}$

Figure 1



q_a
100
20
10
2
1
0.1
1
10
100
1000
$\beta_* \equiv \frac{\beta q_a^2}{\epsilon}$

Figure 2



$\frac{\beta a B_0}{I} = \frac{1}{2\pi}(\beta\beta_p)^{1/2}$
0.1
0.01
0.001
$\beta_* \equiv \frac{\beta q_a^2}{\epsilon}$
0.1
1
10
100
1000

Figure 3

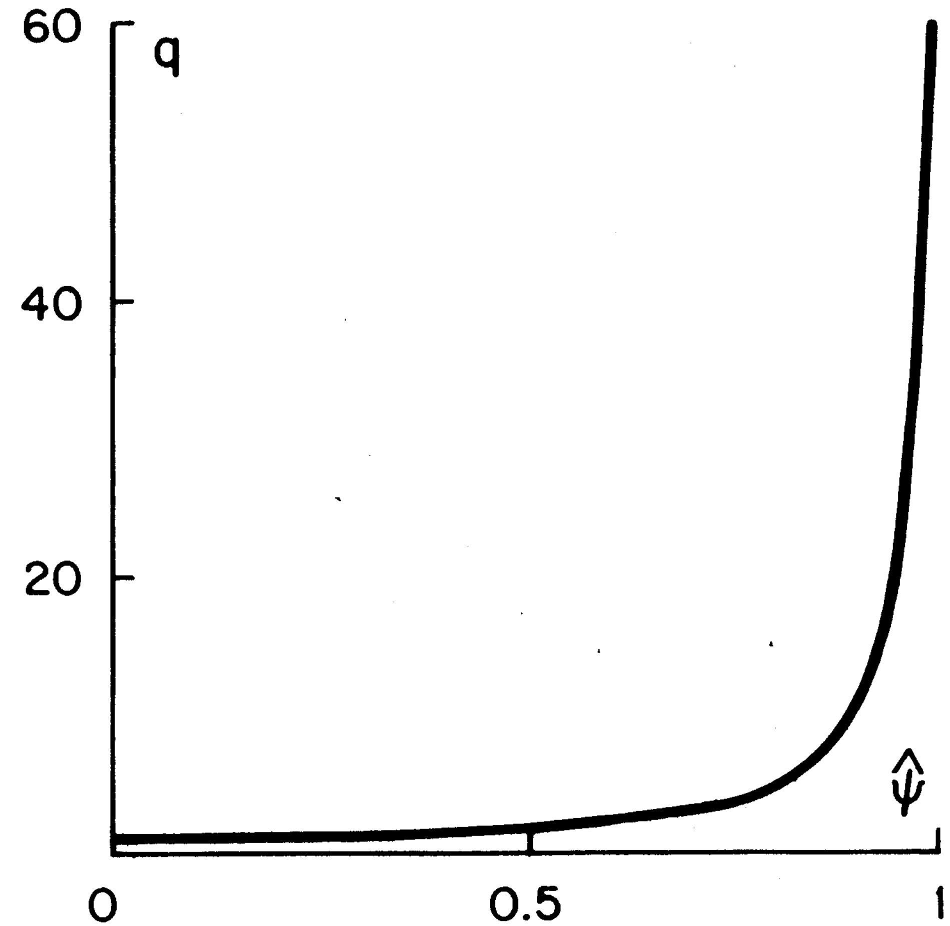


Figure 4

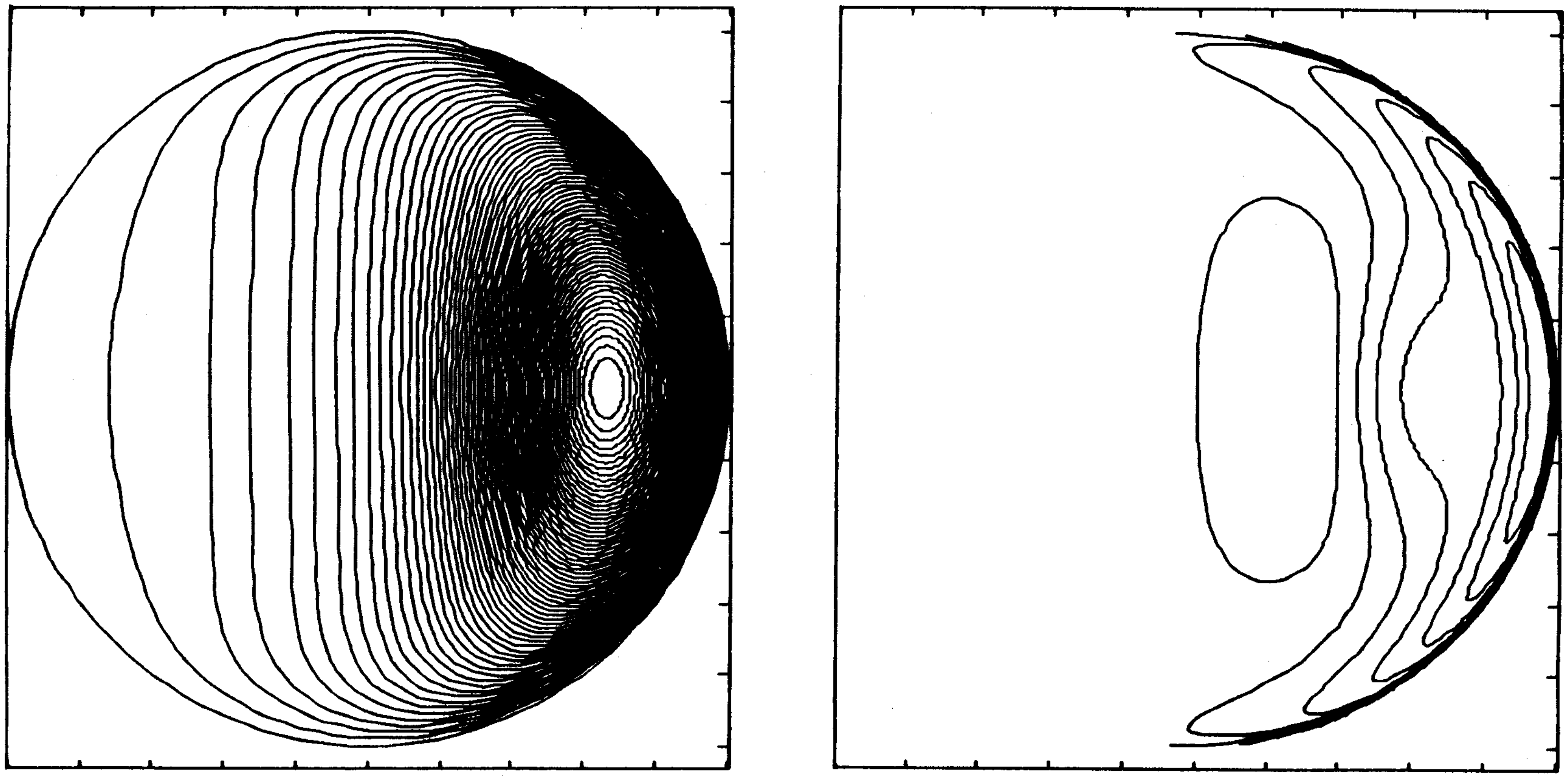


Figure 5