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## Detecting crystal symmetry fractionalization from the ground state: Application to $\mathbb{Z}_2$ spin liquids on the kagome lattice

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In quantum spin liquid states, the fractionalized spinon excitations can carry fractional crystal symmetry quantum numbers, and this symmetry fractionalization distinguishes different symmetry-enriched spin liquid states with identical intrinsic topological order. In this work we propose a simple way to detect signatures of such crystal symmetry fractionalizations from the crystal symmetry representations of the ground state wave function. We demonstrate our method on projected  $\mathbb{Z}_2$  spin liquid wave functions on the kagome lattice, and show that it can be used to classify generic wave functions. Particularly our method can be used to distinguish several proposed candidates of  $\mathbb{Z}_2$  spin liquid states on the kagome lattice.

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It is well known that anyons in topologically ordered phases can carry symmetry quantum numbers that are quantized to fractional values. In the celebrated example of fractional quantum Hall states, Laughlin quasiparticles carry fractional charge—the quantum number of the  $U(1)$  symmetry [1]. In recent years, great progress has been made in understanding the interplay between symmetry and fractionalization in other topologically ordered states. In particular, topological spin liquids exhibit a more subtle kind of symmetry fractionalization, associated with the crystal symmetry of the underlying lattice instead of internal symmetries [2–4]. While some aspects of it have been studied for quite a while, crystal symmetry fractionalization has now received renewed attention, due to an increased interest in the role of crystal symmetry in topological phases of matter. This topic is also becoming timely in view of strong numerical evidence for spin liquids on kagome lattice found in the last few years [5–9]. In order to fully pin down the topological nature of the numerically found spin liquid liquid, the complete pattern of crystal symmetry fractionalization needs to be determined.

In this work, we offer our perspective on crystal symmetry fractionalization in  $\mathbb{Z}_2$  spin liquids. We find that the nontrivial way that crystal symmetry acts on an individual anyon is directly related to the symmetry representation of the topologically ordered ground states, as labeled by the crystal momentum and parity of many-body wave functions. Given that states with different symmetry labels cannot be adiabatically connected, our finding immediately makes it clear that the classification of spin liquids is refined and enriched by taking into account crystal symmetries [10]. Our theoretical result also provides a straightforward method to classify and detect different spin liquids in numerical studies. As a concrete example, we demonstrate that our method can be used to easily distinguish various  $\mathbb{Z}_2$  spin liquids on the kagome lattice [11–13].

We begin by briefly reviewing what is known about crystal symmetry fractionalization in  $\mathbb{Z}_2$  spin liquids, and setting up the terminology for our work. A  $\mathbb{Z}_2$  spin liquid [14,15]

supports three types of anyon excitations: bosonic spinons, fermionic spinons, and visons. As a defining property of topological excitations, anyons of each type can only be created in pairs. This property makes symmetry fractionalization possible. This can be understood by considering a many-body excited state containing two identical anyons that are spatially separated [3,16]. Intuitively speaking, the action of symmetry on this excited state can then be factorized into a product of two independent symmetry actions on the anyons. While the action on a physical state is necessarily described by a *linear* representation of the symmetry group denoted by  $G$ , the action on a single anyon is now allowed to form a *projective* representation  $\tilde{G}$ , such that the tensor product  $\tilde{G} \otimes \tilde{G}$  is a linear representation of  $G$  [17–20]. The projective representation  $\tilde{G}$ , which has a different group algebra than  $G$ , can be regarded as the “square root” of  $G$ . In this sense, symmetry action on anyons can be called “fractionalized.” Throughout this Rapid Communication, a tilde symbol ( $\tilde{\phantom{x}}$ ) placed over a symmetry operation means that it acts on an anyon; otherwise it acts on a physical wave function.

To sharpen the intuitive argument stated above and give a precise definition of symmetry fractionalization is a nontrivial task that requires great care. The main difficulty is that symmetry operations should in principle be performed on a single anyon, and yet any physical wave function necessarily contains an even number of them. (We note that crystal symmetry fractionalization may have implications for excitation spectra that can be detected [2,21,22].) To overcome this difficulty, we take a different approach and give a precise and operational definition of crystal symmetry fractionalization by relating it to the linear symmetry representations of many-body wave functions.

Crystal symmetries of a given lattice form a space group  $G$  generated by translation, rotation, and reflection. Any allowed projective representation of  $G$ , denoted by  $\tilde{G}$ , can be specified by its modified group algebra as compared to  $G$  [2,4]. First, two commuting operations  $X$  and  $Y$  in  $G$ ,  $XY = YX$ , can become anticommuting in  $\tilde{G}$ ,  $\tilde{X}\tilde{Y} = -\tilde{Y}\tilde{X}$ . This will be referred to as

commutation relation fractionalization. Second, an identity of  $X^n = 1$  in  $G$  can become  $\tilde{X}^n = -1$  in  $\tilde{G}$ . This will be referred to as quantum number fractionalization.

As the main result of this work, we find that the symmetry representation of ground states is a diagnosis of crystal symmetry fractionalization in  $\mathbb{Z}_2$  spin liquids. First, the commutation relation between a translation operation  $T_1$  and another symmetry operation  $X$ ,  $\tilde{T}_1\tilde{X} = \pm\tilde{X}\tilde{T}_1$ , can be determined from the difference between eigenvalues of  $X$  for ground states in different topological sectors on a torus geometry with an odd number of unit cells in the direction of  $T_1$ . Second, for an order-two symmetry operation  $X^2 = 1$ ,  $\tilde{X}^2 = \pm 1$  can be determined from the parity eigenvalue of  $X$  acting on ground states on a torus with  $4n + 2$  sites. We find it remarkable that the fractionalized symmetry property of anyons is simply and directly encoded in the symmetry of ground states. Some technical details of our derivation are available in the Supplemental Material [23].

*a. Fractionalized commutation relation.* In the presence of a fractionalization in the commutation relation between  $T_1$  and another symmetry operation  $X$ , ground states in different topological sectors have different eigenvalues of  $X$  on a torus with odd number of unit cells in the  $T_1$  direction [3]. To extract the fractionalization of different anyons, we find it crucial to choose a basis according to the anyon flux along the direction of  $T_1$ .

On a torus, a gapped  $\mathbb{Z}_2$  spin liquid has a fourfold ground state degeneracy, which is protected by its intrinsic topological order. A basis of these four ground states can be chosen such that each state carries a different anyon flux going through the torus in the direction of  $T_1$ , which can be diagnosed by a Wilson loop operator in the direction of  $T_2$  [15]. There are four types of such flux; each corresponds to one type of anyon excitation in the toric code topological order. Therefore the four ground states can be labeled as  $|G_a\rangle$ , where  $a$  denotes the type of anyon. In this Rapid Communication we denote the four types of anyon excitations in the  $\mathbb{Z}_2$  spin liquid state as  $a = 1, b, f$ , and  $v$ , standing for the trivial particle, the bosonic spinon, the fermionic spinon, and the vison, respectively.

Using this basis, the ratio between parity eigenvalues of two ground states can be calculated by considering the Berry phase picked up through the following operations acting on  $|G_1\rangle$ ,

$$X^{-1}(f^a)^{-1}Xf^a|G_1\rangle = e^{i\Delta\Phi}|G_1\rangle, \quad (1)$$

where  $f^a$  denotes the operation of moving one  $a$  anyon across the torus in the direction of  $T_1$  and it maps  $|G_1\rangle$  to  $f^a|G_1\rangle = |G_a\rangle$  [24]. This Berry phase  $\Delta\Phi$  can be obtained from the ground state  $X$ -symmetry representations as the following,

$$\begin{aligned} |G_1\rangle &\xrightarrow{f^a} |G_a\rangle \xrightarrow{X} \lambda_X^a |G_a\rangle \\ &\xrightarrow{(f^a)^{-1}} \lambda_X^a |G_1\rangle \xrightarrow{X^{-1}} \lambda_X^a (\lambda_X^1)^{-1} |G_1\rangle, \end{aligned} \quad (2)$$

where  $\lambda_X^a$  denotes the parity eigenvalue of  $X$  acting on  $|G_a\rangle$ :  $X|G_a\rangle = \lambda_X^a|G_a\rangle$ .

On the other hand, the same Berry phase can be obtained using the projective crystal symmetry representation of the  $a$  anyon. Starting from the ground state  $|G_1\rangle$ , the operation  $f^a$  creates two anyons locally and moves one across the torus

along  $T_1$ , which is equivalent to acting  $T_1$  on one of the two anyons  $n_1$  times, so the end state can be expressed as  $a \otimes \tilde{T}_1^{n_1} a$ , where  $\otimes$  denotes anyon fusion and  $n_1$  is the number of unit cells in the direction of  $T_1$ , which is an odd number according to our setup. Then  $X$  acts on both anyons and maps the state to  $\tilde{X}a \otimes \tilde{X}\tilde{T}_1^{n_1} a$ . The rest of the actions can be calculated similarly,

$$\begin{aligned} 1 &= a \otimes a \xrightarrow{f^a} a \otimes \tilde{T}_1^{n_1} a \xrightarrow{X} \tilde{X}a \otimes \tilde{X}\tilde{T}_1^{n_1} a \\ &\xrightarrow{(f^a)^{-1}} \tilde{X}a \otimes \tilde{T}_1^{-n_1} \tilde{X}\tilde{T}_1^{n_1} a \xrightarrow{X^{-1}} a \otimes \tilde{X}^{-1} \tilde{T}_1^{-n_1} \tilde{X}\tilde{T}_1^{n_1} a. \end{aligned}$$

Hence after the series of operations one anyon is changed into  $\tilde{X}^{-1}\tilde{T}_1^{-n_1}\tilde{X}\tilde{T}_1^{n_1}a = (\tau_X^a)^{n_1}a$ , where  $\tau_X^a = \pm 1$  denotes the fractionalization of the commutation relation,  $\tilde{T}_1\tilde{X}a = \tau_X^a\tilde{X}\tilde{T}_1a$ . This can be simplified as  $\tau_X^a$  because  $n_1$  is odd and  $(\tau_X^a)^2 = 1$ . Comparing this result with Eq. (2) we obtain the following relation between commutation relation fractionalization and ground state parity eigenvalues,

$$\tau_X^a = \lambda_X^a / \lambda_X^1. \quad (3)$$

*b. Fractionalized quantum number.* The action of an order-two crystal symmetry  $X$  on an anyon is fractionalized if acting  $\tilde{X}$  twice on a single anyon yields  $\tilde{X}^2 = -1$ . To detect such fractionalized quantum number, we act  $X$  once on an excited state containing two anyons, whose positions are swapped by  $X$  [25]. Specifically,  $\tilde{X}$  maps an anyon at a site  $i$  to another anyon at the image site  $X(i)$  and vice versa.

The symmetry action on an anyon is accompanied by additional gauge transformations [2],

$$\tilde{X}a_i = U_i a_{X(i)}, \quad \tilde{X}a_{X(i)} = U_{X(i)} a_i. \quad (4)$$

Therefore, acting  $\tilde{X}$  twice on the anyon  $a_i$  leaves anyon at its original position but yields a factor  $\tilde{X}^2 a = U_{X(i)} U_i a$ . Alternatively, if one perform the operation  $X$  once on a physical wave function  $|\Psi\rangle$  that contains a pair of anyons at  $i$  and  $X(i)$ , the same phase factors  $U_i$  and  $U_{X(i)}$  are collected from each anyon, so that the pair of spinons acquires the same total phase of  $U_{X(i)} U_i$ . Importantly, if the anyon under consideration is a fermion, there is an additional statistical sign due to the exchange of two fermions under  $X$ . To summarize, when a bosonic anyon carries a fractionalized quantum number  $\tilde{X}^2 = -1$ , the parity eigenvalue of an excited state  $|\Psi\rangle$  containing a pair of such anyons is opposite to that of the ground state, from which  $|\Psi\rangle$  is created. For fermionic anyons,  $\tilde{X}^2 = -1$  implies that the parity eigenvalues of  $|\Psi\rangle$  and  $|G\rangle$  are identical.

We now show that detecting  $\tilde{X}^2 = \pm 1$  for spinons can be further simplified when the spin liquid is constructed from parton methods (using either the Schwinger-boson or Abrikosov-fermion approach), which put exactly one spinon on every lattice site. Specifically, we choose a lattice geometry with  $4n + 2$  sites, so that the ground state contains an odd number of pairs of spinons. The parity eigenvalue of such a ground state is then equal to the parity of a single pair of spinons, which in turn detects  $\tilde{X}^2 = -1$  as described above. Importantly, the contribution to the parity eigenvalue from the fractional quantum number is independent of the topological sector of the ground state, in contrast to the previous case

involving the commutation relation between  $\tilde{X}$  and  $\tilde{T}_1$ . This will be demonstrated with concrete examples below.

*c.  $\mathbb{Z}_2$  spin liquids on kagome lattices.* We now apply our method of detecting crystal symmetry fractionalization to  $\mathbb{Z}_2$  spin liquid states on the kagome lattice. Recent numerical studies of the spin- $\frac{1}{2}$  Heisenberg model on the kagome lattice [5–9] have found strong evidence for a gapped spin liquid state, likely with a  $\mathbb{Z}_2$  topological order (see however, Ref. [26]). On the other hand, various types of  $\mathbb{Z}_2$  spin liquids on the kagome lattice that differ in symmetry properties have been theoretically constructed using parton methods in early studies [11–13]. In what follows, we will connect the numerical findings to theoretical constructions and show how to determine which of the  $\mathbb{Z}_2$  spin liquid states theoretically proposed so far is consistent with the ground state of the Heisenberg model on the kagome lattice.

To start, we quickly describe the parton construction of various  $\mathbb{Z}_2$  spin liquid states, paying particular attention to the role of crystal symmetry. Parton constructions postulate that the low-energy dynamics of the spin liquid phase is described by gapped spinons (which carry  $\mathbb{Z}_2$  gauge charge) interacting with  $\mathbb{Z}_2$  gauge fields. Depending on the  $\mathbb{Z}_2$  background flux patterns, the spinon exhibits different crystal symmetry fractionalizations, thus leading to distinct spin liquid states. This is because in the presence of gauge flux, crystal symmetries acting on a spinon involve additional  $\mathbb{Z}_2$  gauge transformations. For example, when there is a  $\pi$  flux within a unit cell, translations of spinons correspond to the magnetic translation group with the property  $\tilde{T}_1\tilde{T}_2 = -\tilde{T}_2\tilde{T}_1$ , resulting in a fractionalized commutation relation.

In the parton construction, spin liquids with different  $\mathbb{Z}_2$  background flux patterns are classified using the projective symmetry group (PSG) analysis invented by Wen [2], from which we can derive crystal symmetry fractionalization of bosonic and fermionic spinons. Below we derive the crystal symmetry fractionalization for  $\mathbb{Z}_2$  spin liquids on the kagome lattice that were constructed using the Schwinger-boson approach in previous works [11,12]. The PSG analysis by Wang and Vishwanath [12] has found that there are four spin liquid states with different  $\mathbb{Z}_2$  flux patterns, which are adiabatically connected to nearest-neighbor resonating-valence-bond states and therefore have better variational energy than other states. Hence we use them as examples to demonstrate our method of detecting crystal symmetry fractionalization. These four states are labeled by three  $\mathbb{Z}_2$  variables  $(p_1, p_2, p_3)$ ; for the sake of completeness, this terminology from Ref. [12] is reviewed in Sec. I of the Supplemental Material [23].

The kagome lattice has three independent symmetry operations:  $T_{1,2}$ ,  $\sigma$ , and  $R_{\pi/3}$ , which denote translation, mirror reflection, and rotation, respectively, and their definition is shown in Fig. 1. A straightforward translation of terminology shows that  $(p_1, p_2, p_3)$  in the PSG analysis directly yields the crystal symmetry fractionalizations of the bosonic spinon excitations (denoted by  $b$ ), listed in the first row of Table I. Here,  $\tau_{T_2} = \pm 1$  labels the fractionalization of the commutation relation between  $T_1$  and  $T_2$ , defined by  $\tilde{T}_1\tilde{T}_2 = \tau_{T_1}\tilde{T}_2\tilde{T}_1$ . Likewise,  $\tau_\sigma$  is defined by  $\tilde{T}_1\tilde{\sigma} = \tau_\sigma\tilde{T}_1\tilde{\sigma}$ , and the fractionalization of commutation relation between  $T_1$  and the twofold rotation  $R_\pi \equiv R_{\pi/3}^3$  takes the form of  $\tilde{T}_1\tilde{R}_\pi = \tau_{R_\pi}\tilde{R}_\pi\tilde{T}_1^{-1}$ . All three  $\tau$ 's

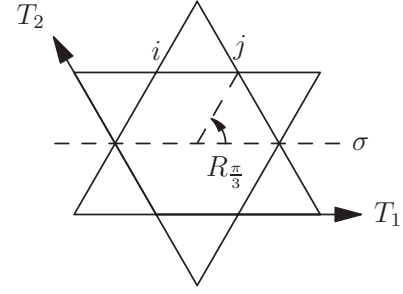


FIG. 1. The definition of crystal symmetry operations.  $T_{1,2}$ ,  $\sigma$ , and  $R_{\pi/3}$  label the symmetry operations of translation, mirror reflection, and sixfold rotation, respectively.

are equal for the four spin liquid states we consider. In addition, quantum number fractionalizations  $\tilde{\sigma}^2 = \pm 1$  and  $\tilde{R}_{\pi/3}^6 = \pm 1$  are listed in the last two columns of Table II.

A limitation of the previous PSG analysis is that it is tied to the Schwinger-boson formalism and hence only gives the crystal symmetry fractionalization of the bosonic spinons. The vison excitation in all four states has the same crystal symmetry fractionalization [4,27]:  $\tau_{T_2} = \tau_\sigma = \tau_{R_\pi} = -1$  and  $\tilde{\sigma}^2 = \tilde{R}_\pi^2 = 1$ . This result can be simply obtained from the charge-flux duality: in a spin liquid state with odd number of spinons per unit cell as is the case for the kagome lattice, the vison always sees a  $\pi$  flux per unit cell because a spinon is a  $\pi$  flux to a vison. As a result, the vison always has the property  $\tilde{T}_1\tilde{T}_2 = -\tilde{T}_2\tilde{T}_1$ .

We now use the method of flux attachment to derive the crystal symmetry fractionalization for the fermionic spinon, which is equivalent to a composite of a bosonic spinon and a vison—the latter is a  $\pi$  flux to the former. Compared to a bosonic spinon, the fermionic spinon always sees an extra  $\pi$  flux per unit cell due to the vison attached to it; hence  $\tau_{T_2}^b = -\tau_{T_2}^f$ . Similarly, the difference in  $\tilde{R}_\pi^2$  between bosonic and fermionic spinons follows from the fact that the attachment of a  $\pi$  flux changes the angular momentum between integer and half-integer values [28]. These results can also be derived using general methods described in Sec. I of the Supplemental Material [23], and are summarized in the second row of Table I [29].

As an independent check of the above results, we find that the above four spin liquid states constructed from the Schwinger-boson approach can be equivalently described using the Abrikosov-fermion approach. To establish the mapping between the two parton constructions, we match the ground states of spin liquids in the nearest-neighbor

TABLE I. Crystal symmetry fractionalizations of different anyon excitations.  $b$ ,  $f$ , and  $v$  denote the bosonic spinon, the fermionic spinon, and the vison, respectively.

Anyon	$\tau_{T_1}^a = \tau_\sigma^a = \tau_{R_\pi}^a$	$\tilde{\sigma}^2$	$\tilde{R}_{\pi/3}^6 = \tilde{R}_\pi^2$
$b$	$(-1)^{p_1}$	$(-1)^{p_2}$	$(-1)^{p_1+p_3}$
$f$	$(-1)^{p_1+1}$	$(-1)^{p_2+1}$	$(-1)^{p_1+p_3+1}$
$v$	$-1$	$+1$	$+1$



TABLE II. Correspondence between Schwinger-boson and Abrikosov-fermion constructions.  $p_i$  labels the PSG solutions of Schwinger-boson construction [12]. The  $Q_1 = \pm Q_2$  labels are used by Sachdev [11], and the labels in the last column are used by Lu [13].

$(p_1, p_2, p_3)$	Label in Ref. [11]	Label in Ref. [13]
(0,0,1)	$Q_1 = -Q_2$	$\mathbb{Z}_2[0, \pi] \alpha$
(0,1,0)	$Q_1 = Q_2$	$\mathbb{Z}_2[0, \pi] \beta$
(1,0,1)		$\mathbb{Z}_2[0, 0] \text{B}$
(1,1,0)		$\mathbb{Z}_2[0, 0] \text{A}$

resonating-valence-bond limit, given by the Gutzwiller projection on the corresponding parton wave functions in the two constructions:

$$|\psi\rangle = P_G \exp \left[ \sum_{\langle ij \rangle} \xi_{ij} \epsilon_{\alpha\beta} b_{i\alpha}^\dagger b_{j\beta}^\dagger \right] |0\rangle, \quad (5)$$

$$|\psi\rangle = P_G \exp \left[ \sum_{\langle ij \rangle} \zeta_{ij} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta}^\dagger \right] |0\rangle. \quad (6)$$

Here  $\xi_{ij}$  and  $\zeta_{ij}$  are antisymmetric and symmetric scalars on the nearest-neighbor bonds  $\langle ij \rangle$  used in Schwinger-boson and Abrikosov-fermion constructions, respectively. The values of  $\xi_{ij}$  or  $\zeta_{ij}$  on bonds of the kagome lattice are given by the corresponding PSG analysis. By equating Eq. (5) and (6), we find explicitly the mapping between  $\xi_{ij}$  and  $\zeta_{ij}$  for each of the four spin liquid states [30,31]. As a by-product, this mapping establishes the correspondence between the notation of the Schwinger-boson [11,12] formalism  $(p_1, p_2, p_3)$  with that of the Abrikosov-fermion [13] formalism, as shown in Table II. From this mapping, we have confirmed the results in Table I (see Sec. II of the Supplemental Material [23] for details).

Having derived the crystal symmetry fractionalization of anyons for all four spin liquids on the kagome lattice, we can now determine the symmetry representations of the ground states for each spin liquid, by using the general relation between the two as described in the first part of this work. First, based on Eq. (3) and the results of commutation relation fractionalizations summarized in Table I, the ratios between symmetry eigenvalues of different ground states are determined and the results are summarized in Table III.

Second, if all other aspects of the symmetry fractionalization are the same, a difference in the quantum number fractionalization results in a uniform parity change in the symmetry representation of ground states in all topological

TABLE III. Ratios between  $X$ -symmetry parity eigenvalues of  $|G_a\rangle$  and  $|G_1\rangle$ . The results depend on the commutation relation fractionalization  $\tau_{T_2}^b = -\tau_{T_2}^f = (-1)^{p_1}$ , and the ratios are the same for  $X = T_2, \tau$ , and  $R_\pi$ .

$p_1$	$\lambda_X^b / \lambda_X^1$	$\lambda_X^v / \lambda_X^1$	$\lambda_X^f / \lambda_X^1$
0	+1	-1	-1
1	-1	-1	+1

TABLE IV. Crystal symmetry representations of ground states in different topological sectors on a torus with odd-by- $(4n+2)$  unit cells.  $|G_a\rangle$  denotes the ground state with an anyon flux  $a$  in the direction of  $T_1$ , where  $a = 1, b, v$ , and  $f$  denote the trivial anyon, bosonic spinon, the vison, and the fermionic spinon, respectively.

$X$	$ G_1\rangle$	$ G_b\rangle$	$ G_v\rangle$	$ G_f\rangle$
$T_2$	1	$(-1)^{p_1}$	-1	$(-1)^{p_1+1}$
$\sigma$	$(-1)^{p_2}$	$(-1)^{p_1+p_2}$	$(-1)^{p_2+1}$	$(-1)^{p_1+p_2+1}$
$R_\pi$	$(-1)^{p_3}$	$(-1)^{p_1+p_3}$	$(-1)^{p_3+1}$	$(-1)^{p_1+p_3+1}$

sectors. This relation can be obtained by explicitly calculating the parity eigenvalues of the model wave functions in Eq. (5) and Eq. (6). The results are summarized in Table III and the details of the derivation is given in Sec. III of the Supplemental Material [23]. These results are determined from projected mean field wave functions but they also apply to general wave functions in the same topologically ordered phase, because the crystal symmetry representations are invariant when the state is smoothly deformed without breaking crystal symmetries.

Summarizing the above results, we see that the PSG parameters  $p_2$  and  $p_3$  determine the parity eigenvalues of the ground state  $|G_1\rangle$ , and then using  $p_1$  the parity eigenvalues of the other sectors are also determined. Hence we can obtain from  $(p_1, p_2, p_3)$  the crystal symmetry representations of all topological sectors on a torus with odd-by- $(4n+2)$  unit cells, as summarized in Table IV.

*d. Conclusions.* In this work we propose a method to detect crystal symmetry fractionalizations from the crystal symmetry representations of the ground states. On a torus with  $4n+2$  sites, the ratio between symmetry parity eigenvalues of ground states in different topological sectors detects the fractionalization of commutation relation with a translation symmetry, and a uniform sign change in all sectors detects the quantum number fractionalization.

Our method can be applied to study the nature of the topological order of the  $\mathbb{Z}_2$  spin liquid states obtained in numerical studies. Particularly using the infinite-size density matrix renormalization group (DMRG) method [32] or an infinite projected entangled pair states (PEPS) ansatz [33] the ground states in different topological orders can be obtained on an infinite cylinder and labeled with the anyon type by calculating the modular matrices [32,34,35]. To study both commutation relation fractionalization and quantum number fractionalization, we suggest using a cylinder  $(4n+2)$  unit cells wide. Then the crystal symmetry fractionalization studied in this work can be determined by putting the system on a torus with a length of an odd number of unit cells and examining the crystal symmetry representation of topologically degenerate ground states. For the  $\mathbb{Z}_2$  spin liquid states discussed above, the results are shown in Table IV.

In this work we explicitly derived the crystal symmetry quantum number of ground states and the fractional symmetry quantum number of anyons for four  $\mathbb{Z}_2$  spin liquid states on the kagome lattice [12]. Our method can be used straightforwardly to study additional spin liquid states that have been theoretically proposed [4,12,13].

*Note added.* Recently we were informed of a related work [36].

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