IONOSPHERIC ION ACCELERATION
BY MULTIPLE ELECTROSTATIC WAVES

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Abstract. Observations by Topaz 3 show ionsopheric O^+ and H^+ ions, of ambient energies around 0.3 eV, to be transversely (to the geomagnetic field) energized, to around 10 eV, within lower-hybrid structures composed of broadband large amplitude (100–200 mV/m) electrostatic waves. In this paper we show that the energization of O^+ ions can be explained by a new nonlinear, coherent interaction mechanism involving multiple electrostatic waves propagating across the magnetic field. Low energy ions, whose velocities are well below the phase velocities of the waves, are shown to gain energy monotonically increasing in time when averaged over a cyclotron orbit, We examine the properties of this coherent energization mechanism numerically and by an analytical, multiple time scale, analysis. We find, in accordance with observations, that the tail of the O^+ distribution is most likely to be energized. The analysis provides the spatial extent, along the geomagnetic field, of the lower-hybrid structures needed for the observed energization.

1. Introduction

The presence of terrestrial ionospheric H^+ and O^+ ions in the magnetosphere has been well documented by observations over nearly the past twenty years. The ambient energies of the ions, through their collisional coupling with the neutrals, is approximately 1/3 eV at an altitude of about 1000 km in the upper auroral ionosphere. The gravitational escape energies from this altitude are about 0.6 eV and 10 eV for H^+ and O^+ , respectively. One of the outstanding problems of space plasma physics is the process of energization whereby the H^+ and O^+ ions are able to gravitationally escape the ionosphere. Rocket observations over the past few years have begun to provide some of the essential features

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of the energization process. Observations by Topaz 3 show that electrostatic, lower-hybrid waves propagating across the geomagnetic field may be one of the energization mechanisms [*Kintner et al.,* 1992; *Vago et al.,* 1992]. The acceleration of the ions is due to wave-particle interactions with the H^+ and O^+ ions interacting with the electrostatic fields [*Kintner et al.,* 1992; *Vago et al.,* 1992]. The fields are not generated by the ions – the source of free-energy generating the waves maybe the precipitating energetic electrons [*Kintner et al.,* 1992; *Vago et al.,* 1992]. Motivated by these observations, we have studied the interactions of ions with a prescribed spectrum of coherent, electrostatic waves propagating across the geomagnetic field. We have not concerned ourselves with the mechanisms responsible for generating the electrostatic fields. Rather, we have assumed that the characeristics of the field spectrum are fixed within the bounds of observations, and the ions interact with this prescribed spectrum of fields. An important characteristic of the measured wave fields is that their lowest phase velocity is greater than the ambient thermal velocity of H^+ ions, and much greater than the ambient thermal velocity of O^+ ions.

The studies we report in this paper are in contrast to some of the previous analyses on ion acceleration by lower-hybrid fields [*Papadopoulos et al.,* 1980; *Lysak,* 1986]. Some of these prior analyses have assumed that the ions interact with a single, coherent, electrostatic wave. While this may be suitable for a frequency spectrum that has a very narrow band (less than the ion-cyclotron frequency), the observed broadband spectra do not satisfy this requirement. Furthermore, as we will show, the single wave analysis falls short of explaining the observed energies that are achieved by H^+ and O^+ . In the single wave analysis, ions whose perpendicular (to the geomagnetic field) velocities are close to the phase velocities of the waves interact strongly with the waves. For O^+ ions the single wave analysis fails because the observed wave phase velocities are not resonant with any significant fraction of the ion. For H^+ ions the analysis fails because the observed wave phase velocities do not extend out to velocities at which the energized H^+ ions are observed. A quasilinear description of the wave-particle interaction applies to broadband spectra [*Kennel and Engelmann,* 1966]. However, in the quasilinear analysis only those ions are affected whose velocities are resonant with the phase velocities of the waves in

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the broadband spectrum [*Kennel and Engelmann*, 1966]. Thus, a quasilinear analysis also fails to describe the observed acceleration of ionospheric ions since it also requires the presence of waves that are resonant with the low energy ions. In this paper we show that it is possible to energize O^+ ions provided that the field spectrum consists of two or more waves with frequencies separated by an integer multiple of the ion-cyclotron frequency. In such a case, we show that there exists a nonlinear, coherent energization mechanism by which low energy O^+ ions are accelerated to observed energies. The phase velocities of the waves in the spectrum can, in accordance with observations, be much greater than the O^+ thermal velocity. We do not have to resort, as in previous theories, to searching for mechanisms that generate low phase-velocity waves of substantial electric field amplitude which can then interact with the ambient O^+ distribution function [*Retterer et al.*, 1986]. The phenomenon of nonlinear, coherent energization does not exist for ion dynamics in a single wave [*Fukuyama et al.*, 1977; *Karney and Bers*, 1977; *Karney*, 1978; *Papadopoulos et al.*, 1980; *Lysak*, 1986], or in any quasilinear description of wave-particle interactions [*Retterer et al.*, 1986].

In this paper we outline the important observations from Topaz 3 in section 2 that are relevant for our analysis. Section 3 contains the dynamical model of the ions, and the results from an analysis in the case of a single wave are in section 4. Included is the shortcoming of such an analysis when trying to explain the observations. In section 5 we give numerical results for the dynamics of ions in two electrostatic waves. The analytical analyses describing these results is given in section 6. The energization of ions in a broadband spectrum and the application to O^+ energization are in sections 7 and 8, respectively.

2. Observations

The theoretical model analysis that we will be discussing is aimed at understanding specific observations reported by *Kintner et al.* [1992] and by *Vago et al.* [1992]. The observations, from Topaz 3 rocket which was launched northward from Poker Flat, Alaska in 1991, show that, at altitudes near 1000 km, there is transverse (to the geomagnetic

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field) energization of auroral ionospheric ions in localized regions of intense lower hybrid waves. The energization occurs in density depleted regions of about 50-100 m across the geomagnetic field $\vec{B}$ (and $\sim$ 100 km along $\vec{B}$ [*Arnoldy et al.*, 1993]) within which exist intense electric fields ranging in amplitude from 50 to 150 mV/m. Within these regions the oxygen O^+ and hydrogen H^+ ions are observed to be accelerated transversely with characteristic energies in the range 6-10 eV. The ambient energies of the ions in this part of the ionosphere is approximately 0.34 eV (corresponding to a plasma temperature of 4000° K). Observations show that, predominantly, the bulk of the H^+ distribution is energized, while for O^+ the tail distribution gets energized. Occasionally, the tail of the H^+ distribution displayed transverse energization. The H^+ ions are the minority ion species with the ambient density of O^+ being larger than the H^+ density by about an order of magnitude. The wave spectrum, observed to be cutoff near the local lower-hybrid frequency, ranges in frequency from about 5 kHz to about 12 kHz. These lower-hybrid waves are primarily coherent, electrostatic, propagating across the geomagnetic field, and ranging in wavelengths from 2 m to 20 m. Very importantly, from a theoretical point of view, the Topaz 3 observations showed that the lower-hybrid waves are inducing transverse energization of ions; the ions are not responsible for the generation of these field structures comprised of lower-hybrid waves. The generation and the physical mechanisms responsible for these lower-hybrid structures is presently not very well understood.

3. Modelling the Motion of Ions

Within the context of the above observations, our aim is to understand the energization of ionospheric ions inside the lower-hybrid field structures. Towards this end we will assume, consistent with observations, that the lower-hybrid waves lead to the energization of ions through wave-particle interactions. Since the ions are not responsible for the generation and presence of the structures with enhanced lower-hybrid electric field amplitudes, we will study the dynamics of ions in prescribed electrostatic fields whose characteristic properties are similar to those observed by Topaz 3.

The motion of an ion interacting with N plane electrostatic waves, propagating per-

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pendicularly (along $\hat{x}$) to an ambient, uniform, magnetic field $\vec{B} = B_0\hat{z}$, is given by:

$$\frac{dx}{dt} = v \tag{1}$$

$$\frac{dv}{dt} = -\Omega^2 x + \sum_{i=1}^{N} \frac{QE_i}{M} \sin(k_i x - \omega_i t) \tag{2}$$

where x and v are the position and velocity, respectively, of an ion of charge Q and mass M, E_i is the electric field amplitude of the i-th plane wave with wavenumber k_i and angular frequency ω_i, and $\Omega = QB_0/M$ is the ion-cyclotron frequency. The Hamiltonian corresponding to equations (1) and (2) is:

$$H(x, v, t) = \frac{1}{2}\left(v^2 + \Omega^2 x^2\right) + \sum_{i=1}^{N} \frac{QE_i}{Mk_i} \cos(k_i x - \omega_i t) \tag{3}$$

which can be expressed in terms of the normalized action-angle variables of the unperturbed ($E_i = 0$ for all i) system:

$$H(\psi, I, \tau) = I + \sum_{i=1}^{N} \frac{\epsilon_i}{k_i} \cos\left\{k_i\sqrt{2I}\sin(\psi) - \nu_i\tau\right\}$$

$$= I + \sum_{i=1}^{N} \frac{\epsilon_i}{k_i} \sum_{n=-\infty}^{\infty} J_n\left(k_i\sqrt{2I}\right) \cos(n\psi - \nu_i\tau) \tag{4}$$

where we have replaced k_i/k_1 by k_i, $I = [(k_1 x)^2 + (k_1 v/\Omega)^2]/2 = \rho^2/2$ and $\psi = \tan^{-1}(x\Omega/v)$ are the normalized action and angle variables, respectively, ρ is the normalized Larmor radius of the ion, $\epsilon_i = QE_i k_1/(M\Omega^2)$, $\nu_1 = \omega_i/\Omega$, and $\tau = \Omega t$. The action I is a measure of the energy of an ion. In order to provide a feeling for the normalized quantities, consider a singly charged O^+ ion with an initial ambient energy of 0.34 eV, in a magnetic field $B_0 = 0.36$G, interacting with a single wave of amplitude 100 mV/m, frequency 5 kHz, and wavelength 2 m, corresponding to the lowest measured phase velocity with substantial electric field amplitude. Then $I \approx 220.7$, $\rho \approx 21$, $\epsilon \approx 40.7$, and $\nu \approx 146.2$ where we have dropped the subscripts on ϵ and ν. For a H^+ ion with an energy of 0.34 eV, the corresponding values would be $I \approx 13.8$, $\rho \approx 5.3$, $\epsilon \approx 2.55$, and $\nu \approx 9.1$.

4. Interaction with a Single Electrostatic Wave

There have been a number of studies on the effect of a single wave on the dynamics of an ion [*Fukuyama et al.*, 1977; *Karney and Bers*, 1977; *Karney*, 1978]. These studies have been applied to understanding wave-particle interactions in space plasmas [*Papadopoulos et al.*, 1980; *Lysak*, 1986]. The need to study the dynamics of O^+ and H^+ ions in more than one electrostatic wave is clearly demonstrated by the results obtained from these previous studies. In general, it has been found that an ion gains energy from a single wave only if its motion becomes chaotic [*Karney and Bers*, 1977; *Karney*, 1978]. Otherwise, on the average, an ion will not gain any energy. The ion motion becomes chaotic if the wave amplitude is above a threshold value [*Karney and Bers*, 1977; *Karney*, 1978]:

$$\epsilon > \epsilon_{th} \approx \frac{1}{4}\nu^{2/3} \tag{5}$$

and if the initial values of the normalized Larmor radius ρ_0 of the ion is within the following bounds [*Karney and Bers*, 1977; *Karney*, 1978]:

$$\nu - \sqrt{\epsilon} \lesssim \rho_0 \lesssim \left(\frac{2}{\pi}\right)^{1/3} (4\epsilon\nu)^{2/3} \tag{6}$$

The left-hand side of the above inequality gives the lower bound, in ρ, of the chaotic phase space, and the right-hand side gives the upper bound. For an ion to get energized by its interaction with a single wave, its ρ_0 has to be within the bounds given by (6). Otherwise, the ion will not gain any energy and its motion will remain coherent.

If we consider a wave with frequency 5 kHz and wavelength 2 m in a magnetic field of 0.36 G, then the threshold amplitude for chaotic motion is:

$$\begin{aligned} \epsilon_{th}^O &\approx 6.94 &\implies& & E_{th}^O &\approx 17.0 \text{ mV/m} \\ \epsilon_{th}^H &\approx 1.1 &\implies& & E_{th}^H &\approx 42.9 \text{ mV/m} \end{aligned} \tag{7}$$

for O^+ and H^+, respectively. Since the electric field amplitudes in the structures observed by Topaz 3 are well above these threshold values, part of the phase space of O^+ and H^+

will be chaotic. The chaotic region of phase space, obtained from (6) for an electric field amplitude of 100 mV/m, for O^+ and H^+ , respectively, is:

$$139.8 \lesssim \rho_0^O \lesssim 712.2 \quad \Longrightarrow \quad 15.3 \text{ eV} \lesssim I_0^O \lesssim 396.1 \text{ eV}$$

$$7.5 \lesssim \rho_0^H \lesssim 17.7 \quad \Longrightarrow \quad 0.7 \text{ eV} \lesssim I_0^H \lesssim 3.9 \text{ eV}$$

$$(8)$$

where the units of I_0 have been reexpressed in terms of energy. In Figure 1, we plot this chaotic part of phase space, in energy units, for O^+ and H^+ and show the range of energies for these ions as observed by Topaz 3. The problems associated with explaining the observed energy ranges of O^+ and H^+ using a single wave model are now readily apparent. The lower bound of the O^+ chaotic phase space is at about fifty times the ambient thermal energy of O^+ and, hence, encompasses a negligible number of O^+ ions. Furthermore, the entire chaotic region for O^+ is well above the observed O^+ energies. The maximum energy of the chaotic region is much larger than the observed energy range for the energized O^+ ions. So even if it were possible to extend the lower part of the O^+ chaotic space into the O^+ thermal distribution function, the energies that O^+ ions could achieve would be significantly larger than the observed energy range. For H^+ ions the problem is a bit different. The lower part of the H^+ chaotic region is at about twice the ambient thermal energy energy of H^+ . So a significant population of H^+ ions have access to the chaotic phase space. However, the maximum energy that these ions can achieve is well below the observed energy range.

Figure 2 shows the surface-of-section obtained by solving, numerically, equations (1) and (2) for two O^+ ions, in a single wave, starting with different initial normalized Larmor radii ($\rho = \sqrt{2I}$). The initial energy of ion 1 is below the chaotic region while that of the ion 2 is in the chaotic region. The orbit of the first ion is completely coherent, mapping out a line in phase space, and gains no energy, while that of the second ion is chaotic and covers an area in phase space indicating an average gain in energy. A distribution of initial O^+ ions started in the chaotic phase space will, eventually, uniformly cover the entire phase space and, consequently, the final distribution of these ions will be flattened out in the chaotic phase space. Figure 3 shows the normalized Larmor radius of the two ions as a

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function of time. It is evident that the motion of the first ion is completely coherent and that it gains no energy. Meanwhile, the second ion exhibits chaotic motion spanning a large range in energy.

5. Interaction with two Electrostatic Waves

Before embarking on an analysis of ion motion in a broad spectrum of electrostatic waves, it is useful to determine if there are any new phenomena that can arise in the presence of two waves. The normalized equation of motion of an ion interacting with two waves is:

$$\frac{d^2x}{d\tau^2} + x = \epsilon_1 \sin(x - \nu_1\tau) + \epsilon_2 \sin(\kappa x - \nu_2\tau) \tag{9}$$

where $\kappa = k_2/k_1$ and $k_1 x \to x$. Whereas the parameter space for the case of one wave is two dimensional – depending on just the wave amplitude and the ratio of the wave frequency to the ion-cyclotron frequency, the parameter space for two waves is five dimensional – the two amplitudes, the ratios of the two wave frequencies to the ion-cyclotron frequency, and the ratio of the two wavelengths. After an extensive numerical analysis spanning various portions of this five-dimensional parameter space, it was found that the dynamics of ions in two waves was not any different from that in one wave except for a special set of wave frequencies. This special set corresponds to the two waves having frequencies separated by an integer multiple of the ion-cyclotron frequency [*Benisti et al.*, 1996; *Benisti et al.*, 1997; *Ram et al.*,, 1996; *Ram et al.*, 1996a] In cases when this frequency separation is not satisfied, the phase space is divided into two distinct regions – coherent and chaotic – just as in the case of one wave. However, the bounds of the chaotic region are modified, relative to the bounds for a single wave, and depend on the phase velocities and amplitudes of the two waves. When the frequencies of the two waves are separated by an integer multiple of the ion-cyclotron frequency, the two regions of phase space – coherent and chaotic – can be connected. In other words, an O^+ ion having its initial energy well below the chaotic region can get energized into the chaotic region. This is illustrated in Figure 4, where we plot the normalized Larmor radius versus time for an ion started below the chaotic region.

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In this case the wave frequencies are separated by one ion-cyclotron frequency. In the initial stage the O^+ ion is seen to monotonically gain energy in a coherent fashion, i.e. it does not exhibit chaotic motion. Eventually, this ion gets energized into the chaotic phase where its motion becomes diffusive. The initial energization of the ion, which we refer to as *nonlinear coherent energization* is a completely new phenomenon which is present in the case of two, specially chosen, waves. In the presence of two waves, an ion can cross from the low-energy coherent region of phase space into the higher-energy chaotic region – this being impossible for one wave. The ion motion in the two regions of phase space – the low-energy coherent region and the higher-energy chaotic region – are not only distinguishable from the motion of the ion and the evolution of its energy, but also by the time scales. In the coherent energization region, as is evident from Figure 4, the rate of increase of energy is much smaller than in the chaotic region. This distinguishing characteristic will be useful in our later discussions on explaining the observations of Topaz 3.

The coherent energization is a nonlinear phenomenon. If we were to set $k_1 = k_2 = 0$, then the motion is completely integrable and the ions do not gain any significant energy. The coherent energization persists even if the frequencies of the two waves are reduced, provided the difference in frequencies is an integer multiple of the ion cyclotron frequency. In order to illustrate some of the features of coherent energization, we will show numerical results for the case of lower frequency waves. Lowering the frequencies reduces the computation time but does not change the basic properties of the ion dynamics. Figure 5 shows the normalized Larmor radius as a function of time for the case when $\nu_1 = 24.43$ and $\nu_2 = 25.43$. At early times, the ion gets coherently enerigized until in reaches the chaotic region of phase space. The upper and lower bounds of the choatic region can be readily identified from this figure. This figure also illustrates that the ion can leave the chaotic region and come back out into the coherent region where it will, for some time, lose energy coherently. After reaching a minimum in energy, the ion gets coherently energized back into the chaotic region. The times at which the ion exits the chaotic region are randomly distributed. The coherent and chaotic energization regions contine to be clearly identifiable at all times by the time scale associated with the change in energy.

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The nonlinearity of the energization process is demonstrated in Figure 6 which shows the dynamics of two different ions. The two ions are initially started with the same energy. However, the initial phases of the two ions are different with the first ion having $\psi_0 = 0.06\pi$ and the second ion having $\psi_0 = 0.26\pi$. The second ion does not make it into the chaotic phase space; its energy is bounded from above by the lower boundary of the chaotic region. However, on the average, the second ion does gain significant energy – its average energy being about half the lower energy bound of the chaotic region.

The dependence of the coherent energization on the wavelengths of the two waves is illustrated in Figures 7 and 8. In these figures we show the dynamics of three ions, with different initial phases but the same initial actions, for $\kappa = 0.98$ (Figure 7) and $\kappa = 1.04$ (Figure 8). In Figure 7 the ions are energized over a small range of energies with no ion making it into the chaotic phase space. Unlike in the case of Figure 6, in Figure 8 the ions are coherently energized till they all make it into the chaotic phase space. From these results we conclude that more effective energization is obtained when the higher frequency wave has the shorter wavelength, as is the case for lower-hybrid waves.

6. Analytical Study of Dynamics in two Waves

Since the nonlinear energization of low energy ions is a coherent process, we expect that it can be analytically explained by an appropriate model. Towards that end, we carry out a perturbation analysis of equation (9) using the method of multiple time scales [*Nayfeh*, 1973]. A more comprehensive and general analytical treatment using the Lie transform perturbation technique has also been developed and will be presented elsewheres [*Benisti et al.*, 1997]. The perturbation parameter, in the method of multiple time scales, is the normalized amplitude of the waves. In our analysis we assume that neither ν_1 nor ν_2 is an integer, i.e. the wave frequencies are not an integer multiple of the ion-cyclotron frequency. However, we will assume that the difference in the frequencies of the two waves is an integer multiple of the ion-cyclotron frequency, i.e. $\nu_1 - \nu_2 = N$, an integer. (The analysis can be easily generalized to the case when $\nu_1 + \nu_2 = N$, and also for ν_1 and ν_2 are integers [*Benisti et al.*, 1997].) Our analysis breaks down in the vicinity of the chaotic

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regime. Upon carrying the multiple time scale analysis to second order in the amplitudes, we find that an approximate solution of (9) is given by:

$$x(\tau) \approx \rho(\tau) \sin \left\{ \tau + \overline{\psi}(\tau) \right\} \tag{10}$$

where ρ is the normalized Larmor radius, and $\overline{\psi}$ is the phase. The evolution equation for $\rho(\tau)$ and $\overline{\psi}(\tau)$ are:

$$\frac{\partial \rho}{\partial \tau} = -\frac{\epsilon_1 \epsilon_2}{2\rho} N \sin(N\overline{\psi}) \sum_{l=-\infty}^{\infty} \frac{J_l(\rho) J_{l-N}(\kappa\rho)}{1 - (l - \nu_1)^2} \tag{11a}$$

$$\frac{\partial \overline{\psi}}{\partial \tau} = -\frac{1}{4\rho} \frac{\partial}{\partial \rho} \sum_{l=-\infty}^{\infty} \frac{\epsilon_1^2 J_l^2(\rho) + \epsilon_2^2 J_{l-N}^2(\kappa\rho)}{1 - (l - \nu_1)^2} - \frac{\epsilon_1 \epsilon_2}{2\rho} \frac{\partial}{\partial \rho} \sum_{l=-\infty}^{\infty} \frac{J_l(\rho) J_{l-N}(\kappa\rho)}{1 - (l - \nu_1)^2} \tag{11b}$$

If $\epsilon_1 = \epsilon_2 = \epsilon$, then the above equations become independent of amplitude if we define a new time variable $\overline{\tau} = \epsilon^2 \tau$. This implies that the change in the Larmor radius of an ion is independent of the amplitude of the two waves. However, the rate at which the Larmor radius changes is inversely proportional to the square of the amplitude.

Upon substituting $I = \rho^2/2$, the above evolutions equations for the amplitude and the phase can be derived from the Hamiltonian:

$$\overline{H}\left(I, \overline{\psi}\right) = S_1(I) + \cos\left(N\overline{\psi}\right) S_2(I) \tag{12}$$

where S_1 and S_2 are the following sums:

$$
\begin{aligned}
S_1(I) &= \frac{1}{4} \sum_{l=-\infty}^{\infty} \frac{\epsilon_1^2 J_l^2(\rho) + \epsilon_2^2 J_{l-N}^2(\kappa\rho)}{1 - (l - \nu_1)^2} \\
S_2(I) &= \frac{1}{2} \epsilon_1 \epsilon_2 \sum_{l=-\infty}^{\infty} \frac{J_l(\rho) J_{l-N}(\kappa\rho)}{1 - (l - \nu_1)^2}
\end{aligned}
\tag{13}
$$

The Hamiltonian (12) is independent of time signifying that it is a new constant, or invariant, (to second order in the amplitude) of the dynamics. This constant is determined by the initial conditions $I(\tau = 0)$ and $\overline{\psi}(\tau = 0)$ of an ion. Note that at $\tau = 0$, $\overline{\psi} = \psi$, where ψ is the angle in (4).

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If we define $S_1 \pm S_2 = \overline{H}_\pm$, then

$$\overline{H}_- (I) = \frac{1}{4} \sum_{l=-\infty}^{\infty} \frac{\{\epsilon_1 J_l (\rho) - \epsilon_2 J_{l-N} (\kappa\rho)\}^2}{1 - (l - \nu_1)^2}$$

$$\overline{H}_+ (I) = \frac{1}{4} \sum_{l=-\infty}^{\infty} \frac{\{\epsilon_1 J_l (\rho) + \epsilon_2 J_{l-N} (\kappa\rho)\}^2}{1 - (l - \nu_1)^2}$$

(14)

It is easy to note that, algebraically,

$$\text{for } S_2 (I) > 0 : \quad \overline{H}_- (I) \leq \overline{H} (I, \overline{\psi}) \leq \overline{H}_+ (I) \tag{15a}$$

$$\text{for } S_2 (I) < 0 : \quad \overline{H}_+ (I) \leq \overline{H} (I, \overline{\psi}) \leq \overline{H}_- (I) \tag{15b}$$

$$\text{for } S_2 (I) = 0 : \quad \overline{H} (I, \overline{\psi}) = \overline{H}_- (I) = \overline{H}_+ (I) \tag{15c}$$

Thus, the orbits of the ions are bound in energy to lie between two consecutive zeros of $S_2 (I)$. The first zero of S_2 occurs at $I = 0$ or, equivalently, at $\rho = 0$. However, this perturbation analysis breaks down as the ions approach the chaotic region which is, approximately, still given by the left-hand side of (6) with $\nu = \min(\nu_1, \nu_2)$.

This perturbation analysis gives results which are in good agreement with those obtained from the exact numerical integration of the orbits. This is illustrated in Figures 9a and 9b. For the parameters corresponding to Figure 6, the bounds of $\overline{H}$ obtained from (14) are plotted in Figures 9a and 9b. Figure 9b is a magnified view of a smaller region from Figure 9a. The orbit of any single ion is a straight horizontal line with the $\overline{H}$ given by the initial value and bound to lie between $\overline{H}_+$ and $\overline{H}_-$ in accordance with (15b). In Figure 6 we noted that ion 1 made it into the chaotic region while ion 2 did not. Figure 9b illustrates why this happens. The "inverse bump" in $\overline{H}_-$ near $\rho \approx 22$ acts as a barrier for low energy ions preventing them from making it into the chaotic region, which is near $\rho \approx 23$. The local minimum, $(\overline{H}_-)_{\min} \approx -1.01 \times 10^{-3}$, of $\overline{H}_-$ occurs at $\rho_m \approx 21.75$. Ions with normalized Larmor radii $\rho < \rho_m$ and with $\overline{H} > (\overline{H}_-)_{\min}$ will not make it into the chaotic region. In fact, for the wave amplitudes considered in Figure 6, no ions with initial $\rho < 4.7$ will make it into the chaotic region, although they could undergo substantial coherent energization. The initial conditions for ions 1 and 2 of Figure 6 are marked by

a cross and a circle, respectively, in Figure 9b. It is now clear that ion 2 will not be able to make it into the chaotic region. However, ion 2 will still undergo substantial coherent energization, almost to near the chaotic region. This is in agreemnent with the results shown in Figure 6.

7. Coherent Energization in a Broadband Spectrum

The results obtained from the study of ion dynamics in two waves are a useful guide for determining some of the necessary conditions for which coherent energization can occur in a broadband spectrum of waves. Figure 10 shows the dynamics of three ions, initially with the same energy but different phases, in a spectrum composed of twelve waves. The waves are composed of six pairs of waves whose frequencies are separated by the ion-cyclotron frequency, while the frequency between any two pairs is randomly selected to be less than the ion-cyclotron frequency. The wavenumbers of each wave is chosen such that the higher frequency wave has a shorter wavelength. As the figure shows, all the ions are coherently energized, from their initial low energies, into the chaotic phase space. This result provides a necessary criterion for coherent energization – the spectrum has to be composed of pairs of waves that are separated by an ion-cyclotron frequency; each pair could be randomly distributed, in frequency, with respect to any other pair. Numerical simulations for a variety of frequencies and frequency bandwidths confirm this necessary criterion.

8. Application to O^+ Energization

In order to estimate the time it would take O^+ ions to get energized, by nonlinear, coherent energization, to 10 eV in a broadband spectrum of waves, of the sort observed by Topaz 3, we consider the dynamics in a spectrum of 162 waves ranging in frequency from $146.2\Omega_O$ to $200\Omega_O$ (Ω_O is the O^+ cyclotron frequency), corresponding to a range from 5 kHz to 6.84 kHz. All the waves are assumed to have the same wavelength of 2 m, and the same amplitude of 25 mV/m. The root mean square amplitude of the electrostatic field is approximately 225 mV/m. The result for two O^+ ions started at different initial energies is plotted in Figure 11. Ion 1 is initially at a transverse energy of approximately

2.3 eV (corresponding to a transverse speed of about 2.6 times the thermal speed), and ion 2 is initially at a transverse energy of approximately 4.6 eV (corresponding to a transverse speed of about 3.6 times the thermal speed). From this figure we find that the time needed for ions 1 and 2 to be transversely energized to 10 eV (the dashed line in Figure 11) is $\tau_E^1 \sim 122$ sec and $\tau_E^2 \sim 65$ sec, respectively. The simulation result shows that both the O^+ ions will make it into the chaotic region of phase space which exists at higher energies. The time taken to reach the chaotic region is about 153 sec and 88 sec, respectively, for the two ions. However, in order to reach the observed energies of 10 eV, the O^+ ions have to be energized by the nonlinear, coherent mechanism discussed in section 4. It is worth comparing the energization time τ_E with some other time scales that are relevant to this problem. At an altitude of 1000 km, and assuming the geomagnetic field to be 0.36 G, the gyrocenter of an O^+ ion, with a parallel energy of 1/3 eV and a perpendicular energy of 10 eV, drifts transversely due to the gradient and curvature of the geomagnetic field [*Lyons and Williams*, 1984] at a speed of about 9×10^{-3} m/sec. In the time it takes the two ions to get energized to 10 eV, their gyrocenters have drifted by about 1.1 m and 0.6 m, respectively. This is significantly less than the transverse width of the lower-hybrid structures observed by Topaz 3. So the ions would not drift out of the interaction region within the time it takes to get transversely energized to 10 eV. If we assume that the speed, along the geomagnetic field, of the two ions is their thermal speed, then the distance travelled along the geomagnetic field by the two ions, before they get transversely energized to 10 eV, is approximately 173 km and 92 km, respectively. This implies that the lower-hybrid structures have to extend up to about 173 km and 92 km, respectively, along the geomagnetic field for the two ions to be transversely energized to 10 eV. These distances are comparable to the estimated lengths of the lower-hybrid structures along the geomagnetic field which are of the order of a few hundred kilometers [*Arnoldy et al.*, 1993]. The fractional change in the strength of the geomagnetic dipolar field over a distance of 173 km is less than 0.01%. Thus, our approximation of a constant magnetic field is very reasonable. These results indicate that the tail of the O^+ distribution function is likely to get accelerated by the nonlinear, coherent energization mechanism for reasonable sizes

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of the lower-hybrid structures. Also, ions with initially small parallel velocities are more likely to get accelerated than those with larger parallel velocities.

The results in Figure 11 also show that the O^+ ions can get energized beyond 10 eV make it into the chaotic region where they can achieve much higher energies. However, the corresponding time scales are long which, in turn, imply that the required parallel length of the lower-hybrid structures has to be larger than the length required to get to 10 eV.

9. Conclusions

We have considered the dynamics of ions in an ambient magnetic field interacting with two or more electrostatic waves propagating across the magnetic field. We showed that low energy ions, whose initial velocities are well below the transverse phase velocities of the waves, can get nonlinearly and coherently energized by two waves if the frequencies of the waves are separated by an integer multiple of the ion-cyclotron frequency. These low energy ions would not be affected if they were interacting with a single wave whose phase velocity was larger than the transverse velocities of the ions. We showed that the single wave analyses fall well short of explaining any results of transverse energization as observed by TOPAZ 3. The nonlinear, coherent energization process exists also in a broad spectrum composed of waves separated by an ion-cyclotron frequency. The dynamics of ions in a spectrum of waves provides a time scale for the energization of O^+ ions. This time scale then gives a measure of the size of the lower-hybrid structures along the geomagnetic field needed to accelerate O^+ ions in the tail of the distribution function to the observed energies.

We have also developed an analytical formulation, based on a multiple time scale analysis, for the case of ions interacting with two waves, which provides an insight into the nonlinear, coherent energization process. It shows that the energies to which ions can be coherently energized depends only on the ratio of the amplitudes of the two waves. For two waves of equal amplitudes, the energies attained by the ions is independent of the amplitudes of the waves. However, the time needed to get energized is inversely proportional to the square of the amplitude. The analysis determines the range of energies to which

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an initial ion distribution function can be energized and the dependence of this range on the wavelengths and frequencies of the waves. All these results are readily obtained from our analytic (perturbation) theory, and were corroborated by detailed computations of ion trajectories from the exact equations of motion. The latter are long, and costly, computations compared to the evaluation of our analytic results. The theoretical analysis is being extended to include the dynamics in a broad spectrum of waves.

We have not addressed the energization of H^+ ions in this paper. That is a subject of a separate paper [*Benisti et al.*, 1997] because the energization of H^+ ions to higher energies requires an extension of the chaotic H^+ phase space. This is in contrast to the nonlinear, coherent energization needed to energize O^+ ions. We have shown that a single wave analysis shows that the bulk of the H^+ distribution function can be energized due to chaotic dynamics. However, the maximum energies that can be attained fall short of the observed energies. In a spectrum of lower-hybrid waves we find that the upper energy boundary of the region of chaotic H^+ dynamics can be extended to higher energies [*Benisti et al.*, 1997].

The concept of nonlinear, coherent energization can be tested in a laboratory experiment by considering the interaction of ions with two electrostatic waves propagating across a d.c. magnetic field. Recent calculations [*Benisti et al.*, 1997] show that the wave frequencies have to be separated only approximately by an integer multiple of the ion-cyclotron frequency. If the separation exceeds twice the ion-cyclotron frequency, the initial energy of the ions has to increase for them to be energized. Low energy ions, whose initial velocities are well below the phase velocities of the waves, should be energized to higher energies if the higher frequency wave has the shorter or the longer wavelength. However, in the former case the ions get energized to higher energies and can make it even into the chaotic phase space.

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FIGURE CAPTIONS

Figure 1: The chaotic part of phase space of H^+ and O^+ ions as obtained from a single wave analysis. The single wave is assumed to have a wavelength of 2 m, frequency of 5 kHz, and an amplitude of 100 mV/m. Also shown is the observed range of ion energies.

Figure 2: Surface-of-section plot for two ions with initial normalized Larmor radii of $\rho_0 = 66$ and 148 interacting with a single wave. The other parameters are: $\epsilon = 40.7$ (corresponding to an electric field amplitude of 100 mV/m for a wavelength of 2 m) and $\nu = 146.2$, corresponding to a frequency of 5 kHz.

Figure 3: The normalized Larmor radius as a function of normalized time for the same two ions shown in Figure 2.

Figure 4: ρ versus τ for an ion interacting with two waves having the following normalized parameters: $\epsilon_1 = \epsilon_2 = 40.7$, $\kappa = 1$, $\nu_1 = 146.2$, and $\nu_2 = 147.2$. Initially the ion is started with $\rho_0 = 94$ and $\psi_0 = 0$. The ion undergoes nonlinear, coherent energization into the chaotic phase space.

Figure 5: ρ versus τ for an ion interacting with two waves having the following normalized parameters: $\epsilon_1 = \epsilon_2 = 3$, $\kappa = 1$, $\nu_1 = 24.43$, and $\nu_2 = 25.43$. Initially the ion is started with $\rho_0 = 5.4$ and $\psi_0 = 0.06\pi$.

Figure 6: ρ versus τ for two ions interacting with two waves having the same parameters as in Figure 5. Initially, for ion 1 $\rho_0^{(1)} = 5.4$ and $\psi_0^1 = 0.06\pi$ and for ion2 $\rho_0^{(2)} = 5.4$ and $\psi_0^2 = 0.26\pi$

Figure 7: ρ versus τ for three ions interacting with two waves having the same parameters as in Figure 5 except $\kappa = 0.98$. Initially, $\rho_0 = 5.4$ for all the ions. Their initial phases are: $\psi_0 = 0$, $\pi/2$, and π. The maximum value of ρ attained by the ions decreases with increasing phase.

Figure 8: Same as Figure 7 except that $\kappa = 1.04$. The ions with initial phases $\psi_0 = \pi/2$ and $\psi_0 = \pi$ reach the chaotic phase space first and last, respectively.

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Figure 9a: $\overline{H}_-$ and $\overline{H}_+$, obtained from (14), plotted as a function of ρ for two waves having the same parameters as in Figure 6.

Figure 9b: A magnified view of Figure 9a. Here $\times$ and $\circ$ mark the values of $\overline{H}$ obtained from the initial conditions for ion 1 and ion 2, respectively, of Figure 6.

Figure 10: The dynamics of three ions interacting with twelve waves having $\nu_i = (24, 24.284, 24.43, 24.531, 24.784, 24.877, 25, 25.284, 25.43, 25.531, 25.784, 25.877)$ and $k_i = (1, 1.012, 1.034, 1.041, 1.076, 1.083, 1.094, 1.1, 1.108, 1.115, 1.122, 1.138)$, respectively. All waves have the same normalized amplitude $\epsilon_i = 0.576$. Initially all the ions have $\rho_0 = 5.4$ but with phases $\psi_0 = (0, \pi/2, \pi)$. The ions with initial phases $\psi_0 = \pi/2$ and $\psi_0 = \pi$ reach the chaotic phase space first and last, respectively.

Figure 11: The dynamics of two ions in 162 waves. All the waves have the same normalized wavenumber $k_i = 1$ and normalized amplitude $\epsilon_i = 10.7$, for $i = 1, 2, ... 162$. The frequencies of these waves are $\nu_{3n-2} = 146.2 + n$, $\nu_{3n-1} = 146.381 + n$, $\nu_{3n} = 146.873 + n$, for $n = 1, 2, ..., 54$. The wavelength and the electric field amplitude of each wave are 2 m and 25 mV/m, respectively. Initially ion 1 has $\rho_0^{(1)} = 54$, corresponding to an energy of 2.3 eV, and ion 2 has $\rho_0^{(2)} = 76.5$, corresponding to an energy of 4.6 eV. Both ions start off with an initial phase $\psi_0 = 0$.
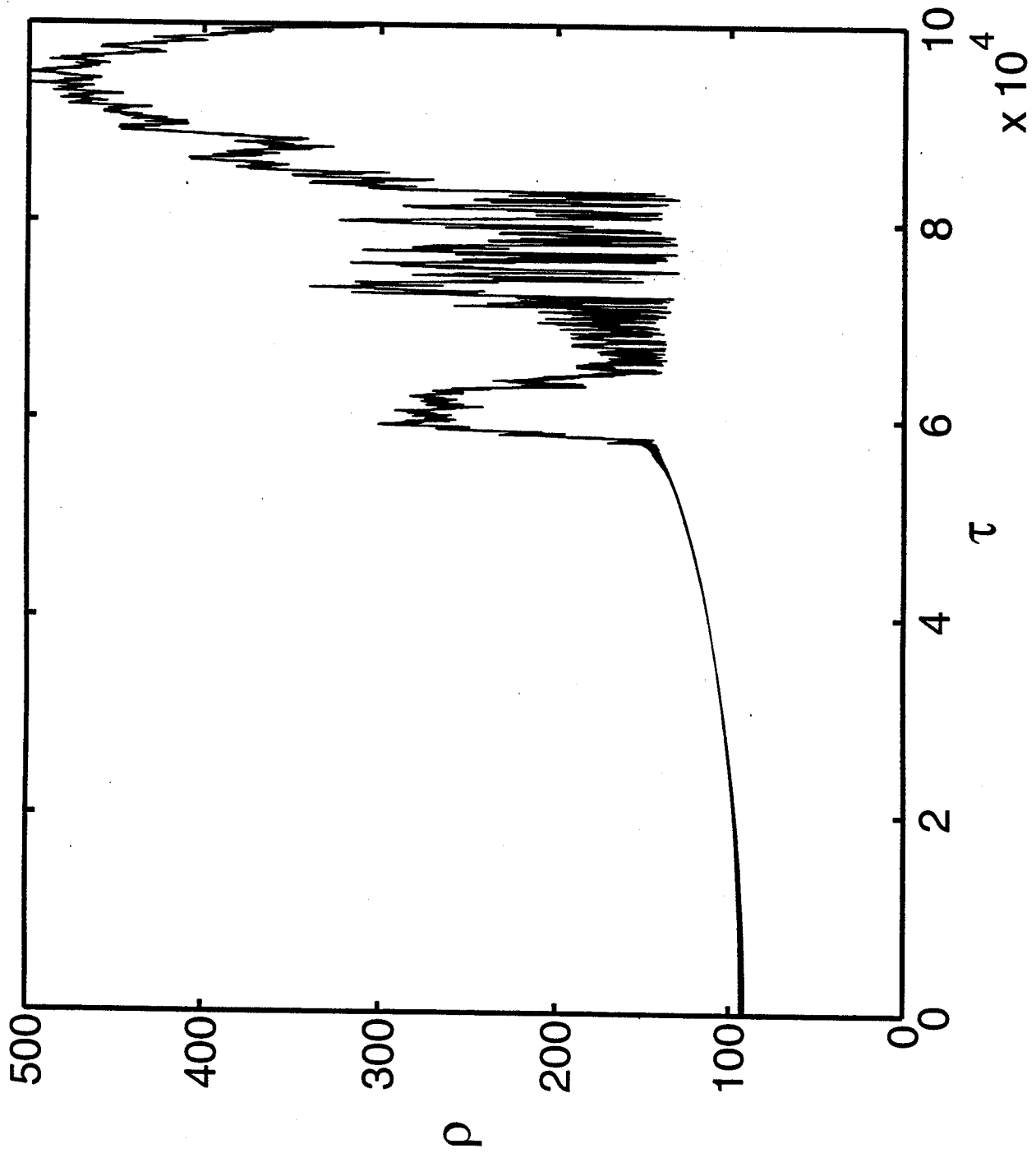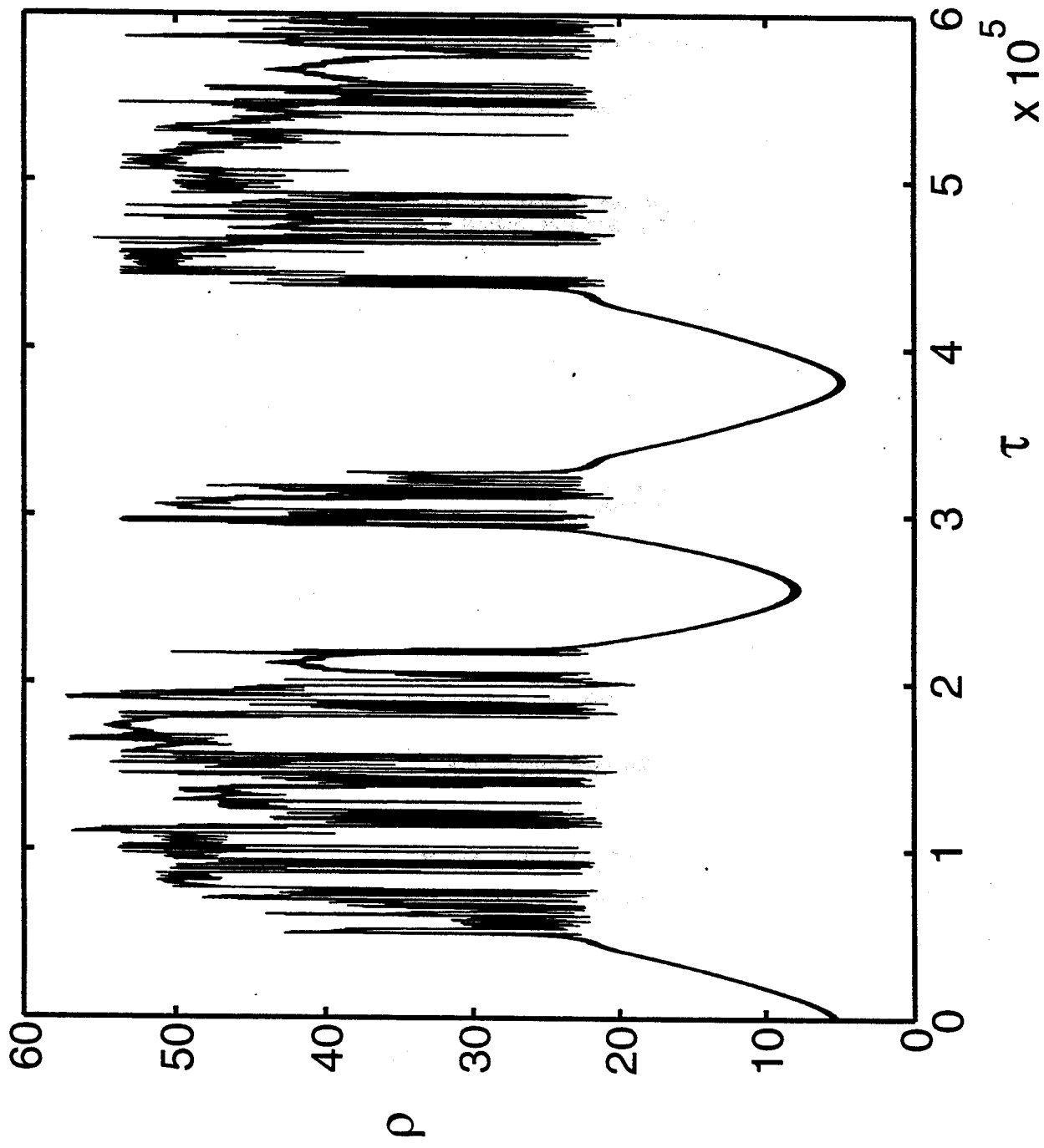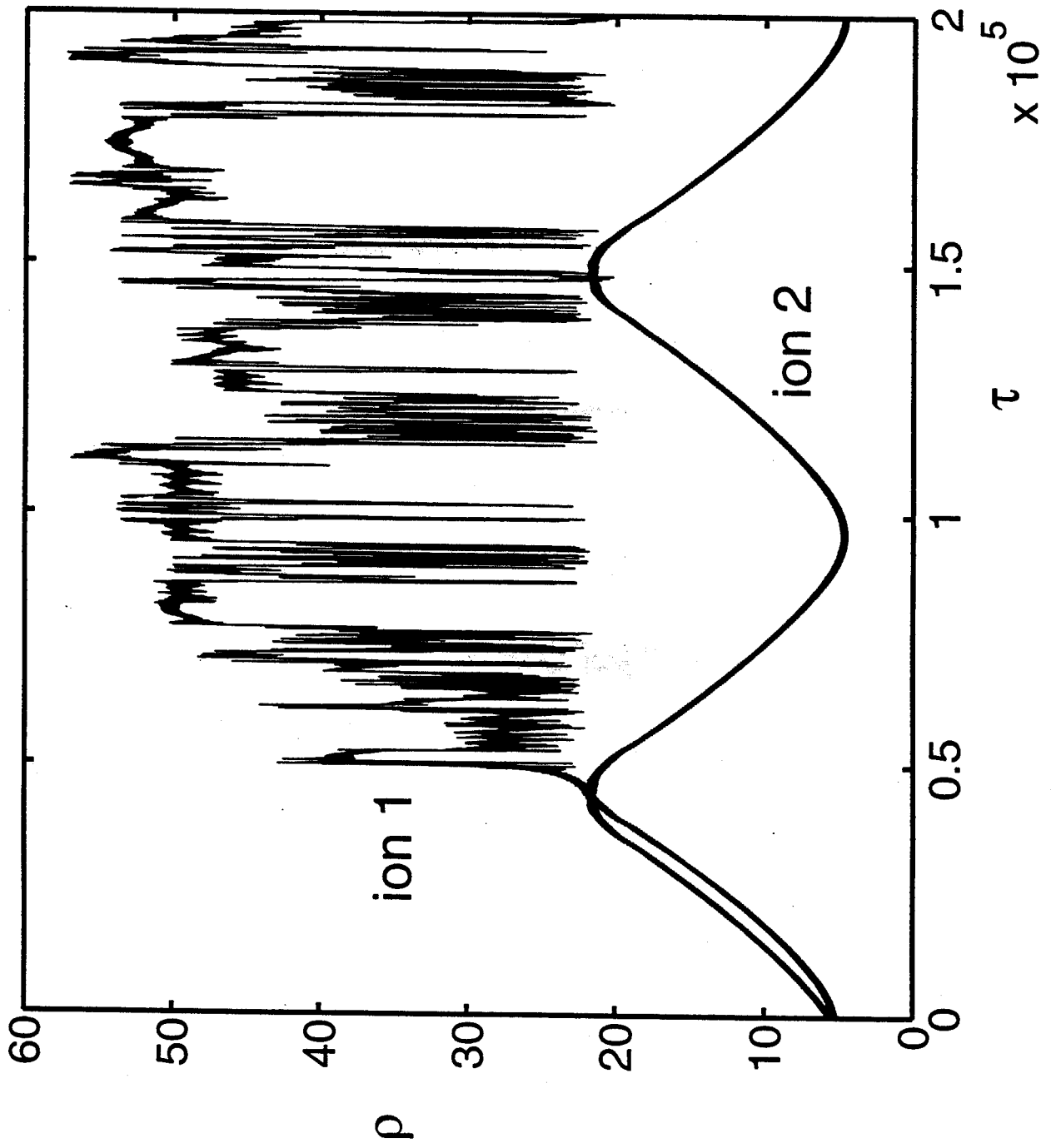
FIGURE 1

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FIGURE 2
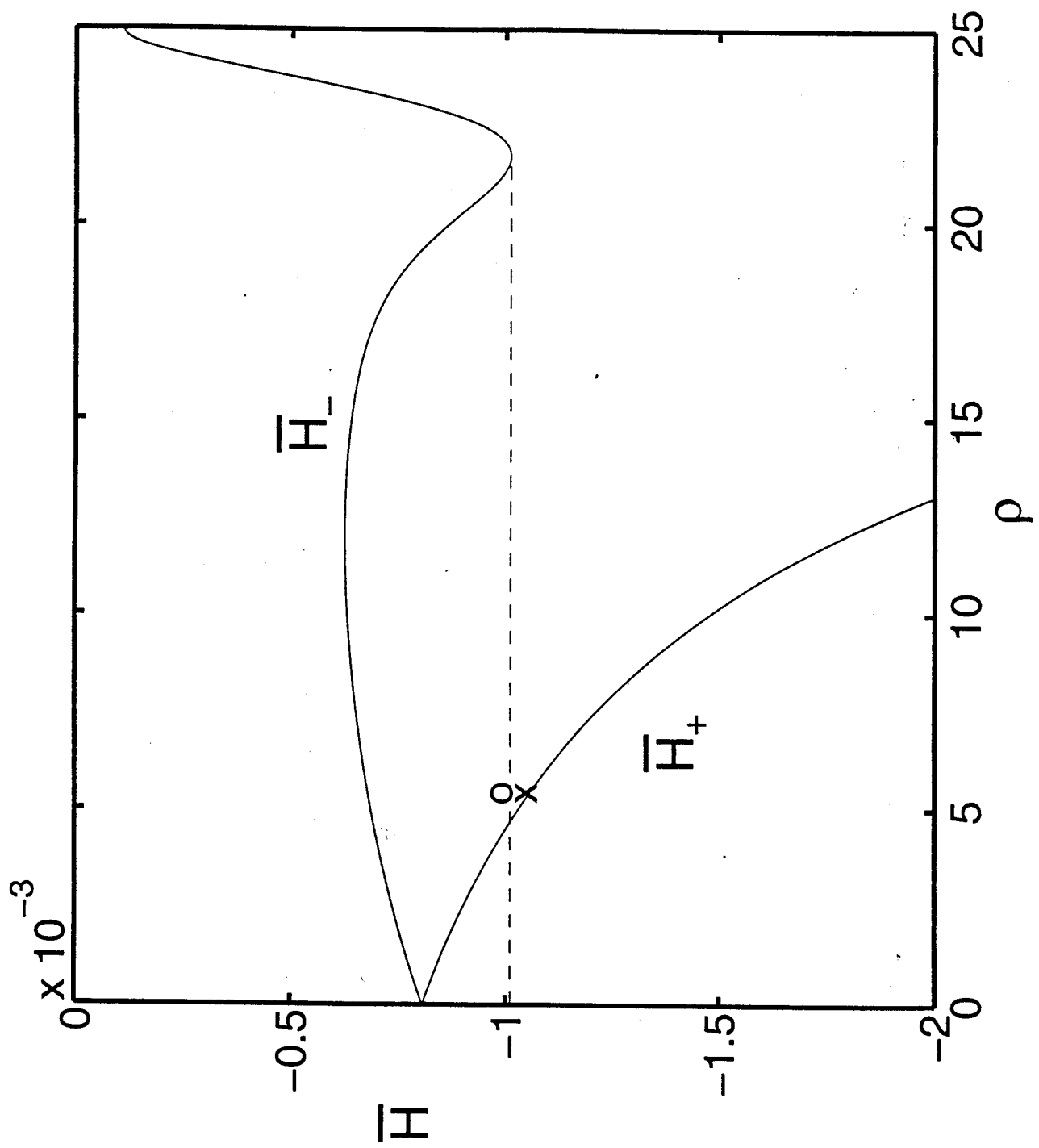
FIGURE 3

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FIGURE 4

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FIGURE 6

FIGURE 7

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FIGURE 8

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FIGURE 9a

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FIGURE 9b

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FIGURE 10

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FIGURE 11

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