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Citation: Demaine, Erik D., Francisco Gomez-Martin, Henk Meijer, David Rappaport, Perouz Taslakian, Godfried T. Toussaint, Terry Winograd, and David R. Wood. "The Distance Geometry of Music." Computational Geometry 42, no. 5 (July 2009): 429–454. © 2009 Elsevier B.V.

As Published: http://dx.doi.org/10.1016/j.comgeo.2008.04.005

Publisher: Elsevier

Persistent URL: http://hdl.handle.net/1721.1/96166

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

Computational Geometry 42 (2009) 429–454

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

The distance geometry of music

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A R T I C L E   I N F O

Article history:
Received 4 May 2006
Received in revised form 18 April 2008
Accepted 19 April 2008
Available online 24 December 2008
Communicated by P. Bose and A. Mukhopadhyay

Keywords:
Maximally even
Euclidean rhythms
Deep rhythms
Euclidean algorithm
Generated rhythms

A B S T R A C T

We demonstrate relationships between the classic Euclidean algorithm and many other fields of study, particularly in the context of music and distance geometry. Specifically, we show how the structure of the Euclidean algorithm defines a family of rhythms which encompass over forty timelines (*ostinatos*) from traditional world music. We prove that these *Euclidean rhythms* have the mathematical property that their onset patterns are distributed as evenly as possible: they maximize the sum of the Euclidean distances between all pairs of onsets, viewing onsets as points on a circle. Indeed, Euclidean rhythms are the unique rhythms that maximize this notion of *evenness*. We also show that essentially all Euclidean rhythms are *deep*: each distinct distance between onsets occurs with a unique multiplicity, and these multiplicities form an interval $1, 2, \ldots, k-1$. Finally, we characterize all deep rhythms, showing that they form a subclass of generated rhythms, which in turn proves a useful property called shelling. All of our results for musical rhythms apply equally well to musical scales. In addition, many of the problems we explore are interesting in their own right as distance geometry problems on the circle; some of the same problems were explored by Erdős in the plane.

1. Introduction

Polygons on a circular lattice, African bell rhythms [86], musical scales [23], spallation neutron source accelerators in nuclear physics [16], linear sequences in mathematics [63], mechanical words and stringology in computer science [62], drawing digital straight lines in computer graphics [59], calculating leap years in calendar design [7,51], and an ancient algorithm for computing the greatest common divisor of two numbers, originally described by Euclid [39,44]—what do these disparate concepts all have in common? The short answer is, "patterns distributed as evenly as possible". For the long answer, please read on.

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[1] Supported by NSERC.
[2] Supported by FQRNT and NSERC.
[3] Supported by the Government of Spain grant MEC SB2003-0270, and by the projects MCYT-FEDER BFM2003-00368 and Gen. Cat 2001SGR00224.

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2008.04.005

Mathematics and music have been intimately intertwined since over 2500 years ago when the famous Greek mathematician, Pythagoras of Samos (circa 500 B.C.), developed a theory of consonants based on ratios of small integers [8,10]. Most of this interaction between the two fields, however, has been in the domain of pitch and scales. For some historical snapshots of this interaction, we refer the reader to H.S.M. Coxeter's delightful account [32]. In music theory, much attention has been devoted to the study of intervals used in pitch scales [43], but relatively little work has been devoted to the analysis of time duration intervals of rhythm. Some notable recent exceptions are the books by Simha Arom [3], Justin London [61] and Christopher Hasty [49].

In this paper, we study various mathematical properties of musical rhythms and scales that are all, at some level, connected to an algorithm of another famous ancient Greek mathematician, Euclid of Alexandria (circa 300 B.C.). We begin (in Section 2) by showing several mathematical connections between musical rhythms and scales, the work of Euclid, and other areas of knowledge such as nuclear physics, calendar design, mathematical sequences, and computer science. In particular, we define the notion of *Euclidean rhythms*, generated by an algorithm similar to Euclid's. Then, in the more technical part of the paper (Sections 3–5.2), we study two important properties of rhythms and scales, called *evenness* and *deepness*, and show how these properties relate to the work of Euclid.

The Euclidean algorithm has been connected to music theory previously by Viggo Brun [20]. Brun used Euclidean algorithms to calculate the lengths of strings in musical instruments between two lengths l and $2l$, so that all pairs of adjacent strings have the same length ratios. In contrast, we relate the Euclidean algorithm to rhythms and scales in world music.

Musical rhythms and scales can both be seen as two-way infinite binary sequences [85]. In a rhythm, each bit represents one unit of time called a *pulse* (for example, the length of a sixteenth note), a one bit represents a played note or *onset* (for example, a sixteenth note), and a zero bit represents a silence (for example, a sixteenth rest). In a scale, each bit represents a pitch (spaced uniformly in log-frequency space), and zero or one represents whether the pitch is absent or present in the scale. Here we suppose that all time intervals between onsets in a rhythm are multiples of a fixed time unit, and that all tone intervals between pitches in a scale are multiples of a fixed tonal unit (in logarithm of frequency).

The time dimension of rhythms and the pitch dimension of scales have an intrinsically cyclic nature, cycling every measure and every octave, respectively. In this paper, we consider rhythms and scales that match this cyclic nature of the underlying space. In the case of rhythms, such cyclic rhythms are also called *timelines*, rhythmic phrases or patterns that are repeated throughout a piece; in the remainder of the paper, we use the term "rhythm" to mean "timeline". The infinite bit sequence representation of a cyclic rhythm or scale is just a cyclic repetition of some n-bit string, corresponding to the timespan of a single measure or the log-frequency span of a single octave. To properly represent the cyclic nature of this string, we imagine assigning the bits to n points equally spaced around a circle of circumference n [64]. A rhythm or scale can therefore be represented as a subset of these n points. We use k to denote the size of this subset; that is, k is the number of onsets in a rhythm or pitches in a scale. For uniformity, the terminology in the remainder of this paper speaks primarily about rhythms, but the notions and results apply equally well to scales.

In this paper, we use four representations of rhythms of timespan n. The first representation is the commonly used *box-like* representation, also known as the Time Unit Box System (TUBS), which is a sequence of n '$\times$'s and '$\cdot$'s where '$\times$' represents an onset and '$\cdot$' denotes a silence (a zero bit) [85]. This notation was used and taught in the West by Philip Harland at the University of California, Los Angeles, in 1962, and it was made popular in the field of ethnomusicology by James Koetting [58]. However, such box notation has been used in Korea for hundreds of years [50]. The second representation of rhythms and scales we use is the *clockwise distance sequence*, which is a sequence of integers that sum up to n and represent the lengths of the intervals between consecutive pairs of onsets, measuring clockwise arc-lengths or distances around the circle of circumference n. The third representation of rhythms and scales writes the onsets as a *subset* of the set of all pulses, numbered $0, 1, \ldots, n-1$, with a subscript of n on the right-hand side of the subset to denote the timespan. Clough and Douthett [22] use this notation to represent scales. For example, the Cuban clave Son rhythm can be represented as $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \times \cdot \cdot \cdot]$ in box-like notation, $(3, 3, 4, 2, 4)$ in clockwise distance sequence notation, and $\{0, 3, 6, 10, 12\}_{16}$ in subset notation. Finally, the fourth representation is a graphical *clock diagram* [85], such as Fig. 1, in which the zero label denotes the start of the rhythm and time flows in a clockwise direction. In such clock diagrams we usually connect adjacent onsets by line segments, forming a polygon. We consider two rhythms distinct if their sequence of zeros and ones differ, starting from the first bit in any of the described representations. We assume that two rhythms that do not have the same sequence are different. However, if a rhythm is a rotated version of another, we say that they are instances of the same *necklace*. Thus a rhythm necklace is a clockwise distance sequence that disregards the starting point in the cycle. Note that the clockwise distance sequence notation requires that the rhythm starts with an onset, so it cannot be used to represent all rhythms; it is most useful for our analysis of necklaces.

Even rhythms. Consider the following three 12/8-time rhythms expressed in box-like notation: $[\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot]$, $[\times \cdot \times \cdot \times \times \cdot \times \cdot \times \cdot \times]$, and $[\times \cdot \cdot \cdot \times \times \cdot \cdot \times \times \times \cdot]$. The first rhythm contains beats that are distributed perfectly (well spaced). Such rhythms are found throughout the world, and are most easily identified and incorporated in music and dance. However, in many cultures where rhythm is more highly developed, rhythms are preferred that are not perfectly even. It is intuitively clear that the first rhythm is more *even* (well spaced) than the second rhythm, and that the second rhythm is more even than the third rhythm. In fact, the second rhythm is the internationally most well known of all African timelines. It is traditionally played on an iron bell, and is known on the world scene mainly by its Cuban name *Bembé* [86]. Traditional rhythms tend to exhibit such properties of evenness to some degree.

Why do many traditional rhythms display such evenness? Many are timelines (also sometimes called *claves*), that is, rhythms repeated throughout a piece that serve as a rhythmic reference point [67,91]. Often these claves have a *call-and-response structure*, meaning that the pattern is divided into two parts: the first poses a rhythmic question, usually by creating rhythmic tension, and the second part answers this question by releasing that tension. A good example of this structure is the popular clave Son [× · · × · · × · · · × · × · · ·]. This clave creates such tension through syncopation, which can be found between the second and third onsets as well as between the third and fourth onsets. The latter is weak syncopation because the strong beat at position 8 lies half-way between the third and fourth onsets. (The strong beats of the underlying 4/4 meter (beat) occur at positions 0, 4, 8, and 12.) On the other hand, the former syncopation is strong because the strong beat at position 4 is closer to the second onset than to the third onset [47]. Claves played with instruments that produce unsustained notes often use syncopation and accentuation to bring about rhythmic tension. Many clave rhythms create syncopation by evenly distributing onsets in contradiction with the pulses of the underlying meter. For example, in the clave Son, the first three onsets are equally spaced at the distance of three sixteenth pulses, which forms a contradiction because 3 does not divide 16. Then, the response of the clave answers with an offbeat onset, followed by an onset on the fourth strong beat of a 4/4 meter, releasing that rhythmic tension.

On the other hand, a rhythm that is too even, such as the example [× · × · × · × · × · × ·], is less interesting from a syncopation point of view. Indeed, in the most interesting rhythms with k onsets and timespan n, k and n are relatively prime (have no common divisor larger than 1). This property is natural because the rhythmic contradiction is easier to obtain if the onsets do not coincide with the strong beats of the meter. Also, we find that many claves have an onset on the last strong beat of the meter, as does the clave Son. This is a natural way to respond in the call-and-response structure. A different case is that of the Bossa-Nova clave [× · · × · · × · · · × · · × · ·]. This clave tries to break the feeling of the pulse and, although it is very even, it produces a cycle that perceptually does not coincide with the beginning of the meter.

This prevalence of evenness in world rhythms has led to the study of mathematical measures of evenness in the new field of mathematical ethnomusicology [26,88,89], where they may help to identify, if not explain, cultural preferences of rhythms in traditional music. Furthermore, evenness in musical chords plays a significant role in the efficacy of voice leading as discussed in the work of Tymoczko [52,90].

The notion of *maximally even sets* with respect to scales represented on a circle was introduced by Clough and Douthett [22]. According to Block and Douthett [12], Douthett and Entringer went further by constructing several mathematical

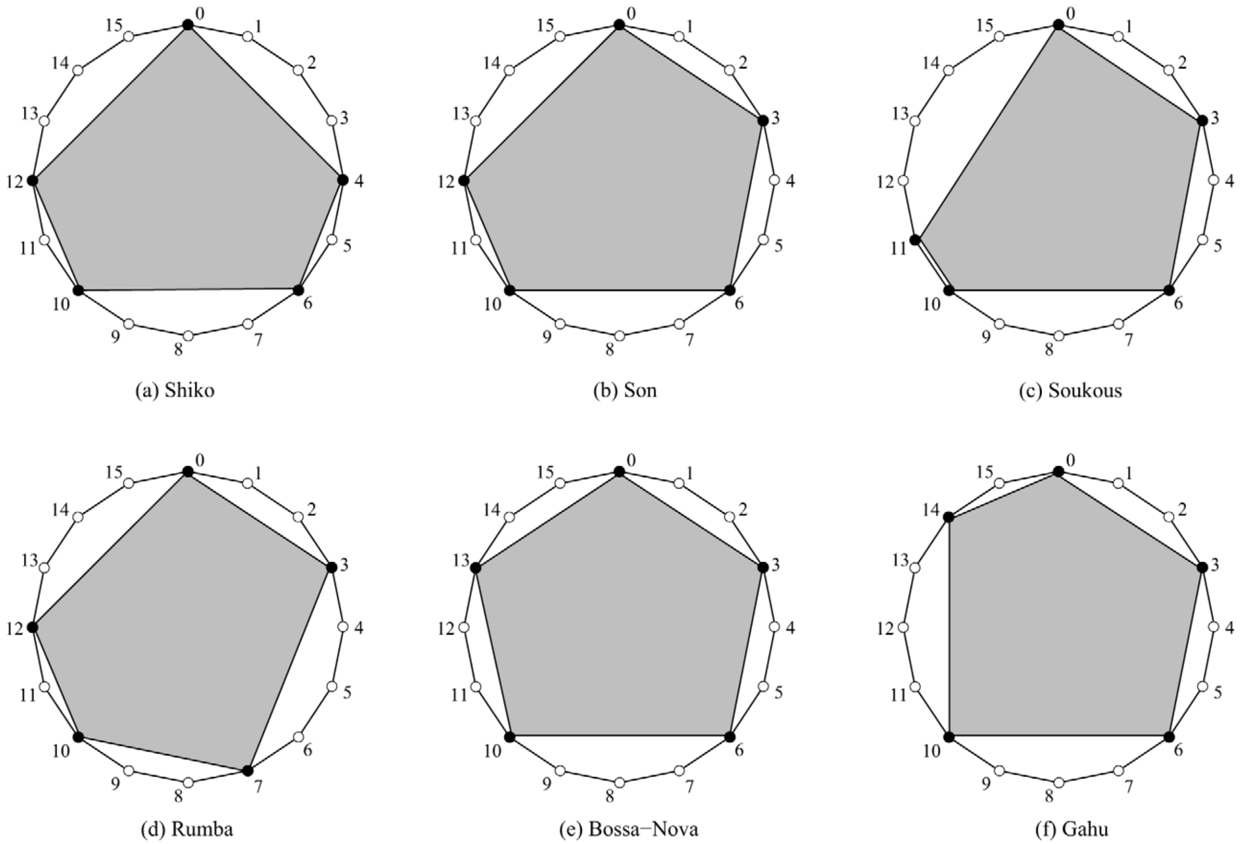

Fig. 1. The six fundamental African and Latin American rhythms which all have equal sum of pairwise geodesic distances; yet intuitively, the Bossa-Nova rhythm is more "even" than the rest.

measures of the amount of *evenness* contained in a scale; see [12, page 40]. One of their evenness measures simply sums the interval arc-lengths (geodesics along the circle) between all pairs of onsets (or more precisely, onset points). This measure differentiates between rhythms that differ widely from each other. For example, the two four-onset rhythms [× · · · × · · · × · · · × · · ·] and [× · × · × · · × · · · · · · · ·] yield evenness values of 32 and 23, respectively, reflecting clearly that the first rhythm is more evenly spaced than the second. However, the measure is too coarse to be useful for comparing rhythm timelines such as those studied in [85,86]. For example, all six fundamental 4/4-time clave/bell patterns discussed in [85] and shown in Fig. 1 have an equal pairwise sum of geodesic distances, namely 48, yet the *Bossa-Nova* clave is intuitively more even than, say, the *Soukous* and *Rumba* claves.

Another distance measure that has been considered is the sum of pairwise chordal distances between adjacent onsets, measured by Euclidean distance between points on the circle. It can be shown that the rhythms maximizing this measure of evenness are precisely the rhythms with maximum possible area. Rappaport [73] shows that many of the most common chords and scales in Western harmony correspond to these maximum-area sets. This evenness measure is finer than the sum of pairwise arc-lengths, but it still does not distinguish half the rhythms in Fig. 1. Specifically, the Son, Rumba, and Gahu claves have the same occurrences of arc-lengths between consecutive onsets, so they also have the same occurrences (and hence total) of distances between consecutive onsets.

The measure of evenness we consider here is the sum of all pairwise Euclidean distances between points on the circle, as described by Block and Douthett [12]. It is worth pointing out that the mathematician Fejes-Tóth [84] showed in 1956 that a configuration of points on a circle maximizes this sum when the points are the vertices of a regular polygon. This measure is also more discriminating than the others, and is therefore the preferred measure of evenness. For example, this measure distinguishes all of the six rhythms in Fig. 1, ranking the Bossa-Nova rhythm as the most even, followed by the Son, Rumba, Shiko, Gahu, and Soukous. Intuitively, the rhythms with a larger sum of pairwise chordal distances have more "well spaced" onsets. It may seem odd that rhythms "lie" in the one-dimensional musical space, while the evenness of the rhythm is measured through chord lengths that "live" in the two-dimensional plane in which the circle is embedded. However, note that two chords are equal if and only if its two corresponding circular arcs are equal. Therefore a polygon is regular if and only if all its circular arcs are equal.

In Section 4, we study the mathematical and computational aspects of rhythms that maximize evenness. We describe three algorithms that generate such rhythms, show that these algorithms are equivalent, and show that in fact the rhythm of maximum evenness is essentially unique. These results characterize rhythms with maximum evenness. One of the algorithms is the Euclidean-like algorithm from Section 2, proving that the rhythms of maximum evenness are precisely the Euclidean rhythms from that section.

Deep rhythms. Another important property of rhythms and scales that we study in this paper is *deepness*. Consider a rhythm with k onsets and timespan n, represented as a set of k points on a circle of circumference n. Now measure the arc-length/geodesic distances along the circle between all pairs of onsets. A musical scale or rhythm is *Winograd-deep* if every distance $1, 2, \dots, \lfloor n/2 \rfloor$ has a unique number of occurrences (called the *multiplicity* of the distance). For example, the rhythm [××× · × ·] is Winograd-deep because distance 1 appears twice, distance 2 appears thrice, and distance 3 appears once.

The notion of deepness in scales was introduced by Winograd in an oft-cited but unpublished class project report from 1966 [94], disseminated and further developed by the class instructor Gamer in 1967 [45,46], and considered further in numerous papers and books, e.g., [23,53]. Equivalently, a scale is *Winograd-deep* if the number of onsets it has in common with each of its cyclic shifts (rotations) is unique. This equivalence is the Common Tone Theorem [53, page 42], and it is originally described by Winograd [94] (who in fact uses this definition as his primary definition of "deep"). Deepness is one property of the ubiquitous Western *diatonic* 12-tone major scale [× · × · ×× · × · × · ×] [53], and it captures some of the rich structure that perhaps makes this scale so attractive.

Winograd-deepness translates directly from scales to rhythms. For example, the diatonic major scale is equivalent to the famous Cuban rhythm *Bembé* [70,86]. Fig. 2 shows a graphical example of a Winograd-deep rhythm. However, the notion of Winograd-deepness is rather restrictive for rhythms, because it requires half of the pulses in a timespan (rounded to a nearest integer) to be onsets. In contrast, for example, the popular Bossa-Nova rhythm [× · · × · · × · · · × · · × · ·] $= \{0, 3, 6, 10, 13\}_{16}$ pictures in Fig. 1 has only five onsets in a timespan of sixteen. Nonetheless, if we focus on just the distances that appear at least once between two onsets, then the multiplicities of occurrence are all unique and form an interval starting at 1: distance 4 occurs once, distance 7 occurs twice, distance 6 occurs thrice, and distance 3 occurs four times.

We therefore define a rhythm (or scale) to be *Erdős-deep* if it has k onsets and, for every multiplicity $1, 2, \dots, k-1$, there is a nonzero arc-length/geodesic distance determined by the points on the circle with exactly that multiplicity. The same definition is made by Toussaint [87]. Every Winograd-deep rhythm is also Erdős-deep, so this definition is strictly more general.

To further clarify the difference between Winograd-deep and Erdős-deep rhythms, it is useful to consider which distances can appear. For a rhythm to be Winograd-deep, all the distances between 1 and $k-1$ must appear a unique number of times. In contrast, to be an Erdős-deep rhythm, it is only required that any distance that appears must have a unique multiplicity. Thus, the Bossa-Nova rhythm is not Winograd-deep because distances 1, 2 and 5 do not appear.

The property of Erdős deepness involves only the distances between points in a set, and is thus a feature of *distance geometry*—in this case, in the discrete space of n points equally spaced around a circle. In 1989, Paul Erdős [37] considered

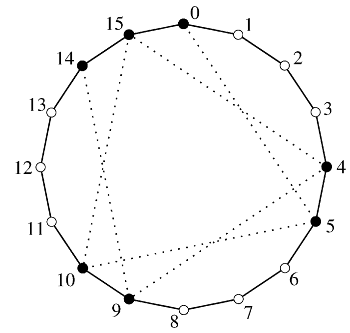

Fig. 2. A rhythm with $k = 7$ onsets and timespan $n = 16$ that is Winograd-deep and thus Erdős-deep. Distances ordered by multiplicity from 1 to 6 are 2, 7, 4, 1, 6, and 5. The dotted line shows how the rhythm is generated by multiples of $m = 5$.

the analogous question in the plane, asking whether there exist n points in the plane (no three on a line and no four on a circle) such that, for every $i = 1, 2, \ldots, n-1$, there is a distance determined by these points that occurs exactly i times. Solutions have been found for n between 2 and 8, but in general the problem remains open. Palásti [68] considered a variant of this problem with further restrictions—no three points form a regular triangle, and no one is equidistant from three others—and solved it for $n = 6$.

In Section 5, we characterize all rhythms that are Erdős-deep. In particular, we prove that all deep rhythms, besides one exception, are *generated*, meaning that the rhythm can be represented as $\{0, m, 2m, \ldots, (k-1)m\}_n$ for some integer m, where all arithmetic is modulo n. In the context of scales, the concept of "generated" was defined by Wooldridge [95] and used by Clough et al. [23]. For example, the rhythm in Fig. 2 is generated with $m = 5$. Our characterization generalizes a similar characterization for Winograd-deep scales proved by Winograd [94], and independently by Clough et al. [23].

In the pitch domain, generated scales are very common. The Pythagorean tuning is a good example: all its pitches are generated from the fifth of ratio $3:2$ modulo the octave. Another example is the equal-tempered scale, which is generated with a half-tone of ratio $\sqrt[12]{2}$ [10]. Generated scales are also of interest in the theory of the well-formed scales [21].

Generated rhythms have an interesting property called *shellability*. If we remove the "last" generated onset 14 from the rhythm in Fig. 2, the resulting rhythm is still generated, and this process can be repeated until we run out of onsets. In general, every generated rhythm has a *shelling* in the sense that it is always possible to remove a particular onset and obtain another generated rhythm.

Most African drumming music consists of rhythms operating on three different strata: the unvarying timeline usually provided by one or more bells, one or more rhythmic motifs played on drums, and an improvised solo (played by the lead drummer) riding on the other rhythmic structures. Shellings of rhythms are relevant to the improvisation of solo drumming in the context of such a rhythmic background. The solo improvisation must respect the style and feeling of the piece which is usually determined by the timeline. One common technique to achieve this effect is to "borrow" notes from the timeline, and to alternate between playing subsets of notes from the timeline and from other rhythms that interlock with the timeline [1, 2]. In the words of Kofi Agawu [1], "It takes a fair amount of expertise to create an effective improvisation that is at the same time stylistically coherent". The borrowing of notes from the timeline may be regarded as a fulfillment of the requirements of style coherence. Another common method is to make parsimonious transformations to the timeline or improvise on a rhythm that is functionally related to the timeline [60]. Although such an approach does not give the performer wide scope for free improvisation, it is efficient in certain drumming contexts. In the words of Christophe Waterman [34], "individuals improvise, but only within fairly strict limits, since varying the constituent parts too much Could unravel the overall texture".

Of course, some subsets of notes of a rhythm may be better choices than others. One might often want to select sets of rhythms that share a common property. For example, if a rhythm is deep, one might want to select subsets of the rhythm that are also deep. Furthermore, a shelling seems a natural way to decrease or increase the density of the notes in an improvisation that respects these constraints. For example, in the *Bembé* bell timeline $[\times \cdot \times \cdot \times\times \cdot \times \cdot \times \cdot \times]$, which is deep, one possible shelling is $[\times \cdot \times \cdot \times\times \cdot \times \cdot \times \cdot \cdot]$, $[\times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot]$, $[\times \cdot \times \cdot \cdot \cdot \cdot \times \cdot \times \cdot \cdot]$, $[\times \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot]$. All five rhythms sound good and are stylistically coherent. In fact the shelled rhythms are used in African drum music [25]. To our knowledge, shellings have not been studied from the musicological point of view. However, they may be useful both for theoretical analysis as well as providing formal rules for "improvisation" techniques.

One of the consequences of our characterization that we obtain in Section 5 is that every Erdős-deep rhythm has a shelling. More precisely, it is always possible to remove a particular onset that preserves the Erdős-deepness property. Thus this is one method of implementing parsimonious transformation of rhythms. Finally, to tie everything together, we show that essentially all Euclidean rhythms (or equivalently, rhythms that maximize evenness) are Erdős-deep.

2. Euclid and evenness in various disciplines

In this section, we first describe Euclid's classic algorithm for computing the greatest common divisor of two integers. Then, through an unexpected connection to timing systems in neutron accelerators, we see how the same type of algorithm can be used as an approach to maximizing “evenness” in a binary string with a specified number of zeroes and ones. This algorithm defines an important family of rhythms, called *Euclidean rhythms*, which we show appear throughout world music. Finally, we see how similar ideas have been used in algorithms for drawing digital straight lines and in combinatorial strings called Euclidean strings.

2.1. The Euclidean algorithm for greatest common divisors

The Euclidean algorithm for computing the greatest common divisor of two integers is one of the oldest known algorithms (circa 300 B.C.). It was first described by Euclid in Proposition 2 of Book VII of *Elements* [39,44]. Indeed, Donald Knuth [57] calls this algorithm the “granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day”.

The idea of the algorithm is simple: repeatedly replace the larger of the two numbers by their difference until both are equal. This final number is then the greatest common divisor. For example, consider the numbers 5 and 13. First, $13 - 5 = 8$; then $8 - 5 = 3$; next $5 - 3 = 2$; then $3 - 2 = 1$; and finally $2 - 1 = 1$. Therefore, the greatest common divisor of 5 and 13 is 1; in other words, 5 and 13 are relatively prime.

The algorithm can also be described succinctly in a recursive manner as follows [29]. Let k and n be the input integers with $k < n$.

Algorithm. $\textsc{Euclid}(k, n)$

1. **if** $k = 0$ **then return** n
2. **else return** $\textsc{Euclid}(n \bmod k, k)$

Running this algorithm with $k = 5$ and $n = 13$, we obtain $\textsc{Euclid}(5, 13) = \textsc{Euclid}(3, 5) = \textsc{Euclid}(2, 3) = \textsc{Euclid}(1, 2) = \textsc{Euclid}(0, 1) = 1$. Note that this division version of Euclid’s algorithm skips one of the steps $(5, 8)$ made by the original subtraction version.

2.2. Evenness and timing systems in neutron accelerators

One of our main musical motivations is to find rhythms with a specified timespan and number of onsets that maximize evenness. Bjorklund [15,16] was faced with a similar problem of maximizing evenness, but in a different context: the operation of components such as high-voltage power supplies of spallation neutron source (SNS) accelerators used in nuclear physics. In this setting, a timing system controls a collection of gates over a time window divided into n equal-length intervals. (In the case of SNS, each interval is 10 seconds.) The timing system can send signals to enable a gate during any desired subset of the n intervals. For a given number n of time intervals, and another given number $k < n$ of signals, the problem is to distribute the pulses as evenly as possible among these n intervals. Bjorklund [16] represents this problem as a binary sequence of k ones and $n - k$ zeroes, where each bit represents a time interval and the ones represent the times at which the timing system sends a signal. The problem then reduces to the following: construct a binary sequence of n bits with k ones such that the k ones are distributed as evenly as possible among the $(n - k)$ zeroes.

One simple case is when k evenly divides n (without remainder), in which case we should place ones every n/k bits. For example, if $n = 16$ and $k = 4$, then the solution is [1000100010001000]. This case corresponds to n and k having a common divisor of k. More generally, if the greatest common divisor between n and k is g, then we would expect the solution to decompose into g repetitions of a sequence of n/g bits. Intuitively, a string of maximum evenness should have this kind of symmetry, in which it decomposes into more than one repetition, whenever such symmetry is possible. This connection to greatest common divisors suggests that a rhythm of maximum evenness might be computed using an algorithm like Euclid’s. Indeed, Bjorklund’s algorithm closely mimics the structure Euclid’s algorithm, although this connection has never been mentioned before.

We describe Bjorklund’s algorithm by using one of his examples. Consider a sequence with $n = 13$ and $k = 5$. Because $13 - 5 = 8$, we start by considering a sequence consisting of 5 ones followed by 8 zeroes which should be thought of as 13 sequences of one bit each:

[1][1][1][1][1][0][0][0][0][0][0][0][0]

If there is more than one zero the algorithm moves zeroes in stages. We begin by taking zeroes one at a time (from right to left), placing a zero after each one (from left to right), to produce five sequences of two bits each, with three zeroes remaining:

[10] [10] [10] [10] [10][0] [0] [0]

Next we distribute the three remaining zeros in a similar manner, by placing a [0] sequence after each [10] sequence:

[100] [100] [100] [10] [10]

Now we have three sequences of three bits each, and a remainder of two sequences of two bits each. Therefore we continue in the same manner, by placing a [10] sequence after each [100] sequence:

[10010] [10010] [100]

The process stops when the remainder consists of only one sequence (in this case the sequence [100]), or we run out of zeroes (there is no remainder). The final sequence is thus the concatenation of [10010], [10010], and [100]:

[1001010010100]

We could proceed further in this process by inserting [100] into [10010] [10010]. However, Bjorklund argues that, because the sequence is cyclic, it does not matter (hence his stopping rule). For the same reason, if the initial sequence has a group of ones followed by only one zero, the zero is considered as a remainder consisting of one sequence of one bit, and hence nothing is done. Bjorklund [16] shows that the final sequence may be computed from the initial sequence using $O(n)$ arithmetic operations in the worst case.

A more convenient and visually appealing way to implement this algorithm by hand is to perform the sequence of insertions in a vertical manner as follows. First take five zeroes from the right and place them under the five ones on the left:

1 1 1 1 1 0 0 0
0 0 0 0 0

Then move the three remaining zeroes in a similar manner:

1 1 1 1 1
0 0 0 0 0
0 0 0

Next place the two remainder columns on the right under the two leftmost columns:

1 1 1
0 0 0
0 0 0
1 1
0 0

Here the process stops because the remainder consists of only one column. The final sequence is obtained by concatenating the three columns from left to right:

1 0 0 1 0 1 0 0 1 0 1 0 0

Bjorklund's algorithm applied to a string of n bits consisting of k ones and $n-k$ zeros has the same structure as running $\textsc{Euclid}(k,n)$. Indeed, Bjorklund's algorithm uses the repeated subtraction form of division, just as Euclid did in his *Elements* [39]. It is also well known that applying the algorithm $\textsc{Euclid}(k,n)$ to two $O(n)$ bit numbers (binary sequences of length n) causes it to perform $O(n)$ arithmetic operations in the worst case [29].

2.3. Euclidean rhythms

The binary sequences generated by Bjorklund's algorithm, as described in the preceding, may be considered as one family of rhythms. Furthermore, because Bjorklund's algorithm is a way of visualizing the repeated-subtraction version of the Euclidean algorithm, we call these rhythms *Euclidean rhythms*. We denote the Euclidean rhythm by $E(k,n)$, where k is the number of ones (onsets) and n (the number of pulses) is the length of the sequence (zeroes plus ones). For example, $E(5,13) = [1001010010100]$. The zero-one notation is not ideal for representing binary rhythms because it is difficult to visualize the locations of the onsets as well as the duration of the inter-onset intervals. In the more iconic box notation, the preceding rhythm is written as $E(5,13) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot]$. It should be emphasized that Euclidean rhythms are merely the result of applying Euclid's algorithm and do not privilege a priori the resulting rhythm over any of its other rotations.

The rhythm $E(5,13)$ is in fact used in Macedonian music [4], but having a timespan of 13 (and defining a measure of length 13), it is rarely found in world music. For contrast, let us consider two widely used values of k and n; in particular, what is $E(3,8)$? Applying Bjorklund's algorithm to the corresponding sequence [11100000], the reader may easily verify that the resulting Euclidean rhythm is $E(3,8) = [\times\cdot\cdot\times\cdot\cdot\times\cdot]$. Fig. 3(a) shows a clock diagram of this rhythm, where the numbers by the sides of the triangle indicate the arc-lengths between those onsets.

The Euclidean rhythm $E(3,8)$ is one of the most famous on the planet. In Cuba, it goes by the name of the *tresillo*, and in the USA, it is often called the *Habanera* rhythm. It was used in hundreds of *rockabilly* songs during the 1950s. It can often be heard in early rock-and-roll hits in the left-hand patterns of the piano, or played on the string bass or saxophone [19,42,65]. A good example is the bass rhythm in Elvis Presley's *Hound Dog* [19]. The tresillo pattern is also found widely in West

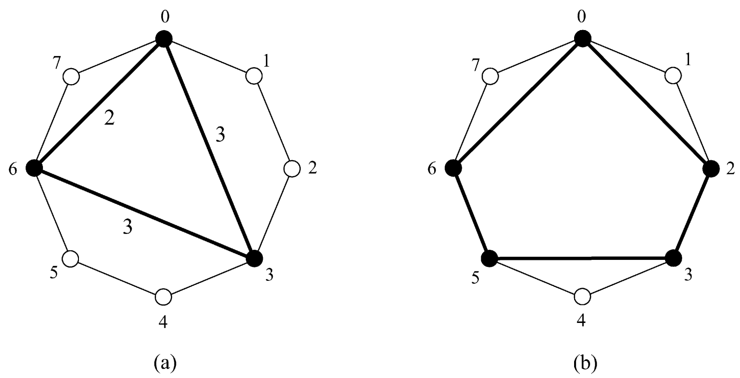

Fig. 3. (a) The Euclidean rhythm $E(3,8)$ is the Cuban *tresillo*. (b) The Euclidean rhythm $E(5,8)$ is the Cuban *cinquillo*.

African traditional music. For example, it is played on the *atoke* bell in the *Sohu*, an *Ewe* dance from Ghana [54]. The tresillo can also be recognized as the first bar (first eight pulses) of the ubiquitous two-bar clave Son shown in Fig. 1(b).

In the two examples $E(5,13)$ and $E(3,8)$, there are fewer ones than zeros. If instead there are more ones than zeros, Bjorklund's algorithm yields the following steps with, for example, $k=5$ and $n=8$:

$$\begin{array}{c} [1\ 1\ 1\ 1\ 1\ 0\ 0\ 0] \\ [10]\ [10]\ [10]\ [1]\ [1] \\ [101]\ [101]\ [10] \\ [1\ 0\ 1\ 1\ 0\ 1\ 1\ 0] \end{array}$$

The resulting Euclidean rhythm is $E(5,8) = [\times \cdot \times\times \cdot \times\times \cdot\,]$. Fig. 3(b) shows a clock diagram for this rhythm. It is another famous rhythm on the world scene. In Cuba, it goes by the name of the *cinquillo* and it is intimately related to the tresillo [42]. It has been used in jazz throughout the 20th century [72], and in rockabilly music. For example, it is the hand-clapping pattern in Elvis Presley's *Hound Dog* [19]. The cinquillo pattern is also widely used in West African traditional music [71,85], as well as Egyptian [48] and Korean [50] music.

We show in this paper that Euclidean rhythms have two important properties: they maximize evenness and they are deep. The evenness property should come as no surprise, given how we designed the family of rhythms. To give some feeling for the deepness property, we consider the two examples in Fig. 3, which have been labeled with the distances between all pairs of onsets, measured as arc-lengths. The *tresillo* in Fig. 3(a) has one occurrence of distance 2 and two occurrences of distance 3. The *cinquillo* in Fig. 3(b) contains one occurrence of distance 4, two occurrences of distance 1, three occurrences of distance 2, and four occurrences of distance 3. Thus, every distance has a unique multiplicity, making these rhythms Erdős-deep.

2.4. Euclidean rhythms in traditional world music

In this section, we list all the Euclidean rhythms found in world music that we have collected so far, restricting attention to those in which k and n are relatively prime. In some cases, the Euclidean rhythm is a rotated version of a commonly used rhythm; this makes the two rhythms instances of the same necklace. Fig. 4 illustrates an example of two rhythms that are instances of the same necklace. We provide this list because it is interesting ethnomusicological data on rhythms. We make no effort in this paper to establish that Euclidean rhythms are more common than their rotations. We leave the problem of defining which rhythms are preferred over others as an open problem to ethnomusicologists.

Rhythms in which k and n have a common divisor larger than 1 are common all over the planet in traditional, classical, and popular genres of music. For example, $E(4,12) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot\,]$ is the 12/8-time *Fandango* clapping pattern in the Flamenco music of southern Spain, where '$\times$' denotes a loud clap and '$\cdot$' denotes a soft clap [35]. However, the string itself is periodic: $E(4,12)$ has period 3, even though it appears in a timespan of 12. For this reason, we restrict ourselves to the more interesting Euclidean rhythms that do not decompose into repetitions of shorter Euclidean rhythms. We are also not concerned with rhythms that have only one onset ($[\times \cdot\,]$, $[\times \cdot \cdot\,]$, etc.), and similarly with any repetitions of these rhythms (for example, $[\times \cdot \times \cdot\,]$).

There are surprisingly many Euclidean rhythms with k and n relatively prime that are found in world music. Appendix A includes more than 40 such rhythms uncovered so far.

2.5. Aksak rhythms

Euclidean rhythms are closely related to a family of rhythms known as *aksak* rhythms, which have been studied from the combinatorial point of view for some time [4,17,28]. Béla Bartók [9] and Constantin Brăiloiu [17], respectively, have used

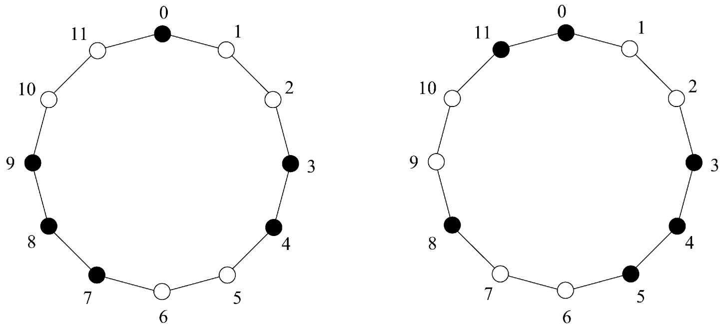

Fig. 4. These two rhythms are instances of the same rhythm necklace.

the terms *Bulgarian rhythm* and *aksak* to refer to those meters that use units of durations 2 and 3, and no other durations. Furthermore, the rhythm or meter must contain at least one duration of length 2 and at least one duration of length 3. Arom [4] refers to these durations as *binary cells* and *ternary cells*, respectively.

Arom [4] generated an inventory of all the theoretically possible *aksak* rhythms for values of n ranging from 5 to 29, as well as a list of those that are actually used in traditional world music. He also proposed a classification of these rhythms into several classes, based on structural and numeric properties. Three of his classes are considered here:

1. An *aksak* rhythm is *authentic* if n is a *prime* number.
2. An *aksak* rhythm is *quasi-aksak* if n is an *odd* number that is not prime.
3. An *aksak* rhythm is *pseudo-aksak* if n is an *even* number.

A quick perusal of the Euclidean rhythms listed in the preceding reveals that *aksak* rhythms are well represented. Indeed, all three of Arom's classes (authentic, quasi-aksak, and pseudo-aksak) make their appearance. There is a simple characterization of those Euclidean rhythms that are *aksak*. From the iterative subtraction algorithm of Bjorklund it follows that if $n = 2k$ all cells are binary (duration 2). Similarly, if $n = 3k$ all cells are ternary (duration 3). Therefore, to ensure that the Euclidean rhythm contains both binary and ternary cells, and no other durations, it follows that n must be between $2k$ and $3k$.

Of course, not all *aksak* rhythms are Euclidean. Consider the Bulgarian rhythm with interval sequence (3322) [4], which is also the metric pattern of *Indian Lady* by Don Ellis [55]. Here $k = 4$ and $n = 10$, and $E(4,10) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot]$ or (3232), a periodic rhythm.

The following Euclidean rhythms are *authentic aksak*:

$E(2,5) = [\times\cdot\times\cdot\cdot] = (23)$ (classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda).

$E(3,7) = [\times\cdot\times\cdot\times\cdot\cdot] = (223)$ (Bulgaria, Greece, Sudan, Turkestan).

$E(4,11) = [\times\cdot\cdot\times\cdot\cdot\times\cdot\cdot\times\cdot] = (3332)$ (Southern India rhythm), (Serbian necklace).

$E(5,11) = [\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\cdot] = (22223)$ (classical music, Bulgaria, Northern India, Serbia).

$E(5,13) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot] = (32323)$ (Macedonia).

$E(6,13) = [\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\cdot] = (222223)$ (Macedonia).

$E(7,17) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot] = (3232322)$ (Macedonian necklace).

$E(8,17) = [\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\cdot] = (22222223)$ (Bulgaria).

$E(8,19) = [\times\cdot\cdot\times\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\times\cdot\cdot\times\cdot] = (32232232)$ (Bulgaria).

$E(9,23) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot] = (323232323)$ (Bulgaria).

The following Euclidean rhythms are *quasi-aksak*:

$E(4,9) = [\times\cdot\times\cdot\times\cdot\times\cdot\cdot] = (2223)$ (Greece, Macedonia, Turkey, Zaïre).

$E(7,15) = [\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\cdot] = (2222223)$ (Bulgarian necklace).

The following Euclidean rhythms are *pseudo-aksak*:

$E(3,8) = [\times\cdot\cdot\times\cdot\cdot\times\cdot] = (332)$ (Central Africa, Greece, India, Latin America, West Africa, Sudan).

$E(5,12) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot] = (32322)$ (Macedonia, South Africa).

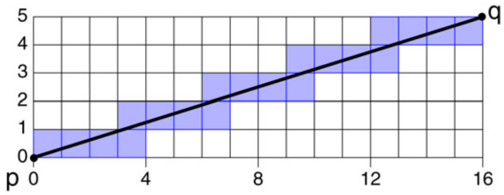

Fig. 5. The shaded pixels form a digital straight line determined by the points p and q.

$E(7,16) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot] = (3223222)$ (Brazilian, Macedonian, West African necklaces).

$E(7,18) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (3232323)$ (Bulgaria).

$E(9,22) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (323232322)$ (Bulgarian necklace).

$E(11,24) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (32222322222)$
(Central African and Bulgarian necklaces).

$E(15,34) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (322232223222322)$
(Bulgarian necklace).

2.6. Drawing digital straight lines

Euclidean rhythms and necklace patterns also appear in the computer graphics literature on drawing digital straight lines [59]. The problem here consists of efficiently converting a mathematical straight line segment defined by the x and y integer coordinates of its endpoints, to an ordered sequence of pixels that most faithfully represents the given straight line segment. Fig. 5 illustrates an example of a digital straight line (shaded pixels) determined by the two given endpoints p and q. All the pixels intersected by the segment (p,q) are shaded. If we follow either the lower or upper boundary of the shaded pixels from left to right we obtain the interval sequences (43333) or (33334), respectively. Note that the upper pattern corresponds to $E(5,16)$, a *Bossa-Nova* variant. Indeed, Harris and Reingold [51] show that the well-known Bresenham algorithm [18] is described by the Euclidean algorithm.

2.7. Calculating leap years in calendar design

For thousands of years human beings have observed and measured the time it takes between two consecutive sunrises, and between two consecutive spring seasons. These measurements inspired different cultures to design calendars [7,74]. Let T_y denote the duration of one revolution of the earth around the sun, more commonly known as a year. Let T_d denote the duration of one complete rotation of the earth, more commonly known as a day. The values of T_y and T_d are of course continually changing, because the universe is continually reconfiguring itself. However the ratio T_y/T_d is approximately 365.242199.... It is very convenient therefore to make a year last 365 days. The problem that arises both for history and for predictions of the future, is that after a while the 0.242199.... starts to contribute to a large error. One simple solution is to add one extra day every 4 years: the so-called Julian calendar. A day with one extra day is called a leap year. But this assumes that a year is 365.25 days long, which is still slightly greater than 365.242199.... So now we have an error in the opposite direction albeit smaller. One solution to this problem is the Gregorian calendar [78]. The Gregorian calendar defines a leap year as one divisible by 4, except not those divisible by 100, except not those divisible by 400. With this rule a year becomes $365 + 1/4 - 1/100 + 1/400 = 365.2425$ days long, not a bad approximation.

Another solution is provided by the Jewish calendar which uses the idea of cycles [7]. Here a regular year has 12 months and a leap year has 13 months. The cycle has 19 years including 7 leap years. The 7 leap years must be distributed as evenly as possible in the cycle of 19. The cycle is assumed to start with Creation as year 1. If the year modulo 19 is one of 3, 6, 8, 11, 14, 17, or 19, then it is a leap year. For example, the year $5765 = 303 \cdot 19 + 8$ and so is a leap year. The year 5766, which begins at sundown on the Gregorian date of October 3, 2005, is $5766 = 303 \times 19 + 9$, and is therefore not a leap year. Applying Bjorklund's algorithm to the integers 7 and 19 yields $E(7,19) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \cdot]$. If we start this rhythm at the 7th pulse we obtain the pattern $[\cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times]$, which describes precisely the leap year pattern 3, 6, 8, 11, 14, 17, and 19 of the Jewish calendar. In this sense the Jewish calendar is an instance of a Euclidean necklace.

2.8. Euclidean strings

In the study of the combinatorics of words and sequences, there exists a family of strings called Euclidean strings [38]. In this section we explore the relationship between Euclidean strings and Euclidean rhythms. We use the same terminology and notation introduced in [38]. Euclidean strings result from a mathematical algorithm and represent a different arbitrary

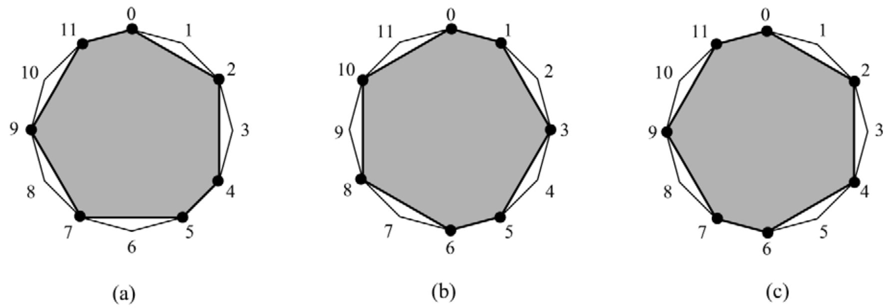

Fig. 6. Two right-rotations of the *Bembé* string: (a) the *Bembé*, (b) rotation by one unit, (c) rotation by seven units.

convention as to how to choose a canonical rhythm that represents the necklace. Whether there is anything musically meaningful about these conventions is left to ethnomusicologists to decide.

Let $P = (p_0, p_1, \ldots, p_{n-1})$ denote a string of nonnegative integers. Let $\rho(P)$ denote the right rotation of P by one position; that is, $\rho(P) = (p_{n-1}, p_0, p_1, \ldots, p_{n-2})$. Let $\rho^d(P)$ denote the right rotation of P by d positions. If P is considered as a cyclic string, a right rotation corresponds to a clockwise rotation. Fig. 6 illustrates the $\rho(P)$ operator with P equal to the *Bembé* bell-pattern of West Africa [86]. Fig. 6(a) shows the *Bembé* bell-pattern, Fig. 6(b) shows $\rho(P)$, which is a hand-clapping pattern from West Africa [70], and Fig. 6(c) shows $\rho^7(P)$, which is the *Tambú* rhythm of Curaçao [77].

Ellis et al. [38] define a string $P = (p_0, p_1, \ldots, p_{n-1})$ to be *Euclidean* if incrementing p_0 by 1 and decrementing p_{n-1} by 1 yields a new string $\tau(P)$ that is the rotation of P. In other words, P and $\tau(P)$ are instances of the same necklace. Therefore, if we represent rhythms as binary sequences, Euclidean rhythms cannot be Euclidean strings because all Euclidean rhythms begin with a 'one'. Increasing p_0 by one makes it a 'two', which is not a binary string. Therefore, to explore the relationship between Euclidean strings and Euclidean rhythms, we will represent rhythms by their clockwise distance sequences, which are also strings of nonnegative integers. As an example, consider $E(4, 9) = [\times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2223)$. Now $\tau(2223) = (3222)$, which is a rotation of $E(4, 9)$, and thus (2223) is a Euclidean string. Indeed, for $P = E(4, 9)$, $\tau(P) = \rho^3(P)$. As a second example, consider the West African clapping-pattern shown in Fig. 6(b) given by $P = (1221222)$. We have that $\tau(P) = (2221221) = \rho^6(P)$, the pattern shown in Fig. 6(c), which also happens to be the mirror image of P about the $(0, 6)$ axis. Therefore P is a Euclidean string. However, note that P is not a Euclidean rhythm. Nevertheless, P is a rotation of the Euclidean rhythm $E(7, 12) = (2122122)$.

Ellis et al. [38] have many beautiful results about Euclidean strings. They show that Euclidean strings exist if, and only if, n and $(p_0 + p_1 + \cdots + p_{n-1})$ are relatively prime numbers, and that when they exist they are unique. They also show how to construct Euclidean strings using an algorithm that has the same structure as the Euclidean algorithm. In addition they relate Euclidean strings to many other families of sequences studied in the combinatorics of words [5,62].

Let $R(P)$ denote the reversal (or mirror image) of P; that is, $R(P) = (p_{n-1}, p_{n-2}, \ldots, p_1, p_0)$. Now we may determine which of the Euclidean rhythms used in world music listed in the preceding, are Euclidean strings or *reverse* Euclidean strings. The length of a Euclidean string is defined as the number of integers it has. This translates in the rhythm domain to the number of onsets a rhythm contains. Furthermore, strings of length one are Euclidean strings, trivially. Therefore all the trivial Euclidean rhythms with only one onset, such as $E(1, 2) = [\times \cdot] = (2)$, $E(1, 3) = [\times \cdot \cdot] = (3)$, and $E(1, 4) = [\times \cdot \cdot \cdot] = (4)$, etc., are both Euclidean strings as well as reverse Euclidean strings. In the lists that follow the Euclidean rhythms are shown in their box-notation format as well as in the clockwise distance sequence representation. The styles of music that use these rhythms is also included. Finally, if only a rotated version of the Euclidean rhythm is played, then it is still included in the list but referred to as a necklace.

The following Euclidean rhythms are Euclidean strings:

$E(2, 3) = [\times \times \cdot] = (12)$ (West Africa, Latin America, Nubia, Northern Canada).

$E(2, 5) = [\times \cdot \times \cdot \cdot] = (23)$ (classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda), (*authentic aksak*).

$E(3, 4) = [\times \times \times \cdot] = (112)$ (Brazil, Bali rhythms), (Colombia, Greece, Spain, Persia, Trinidad necklaces).

$E(3, 7) = [\times \cdot \times \cdot \times \cdot \cdot] = (223)$ (Bulgaria, Greece, Sudan, Turkestan), (*authentic aksak*).

$E(4, 5) = [\times \times \times \times \cdot] = (1112)$ (Greece).

$E(4, 9) = [\times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2223)$ (Greece, Macedonia, Turkey, Zaïre), (*quasi-aksak*).

$E(5, 6) = [\times \times \times \times \times \cdot] = (11112)$ (Arab).

$E(5, 11) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22223)$ (classical music, Bulgaria, Northern India, Serbia), (*authentic aksak*).

$E(5, 16) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \cdot] = (33334)$ (Brazilian, West African necklaces).

$E(6, 7) = [\times \times \times \times \times \times \cdot] = (111112)$ (Greek necklace).

$E(6,13) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (222223)$ (Macedonia), (*authentic aksak*).

$E(7,8) = [\times \times \times \times \times \times \times \cdot] = (1111112)$ (Libyan necklace).

$E(7,15) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2222223)$ (Bulgarian necklace), (*quasi-aksak*).

$E(8,17) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22222223)$ (Bulgaria), (*authentic aksak*).

The following Euclidean rhythms are reverse Euclidean strings:

$E(3,5) = [\times \cdot \times \cdot \times] = (221)$ (Korean, Rumanian, Persian necklaces).

$E(3,8) = [\times \cdot \cdot \times \cdot \cdot \times \cdot] = (332)$ (Central Africa, Greece, India, Latin America, West Africa, Sudan), (*pseudo-aksak*).

$E(3,11) = [\times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot] = (443)$ (North India).

$E(3,14) = [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot] = (554)$ (North India).

$E(4,7) = [\times \cdot \times \cdot \times \cdot \times] = (2221)$ (Bulgaria).

$E(4,11) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot] = (3332)$ (Southern India rhythm), (Serbian necklace), (*authentic aksak*).

$E(4,15) = [\times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot] = (4443)$ (North India).

$E(5,7) = [\times \cdot \times \times \cdot \times \times] = (21211)$ (Arab).

$E(5,9) = [\times \cdot \times \cdot \times \cdot \times \cdot \times] = (22221)$ (Arab).

$E(5,12) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (32322)$ (Macedonia, South Africa), (*pseudo-aksak*).

$E(7,9) = [\times \cdot \times \times \times \cdot \times \times \times] = (2112111)$ (Greece).

$E(7,10) = [\times \cdot \times \times \cdot \times \times \cdot \times \times] = (2121211)$ (Turkey).

$E(7,16) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot] = (3223222)$ (Brazilian, Macedonian, West African necklaces), (*pseudo-aksak*).

$E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (3232322)$ (Macedonian necklace), (*authentic aksak*).

$E(9,22) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (323232322)$ (Bulgarian necklace), (*pseudo-aksak*).

$E(11,12) = [\times \cdot \times \times \times \times \times \times \times \times \times \times] = (11111111112)$ (Oman necklace).

$E(11,24) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (32222322222)$ (Central African and Bulgarian necklaces), (*pseudo-aksak*).

The following Euclidean rhythms are neither Euclidean nor reverse Euclidean strings:

$E(5,8) = [\times \cdot \times \times \cdot \times \times \cdot] = (21212)$ (Egypt, Korea, Latin America, West Africa).

$E(5,13) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (32323)$ (Macedonia), (*authentic aksak*).

$E(7,12) = [\times \cdot \times \times \cdot \times \cdot \times \times \cdot \times \cdot] = (2122122)$ (West Africa), (Central African, Nigerian, Sierra Leone necklaces).

$E(7,18) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (3232323)$ (Bulgaria), (*pseudo-aksak*).

$E(8,19) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot] = (32232232)$ (Bulgaria), (*authentic aksak*).

$E(9,14) = [\times \cdot \times \times \cdot \times \times \cdot \times \times \cdot \times \times \cdot] = (212121212)$ (Algerian necklace).

$E(9,16) = [\times \cdot \times \times \cdot \times \cdot \times \cdot \times \times \cdot \times \cdot \times \cdot] = (212221222)$ (West and Central African, and Brazilian necklaces).

$E(9,23) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (323232323)$ (Bulgaria), (*authentic aksak*).

$E(13,24) = [\times \cdot \times \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (2122222122222)$ (Central African necklace).

$E(15,34) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (322232223222322)$ (Bulgarian necklace), (*pseudo-aksak*).

The Euclidean rhythms that appear in classical music and jazz are also Euclidean strings (the first group). Furthermore, this group is not popular in African music. The Euclidean rhythms that are neither Euclidean strings nor reverse Euclidean strings (group three) fall into two categories: those consisting of clockwise distances 1 and 2, and those consisting of clockwise distances 2 and 3. The latter group is used only in Bulgaria, and the former is used in Africa. Finally, the Euclidean rhythms that are reverse Euclidean strings (the second group) appear to have a much wider use. Finding musicological explanations for these mathematical properties raises interesting ethnomusicological questions.

The Euclidean strings defined in [38] determine another family of rhythms, many of which are also used in world music but are not necessarily Euclidean rhythms. For example, (1221222) is an Afro–Cuban bell pattern. Therefore it would be interesting to explore empirically the relation between Euclidean strings and world music rhythms, and to determine formally the exact mathematical relation between Euclidean rhythms and Euclidean strings.

3. Definitions and notation

Before we begin the more technical part of the paper, we need to define some precise mathematical notation for describing rhythms.

Let $\mathbb{Z}^+$ denote the set of positive integers. For $k, n \in \mathbb{Z}^+$, let $\gcd(k,n)$ denote the greatest common divisor of k and n. If $\gcd(k,n) = 1$, we call k and n *relatively prime*. For integers $a < b$, let $[a,b] = \{a, a+1, a+2, \ldots, b\}$.

Let C be a circle in the plane, and consider any two points x, y on C. The *chordal distance* between x and y, denoted by $\bar{d}(x,y)$, is the length of the line segment $\overline{xy}$; that is, $\bar{d}(x,y)$ is the Euclidean distance between x and y. The *clockwise distance* from x to y, or of the ordered pair (x,y), is the length of the clockwise arc of C from x to y, and is denoted by $\overset{\frown}{d}(x,y)$. Finally, the *geodesic distance* between x and y, denoted by $\overset{\frown}{d}(x,y)$, is the length of the shortest arc of C between x and y; that is, $\overset{\frown}{d}(x,y) = \min\{\overset{\frown}{d}(x,y), \overset{\frown}{d}(y,x)\}$.

A *rhythm of timespan n* is a subset of $\{0, 1, \ldots, n-1\}$, representing the set of pulses that are onsets in each repetition. For clarity, we write the timespan n as a subscript after the subset: $\{\ldots\}_n$. Geometrically, if we locate n equally spaced points clockwise around a circle C_n of circumference n, then we can view a rhythm of timespan n as a subset of these n points. We consider an element of C_n to simultaneously be a point on the circle and an integer in $\{0, 1, \ldots, n-1\}$.

The *rotation* of a rhythm R of timespan n by an integer $\Delta \geqslant 0$ is the rhythm $\{(i+\Delta) \bmod n\colon\ i \in R\}_n$ of the same timespan n. The *scaling* of a rhythm R of timespan n by an integer $\alpha \geqslant 1$ is the rhythm $\{\alpha i\colon\ i \in R\}_{\alpha n}$ of timespan αn.

Let $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ be a rhythm of timespan n with k onsets sorted in clockwise order. Throughout this paper, an onset r_i will mean $(r_{i \bmod k}) \bmod n$. Observe that the clockwise distance $\overset{\frown}{d}(r_i, r_j) = (r_j - r_i) \bmod n$. This is the number of points on C_n that are contained in the clockwise arc $(r_i, r_j]$ and is also known as the *chromatic length* [22].

The *geodesic distance multiset* of a rhythm R is the multiset of all nonzero pairwise geodesic distances; that is, it is the multiset $\{\overset{\frown}{d}(r_i, r_j)\colon\ r_i, r_j \in R, r_i \neq r_j\}$. The geodesic distance multiset has cardinality $\binom{k}{2}$. The *multiplicity* of a distance d is the number of occurrences of d in the geodesic distance multiset.

A rhythm is *Erdős-deep* if it has (exactly) one distance of multiplicity i, for each $i \in [1, k-1]$. Note that these multiplicities sum to $\sum_{i=1}^{k-1} i = \binom{k}{2}$, which is the cardinality of the geodesic distance multiset, and hence these distances are all the distances in the rhythm. Every geodesic distance is between 0 and $\lfloor n/2 \rfloor$. A rhythm is *Winograd-deep* if every two distances from $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ have different multiplicity.

A *shelling* of an Erdős-deep rhythm R is an ordering $s_1, s_2, \ldots, s_k$ of the onsets in R such that $R - \{s_1, s_2, \ldots, s_i\}$ is an Erdős-deep rhythm for $i = 0, 1, \ldots, k$. (Every rhythm with at most two onsets is Erdős-deep.)

The *evenness* of rhythm R is the sum of all inter-onset chordal distances in R; that is, $\sum_{0 \leqslant i < j \leqslant k-1} \bar{d}(r_i, r_j)$.

The *clockwise distance sequence* of R is the circular sequence $(d_0, d_1, \ldots, d_{k-1})$ where $d_i = \overset{\frown}{d}(r_i, r_{i+1})$ for all $i \in [0, k-1]$. Observe that each $d_i \in \mathbb{Z}^+$ and $\sum_i d_i = n$.

Observation 1. *There is a one-to-one relationship between rhythms with k onsets and timespan n and circular sequences* $(d_0, d_1, \ldots, d_{k-1})$ *where each* $d_i \in \mathbb{Z}^+$ *and* $\sum_i d_i = n$.

4. Even rhythms

In this section we first describe three algorithms that generate even rhythms. We then characterize rhythms with maximum evenness and show that, for given numbers of pulses and onsets, the three described algorithms generate the unique rhythm with maximum evenness. As mentioned in the introduction, the measure of evenness considered here is the pairwise sum of chordal distances.

The even rhythms characterized in this section were studied by Clough and Myerson [30,31] for the case where the numbers of pulses and onsets are relatively prime. This was subsequently expanded upon by Clough and Douthett [22]. We revisit these results and provide an additional connection to rhythms (and scales) that are obtained from the Euclidean algorithm. Most of these results are stated in [22]. However our proofs are new, and in many cases are much more streamlined.

4.1. Characterization

We first present three algorithms for computing a rhythm with k onsets, timespan n, for any $k \leqslant n$, that possess large evenness.

The first algorithm is by Clough and Douthett [22]:

Algorithm. Clough-Douthett(k, n)

1. **return** $\{\lfloor \frac{in}{k} \rfloor:\ i \in [0, k-1]\}$.

Because $k \leqslant n$, the rhythm output by Clough-Douthett(k, n) has k onsets as desired.

The second algorithm is a geometric heuristic implicit in the work of Clough and Douthett [22]:

Algorithm. Snap(k, n)

1. Let D be a set of k evenly spaced points on C_n such that $D \cap C_n = \emptyset$.
2. For each point $x \in D$, let x' be the first point in C_n clockwise from x.
3. **return** $\{x':\ x \in D\}$.

Because $k \leqslant n$, the clockwise distance between consecutive points in D in the execution of Snap(k, n) is at least that of consecutive points in C_n. Thus, $x' \neq y'$ for distinct $x, y \in D$, so Snap returns a rhythm with k onsets as desired.

The third algorithm is a recursive algorithm in the same mold as Euclid's algorithm for greatest common divisors. The algorithm uses the clockwise distance sequence notation described in the introduction. The resulting rhythm always defines the same necklace as the Euclidean rhythms from Section 2.3; that is, the only difference is a possible rotation.

Algorithm. Euclidean(k, n)

1. **if** k evenly divides n **then return** $(\underbrace{\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k}}_{k})$.
2. $a \leftarrow n \bmod k$.
3. $(x_1, x_2, \ldots, x_a) \leftarrow$ Euclidean(k, a).
4. **return** $(\underbrace{\lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor}_{x_1-1}, \lceil \frac{n}{k} \rceil; \underbrace{\lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor}_{x_2-1}, \lceil \frac{n}{k} \rceil; \ldots; \underbrace{\lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor}_{x_a-1}, \lceil \frac{n}{k} \rceil)$.

As a simple example, consider $k = 5$ and $n = 13$. The sequence of calls to Euclidean(k, n) follows the same pattern as the Euclid algorithm for greatest common divisors from Section 2.1, except that it now stops one step earlier: $(5, 13)$, $(3, 5)$, $(2, 3)$, $(1, 2)$. At the base of the recursion, we have Euclidean$(1, 2) = (2) = [\times\ \cdot\ ]$. At the next level up, we obtain Euclidean$(2, 3) = (1, 2) = [\times \times\ \cdot\ ]$. Next we obtain Euclidean$(3, 5) = (2; 1, 2) = [\times \cdot \times \times \cdot\ ]$. Finally, we obtain Euclidean$(5, 13) = (2, 3; 3; 2, 3) = [\times \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot\ ]$. (For comparison, the Euclidean rhythm from Section 2.2 is $E(5, 13) = (2, 3, 2, 3, 3)$, a rotation by 5.)

We now show that algorithm Euclidean(k, n) outputs a circular sequence of k integers that sum to n (which is thus the clockwise distance sequence of a rhythm with k onsets and timespan n). We proceed by induction on k. If k evenly divides n, then the claim clearly holds. Otherwise a $(= n \bmod k) > 0$, and by induction $\sum_{i=1}^{a} x_i = k$. Thus the sequence that is output has k terms and sums to

$$\begin{aligned} a\left\lceil \frac{n}{k} \right\rceil + \left\lfloor \frac{n}{k} \right\rfloor \sum_{i=1}^{a} (x_i - 1) &= a\left\lceil \frac{n}{k} \right\rceil + (k-a)\left\lfloor \frac{n}{k} \right\rfloor \\ &= a\left(1 + \left\lfloor \frac{n}{k} \right\rfloor\right) + (k-a)\left\lfloor \frac{n}{k} \right\rfloor \\ &= a + k\left\lfloor \frac{n}{k} \right\rfloor \\ &= n. \end{aligned}$$

The following theorem is one of the main contributions of this paper.

Theorem 4.1.

Let $n \geqslant k \geqslant 2$ be integers. The following are equivalent for a rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ with k onsets and timespan n:

(A) *R has maximum evenness (sum of pairwise inter-onset chordal distances);*
(B) *R is a rotation of the* Clough-Douthett(k, n) *rhythm;*
(C) *R is a rotation of the* Snap(k, n) *rhythm;*
(D) *R is a rotation of the* Euclidean(k, n) *rhythm; and*
($\star$) *for all $\ell \in [1, k]$ and $i \in [0, k-1]$, the ordered pair $(r_i, r_{i+\ell})$ has clockwise distance $\overset{\frown}{d}(r_i, r_{i+\ell}) \in \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$.*

Moreover, up to a rotation, there is a unique rhythm that satisfies these conditions.

Note that the evenness of a rhythm equals the evenness of the same rhythm played backwards. Thus, if R is the unique rhythm with maximum evenness, then R is the same rhythm as R played backwards (up to a rotation).

The proof of Theorem 4.1 proceeds as follows. In Section 4.2 we prove that each of the three algorithms produces a rhythm that satisfies property ($\star$). Then in Section 4.3 we prove that there is a unique rhythm that satisfies property ($\star$). Thus the three algorithms produce the same rhythm, up to rotation. Finally in Section 4.4 we prove that the unique rhythm that satisfies property ($\star$) maximizes evenness.

4.2. Properties of the algorithms

We now prove that each of the algorithms has property ($\star$). Clough and Douthett [22] proved the following.

Proof (B) $\Rightarrow$ ($\star$). Say $R = \{r_0, r_1, \dots, r_{k-1}\}_n$ is the CLOUGH-DOUTHETT(k,n) rhythm. Consider an ordered pair $(r_i, r_{i+\ell})$ of onsets in R. Let $p_i = in \bmod k$ and let $p_\ell = \ell n \bmod k$. By symmetry we can suppose that $r_i \leqslant r_{(i+\ell) \bmod k}$. Then the clockwise distance $\widehat{d}(r_i, r_{i+\ell})$ is

$$\left\lfloor \frac{(i+\ell)n}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor = \left\lfloor \frac{in}{k} \right\rfloor + \left\lfloor \frac{\ell n}{k} \right\rfloor + \left\lfloor \frac{p_i + p_\ell}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor = \left\lfloor \frac{\ell n}{k} \right\rfloor + \left\lfloor \frac{p_i + p_\ell}{k} \right\rfloor,$$

which is $\lfloor \frac{\ell n}{k} \rfloor$ or $\lceil \frac{\ell n}{k} \rceil$, because $\lfloor \frac{p_i + p_\ell}{k} \rfloor \in \{0, 1\}$. □

A similar proof shows that the rhythm $\{\lceil \frac{in}{k} \rceil\colon\ i \in [0, k-1]\}$ satisfies property ($\star$). Observe that ($\star$) is equivalent to the following property.

($\star\star$) If $(d_0, d_1, \dots, d_{k-1})$ is the clockwise distance sequence of R, then for all $\ell \in [1, k]$, the sum of any ℓ consecutive elements in $(d_0, d_1, \dots, d_{k-1})$ equals $\lceil \frac{\ell n}{k} \rceil$ or $\lfloor \frac{\ell n}{k} \rfloor$.

Proof (C) $\Rightarrow$ ($\star\star$). Let $(d_0, d_1, \dots, d_{k-1})$ be the clockwise distance sequence of the rhythm determined by SNAP(k,n). For the sake of contradiction, suppose that for some $\ell \in [1, k]$, the sum of ℓ consecutive elements in $(d_0, d_1, \dots, d_{k-1})$ is greater than $\lceil \frac{\ell n}{k} \rceil$. The case in which the sum is less than $\lfloor \frac{\ell n}{k} \rfloor$ is analogous. We can assume that these ℓ consecutive elements are $(d_0, d_1, \dots, d_{\ell-1})$. Using the notation defined in the statement of the algorithm, let $x_0, x_1, \dots, x_\ell$ be the points in D such that $\widehat{d}(x'_i, x'_{i+1}) = d_i$ for all $i \in [0, \ell-1]$. Thus $\widehat{d}(x'_1, x'_{\ell+1}) \geqslant \lceil \frac{\ell n}{k} \rceil + 1$. Now $\widehat{d}(x_{\ell+1}, x'_{\ell+1}) < 1$. Thus $\widehat{d}(x'_1, x_{\ell+1}) > \lceil \frac{\ell n}{k} \rceil \geqslant \frac{\ell n}{k}$, which implies that $\widehat{d}(x_1, x_{\ell+1}) > \frac{\ell n}{k}$. This contradicts the fact that the points in D were evenly spaced around C_n in the first step of the algorithm. □

Proof (D) $\Rightarrow$ ($\star\star$). We proceed by induction on k. Let $R =$ EUCLIDEAN(k,n). If k evenly divides n, then $R = (\frac{n}{k}, \frac{n}{k}, \dots, \frac{n}{k})$, which satisfies (D). Otherwise, let $a = n \bmod k$ and let $(x_1, x_2, \dots, x_a) =$ EUCLIDEAN(k,a). By induction, for all $\ell \in [1, a]$, the sum of any ℓ consecutive elements in $(x_1, x_2, \dots, x_a)$ equals $\lfloor \frac{\ell k}{a} \rfloor$ or $\lceil \frac{\ell k}{a} \rceil$. Let S be a sequence of m consecutive elements in R. By construction, for some $1 \leqslant i \leqslant j \leqslant a$, and for some $0 \leqslant s \leqslant x_i - 1$ and $0 \leqslant t \leqslant x_j - 1$, we have

$$S = \Bigg(\underbrace{\left\lfloor \frac{n}{k} \right\rfloor, \dots, \left\lfloor \frac{n}{k} \right\rfloor}_{s}, \left\lceil \frac{n}{k} \right\rceil, \underbrace{\left\lfloor \frac{n}{k} \right\rfloor, \dots, \left\lfloor \frac{n}{k} \right\rfloor}_{x_{i+1}-1}, \left\lceil \frac{n}{k} \right\rceil, \dots, \underbrace{\left\lfloor \frac{n}{k} \right\rfloor, \dots, \left\lfloor \frac{n}{k} \right\rfloor}_{x_{j-1}-1}, \left\lceil \frac{n}{k} \right\rceil, \underbrace{\left\lfloor \frac{n}{k} \right\rfloor, \dots, \left\lfloor \frac{n}{k} \right\rfloor}_{t} \Bigg).$$

It remains to prove that $\lfloor \frac{mn}{k} \rfloor \leqslant \sum S \leqslant \lceil \frac{mn}{k} \rceil$.

We first prove that $\sum S \geqslant \lfloor \frac{mn}{k} \rfloor$. We can assume the worst case for $\sum S$ to be minimal, which is when $s = x_i - 1$ and $t = x_j - 1$. Thus by induction,

$$m + 1 = \sum_{\alpha=i}^{j} x_\alpha \leqslant \left\lceil \frac{(j-i+1)k}{a} \right\rceil.$$

Hence

$$\frac{am}{k} \leqslant \frac{a}{k} \left\lceil \frac{(j-i+1)k}{a} \right\rceil - \frac{a}{k} \leqslant \frac{a}{k} \left(\frac{(j-i+1)k + a - 1}{a} \right) - \frac{a}{k} = j - i + 1 - \frac{1}{k}.$$

Thus $\lfloor \frac{am}{k} \rfloor \leqslant j - i$ and

$$\sum S = m \left\lfloor \frac{n}{k} \right\rfloor + j - i \geqslant m \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{am}{k} \right\rfloor = \left\lfloor m \left\lfloor \frac{n}{k} \right\rfloor + \frac{am}{k} \right\rfloor = \left\lfloor \frac{m}{k} \left(k \left\lfloor \frac{n}{k} \right\rfloor + a \right) \right\rfloor = \left\lfloor \frac{mn}{k} \right\rfloor.$$

Now we prove that $\sum S \leqslant \lfloor \frac{mn}{k} \rfloor$. We can assume the worst case for $\sum S$ to be maximal, which is when $s=0$ and $t=0$. Thus by induction,

$$m-1=\sum_{\alpha=i+1}^{j-1} x_\alpha \geqslant \left\lfloor \frac{(j-i-1)k}{a} \right\rfloor.$$

Hence

$$\frac{am}{k} \geqslant \frac{a}{k}\left\lfloor \frac{(j-i-1)k}{a} \right\rfloor + \frac{a}{k} \geqslant \frac{a}{k}\left(\frac{(j-i-1)k-a+1}{a}\right) + \frac{a}{k} = j-i-1+\frac{1}{k}.$$

Thus $\lceil \frac{am}{k} \rceil \geqslant j-i$ and

$$\sum S = m\left\lfloor \frac{n}{k} \right\rfloor + j-i \leqslant m\left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{am}{k} \right\rceil = \left\lceil m\left\lfloor \frac{n}{k} \right\rfloor + \frac{am}{k} \right\rceil = \left\lceil \frac{m}{k}\left(k\left\lfloor \frac{n}{k} \right\rfloor + a\right) \right\rceil = \left\lceil \frac{mn}{k} \right\rceil. \qquad \square$$

4.3. Uniqueness

In this section we prove that there is a unique rhythm satisfying the conditions in Theorem 4.1. The following well-known number-theoretic lemmas will be useful. Two integers x and y are *inverses* modulo m if $xy \equiv 1 \pmod{m}$.

Lemma 4.2. *(See [82], page 55.) An integer x has an inverse modulo m if and only if x and m are relatively prime. Moreover, if x has an inverse modulo m, then it has an inverse* $y \in [1, m-1]$.

Lemma 4.3. *If x and m are relatively prime, then* $ix \not\equiv jx \pmod{m}$ *for all distinct* $i, j \in [0, m-1]$.

Proof. Suppose that $ix \equiv jx \pmod{m}$ for some $i, j \in [0, m-1]$. By Lemma 4.2, x has an inverse modulo m. Thus $i \equiv j \pmod{m}$, and $i = j$ because $i, j \in [0, m-1]$. $\square$

Lemma 4.4. *For all relatively prime integers n and k with* $2 \leqslant k \leqslant n$, *there is an integer* $\ell \in [1, k-1]$ *such that*:

(a) $\ell n \equiv 1 \pmod{k}$,
(b) $i\ell \not\equiv j\ell \pmod{k}$ *for all distinct* $i, j \in [0, k-1]$, *and*
(c) $i\lfloor \frac{\ell n}{k} \rfloor \not\equiv j\lfloor \frac{\ell n}{k} \rfloor \pmod{n}$ *for all distinct* $i, j \in [0, k-1]$.

Proof. By Lemma 4.2 with $x = n$ and $m = k$, n has an inverse ℓ modulo k. This proves (a). Thus k and ℓ are relative prime by Lemma 4.2 with $x = \ell$ and $m = k$. Hence (b) follows from Lemma 4.3. Let $t = \lfloor \frac{\ell n}{k} \rfloor$. Then $\ell n = kt + 1$. By Lemma 4.3 with $m = n$ and $x = t$ (and because $k \leqslant n$), to prove (c) it suffices to show that t and n are relatively prime. Let $g = \gcd(t, n)$. Thus $\ell \frac{n}{g} = k\frac{t}{g} + \frac{1}{g}$. Because $\frac{n}{g}$ and $\frac{t}{g}$ are integers, $\frac{1}{g}$ is an integer and $g = 1$. This proves (c). $\square$

The following theorem is the main result of this section.

Theorem 4.5. *For all integers n and k with* $2 \leqslant k \leqslant n$, *there is a unique rhythm with k onsets and timespan n that satisfies property* ($\star$), *up to a rotation.*

Proof. Let $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ be a k-onset rhythm that satisfies ($\star$). Recall that the index of an onset is taken modulo k, and that the value of an onset is taken modulo n. That is, $r_i = x$ means that $r_{i \bmod k} = x \bmod n$.

Let $g = \gcd(k, n)$. We consider three cases for the value of g.

Case 1. $g = k$: Because R satisfies property ($\star$) for $\ell = 1$, every ordered pair (r_i, r_{i+1}) has clockwise distance $\frac{n}{k}$. By a rotation of R we can assume that $r_0 = 0$. Thus $r_i = \frac{in}{k}$ for all $i \in [0, k-1]$. Hence R is uniquely determined in this case.

Case 2. $g = 1$ (see Fig. 7): By Lemma 4.4(a), there is an integer $\ell \in [1, k-1]$ such that $\ell n \equiv 1 \pmod{k}$. Thus $\ell n = (k-1)\lfloor \frac{\ell n}{k} \rfloor + \lceil \frac{\ell n}{k} \rceil$. Hence, of the k ordered pairs $(r_i, r_{i+\ell})$ of onsets, $k-1$ have clockwise distance $\lfloor \frac{\ell n}{k} \rfloor$ and one has clockwise distance $\lceil \frac{\ell n}{k} \rceil$. By a rotation of R we can assume that $r_0 = 0$ and $r_{k-\ell} = n - \lceil \frac{\ell n}{k} \rceil$. Thus $r_{i\ell} = i\lfloor \frac{\ell n}{k} \rfloor$ for all $i \in [0, k-1]$; that is, $r_{(i\ell) \bmod k} = (i\lfloor \frac{\ell n}{k} \rfloor) \bmod n$. By Lemma 4.4(b) and (c), this defines the k distinct onsets of R. Hence R is uniquely determined in this case.

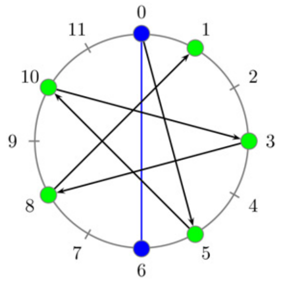

Fig. 7. Here we illustrate Case 2 with $n = 12$ and $k = 7$. Thus $\ell = 3$ because $3 \times 12 \equiv 1 \pmod 7$. We have $\lceil \frac{\ell n}{k} \rceil = 6$ and $\lfloor \frac{\ell n}{k} \rfloor = 5$. By a rotation we can assume that $r_0 = 0$ and $r_{k-\ell} = r_4 = 6$ (the darker dots). Then as shown by the arrows, the positions of the other onsets are implied.

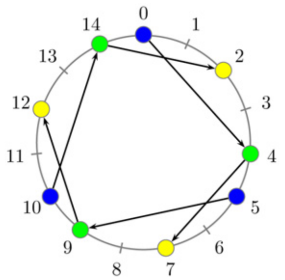

Fig. 8. Here we illustrate Case 3 with $n = 15$ and $k = 9$. Thus $g = 3$, $n' = 5$ and $k' = 3$. We have $\ell' = 2$ because $2 \times 5 \equiv 1 \pmod 3$. Thus $\lceil \frac{\ell' n'}{k'} \rceil = 4$ and $\lfloor \frac{\ell' n'}{k'} \rfloor = 3$. We have $L_0 = 0$, $L_1 = 4$ and $L_2 = 7$. A rotation fixes the first $g = 3$ onsets (the darker or blue dots). As shown by the arrows, these onsets imply the positions of the next three onsets (medium or green dots), which in turn imply the positions of the final three onsets (the light or yellow dots). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Case 3. $g \in [2, k-1]$ (see Fig. 8): Let $k' = \frac{k}{g}$ and let $n' = \frac{n}{g}$. Observe that both k' and n' are integers. Because R satisfies ($\star$) and $\lceil \frac{k'n}{k} \rceil = \lfloor \frac{k'n}{k} \rfloor = n'$, we have $\widehat{d}(r_i, r_{i+k'}) = n'$ for all $i \in [0, k-1]$. Thus

$$r_{ik'+j} = in' + r_j \tag{1}$$

for all $i \in [0, g-1]$ and $j \in [0, n'-1]$.

Now $\gcd(n', k') = 1$ by the maximality of g. By Lemma 4.4(a), there is an integer $\ell' \in [1, k'-1]$ such that $\ell' n' \equiv 1 \pmod{k'}$. Thus $\ell' n' = (k'-1)\lfloor \frac{\ell' n'}{k'} \rfloor + \lceil \frac{\ell' n'}{k'} \rceil$, implying $\ell' n = (k-g)\lfloor \frac{\ell' n'}{k'} \rfloor + g\lceil \frac{\ell' n'}{k'} \rceil$. Hence, of the k ordered pairs $(r_i, r_{i+\ell'})$ of onsets, $k-g$ have clockwise distance $\lfloor \frac{\ell' n'}{k'} \rfloor$ and g have clockwise distance $\lceil \frac{\ell' n'}{k'} \rceil$. By a rotation of R we can assume that $r_0 = 0$ and $r_{\ell'} = \lceil \frac{\ell' n'}{k'} \rceil$. By Eq. (1) with $j = 0$ and $j = \ell'$, we have

$$r_{ik'} = in' \quad \text{and} \quad r_{ik'+\ell'} = in' + \left\lceil \frac{\ell' n'}{k'} \right\rceil \tag{2}$$

for all $i \in [0, g-1]$. This accounts for the g ordered pairs $(r_i, r_{i+\ell'})$ with clockwise distance $\lceil \frac{\ell' n'}{k'} \rceil$. The other $k-g$ ordered pairs $(r_i, r_{i+\ell'})$ have clockwise distance $\lfloor \frac{\ell' n'}{k'} \rfloor$. Define

$$L_0 = 0 \quad \text{and} \quad L_j = \left\lceil \frac{\ell' n'}{k'} \right\rceil + (j-1)\left\lfloor \frac{\ell' n'}{k'} \right\rfloor \quad \text{for all } j \in [1, k'-1].$$

Thus by Eq. (2),

$$r_{ik'+j\ell'} = in' + L_j$$

for all $i \in [0, g-1]$ and $j \in [0, k'-1]$; that is, $r_{(ik'+j\ell') \bmod k} = (in' + L_j) \bmod n$.

To conclude that R is uniquely determined, we must show that over the range $i \in [0, g-1]$ and $j \in [0, k'-1]$, the numbers $ik' + j\ell'$ are distinct modulo k, and the numbers $in' + L_j$ are distinct modulo n.

First we show that the numbers $ik' + j\ell'$ are distinct modulo k. Suppose that

$$ik' + j\ell' \equiv pk' + j\ell' \pmod k \tag{3}$$

for some $i, p \in [0, g-1]$ and $j, q \in [0, k'-1]$. Because $k = k' \cdot g$, we can write $(ik' + j\ell') \bmod k$ as a multiple of k' plus a residue modulo k'. In particular,

$$(ik' + j\ell') \bmod k = k'\left(\left(i + \left\lfloor \frac{j\ell'}{k'} \right\rfloor\right) \bmod g\right) + (j\ell' \bmod k').$$

Thus Eq. (3) implies that

$$k'\left(\left(i+\left\lfloor\frac{j\ell'}{k'}\right\rfloor\right) \bmod g\right)+(j\ell' \bmod k')=k'\left(\left(p+\left\lfloor\frac{q\ell'}{k'}\right\rfloor\right) \bmod g\right)+(q\ell' \bmod k'). \tag{4}$$

Hence $j\ell' \equiv q\ell' \pmod{k'}$. Thus $j=q$ by Lemma 4.4(c). By substituting $j=q$ into Eq. (4), it follows that $i \equiv p \pmod g$. Thus $i=p$ because $i, p \in [0, g-1]$. This proves that the numbers $ik'+j\ell'$ are distinct modulo k.

Now we show that the numbers $in'+L_j$ are distinct modulo n. The proof is similar to the above proof that the numbers $ik'+j\ell'$ are distinct modulo k.

Suppose that

$$in'+L_j \equiv pn'+L_q \pmod n \tag{5}$$

for some $i, p \in [0, g-1]$ and $j, q \in [0, k'-1]$. Because $n=n'\cdot g$, we can write $(in'+L_j) \bmod n$ as a multiple of n' plus a residue modulo n'. In particular,

$$(in'+L_j) \bmod n=n'\left(\left(i+\left\lfloor\frac{L_j}{n'}\right\rfloor\right) \bmod g\right)+(L_j \bmod n').$$

Thus Eq. (5) implies that

$$n'\left(\left(i+\left\lfloor\frac{L_j}{n'}\right\rfloor\right) \bmod g\right)+(L_j \bmod n')=n'\left(\left(p+\left\lfloor\frac{L_q}{n'}\right\rfloor\right) \bmod g\right)+(L_q \bmod n'). \tag{6}$$

Hence $L_j \equiv L_q \pmod{n'}$. We claim that $j=q$. If $j=0$ then $L_j=0$, implying $L_q=0$ and $q=0$. Now assume that $j, q \geqslant 1$. In this case, $L_j=j\lfloor\frac{\ell'n'}{k'}\rfloor+1$ and $L_q=q\lfloor\frac{\ell'n'}{k'}\rfloor+1$. Thus

$$j\left\lfloor\frac{\ell'n'}{k'}\right\rfloor \equiv q\left\lfloor\frac{\ell'n'}{k'}\right\rfloor \pmod{k'}.$$

Hence $j=q$ by Lemma 4.4(c). By substituting $j=q$ into Eq. (6), it follows that $i \equiv p \pmod g$. Thus $i=p$ because $i, p \in [0, g-1]$. This proves that the numbers $in'+L_j$ are distinct modulo n.

Therefore R is uniquely determined. □

We have shown that each of the three algorithms generates a rhythm with property (⋆), and that there is a unique rhythm with property (⋆). Thus all of the algorithms produce the same rhythm, up to rotation. It remains to prove that this rhythm has maximum evenness.

4.4. Rhythms with maximum evenness

We start with a technical lemma. Let v, w be points at geodesic distance d on a circle C. Obviously $\bar{d}(v, w)$ is a function of d, independent of v and w. Let $f(C, d)=\bar{d}(v, w)$.

Lemma 4.6. *For all geodesic lengths $x \leqslant d$ on a circle C, we have $f(C, x)+f(C, d-x) \leqslant 2\cdot f(C, \frac{d}{2})$, with equality only if $d=2x$.*

Proof. We can assume that C is a unit circle. Consider the isosceles triangle formed by the center of C and a geodesic of length d ($\leqslant \pi$). We have $\frac{1}{2}f(C, d)=\sin\frac{d}{2}$. Thus $f(C, d)=2\sin\frac{d}{2}$. Thus our claim is equivalent to $\sin x+\sin(d-x) \leqslant 2\sin\frac{d}{2}$ for all $x \leqslant d$ ($\leqslant \pi/2$). In the range $0 \leqslant x \leqslant d$, $\sin x$ is increasing, and $\sin(d-x)$ is decreasing at the opposite rate. Thus $\sin x+\sin(d-x)$ is maximized when $x=d-x$. That is, when $d=2x$. The result follows. □

For a rhythm $R=\{r_0, r_1, \ldots, r_{k-1}\}_n$, for each $\ell \in [1, k]$, let $S(R, \ell)$ be the sum of chordal distances taken over all ordered pairs $(r_i, r_{i+\ell})$ in R. That is, let $S(R, \ell)=\sum_{i=0}^{k-1}\bar{d}(r_i, r_{i+\ell})$. Property (A) says that R maximizes $\sum_{\ell=1}^{k}S(R, \ell)$. Before we characterize rhythms that maximize the sum of $S(R, \ell)$, we first concentrate on rhythms that maximize $S(R, \ell)$ for each particular value of ℓ. Let $D(R, \ell)$ be the multiset of clockwise distances $\{\widehat{d}(r_i, r_{i+\ell}):\ i \in [0, k-1]\}$. Then $S(R, \ell)$ is determined by $D(R, \ell)$. In particular, $S(R, \ell)=\sum\{f(C_n, d):\ d \in D(R, \ell)\}$ (where $\{f(C_n, d):\ d \in D(R, \ell)\}$ is a multiset).

Lemma 4.7. *Let $1 \leqslant \ell \leqslant k \leqslant n$ be integers. A k-onset rhythm $R=\{r_0, r_1, \ldots, r_{k-1}\}_n$ maximizes $S(R, \ell)$ if and only if $|\widehat{d}(r_i, r_{i+\ell})-\widehat{d}(r_j, r_{j+\ell})| \leqslant 1$ for all $i, j \in [0, k-1]$.*

Proof. Suppose that $R=\{r_0, r_1, \ldots, r_{k-1}\}_n$ maximizes $S(R, \ell)$. Let $d_i=\widehat{d}(r_i, r_{i+\ell})$ for all $i \in [0, k-1]$. Suppose on the contrary that $d_p \geqslant d_q+2$ for some $p, q \in [0, k-1]$. We can assume that $q<p$, $d_p=d_q+2$, and $d_i=d_q+1$ for all $i \in [q+1, p-1]$. Define $r'_i=r_i+1$ for all $i \in [q+1, p]$, and define $r'_i=r_i$ for all other i. Let R' be the rhythm $\{r'_0, r'_1, \ldots, r'_{k-1}\}_n$. Thus $D(R, \ell)\setminus D(R', \ell)=\{d_p, d_q\}$ and $D(R', \ell)\setminus D(R, \ell)=\{d_p-1, d_q+1\}$. Now $d_p-1=d_q+1=\frac{1}{2}(d_p+d_q)$. By Lemma 4.6, $f(C_n, d_p)+f(C_n, d_q)<2\cdot f(C_n, \frac{1}{2}(d_p+d_q))$. Thus $S(R, \ell)<S(R', \ell)$, which contradicts the maximality of $S(R, \ell)$.

For the converse, let R be a rhythm such that $|\widehat{d}(r_i, r_{i+\ell}) - \widehat{d}(r_j, r_{j+\ell})| \leqslant 1$ for all $i, j \in [0, k-1]$. Suppose on the contrary that R does not maximize $S(R, \ell)$. Thus some rhythm $T = (t_0, t_1, \ldots, t_{k-1})$ maximizes $S(T, \ell)$ and $T \neq R$. Hence $D(T, \ell) \neq D(R, \ell)$. Because $\sum D(R, \ell) = \sum D(T, \ell)$ $(= \ell n)$, we have $\widehat{d}(t_i, t_{i+\ell}) - \widehat{d}(t_j, t_{j+\ell}) \geqslant 2$ for some $i, j \in [0, k-1]$. As we have already proved, this implies that T does not maximize $S(T, \ell)$. This contradiction proves that R maximizes $S(R, \ell)$. □

Because $\sum_{i=0}^{k-1} \widehat{d}(r_i, r_{i+\ell}) = \ell n$ for any rhythm with k onsets and timespan n, Lemma 4.7 can be restated as follows.

Corollary 4.8. *Let $1 \leqslant \ell \leqslant k \leqslant n$ be integers. A k-onset rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ maximizes $S(R, \ell)$ if and only if $\widehat{d}(r_i, r_{i+\ell}) \in \{\lceil \frac{\ell n}{k} \rceil, \lfloor \frac{\ell n}{k} \rfloor\}$ for all $i \in [0, k-1]$.* □

Proof $(\star) \Rightarrow$ **(A).** If $(\star)$ holds for some rhythm R, then by Corollary 4.8, R maximizes $S(R, \ell)$ for *every* ℓ. Thus R maximizes $\sum_\ell S(R, \ell)$. □

Proof (A) $\Rightarrow (\star)$**.** By Theorem 4.5, there is a unique rhythm R that satisfies property $(\star)$. Let R denote the unique rhythm that satisfies property $(\star)$. Suppose on the contrary that there is a rhythm $T = (t_0, t_1, \ldots, t_{k-1})$ with property (A) but $R \neq T$. Thus there exists an ordered pair $(t_i, t_{i+\ell})$ in T with clockwise distance $\widehat{d}(t_i, t_{i+\ell}) \notin \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$. By Corollary 4.8, $S(T, \ell) < S(R, \ell)$. Because T has property (A), $\sum_{\ell=1}^{k} S(T, \ell) \geqslant \sum_{\ell=1}^{k} S(R, \ell)$. Thus for some ℓ' we have $S(T, \ell') > S(R, \ell')$. But this is a contradiction, because $S(R, \ell') \geqslant S(T, \ell')$ by Corollary 4.8. □

This completes the proof of Theorem 4.1. We now show that Theorem 4.1 can be generalized for other metrics that satisfy Lemma 4.6. To formalize this idea we introduce the following definition. A function $g : [0, \pi] \to \mathbb{R}^+ \cup \{0\}$ is *halving* if for all geodesic lengths $x \leqslant d \leqslant \pi$ on the unit circle, $g(x) + g(d - x) \leqslant 2 \cdot g(\frac{d}{2})$, with equality only if $d = 2x$. For example, chord length is halving, but geodesic distance is not (because we have equality for all x). Observe that the proof of Lemma 4.7 and Corollary 4.8 depend on this property alone. Thus we have the following generalization of Theorem 4.1.

Theorem 4.9. *Let $n \geqslant k \geqslant 2$ be integers. Let g be a halving function. The following are equivalent for a rhythm $R = (r_0, r_1, \ldots, r_{k-1})$ with n pulses and k onsets*:

(A) *R maximizes $\sum_{i=0}^{k-1} \sum_{j=i+1}^{k-1} g(\widehat{d}(r_i, r_j))$,*
(B) *R is determined by the* Clough-Douthett(k, n) *algorithm,*
(C) *R is determined by the* Snap(k, n) *algorithm,*
(D) *R is determined by the* Euclidean(k, n) *algorithm,*
$(\star)$ *for all $\ell \in [1, k]$ and $i \in [0, k-1]$, the ordered pair $(r_i, r_{i+\ell})$ has clockwise distance $d(r_i, r_{i+\ell}) \in \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$.*

Moreover, up to a rotation, there is a unique rhythm that satisfies these conditions.

5. Deep rhythms

Recall that a rhythm is Winograd-deep if every geodesic distance $1, 2, \ldots, \lfloor n/2 \rfloor$ has a unique multiplicity; it is Erdős-deep if the multiplicity of every geodesic distance defined by pairs of onsets in unique. Winograd [94], and independently Clough et al. [23], characterize all Winograd-deep scales: up to rotation, they are the scales that can be generated by the first $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ multiples (modulo n) of a value that is relatively prime to n, plus one exceptional scale $\{0, 1, 2, 4\}_6$. In this section, we prove a similar (but more general) characterization of Erdős-deep rhythms: up to rotation and scaling, they are the rhythms generable as the first k multiples (modulo n) of a value that is relatively prime to n, plus the same exceptional rhythm $\{0, 1, 2, 4\}_6$. The key difference is that the number of onsets k is now a free parameter, instead of being forced to be either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$. Our proof follows Winograd's, but differs in one case (the second case of Theorem 5.3).

We later prove that every Erdős-deep rhythm has a shelling and that maximally even rhythms with n and k relatively prime are Erdős-deep.

5.1. Characterization of deep rhythms

Our characterization of Erdős-deep rhythms is in terms of two families of rhythms. The main rhythm family consists of the generated rhythms $D_{k,n,m} = \{im \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$ of timespan n, for certain values of k, n, and m. The one exceptional rhythm is $F = \{0, 1, 2, 4\}_6$ of timespan 6.

Fact 5.1. *F is Erdős-deep.*

Lemma 5.2. *If $k \leqslant \lfloor n/2 \rfloor + 1$ and m and n are relatively prime, then $D_{k,n,m}$ is Erdős-deep.*

Proof. The multiset of clockwise distances in $D_{k,n,m}$ is $\{(jm - im) \bmod n\colon\ i < j\} = \{(j-i)m \bmod n\colon\ i < j\}$. There are $k - p$ choices of i and j such that $j - i = p$, so there are exactly p occurrences of the clockwise distance $(pm) \bmod n$ in the multiset. Each of these clockwise distances corresponds to a geodesic distance—either $(pm) \bmod n$ or $(-pm) \bmod n$, whichever is smaller (at most $n/2$). We claim that these geodesic distances are all distinct. Then the multiplicity of each geodesic distance $(\pm pm) \bmod n$ is exactly p, establishing that the rhythm is Erdős-deep.

For two geodesic distances to be equal, we must have $\pm pm \equiv \pm qm \pmod n$ for some (possibly different) choices for the $\pm$ symbols, and for some $p \neq q$. By (possibly) multiplying both sides by -1, we obtain two cases: (1) $pm \equiv qm \pmod n$ and (2) $pm \equiv -qm \pmod n$. Because m is relatively prime to n, by Lemma 4.2, m has a multiplicative inverse modulo n. Multiplying both sides of the congruence by this inverse, we obtain (1) $p \equiv q \pmod n$ and (2) $p \equiv -q \pmod n$. Because $0 \leqslant i < j < k \leqslant \lfloor n/2 \rfloor + 1$, we have $0 \leqslant p = j - i < \lfloor n/2 \rfloor + 1$, and similarly for q: $0 \leqslant p, q \leqslant \lfloor n/2 \rfloor$. Thus, the first case of $p \equiv q \pmod n$ can happen only when $p = q$, and the second case of $p + q \equiv 0 \pmod n$ can happen only when $p = q = 0$ or when $p = q = n/2$. Either case contradicts that $p \neq q$. Therefore the geodesic distances arising from different values of p are indeed distinct, proving the lemma. □

We now state and prove our characterization of Erdős-deep rhythms, which is up to rotation and scaling. Rotation preserves the geodesic distance multiset and therefore Erdős-deepness (and Winograd-deepness). Scaling maps each geodesic distance d to αd, and thus preserves multiplicities and therefore Erdős-deepness (but not Winograd-deepness). Note that the rhythm $D_{k,n,m}$ is a rotation by $-m(k-1) \bmod n$ of the rhythm $D_{k,n,n-m}$; to avoid this duplication we restrict m to be equal to at most $\lfloor n/2 \rfloor$.

Theorem 5.3. *A rhythm is Erdős-deep if and only if it is a rotation of a scaling of either the rhythm F or the rhythm $D_{k,n,m}$ for some k, n, m with $k \leqslant \lfloor n/2 \rfloor + 1$, $1 \leqslant m \leqslant \lfloor n/2 \rfloor$, and m and n are relatively prime.*

Proof. Because a rotation of a scaling of an Erdős-deep rhythm is Erdős-deep, the "if" direction of the theorem follows from Fact 5.1 and Lemma 5.2.

Consider an Erdős-deep rhythm R with k onsets. By the definition of Erdős-deepness, R has one nonzero geodesic distance with multiplicity i for each $i = 1, 2, \ldots, k-1$. Let m be the geodesic distance with multiplicity $k - 1$. Because m is a geodesic distance, $1 \leqslant m \leqslant \lfloor n/2 \rfloor$. Also, $k \leqslant \lfloor n/2 \rfloor + 1$ (for any Erdős-deep rhythm R), because all nonzero geodesic distances are between 1 and $\lfloor n/2 \rfloor$ and therefore at most $\lfloor n/2 \rfloor$ nonzero geodesic distances occur. Thus k and m are suitable parameter choices for $D_{k,n,m}$.

Consider the graph $G_m = (R, E_m)$ with vertices corresponding to onsets in R and with an edge between two onsets of geodesic distance m. By the definition of geodesic distance, every vertex i in G_m has degree at most 2: the only onsets at geodesic distance exactly m from i are $(i - m) \bmod n$ and $(i + m) \bmod n$. Thus, the graph G_m is a disjoint union of paths and cycles. The number of edges in G_m is the multiplicity of m, which we supposed was $k - 1$, which is 1 less than the number of vertices in G_m. Thus, the graph G_m consists of exactly one path and any number of cycles.

The cycles of G_m have a special structure because they correspond to subgroups generated by single elements in the cyclic group $(\mathbb{Z}/(n), +)$. Namely, the onsets corresponding to vertices of a cycle in G_m form a regular (n/a)-gon, with a geodesic distance of $a = \gcd(m, n)$ between consecutive onsets. (a is called the index of the subgroup generated by m.) In particular, every cycle in G_m has the same length $r = n/a$. Because G_m is a simple graph, every cycle must have at least 3 vertices, so $r \geqslant 3$.

The proof partitions into four cases depending on the length of the path and on how many cycles the graph G_m has. The first two cases will turn out to be impossible; the third case will lead to a rotation of a scaling of rhythm F; and the fourth case will lead to a rotation of a scaling of rhythm $D_{k,n,m}$.

First suppose that the graph G_m consists of a path of length at least 1 and at least one cycle. We show that this case is impossible because the rhythm R can have no geodesic distance with multiplicity 1. Suppose that there is a geodesic distance with multiplicity 1, say between onsets i_1 and i_2. If i is a vertex of a cycle, then both $(i + m) \bmod n$ and $(i - m) \bmod n$ are onsets in R. If i is a vertex of the path, then one or two of these are onsets in R, with the case of one occurring only at the endpoints of the path. If $(i_1 + m) \bmod n$ and $(i_2 + m) \bmod n$ were both onsets in R, or $(i_1 - m) \bmod n$ and $(i_2 - m) \bmod n$ were both onsets in R, then we would have another occurrence of the geodesic distance between i_1 and i_2, contradicting that this geodesic distance has multiplicity 1. Thus, i_1 and i_2 must be opposite endpoints of the path. If the path has length ℓ, then the clockwise distance between i_1 and i_2 is $(\ell m) \bmod n$. This clockwise distance (and hence the corresponding geodesic distance) appears in every cycle, of which there is at least one, so the geodesic distance has multiplicity more than 1, a contradiction. Therefore this case is impossible.

Second suppose that the graph G_m consists of a path of length 0 and at least two cycles. We show that this case is impossible because the rhythm R has two geodesic distances with the same multiplicity. Pick any two cycles C and C', and let d be the smallest positive clockwise distance from a vertex of C to a vertex of C'. Thus i is a vertex of C if and only if $(i + d) \bmod n$ is a vertex of C'. Because the cycles are disjoint, $d < a$. Because $r \geqslant 3$, $d < n/3$, so clockwise distances of d are also geodesic distances of d. The number of occurrences of geodesic distance d between a vertex of C and a vertex of

C' is either r or $2r$, the case of $2r$ arising when $d = a/2$ (that is, C' is a "half-rotation" of C). The number of occurrences of geodesic distance $d' = \min\{d+m, n-(d+m)\}$ is the same—either r or $2r$, in the same cases. (Note that $d < a \leqslant n-m$, so $d+m < n$, so the definition of d' correctly captures a geodesic distance modulo n.) The same is true of geodesic distance $d'' = \min\{d-m, n-(d-m)\}$. If other pairs of cycles have the same smallest positive clockwise distance d, then the number of occurrences of d, d', and d'' between those cycles are also equal. Because the cycles are disjoint, geodesic distance d and thus $d+m$ and $d-m$ cannot be $(pm) \bmod n$ for any p, so these geodesic distances cannot occur between two vertices of the same cycle. Finally, the sole vertex x of the path has geodesic distance d to onset i (which must be a vertex of some cycle) if and only if x has geodesic distance d' to onset $(i+m) \bmod n$ (which must be a vertex of the same cycle) if and only if x has geodesic distance d'' to onset $(i-m) \bmod n$ (which also must be a vertex of the same cycle). Therefore the multiplicities of geodesic distances d, d', and d'' must be equal. Because R is Erdős-deep, we must have $d = d' = d''$. To have $d = d'$, either $d = d+m$ or $d = n-(d+m)$, but the first case is impossible because $d > 0$ by nonoverlap of cycles, so $2d+m = n$. Similarly, to have $d = d''$, we must have $2d-m = n$. Subtracting these two equations, we obtain that $2m = 0$, contradicting that $m > 0$. Therefore this case is also impossible.

Third suppose that the graph G_m consists of a path of length 0 and exactly one cycle. We show that this case forces R to be a rotation of a scaling of rhythm F because otherwise two geodesic distances m and m' have the same multiplicity. The number of occurrences of geodesic distance m in the cycle is precisely the length r of the cycle. Similarly, the number of occurrences of geodesic distance $m' = \min\{2m, n-2m\}$ in the cycle is r. The sole vertex x on the path cannot have geodesic distance m or m' to any other onset (a vertex of the cycle) because then x would then be on the cycle. Therefore the multiplicities of geodesic distances m and m' must be equal. Because R is Erdős-deep, m must equal m', which implies that either $m = 2m$ or $m = n-2m$. The first case is impossible because $m > 0$. In the second case, $3m = n$, that is, $m = \frac{1}{3}n$. Therefore, the cycle has $r = 3$ vertices, say at $\Delta, \Delta + \frac{1}{3}n, \Delta + \frac{2}{3}n$. The fourth and final onset x must be midway between two of these three onsets, because otherwise its geodesic distance to the three vertices are all distinct and therefore unique. No matter where x is so placed, the rhythm R is a rotation by $\Delta + c\frac{1}{3}n$ (for some $c \in \{0,1,2\}$) of a scaling by $n/6$ of the rhythm F.

Finally suppose that G_m has no cycles, and consists solely of a path. We show that this case forces R to be a rotation of a scaling of a rhythm $D_{k,n',m'}$ with $1 \leqslant m' \leqslant \lfloor n'/2 \rfloor$ and with m' and n' relatively prime. Let i be the onset such that $(i-m) \bmod n$ is not an onset (the "beginning" vertex of the path). Consider rotating R by $-i$ so that 0 is an onset in the resulting rhythm $R-i$. The vertices of the path in $R-i$ form a subset of the subgroup of the cyclic group $(\mathbb{Z}/(n), +)$ generated by the element m. Therefore the rhythm $R - i = D_{k,n,m} = \{(im) \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$ is a scaling by a of the rhythm $D_{k,n/a,m/a} = \{(im/a) \bmod (n/a)\colon\ i = 0, 1, \ldots, k-1\}_n$. The rhythm $D_{k,n/a,m/a}$ has an appropriate value for the third argument: m/a and n/a are relatively prime ($a = \gcd(m,n)$) and $1 \leqslant m/a \leqslant \lfloor n/2 \rfloor / a \leqslant \lfloor (n/a)/2 \rfloor$. Also, $k \leqslant \lfloor (n/a)/2 \rfloor + 1$ because the only occurring geodesic distances are multiples of a and therefore the number $k-1$ of distinct geodesic distances is at most $\lfloor (n/a)/2 \rfloor$. Therefore R is a rotation by i of a scaling by a of $D_{k,n/a,m/a}$ with appropriate values of the arguments. □

Corollary 5.4. *A rhythm is Erdős-deep if and only if it is a rotation of a scaling of the rhythm F or it is a rotation of a rhythm $D_{k,n,m}$ for some k, n, m satisfying $k \leqslant \lfloor n/2g \rfloor + 1$ where $g = \gcd(m,n)$.*

Proof. First we show that any Erdős-deep rhythm has one of the two forms in the corollary. By Theorem 5.3, there are two flavors of Erdős-deep rhythms, and the corollary directly handles rotations of scalings of F. Thus it suffices to consider a rhythm R that is a rotation by Δ of a scaling by α of $D_{k,n,m}$ where $k \leqslant \lfloor n/2 \rfloor + 1$, $1 \leqslant m \leqslant \lfloor n/2 \rfloor$, and m and n are relatively prime. Equivalently, R is a rotation by Δ of $D_{k,n',m'}$ where $n' = \alpha n$ and $m' = \alpha m$. Now $g = \gcd(n', m') = \alpha$, so $n'/g = n$. Hence, $k \leqslant \lfloor n'/2g \rfloor + 1$ as desired. Thus we have rewritten R in the desired form.

It remains to show that every rhythm in one of the two forms in the corollary is Erdős-deep. Again, rotations of scalings of F are handled directly by Theorem 5.3. So consider a rotation of $D_{k,n,m}$ where $k \leqslant \lfloor n/2g \rfloor + 1$. The value of m matters only modulo n, so we assume that $0 \leqslant m \leqslant n-1$.

First we show that, if $\lfloor n/2 \rfloor + 1 \leqslant m \leqslant n-1$, then $D_{k,n,m}$ can be rewritten as a rotation of the rhythm $D_{k,n,m'}$ where $m' = n - m \leqslant \lfloor n/2 \rfloor$. By reversing the order in which we list the onsets in $D_{k,n,m} = \{im \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$, we can write $D_{k,n,m} = \{(k-1-i)m \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$. Now consider rotating the rhythm $D_{k,n,n-m} = \{i(n-m) \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$ by $(k-1)m$. We obtain the rhythm $\{[i(n-m) + (k-1)m] \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n = \{[(k-1-i)m + in] \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n = \{(k-1-i)m \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n = D_{k,n,m}$ as desired.

Thus it suffices to consider rotations of $D_{k,n,m}$ where $1 \leqslant m \leqslant \lfloor n/2 \rfloor$ and $k \leqslant \lfloor n/2g \rfloor$. The rhythm $D_{k,n',m'}$, where $n' = n/g$ and $m' = m/g$, is Erdős-deep by Theorem 5.3 because n' and m' are relatively prime, $k \leqslant \lfloor n'/2 \rfloor + 1$, and $1 \leqslant m' \leqslant \lfloor n'/2 \rfloor$. But $D_{k,n,m}$ is the scaling of $D_{k,n',m'}$ by the integer g, so $D_{k,n,m}$ is also Erdős-deep. □

An interesting consequence of this characterization is the following:

Corollary 5.5. *Every Erdős-deep rhythm has a shelling.*

Proof. If the Erdős-deep rhythm is $D_{k,n,m}$, we can remove the last onset from the path, resulting in $D_{k-1,n,m}$, and repeat until we obtain the empty rhythm $D_{0,n,m}$. At all times, k remains at most $\lfloor n/2 \rfloor + 1$ (assuming it was originally) and m

remains between 1 and $\lfloor n/2 \rfloor$ and relatively prime to n. On the other hand, $F = \{0, 1, 2, 4\}_6$ has the shelling 4, 2, 1, 0 because $\{0, 1, 2\}_6$ is Erdős-deep. □

We can generalize this characterization of Erdős-deep rhythms to the continuous case where n is an arbitrary real number, and onsets can be at arbitrary (not necessarily integer) points along the circle. We will call such rhythms *continuous rhythms*. In this case we have two kinds of rhythms. First, if m and n are rational multiples of each other, we can scale the rhythm by some rational p such that pm and pn are integers, and apply Theorem 5.3 using pn and pm to characterize all deep rhythms where m is a rational multiple of n. Second, if m and n are irrational multiples of each other, we can show that every $D_{k,n,m}$ is Erdős-deep. The complete characterization of continuous Erdős-deep rhythms is as follows:

Theorem 5.6. *A continuous rhythm is Erdős-deep if and only if it is a rotation of a scaling of $D_{k,n,m}$ with $k \leqslant \lfloor n/2 \rfloor + 1$, $0 < m \leqslant n/2$, and where m and n are either* (1) *irrational multiples of each other, or* (2) *rational multiples such that for some rational p, integers pm and pn are relatively prime.*

Proof. To prove the "if" direction, we show that all geodesic distances defined by $D_{k,n,m}$ are distinct; hence we need to prove that the multiplicity of each geodesic distance $(\pm pm) \bmod n$ is exactly p. First assume that m and n are irrational multiples of each other, i.e., there is no rational number that divides both m and n. Suppose two geodesic distances $\pm pm \equiv \pm qm \pmod n$ for some (possibly different) choices for the $\pm$ symbols, and for some $p \neq q$. Then we can write $\pm pm = \pm qm + rn$ for some integer r. This in turn implies that $m = \frac{r}{\pm p \mp q} n$, which contradicts the fact that m and n are irrational multiples of each other. Therefore, when m and n are irrational multiples of each other, the geodesic distances arising from different values of p are distinct, proving that $D_{k,n,m}$ is Erdős-deep.

If m and n are rational multiples of each other, then so are each of the geodesic distances $2m, 3m, \ldots, (k-1)m \pmod n$ with n. In this case, there exists a rational p such that pn and pm are both integers. We can now apply Theorem 5.3 using pn and pm, and generate all deep rhythms where m is a rational multiple of n.

For the "only if" direction, consider a continuous Erdős-deep rhythm R with k onsets and period n, and with some geodesic distance m having multiplicity $k-1$. Consider the graph $G_m = (R, E_m)$ as defined in the proof of Theorem 5.3 (with vertices corresponding to onsets in R and with an edge between two onsets of geodesic distance m). If m and n are rational multiples of each other, then we can scale R by some rational p and apply Theorem 5.3 to show that R is a scaling by $1/p$ of $D_{k,pn,pm}$ where pm and pn are relatively prime integers and $1 \leqslant pm \leqslant \lfloor pn/2 \rfloor$, so $0 < m \leqslant n/2$.

If m and n are irrational multiples of each other, then there is no rational number r such that $n = rm$. This means that G_m cannot contain a cycle, so consists of a single path of length $k-1$. As in the proof of Theorem 5.3, we can rotate R by $-i$ so that 0 is an onset in the resulting rhythm $R - i$. The vertices of the path in $R - i$ form a subset of the subgroup of the cyclic group $(\mathbb{Z}/(n), +)$ generated by the element m. Therefore the rhythm $R - i = D_{k,n,m} = \{(im) \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$ where m and n are irrational multiples of each other and $0 < m \leqslant n/2$. □

5.2. Connection between deep and even rhythms

A connection between maximally even scales and Winograd-deep scales is shown by Clough et al. [23]. They define a *diatonic scale* to be a maximally even scale with $k = (n+2)/2$ and n a multiple of 4. They show that diatonic scales are Winograd-deep. We now prove a similar result for Erdős-deep rhythms.

Lemma 5.7. *A rhythm R of maximum evenness satisfying $k \leqslant \lfloor n/2 \rfloor + 1$ is Erdős-deep if and only if k and n are relatively prime.*

Proof. Recall that by property $(\star)$ one of the unique characterizations of an even rhythm of maximum evenness can be stated as follows. For all $1 \leqslant \ell \leqslant k$, and for every ordered pair $(r_i, r_{i+\ell})$ of onsets in R, the clockwise distance $\widehat{d}(r_i, r_{i+\ell}) \in \{\lfloor \frac{\ell n}{k} \rfloor, \lceil \frac{\ell n}{k} \rceil\}$.

For the case in which k and n are relatively prime, by Lemma 4.2, there exists a value $\ell < k$ such that $\ell n \equiv 1 \pmod k$. Thus we can write $\ell n = k\lfloor \ell n/k \rfloor + 1$. Let $m = \lfloor \ell n/k \rfloor$. Now consider the set $\{im \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$. By Lemma 4.4(c), we get k distinct values, so R can be realized as $D_{k,n,m} = \{im \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$. Thus, by Lemma 5.2, R is Erdős-deep.

Observe that $F = \{0, 1, 2, 4\}_6$ does not maximize evenness because $\widehat{d}(0, 2) = 2$ and $\widehat{d}(2, 0) = 4$ yet $\ell = 2$. Hence, any rhythm that maximizes evenness and that is deep must also be generated.

Now consider the case in which n and k are not relatively prime. We show that the assumption that R is deep leads to a contradiction. Thus, assuming that R is deep implies that there is a value m such that R can be realized as $D_{k,n,m} = \{im \bmod n\colon\ i = 0, 1, \ldots, k-1\}_n$. This in turn implies that there exists an integer ℓ such that $km = \ell n + 1$, that is, $\ell n \equiv 1 \pmod k$. However, for this to happen, n and k must be relatively prime, a contradiction.

Thus we have shown that R is Erdős-deep if and only if k and n are relatively prime. □

Acknowledgements

This work was initiated at the 20th Bellairs Winter Workshop on Computational Geometry held January 28–February 4, 2005 at Bellairs Research Institute in Barbados. We thank the other participants of that workshop—Greg Aloupis, David Bremner, Justin Colannino, Mirela Damian, Vida Dujmović, Jeff Erickson, Ferran Hurtado, John Iacono, Danny Krizanc, Stefan Langerman, Erin McLeish, Pat Morin, Mark Overmars, Suneeta Ramaswami, Diane Souvaine, Ileana Streinu, Remco Veltcamp, and Sue Whitesides—for helpful discussions and contributing to a fun and creative atmosphere.

Thanks to Simha Arom for providing us with a copy of his paper on the classification of *aksak* rhythms [4], which provided several examples of Euclidean rhythms of which we were not aware. Thanks to Jeff Erickson for bringing to our attention the wonderful paper by Mitchell Harris and Ed Reingold [51] on digital line drawing, leap year calculations, and Euclid's algorithm. Finally, thanks to Marcia Ascher for pointing out that the leap-year pattern of the Jewish calendar is a Euclidean rhythmic necklace.

Finally, we would like to thank the two unknown referees for useful comments.

Appendix A. Euclidean rhythms in traditional world music

Below is a list of Euclidean rhythms that can be found in traditional world music. We restrict out attention to rhythms where k and n are relatively prime.

$E(2,3) = [\times\times\cdot\,] = (12)$ is a common Afro–Cuban drum pattern when started on the second onset as in $[\times\cdot\times]$. For example, it is the conga rhythm of the (6/8)-time *Swing Tumbao* [56]. It is common in Latin American music, as for example in the *Cueca* [92], and the *coros de clave* [76]. It is common in Arabic music, as for example in the *Al Táer* rhythm of Nubia [48]. It is also a rhythmic pattern of the Drum Dance of the Slavey Indians of Northern Canada [6].

$E(2,5) = [\times\cdot\times\cdot\cdot\,] = (23)$ is a rhythm found in Greece, Namibia, Rwanda and Central Africa [4]. It is also a 13th century Persian rhythm called *Khafif-e-ramal* [96], as well as the rhythm of the Macedonian dance *Makedonka* [79]. Tchaikovsky used it as the metric pattern in the second movement of his *Symphony No. 6* [55]. Started on the second onset as in $[\times\cdot\cdot\times\cdot]$ it is a rhythm found in Central Africa, Bulgaria, Turkey, Turkestan and Norway [4]. It is also the metric pattern of Dave Brubeck's *Take Five*, as well as *Mars* from *The Planets* by Gustav Holst [55]. B as in $[\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\cdot]$, it is a Serbian rhythmic pattern [4]. When it is started on the fourth (last) onset it is the *Daasa al kbiri* rhythmic pattern of Yemen [48].

$E(4,15) = [\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\cdot\times\cdot\cdot\,] = (4443)$ is the metric pattern of the *pañcam savārī tāl* of North Indian music [27].

$E(5,6) = [\times\times\times\times\times\cdot\,] = (11112)$ yields the *York-Samai* pattern, a popular Arabic rhythm [81]. It is also a handclapping rhythm used in the *Al Medēmi* songs of Oman [36].

$E(5,7) = [\times\cdot\times\times\cdot\times\times] = (21211)$ is the *Nawakhat* pattern, another popular Arabic rhythm [81]. In Nubia it is called the *Al Noht* rhythm [48].

$E(5,8) = [\times\cdot\times\times\cdot\times\times\cdot\,] = (21212)$ is the Cuban *cinquillo* pattern discussed in the preceding [42], the *Malfuf* rhythmic pattern of Egypt [48], as well as the Korean *Nong P'yŏn* drum pattern [50]. Started on the second onset, it is a popular Middle Eastern rhythm [93], as well as the *Timini* rhythm of Senegal, the *Adzogbo* dance rhythm of Benin [24], the *Spanish Tango* [40], the *Maksum* of Egypt [48], and a 13th century Persian rhythm, the *Al-saghil-al-sani* [96]. When it is started on the third onset it is the *Müsemmen* rhythm of Turkey [14]. When it is started on the fourth onset it is the *Kromanti* rhythm of Surinam.

$E(5,9) = [\times\cdot\times\cdot\times\cdot\times\cdot\times] = (22221)$ is a popular Arabic rhythm called *Agsag-Samai* [81]. Started on the second onset, it is a drum pattern used by the *Venda* in South Africa [71], as well as a Rumanian folk-dance rhythm [69]. It is also the rhythmic pattern of the *Sigaktistos* rhythm of Greece [48], and the *Samai aktsak* rhythm of Turkey [48]. Started on the third onset, it is the rhythmic pattern of the *Nawahiid* rhythm of Turkey [48].

$E(5,11) = [\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\cdot\,] = (22223)$ is the metric pattern of the Savārī tāla used in the Hindustani music of India [61]. It is also a rhythmic pattern used in Bulgaria and Serbia [4]. In Bulgaria is used in the *Kopanitsa* [75]. This metric pattern has been used by Moussorgsky in *Pictures at an Exhibition* [55]. Started on the third onset, it is the rhythm of the Macedonian dance *Kalajdzijsko Oro* [79], and it appears in Bulgarian music as well [4].

$E(5,12) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\,] = (32322)$ is a common rhythm played in the Central African Republic by the *Aka* Pygmies [3,26,33]. It is also the *Venda* clapping pattern of a South African children's song [70], and a rhythm pattern used in Macedonia [4]. Started on the second onset, it is the *Columbia* bell pattern popular in Cuba and West Africa [56], as well as a drumming pattern used in the *Chakacha* dance of Kenya [11] and also used in Macedonia [4]. Started on the third onset, it is the *Bemba* bell pattern used in Northern Zimbabwe [70], and the rhythm of the Macedonian dance *Ibraim Odža Oro* [79]. Started on the fourth onset, it is the *Fume Fume* bell pattern popular in West Africa [56], and is a rhythm used in the former Yugoslavia [4]. Finally, when started on the fifth onset it is the *Salve* bell pattern used in the Dominican Republic in a rhythm called *Canto de Vela* in honor of the Virgin Mary [41], as well as the drum rhythmic pattern of the Moroccan *Al Kudám* [48].

$E(5,13) = [\times\cdot\cdot\times\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot\,] = (32323)$ is a Macedonian rhythm which is also played by starting it on the fourth onset as follows: $[\times\cdot\times\cdot\cdot\times\cdot\cdot\times\cdot\times\cdot\cdot\,]$ [4].

$E(5,16) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot] = (33334)$ is the *Bossa-Nova* rhythm necklace of Brazil. The actual Bossa-Nova rhythm usually starts on the third onset as follows: $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \times \cdot \cdot]$ [85]. However, other starting places are also documented in world music practices, such as $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot]$ [13].

$E(6,7) = [\times \times \times \times \times \times \cdot] = (111112)$ is the *Póntakos* rhythm of Greece when started on the sixth (last) onset [48].

$E(6,13) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (222223)$ is the rhythm of the Macedonian dance *Mama Cone pita* [79]. Started on the third onset, it is the rhythm of the Macedonian dance *Postupano Oro* [79], as well as the *Krivo Plovdivsko Horo* of Bulgaria [75].

$E(7,8) = [\times \times \times \times \times \times \times \cdot] = (1111112)$, when started on the seventh (last) onset, is a typical rhythm played on the *Bendir* (frame drum), and used in the accompaniment of songs of the *Tuareg* people of Libya [81].

$E(7,9) = [\times \cdot \times \times \times \cdot \times \times \times] = (2112111)$ is the *Bazaragana* rhythmic pattern of Greece [48].

$E(7,10) = [\times \cdot \times \times \cdot \times \times \cdot \times \times] = (2121211)$ is the *Lenk fahhte* rhythmic pattern of Turkey [48].

$E(7,12) = [\times \cdot \times \times \cdot \times \cdot \times \times \cdot \times \cdot] = (2122122)$ is a common West African bell pattern. For example, it is used in the *Mpre* rhythm of the *Ashanti* people of Ghana [86]. Started on the seventh (last) onset, it is a *Yoruba* bell pattern of Nigeria, a *Babenzele* pattern of Central Africa, and a *Mende* pattern of Sierra Leone [83].

$E(7,15) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (2222223)$ is a Bulgarian rhythm when started on the third onset [4].

$E(7,16) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot] = (3223222)$ is a *Samba* rhythm necklace from Brazil. The actual Samba rhythm is $[\times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot]$ obtained by starting $E(7,16)$ on the last onset, and it coincides with a Macedonian rhythm [4]. When $E(7,16)$ is started on the fifth onset it is a clapping pattern from Ghana [70]. When it is started on the second onset it is a rhythmic pattern found in the former Yugoslavia [4].

$E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (3232322)$ is a Macedonian rhythm when started on the second onset [79].

$E(7,18) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (3232323)$ is a Bulgarian rhythmic pattern [4].

$E(8,17) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (22222223)$ is a Bulgarian rhythmic pattern which is also started on the fifth onset [4].

$E(8,19) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot] = (32232232)$ is a Bulgarian rhythmic pattern when started on the second onset [4].

$E(9,14) = [\times \cdot \times \times \cdot \times \times \cdot \times \times \cdot \times \times \cdot] = (212121212)$, when started on the second onset, is the rhythmic pattern of the *Tsofyan* rhythm of Algeria [48].

$E(9,16) = [\times \cdot \times \times \cdot \times \cdot \times \cdot \times \times \cdot \times \cdot \times \cdot] = (212221222)$ is a rhythm necklace used in the Central African Republic [3]. When it is started on the second onset it is a bell pattern of the *Luba* people of Congo [66]. When it is started on the fourth onset it is a rhythm played in West and Central Africa [42], as well as a cow-bell pattern in the Brazilian *samba* [80]. When it is started on the penultimate onset it is the bell pattern of the *Ngbaka-Maibo* rhythms of the Central African Republic [3].

$E(9,22) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (323232322)$ is a Bulgarian rhythmic pattern when started on the second onset [4].

$E(9,23) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (323232323)$ is a Bulgarian rhythm [4].

$E(11,12) = [\times \times \times \times \times \times \times \times \times \times \times \cdot] = (11111111112)$, when started on the second onset, is the drum pattern of the *Rahmāni* (a cylindrical double-headed drum) used in the *Sōt silām* dance from *Mirbāt* in the South of Oman [36].

$E(11,24) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (32222322222)$ is a rhythm necklace of the *Aka* Pygmies of Central Africa [3]. It is usually started on the seventh onset. Started on the second onset, it is a Bulgarian rhythm [4].

$E(13,24) = [\times \cdot \times \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (2122222122222)$ is another rhythm necklace of the *Aka* Pygmies of the upper *Sangha* [3]. Started on the penultimate onset, it is the *Bobangi* metal-blade pattern used by the *Aka* Pygmies.

$E(15,34) = [\times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot] = (322232223222322)$ is a Bulgarian rhythmic pattern when started on the penultimate onset [4].

References

[1] V. Kofi Agawu, Gi Dunu, Nyekpadudo, and the study of West African rhythm, Ethnomusicology 30 (1) (Winter 1986) 64–83.
[2] W. Anku, Principles of rhythm integration in African music, Black Music Research Journal 17 (2) (Autumn 1997) 211–238.
[3] S. Arom, African Polyphony and Polyrhythm, Cambridge University Press, Cambridge, England, 1991.
[4] S. Arom, L'aksak: Principes et typologie, Cahiers de Musiques Traditionnelles 17 (2004) 12–48.
[5] J.-P. Allouche, J.O. Shallit, Automatic Sequences, Cambridge University Press, Cambridge, England, 2002.
[6] M.I. Asch, Social context and the musical analysis of Slavey drum dance songs, Ethnomusicology 19 (2) (May 1975) 245–257.
[7] M. Ascher, Mathematics Elsewhere: An Exploration of Ideas Across Cultures, Princeton University Press, Princeton and Oxford, 2002.
[8] A. Ashton, Harmonograph – A Visual Guide to the Mathematics of Music, Walker and Company, New York, 2003.
[9] B. Bartók, Ce qu'on appelle le rythme bulgare, in: Musique de la vie, Stock, Paris, 1981, pp. 142–155.
[10] J.M. Barbour, Tuning and Temperament: A Historical Survey, Dover, New York, 2004.
[11] G. Barz, Music in East Africa, Oxford University Press, Oxford, England, 2004.
[12] S. Block, J. Douthett, Vector products and intervallic weighting, Journal of Music Theory 38 (1994) 21–41.
[13] G. Behague, Bossa and bossas: recent changes in Brazilian urban popular music, Ethnomusicology 17 (2) (1973) 209–233.
[14] T. Bektaş, Relationships between prosodic and musical meters in the Beste form of classical Turkish music, Asian Music 36 (1) (Winter/Spring 2005) 1–26.
[15] E. Bjorklund, A metric for measuring the evenness of timing system rep-rate patterns, SNS ASD Technical Note SNS-NOTE-CNTRL-100, Los Alamos National Laboratory, Los Alamos, USA, 2003.

[16] E. Bjorklund, The theory of rep-rate pattern generation in the SNS timing system, SNS ASD Technical Note SNS-NOTE-CNTRL-99, Los Alamos National Laboratory, Los Alamos, USA, 2003.
[17] C. Brăiloiu, Le rythme aksak, Revue de Musicologie 33 (1951) 71–108.
[18] J.E. Bresenham, Algorithm for computer control of digital plotter, IBM Systems Journal 4 (1965) 25–30.
[19] R. Brewer, The use of Habanera rhythm in rockabilly music, American Music 17 (Autumn 1999) 300–317.
[20] V. Brun, Euclidean algorithms and musical theory, Enseignement Mathématique 10 (1964) 125–137.
[21] N. Carey, Distribution modulo 1 and musical scales, PhD thesis, University of Rochester, Rochester, New York, 1998.
[22] J. Clough, J. Douthett, Maximally even sets, Journal of Music Theory 35 (1991) 93–173.
[23] J. Clough, N. Engebretsen, J. Kochavi, Scales, sets, and interval cycles: a taxonomy, Music Theory Spectrum 21 (1) (Spring 1999) 74–104.
[24] J.M. Chernoff, African Rhythm and African Sensibility, The University of Chicago Press, Chicago, 1979.
[25] J.M. Chernoff, The rhythmic medium in African music, New Literary History 22 (4) (Autumn 1991) 1093–1102, Papers from the Commonwealth Center for Literary and Cultural Change.
[26] M. Chemillier, Ethnomusicology, ethnomathematics. The logic underlying orally transmitted artistic practices, in: G. Assayag, H.G. Feichtinger, J.F. Rodrigues (Eds.), Mathematics and Music, Springer-Verlag, 2002, pp. 161–183.
[27] M. Clayton, Time in Indian Music, Oxford University Press, New York, 2000.
[28] J. Cler, Pour une théorie de l'aksak, Revue de Musicologie 80 (1994) 181–210.
[29] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein, Introduction to Algorithms, The MIT Press, Cambridge, Massachusetts, 2001.
[30] J. Clough, G. Myerson, Variety and multiplicity in diatonic systems, Journal of Music Theory 29 (1985) 249–270.
[31] J. Clough, G. Myerson, Musical scales and the generalized circle of fifths, American Mathematical Monthly 93 (9) (1986) 695–701.
[32] H.S.M. Coxeter, Music and mathematics, The Canadian Music Journal VI (1962) 13–24.
[33] M. Chemillier, C. Truchet, Computation of words satisfying the "rhythmic oddity property" (after Simha Arom's works), Information Processing Letters 86 (2003) 255–261.
[34] P.S. Campbell, C. Waterman, Christopher Waterman on Yoruba music of Africa, Music Educator's Journal 81 (6) (1995) 35–43.
[35] M. Díaz-Bañez, G. Farigu, F. Gómez, D. Rappaport, G.T. Toussaint, El compás flamenco: a phylogenetic analysis, in: Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science, Southwestern College, Winfield, Kansas, July 30–August 1 2004.
[36] I. El-Mallah, K. Fikentscher, Some observations on the naming of musical instruments and on the rhythm in Oman, Yearbook for Traditional Music 22 (1990) 123–126.
[37] P. Erdős, Distances with specified multiplicities, American Mathematical Monthly 96 (1989) 447.
[38] J. Ellis, F. Ruskey, J. Sawada, J. Simpson, Euclidean strings, Theoretical Computer Science 301 (2003) 321–340.
[39] Euclid, Elements, Dover, 1956, Translated by Sir Thomas L. Heath.
[40] B. Evans, Authentic Conga Rhythms, Belwin Mills Publishing Corporation, Miami, 1966.
[41] M. Farquharson, Africa in America, Discos Corazon, Mexico, 1992 [CD].
[42] S.A. Floyd Jr., Black music in the circum-Caribbean, American Music 17 (1) (1999) 1–38.
[43] A. Forte, The Structure of Atonal Music, Yale Univ. Press, New Haven, Connecticut, USA, 1973.
[44] P. Franklin, The Euclidean algorithm, The American Mathematical Monthly 63 (9) (November 1956) 663–664.
[45] C. Gamer, Deep scales and difference sets in equal-tempered systems, in: Proceedings of the 2nd Annual Conference of the American Society of University Composers, 1967, pp. 113–122.
[46] C. Gamer, Some combinational resources of equal-tempered systems, Journal of Music Theory 11 (1967) 32–59.
[47] F. Gómez, A. Melvin, D. Rappaport, G.T. Toussaint, Mathematical measures of syncopation, in: Proc. BRIDGES: Mathematical Connections in Art, Music and Science, Banff, Alberta, Canada, July 31–August 3 2005, pp. 73–84.
[48] K. Hagoel, The Art of Middle Eastern Rhythm, OR-TAV Music Publications, Kfar Sava, Israel, 2003.
[49] C.F. Hasty, Meter as Rhythm, Oxford University Press, Oxford, England, 1997.
[50] X. Lee Hye-Ku, Quintuple meter in Korean instrumental music, Asian Music 13 (1) (1981) 119–129.
[51] M.A. Harris, E.M. Reingold, Line drawing, leap years, and Euclid, ACM Computing Surveys 36 (1) (March 2004) 68–80.
[52] R. Hall, D. Tymoczko, Voice leading, submajorization, and the distribution constraint, Preprint, 2007.
[53] T.A. Johnson, Foundations of Diatonic Theory: A Mathematically Based Approach to Music Fundamentals, Key College Publishing, Emeryville, California, 2003.
[54] R. Kauffman, African rhythm: A reassessment, Ethnomusicology 24 (3) (September 1980) 393–415.
[55] M. Keith, From Polychords to Pólya: Adventures in Musical Combinatorics, Vinculum Press, Princeton, 1991.
[56] T. Klőwer, The Joy of Drumming: Drums and Percussion Instruments from Around the World, Binkey Kok Publications, Diever, Holland, 1997.
[57] D.E. Knuth, The Art of Computer Programming, vol. 2, 3rd ed., Addison Wesley, Reading, Massachusetts, 1998.
[58] J. Koetting, Analysis and notation of West African drum ensemble music, Publications of the Institute of Ethnomusicology 1 (3) (1970).
[59] R. Klette, A. Rosenfeld, Digital straightness – a review, Discrete Applied Mathematics 139 (2004) 197–230.
[60] W. Logan, The ostinato idea in black improvised music: A preliminary investigation, The Black Perspective in Music 12 (2) (1984) 193–215.
[61] J. London, Hearing in Time: Psychological Aspects of Musical Meter, Oxford University Press, New York, 2004.
[62] M. Lothaire, Algebraic Combinatorics on Words, Cambridge University Press, Cambridge, England, 2002.
[63] W.F. Lunnon, P.A.B. Pleasants, Characterization of two-distance sequences, Journal of the Australian Mathematical Society (Series A) 53 (1992) 198–218.
[64] B.J. McCartin, Prelude to musical geometry, The College Mathematics Journal 29 (5) (1998) 354–370.
[65] C. Morrison, Go Cat Go: Rockabilly Music and Its Makers, University of Illinois Press, Urbana, 1996.
[66] Putumayo World Music, Congo to Cuba, Music CD, 2002.
[67] F. Ortiz, La Clave, Editorial Letras Cubanas, La Habana, Cuba, 1995.
[68] I. Palásti, A distance problem of P. Erdős with some further restrictions, Discrete Mathematics 76 (2) (1989) 155–156.
[69] V. Proca-Ciortea, On rhythm in Rumanian folk dance, Yearbook of the International Folk Music Council 1 (1969) 176–199.
[70] J. Pressing, Cognitive isomorphisms between pitch and rhythm in world musics: West Africa, the Balkans and Western tonality, Studies in Music 17 (1983) 38–61.
[71] J. Rahn, Asymmetrical ostinatos in sub-Saharan music: time, pitch, and cycles reconsidered, In Theory Only: Journal of the Michigan Music Theory Society 9 (7) (1987) 23–37.
[72] J. Rahn, Turning the analysis around: African-derived rhythms and Europe-derived music theory, Black Music Research Journal 16 (1) (1996) 71–89.
[73] D. Rappaport, Geometry and harmony, in: Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science, Banff, Alberta, Canada, 2005, pp. 67–72.
[74] E.M. Reingold, N. Dershowitz, Calendrical Calculations: The Millennium Edition, Cambridge University Press, Cambridge, England, 2001.
[75] T. Rice, Music in Bulgaria, Oxford University Press, Oxford, England, 2004.
[76] O.A. Rodríguez, Instrumentos de la Musica Folklórico-Popular de Cuba, Centro de Investigación y Desarrollo de la Musica Cubana, Havana, Cuba, 1997.
[77] R.V. Rosalia, Migrated Rhythm: The Tambú of Curaçao, CaribSeek, 2002.

[78] J.O. Shallit, Pierce expansions and rules for the determination of leap years, Fibonacci Quarterly 32 (5) (1994) 416–423.
[79] A. Singer, The metrical structure of Macedonian dance, Ethnomusicology 18 (3) (September 1974) 379–404.
[80] D. Sole, The Soul of Hand Drumming, Mel Bay Productions Inc., Toronto, 1996.
[81] J.A. Standifer, The Tuareg: their music and dances, The Black Perspective in Music 16 (1) (Spring 1988) 45–62.
[82] J. Stillwell, Elements of Number Theory, Springer, 2003.
[83] R.M. Stone, Music in West Africa, Oxford University Press, Oxford, England, 2005.
[84] L. Fejes Tóth, On the sum of distances determined by a pointset, Acta Mathematica Hungarica 7 (3/4) (1956) 397–401.
[85] G.T. Toussaint, A mathematical analysis of African, Brazilian, and Cuban *clave* rhythms, in: Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science, July 27–29 2002, Towson University, Towson, Maryland, USA, 2002, pp. 157–168.
[86] G.T. Toussaint, Classification and phylogenetic analysis of African ternary rhythm timelines, in: Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science, Granada, Spain, July 23–27 2003, University of Granada, 2003, pp. 25–36.
[87] G.T. Toussaint, Computational geometric aspects of musical rhythm, in: Abstracts of the 14th Annual Fall Workshop on Computational Geometry, Cambridge, Massachusetts, November, 2004, pp. 47–48.
[88] G.T. Toussaint, A mathematical measure of preference in African rhythm, in: Phoenix, Arizona, USA, January 7–10 2004, in: Abstracts of Papers Presented to the American Mathematical Society, vol. 25, American Mathematical Society, 2004, p. 248.
[89] G.T. Toussaint, Mathematical features for recognizing preference in Sub-Saharan African traditional rhythm timelines, in: Proceedings of the 3rd International Conference on Advances in Pattern Recognition, University of Bath, United Kingdom, August 2005, pp. 18–27.
[90] D. Tymoczko, The geometry of musical chords, Science 313 (5783) (2006) 72–74.
[91] E. Uribe, The Essence of Afro–Cuban Persussion and Drum Set, Warner Brothers Publications, Miami, Florida, 1996.
[92] P. van der Lee, Zarabanda: Esquemas rítmicos de acompañamiento en 6/8, Latin American Music Review 16 (2) (Autumn–Winter 1995) 199–220.
[93] B.C. Wade, Thinking musically, Oxford University Press, Oxford, England, 2004.
[94] T. Winograd, An analysis of the properties of 'deep' scales in a t-tone system, Unpublished, May 17 1966. Term paper for music theory course at Colorado College.
[95] M. Wooldridge, Rhythmic implications of diatonic theory: A study of Scott Joplin's ragtime piano works, PhD thesis, State University of New York, Buffalo, 1993.
[96] O. Wright, The Modal System of Arab and Persian Music AD 1250–1300, Oxford University Press, Oxford, England, 1978.