LIMITATIONS ON POWER GAIN

IN FEEDBACK AMPLIFIERS

by

RONALD LESLIE THOMPSON

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ABSTRACT

A method of maximizing the power gain in feedback amplifiers, subject to a given stability margin, is investigated. As an example, this method is applied to a grounded base junction transistor amplifier and to a grounded emitter junction transistor amplifier.

A second function, the unilateral amplification factor, U, is also considered. This function is plotted for the junction transistor amplifier, and this curve is compared with the gain curves of the transistor feedback circuit.

Derivation of the equations in this paper is given in the appendix at the end of chapter 3.

Thesis Supervisor: S. J. Mason
Title: Assistant Professor of Electrical Engineering

ACKNOWLEDGMENTS

The author is indebted to the supervisor of this thesis, Professor S. J. Mason, for his guidance in the choice of a subject which proved to be very interesting. I would also like to thank him for the many discussions we had, in which the general direction of the research was established.

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CHAPTER I

INTRODUCTION

1.1 Gain and Stability

Basic considerations in designing any network are the amount of gain that is available from the given network configuration and whether the circuit will be stable when put into operation. As instability is approached in any network, the gain approaches infinity, so that some compromise between the two must be reached.

For analysis of a circuit, the network can be represented as a two-terminal-pair device. These devices can be passive circuits, unilateral active circuits, or non-unilateral active circuits. It is this latter type circuit that is of interest.

The non-unilateral active circuit is characterized by the fact that it has appreciable internal coupling between the output and input terminals of the amplifier. This coupling, if it is of the right phase and magnitude, leads to regenerative feedback and instability of the network.

With the advent of the transistor, interest in non-unilateral active networks has greatly increased. Very little has been written on gain maximization in networks of this type, and what has been written has considered only those cases in which stability was not a problem. With an unconditionally stable network, the gain was maximized by synthesizing the load and source impedances on a conjugate image impedance match basis.

Professor S. J. Mason, of the Massachusetts Institute of Technology, has written a paper[1] concerning the problem of power gain maximization in non-unilateral active networks in which there is a stability problem.

1. Mason, S. J., "Power Gain in Feedback Amplifiers", RLE Report, July 1952.

1.2 Purpose of this Investigation

The purpose of this investigation is to see how the methods of analysis that Professor Mason has put forward are applied, with specific reference to transistor circuits. The small signal incremental equivalent circuit for a transistor will be used in all the discussions of the specific circuit.

CHAPTER II

GAIN AND STABILITY CONSIDERATIONS

2.1 General Representation of Two-Terminal-Pair Device

The two-terminal-pair network that will be considered here can be represented as shown in Figure 1. The impedances that specify the network

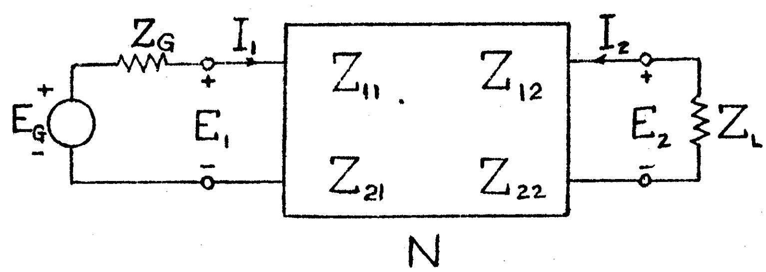

FIGURE 1. GENERAL AMPLIFIER CIRCUIT

N are complex quantities. The equations characterizing such a device are

$$E_1 = I_1Z_{11} + I_2Z_{12} = E_G - I_1Z_G \tag{2-1}$$

$$E_2 = I_1Z_{21} + I_2Z_{22} = - I_2Z_L \tag{2-2}$$

Because of the active nature of the network N, Z_{21} does not equal Z_{12}.

Equations (2-1) and (2-2) can be represented graphically by a signal flow graph,[1] as in Figure 2.

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1. Mason, S. J., "On the Logic of Feedback", MIT E.E. Sc. D. Thesis, 1951.

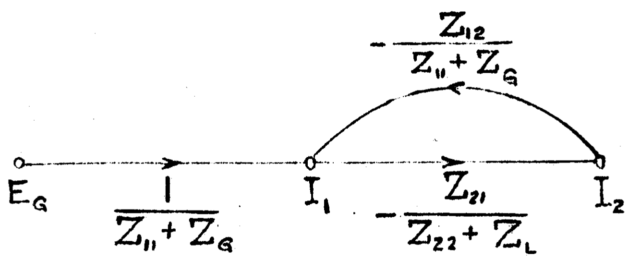


FIGURE 2. FLOW GRAPH FOR GENERAL AMPLIFIER

This graph gives a clear picture of equations (2-1) and (2-2), and lends itself to an easy solution of them.

2.2 Stability

In any amplifier, the gain is a function of the transfer admittance or impedance of the network. By simplifying the graph of Figure 2, the transfer admittance of the network is plainly shown. This simplification is done in two steps in Figure 3. The resulting solution for the transfer admittance

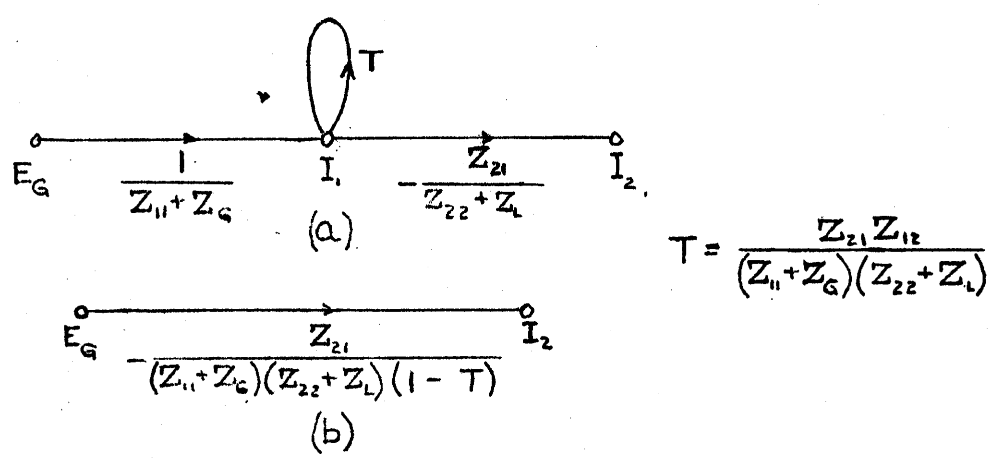


FIGURE 3. REDUCTION OF FLOW GRAPH

is easily seen from Figure 3(b) to be

$$\frac{I_2}{E_G} = -\frac{1}{Z_{12}}\left(\frac{T}{1-T}\right) \tag{2-3}$$

An examination of equation (2-3) shows that when T = + 1, the gain of the network becomes infinite and the circuit is unstable. A closer examination of T must be made, in order to determine what measures must be taken to avoid the point +1.

T is defined as the loop transmission or loop gain of the feedback network and is of the form

$$T = \frac{Z_{12}\,Z_{21}}{(Z_{11} + Z_G)(Z_{22} + Z_L)} \tag{2-4}$$

Assume for the moment that the circuit parameters are a known function of frequency. A Nyquist plot, in the T plane, of T ($j\omega$) can then be constructed, as ω varies from zero to infinity. If this plot runs too near the critical point, + 1, then a change in any of the parameters may sweep T across the critical point. In order to take care of such a contingency, a stability margin will be imposed. This margin will be defined as

$$M \geq m = \left|\frac{T}{1-T}\right| \tag{2-5}$$

In this case M is in an upper bound, which is determined by design considerations, such as allowable tolerances in the system parameters. The larger the value that is assigned to M, the closer T is to +1. This reduces the limits of design tolerances and increases the probability of instability.

2.3 Gain Maximization

With this stability margin imposed on the network, gain maximization must be considered. The gain of the network is defined as the ratio of some output quantity to some input quantity, such as power, current, or voltage. The magnitude of the gain can be either greater than, less than, or equal to unity. The problem here is that of maximizing the power gain in a feedback circuit.

The power gain of a network is defined as the ratio of the power delivered to the load

$$P_L = \left| I_2 \right|^2 R_L \tag{2-6}$$

to the available source power

$$P_S = \frac{\left|E_G\right|^2}{4R_G} \tag{2-7}$$

where E_G is the peak value of a sine curve. The maximum available gain is therefore

$$G_a = 4 \, R_G R_L \left| \frac{I_2}{E_G} \right|^2 \tag{2-8}$$

Combining equations (2-3) and (2-5), the transfer impedance of the network can be put in the form

$$\frac{I_2}{E_G} = - \frac{1}{Z_{12}} \; m \tag{2-9}$$

Substituting this result into equation (2-8), the expression for the maximum available power gain becomes

$$G_a = R_G R_L \left| \frac{2m}{Z_{12}} \right|^2 \tag{2-10}$$

Examination of equation (2-10) shows the relationship between gain and stability margin. As the margin of stability is decreased (m becomes larger) the gain is increased. This is logical, since an oscillator, which is an unstable feedback circuit, has infinite gain. The problem then is to maximize this gain at every frequency by choosing an optimum value of Z_G and Z_L, while still maintaining a specified stability margin. The problem can be partitioned into three distinct cases:

<u>One</u> R_{11} and R_{22} greater than zero and $M > m$

<u>Two</u> R_{11} and R_{22} greater than zero and $M = m$

<u>Three</u> R_{11} and/or R_{22} less than zero and $M = m$

In case one, with $M > m$, the power gain is maximized at values of Z_G and Z_L for which a margin of stability is safely maintained. For this condition, a conjugate image impedance match is used for both the generator and load impedances. Case three is complicated by possible instability of the amplifier and the fact that another parameter, N, must be chosen so that

$$N \geq n = \frac{R_{11}}{R_{11} + R_G} \tag{2-11}$$

The choice of N is dictated by design tolerances. For maximum permissible gain n = N. The conjugate image impedance method is not applicable in case three, and the values of the source and load impedances are based on the value that is chosen for N. The method of solution in both these cases is a semi-graphical one, which will not be considered in this paper.[1]

A junction transistor, connected as a grounded base amplifier and as a grounded emitter amplifier, is considered in this paper. The input and output impedances of these devices are positive for all frequencies, and over part of the frequency spectrum, $M > m$. Therefore, case two is of interest.

Again the process of gain maximization is a semi-graphical one, but now the maximum permissible gain, taking stability into account, will be considered.

The method of gain maximization will be investigated in more detail, but in order to do this, certain quantities must be defined. These are

1. For further information on the design solution for cases one and three see Mason, S. J., "Power Gain in Feedback Amplifiers", RLE Report, July 1952.

$$H = \frac{Z_{12}\ Z_{21}}{R_{11}\ R_{22}} \qquad h = \frac{H}{(1+r)^2} \tag{2-12a}$$

$$F = \frac{\Delta}{R_{11}\ R_{22}} \qquad f = \frac{F}{(1+r)^2} \tag{2-12b}$$

where

$$\Delta = (Z_{11} + Z_G)(Z_{22} + Z_L) - Z_{12}\ Z_{21} \tag{2-13a}$$

$$r = \frac{R_G}{R_{11}} = \frac{R_L}{R_{22}} \tag{2-13b}$$

H is a complex number which can be written as

$$H = |H| \angle \theta_H \tag{2-14}$$

With the quantities given in equations (2-12) to (2-14) and the relationship

$$\left|\frac{h}{f}\right| = M \tag{2-15}$$

a vector diagram for the determination of Z_G and Z_L can be drawn. This diagram is shown in Figure 4.

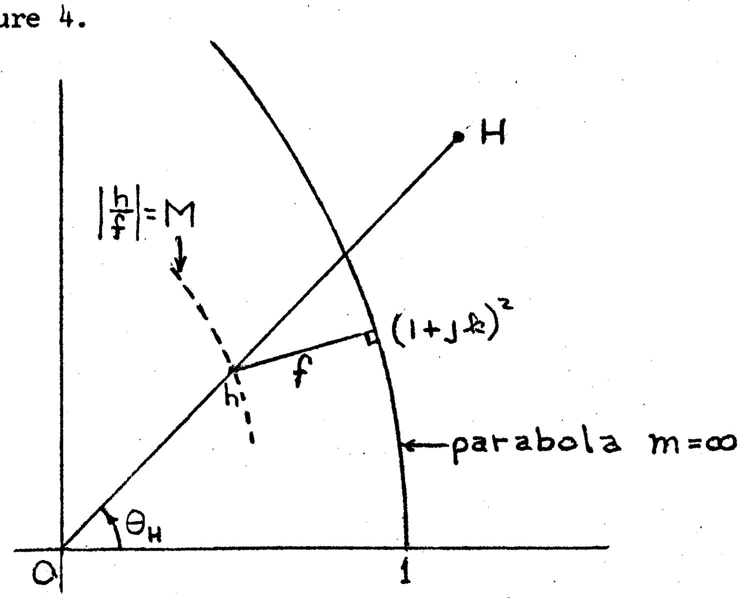


FIGURE 4. VECTOR DIAGRAM SHOWING STABILITY MARGIN

In order that a network be considered in case two, H must lie outside the stability margin curve, M, at some frequency or frequencies. If, however, H lies between this M curve and the curve $m = \infty$, conjugate image impedance matching (case one) can be used for maximizing the gain. This violates the given stability margin but the circuit will be stable as long as its parameters stay constant. When H lies outside the curve $m = \infty$, then case two must always be considered, since a conjugate image impedance match will result in an unstable circuit.

With a chosen value of M, the method of maximization becomes a graphical one. A value of r must be chosen so that the construction of Figure 4 can be made. From the intersection of f with the $m = \infty$ curve a value of k can be determined. Using this value of k, and that of r which was previously determined, equations (2-16) give the optimum values of Z_G and Z_L at any frequency.

$$R_G = r\,R_{11} \tag{2-16a}$$

$$R_L = r\,R_{22} \tag{2-16b}$$

$$X_G = (1+r)\,k\,R_{11} - X_{11} \tag{2-16c}$$

$$X_L = (1+r)\,k\,R_{22} - X_{22} \tag{2-16d}$$

With these values for the generator and load impedances, the maximum permissible gain becomes

$$G_P = R_{11}\,R_{22}\,\left|\frac{2rM}{Z_{12}}\right|^2 \tag{2-17}$$

This process of maximization is a long and tedious one, requiring considerable geometrical construction. However, if considerable computation is expected, construction of curves of constant M and constant k in the 10 log $|h|$ versus θ_H

plane would be worthwhile. A second curve of 10 log $(1+r)^2$ versus r would be very helpful for obtaining values of r, since 10 log $(1+r)^2$ can be obtained from a plot of 10 log $|H|$ versus θ_H and the constant M curves. The method of obtaining r, and the use of these curves, will be explained in Chapter 3.

It must be remembered that although the circuit is in case two for some frequency range, this is not necessarily true for all frequencies. Therefore, when H lies inside the given M curve, the maximization process reverts back to case one and a conjugate image impedance match.

2.4 Unilateral Amplification Factor

In the preceding discussion, the feedback in the network was only that which the network contained as an integral part. If the network input and output are coupled through an arbitrary lossless reciprocal network, N, as shown in Figure 5, any amount or kind of feedback can be introduced into the

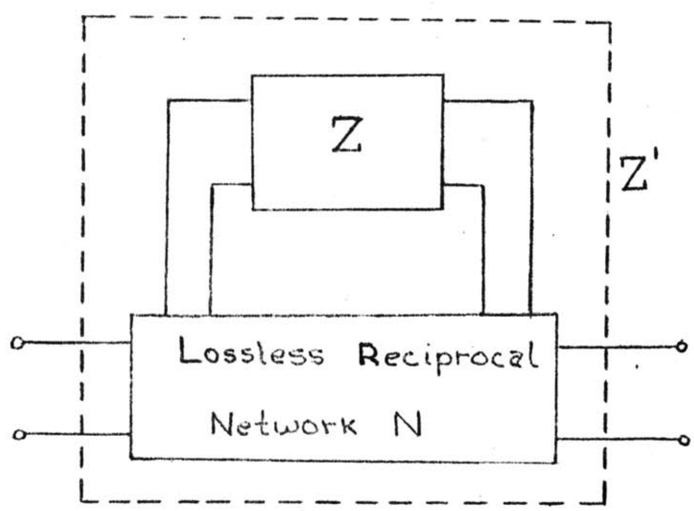


FIGURE 5. NETWORK WITH ARBITRARY FEEDBACK

network. The problem with the network N is to adjust it so that the overall power gain is maximized, subject to the same stability margin that was previously imposed.

With this type of coupling a parameter U, called the unilateral amplification factor, can be defined. It is of the form

$$U = \frac{\left|Z_{21} - Z_{12}\right|^2}{4(R_{11}R_{22} - R_{12}R_{21})} \tag{2-18}$$

It can be shown[1] that this parameter is invariant under permutations of the network terminals and under the addition of a lossless coupling network. Therefore

$$U(Z) = U(Z') \tag{2-19}$$

If $|U| > 1$, a network N can be chosen which will make R_{11} and R_{22} greater than zero. Therefore, this network,. N, has operated on the determinant of the network, Z, in such a manner that a zero has appeared in the 1-2 position. The resulting network for such a situation is shown in Figure 6.

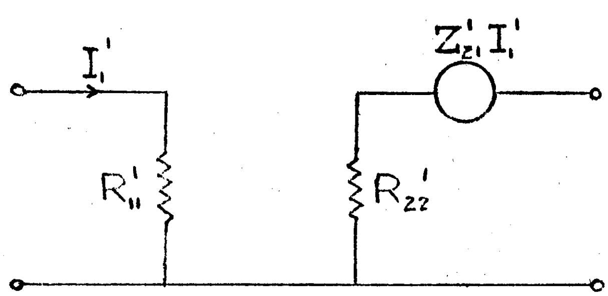


FIGURE 6. TRANSFORMED UNILATERAL NETWORK.

For the network that is shown in Figure 6 the gain is

$$G = \frac{\left|Z'_{12}\right|^2}{4\, R'_{11}R'_{22}} = U(Z') \tag{2-20}$$

Equation (2-20) shows that under the foregoing conditions, $Z_{12} = 0$, that U represents the gain of a two-terminal-pair network, regardless of the way the terminals are connected.

1. Mason, S.J., "Power Gain and Stability Margin in Feedback Amplifiers", RLE Group 35 Report, May 1953.

CHAPTER III

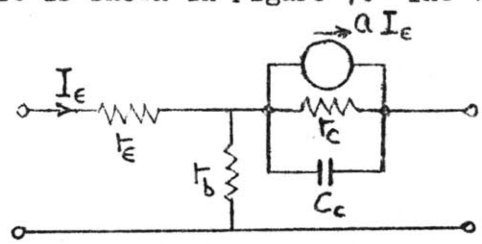
TRANSISTOR GAIN MAXIMIZATION

3.1 Introduction

In the preceding chapter, a method of determining the maximum permissible gain in a two-terminal-pair network was discussed. This method considered stability margin requirements, in order that instability would not be encountered due to parameter changes. In this chapter these methods will be applied to a junction transistor in both the grounded base and grounded emitter connections. The function U will then be examined to see how its value compares with the gain computed for the unit without coupling.

3.2 Grounded Base Connection

The first circuit to be considered is a grounded base junction transistor amplifier. The circuit is shown in Figure 7. The values chosen for the



r_ϵ = 20 ohms

r_b = 350 ohms

r_c = 1.5×10^6 ohms

C_c = 50 μμf

$$a = \frac{.95}{1 + \frac{s}{2\pi \times 10^6}}$$

FIGURE 7. GROUNDED BASE JUNCTION TRANSISTOR

parameters are the average values obtained from a number of transistors.

The gain of the grounded base amplifier was maximized over the frequency range for which the network was in case two. For this work, a value of 2 was arbitrarily chosen for M. Other values could have been chosen since the consideration here is to show how this method applies to a specific circuit and the results that can be obtained from it.

Before going into the analysis the values of the circuit determinant of the network should be obtained for the grounded base transistor. These are in the form

$$Z_{11} = r_b + r_c \tag{3-1a}$$

$$Z_{12} = r_b \tag{3-1b}$$

$$Z_{21} = r_b + \frac{ar_c}{1 + \Upsilon_c s} \tag{3-1c}$$

$$Z_{22} = r_b + \frac{r_c}{1 + \Upsilon_c s} \tag{3-1d}$$

where

$$\Upsilon_c = \frac{1}{r_c C_c} \tag{3-2a}$$

$$s = j\omega \tag{3-2b}$$

The first step in the maximization procedure is to obtain a plot of 10 log$|H|$ versus θ_H. Combining equations (2-12a) and (3-1), and inserting the values of the circuit parameters, H becomes

$$H = 1.18 \frac{(s+3.12x10^6+j1.82x10^7)(s+3.12x10^6-j1.82x10^7)(s-1.3x10^4)}{(s+8.75x10^6)(s-8.75x10^6)(s+6.25x10^6)} \tag{3-3}$$

From equation (3-3), by visualizing a pole-zero plot of H, in the s-plane, curves of 10 log $|H|$ and θ_H versus log ω can be drawn. These curves are shown in Figure 8. From these two plots a curve of 10 log $|H|$ versus θ_H can be drawn, as shown in Figure 9.

Before continuing further, the use of Figure 9 will be explained. From these curves the values of r and k, which are needed for the determination of

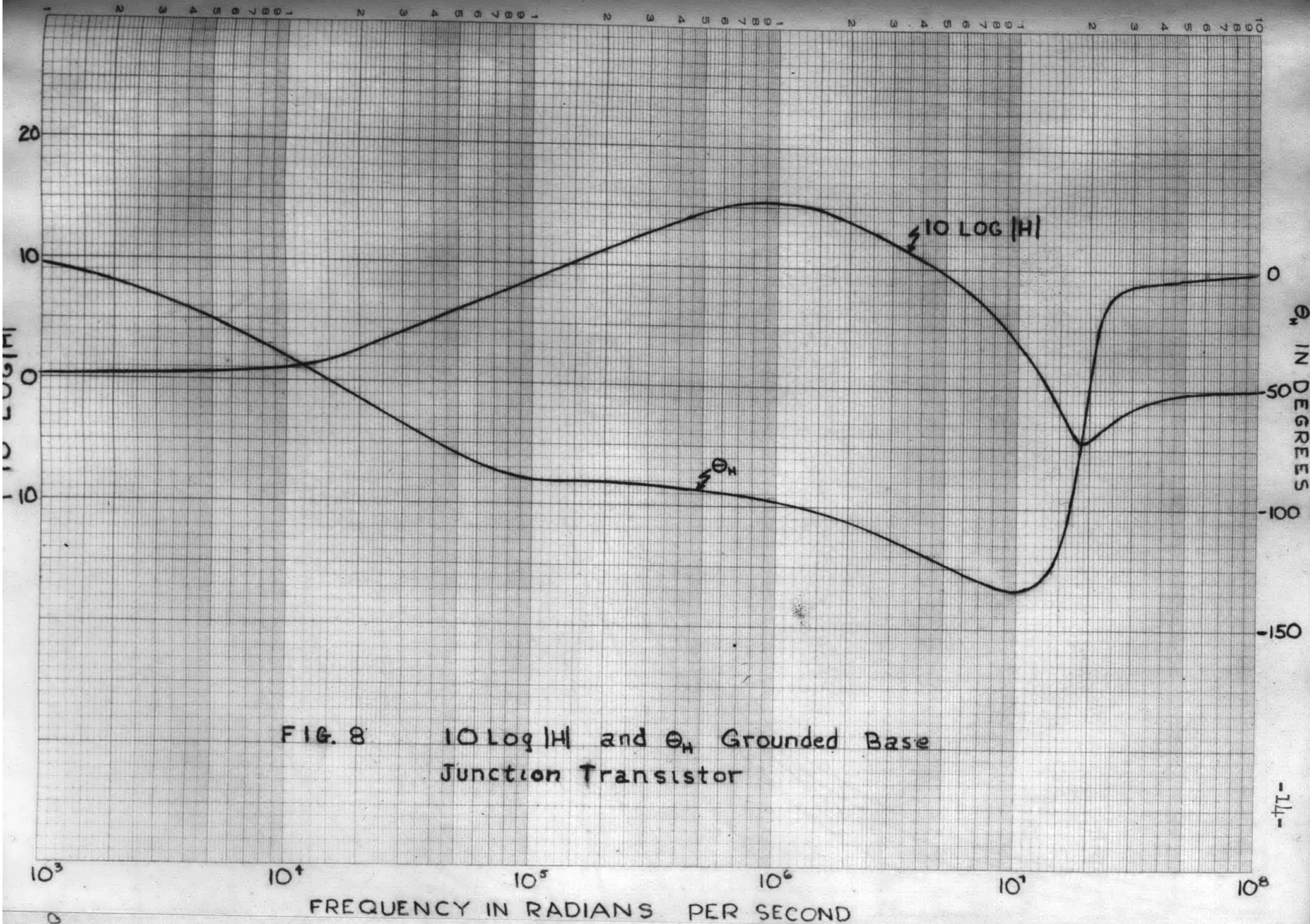


FIG. 8 10 Log |H| and Θ_H Grounded Base Junction Transistor

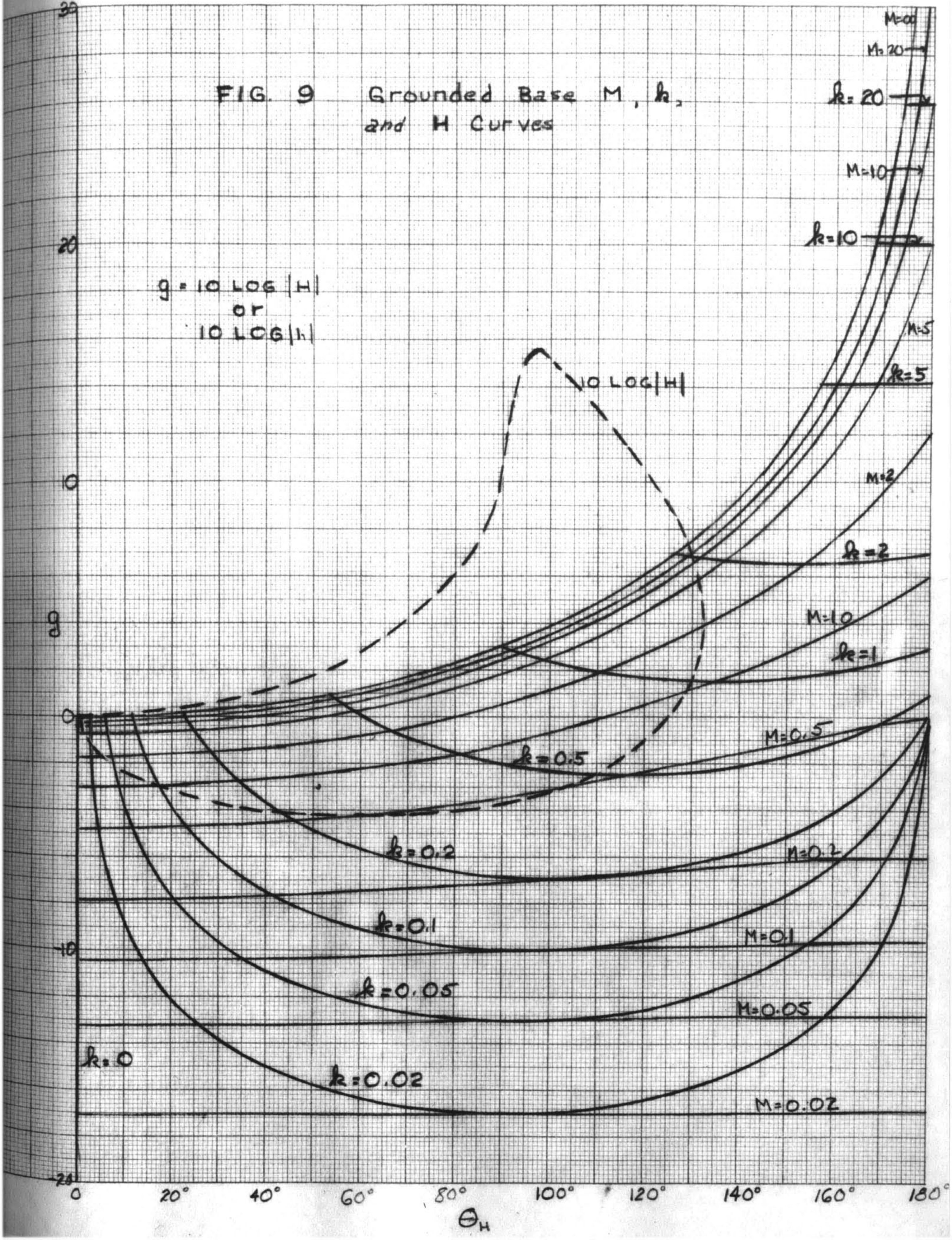


FIG. 9 Grounded Base M, k, and H Curves
g = 10 LOG |H|
or
10 LOG |h|
10 LOG |H|
M=∞
M=20
k=20
M=10
k=10
M=5
k=5
M=2
k=2
M=1.0
k=1
M=0.5
k=0.5
k=0.2
M=0.2
k=0.1
M=0.1
k=0.05
M=0.05
k=0
k=0.02
M=0.02
g
30
20
10
0
-10
-20
0
20°
40°
60°
80°
100°
120°
140°
160°
180°
Θ_H

Z_G and Z_L, can be obtained.

Evaluation of k is an easy process, since curves of constant k are included in Figure 9. At any frequency, the value of k is found where the value of θ_H at this frequency crosses the given M curve.

Determination of r requires the use of an auxiliary curve, Figure 10. Figure 9 is a plot in the 10 log $|h|$ versus θ_H plane, so that any point on the M curve corresponds to a value of 10 log $|h|$. If logarithms are taken of equation (2-12a), it becomes

$$10 \log |H| - 10 \log |h| = 10 \log (1+r)^2 \qquad (3\text{-}4)$$

The evaluation of equation (3-4) can be made by taking the difference between the 10 log $|H|$ curve and the curve for the given stability margin. This value can be used in conjunction with Figure 10 for obtaining r.

With r now determined, relationship (2-17) can be evaluated at every frequency for which case two applies. A curve of this function, G_P, versus log frequency can then be plotted. Substituting parameter values, equation (2-17) becomes

$$G_P = 16.92\, r^2 \, \frac{(s+8.75\times10^5)(s-8.75\times10^5)}{(s+1.3\times10^4)(s-1.3\times10^4)} \qquad (3\text{-}5)$$

Because r is not a known function of s, a break-point method, using the s-plane poles and zeros of G_P, can not be used for plotting this curve. The plot of G_P, for the frequency range in which the amplifier is in case two, is found in Figure 11.

At high frequencies, G_P falls off very rapidly. It is here that the s-plane factors in equation (3-5) become approximately equal and r becomes much less than one. Under these conditions, it is probably better to drop the M = 2 boundary conditions and revert to a conjugate-image-impedance match to improve

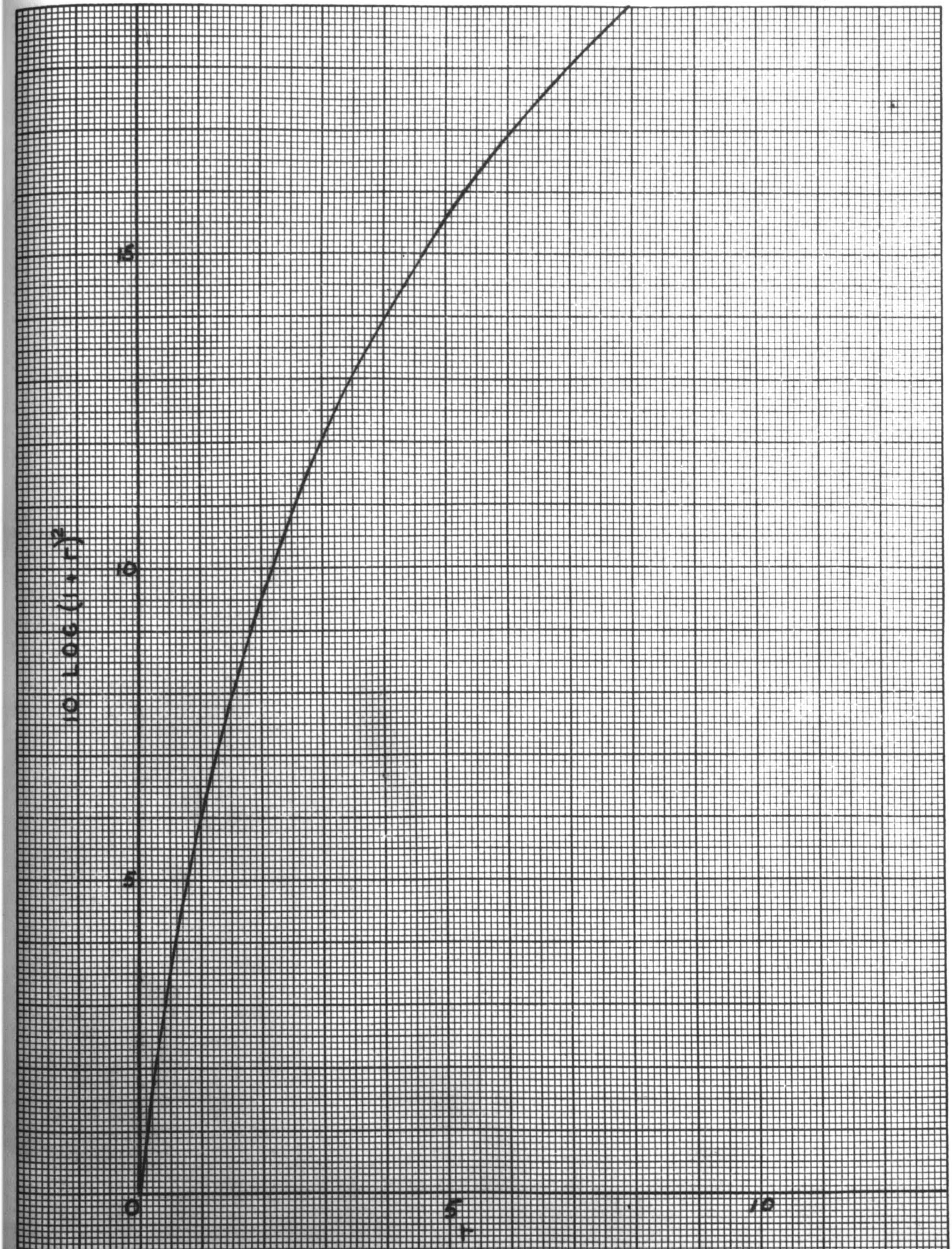


FIG. 10 $10 \log(1+r)^2$ VS. r

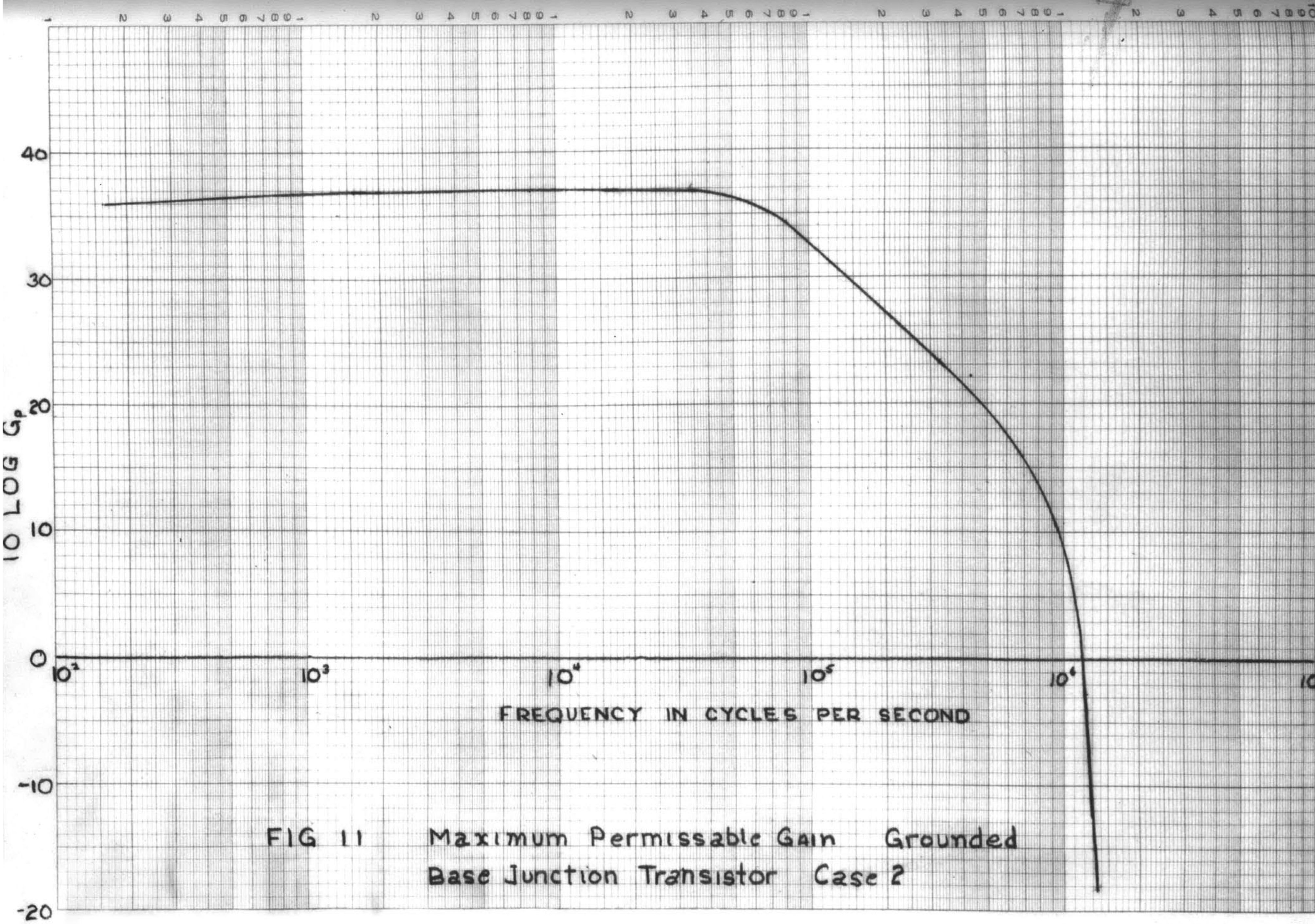


FIG 11 Maximum Permissable Gain Grounded Base Junction Transistor Case 2

the gain. The reason for this is that the H curve now lies within the $m = \infty$ boundary and stability can be maintained by matching in this manner.

The only remaining problem is to evaluate the generator and load impedances which will give the maximum gain. With values obtained from Figure 9 for r and k, equations (2-16) can be evaluated. The values of Z_G and Z_L at two frequencies are listed in Table 1.

Frequency cps	Z_G ohms	Z_L ohms	Gain db
1.59×10^3	118+j146	$318 \times 10^3 - j349 \times 10^3$	37
8.75×10^5	362+j952	354-j2698	12

Table 1. Values of Z_G and Z_L for Grounded Base Amplifier

In synthesizing the impedances found above, a series R - L connection can be used for the source impedance and a series R - C connection for the load impedance, as long as the values can be practically obtained.

3.2 Grounded Emitter

A second connection to be considered is that of a grounded emitter junction transistor amplifier. The method of gain maximization is the same as that for the grounded base amplifier. The circuit under consideration is shown in Figure 12. The values of the circuit parameters are the same as those for the grounded base connection, as given in Figure 7. Here again, a value of $M = 2$ was chosen for the stability margin.

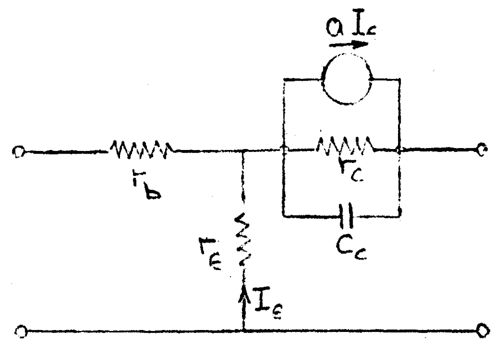


FIGURE 12. GROUNDED EMITTER JUNCTION TRANSISTOR

A set of equations similar to equations (3-1) can be written for this connection. These are

$$Z_{11} = r_b + r_\epsilon \tag{3-6a}$$

$$Z_{12} = r_\epsilon \tag{3-6b}$$

$$Z_{21} = r_\epsilon + \frac{ar_c}{1+\tau_c s} \tag{3-6c}$$

$$Z_{22} = r_\epsilon + \frac{(1-a)r_c}{1+\tau_c s} \tag{3-6d}$$

where τ_c and s have the same meaning as before.

With the use of equation (2-12a), and the 4 preceding equations, H can be written as a polynomial of s-plane factors, from which Figure 13 was drawn. H is of the form

$$H \approx .056 \frac{(s-6.25x10^6)(s-1.3x10^4)}{(s+6.6x10^4)(s-6.6x10^4)} \tag{3-7}$$

The approximation comes about because H has two poles and two zeros at high frequencies (approximately 12 megacycles) which have little effect.[1] Combining

1. See Appendix II, page 36.

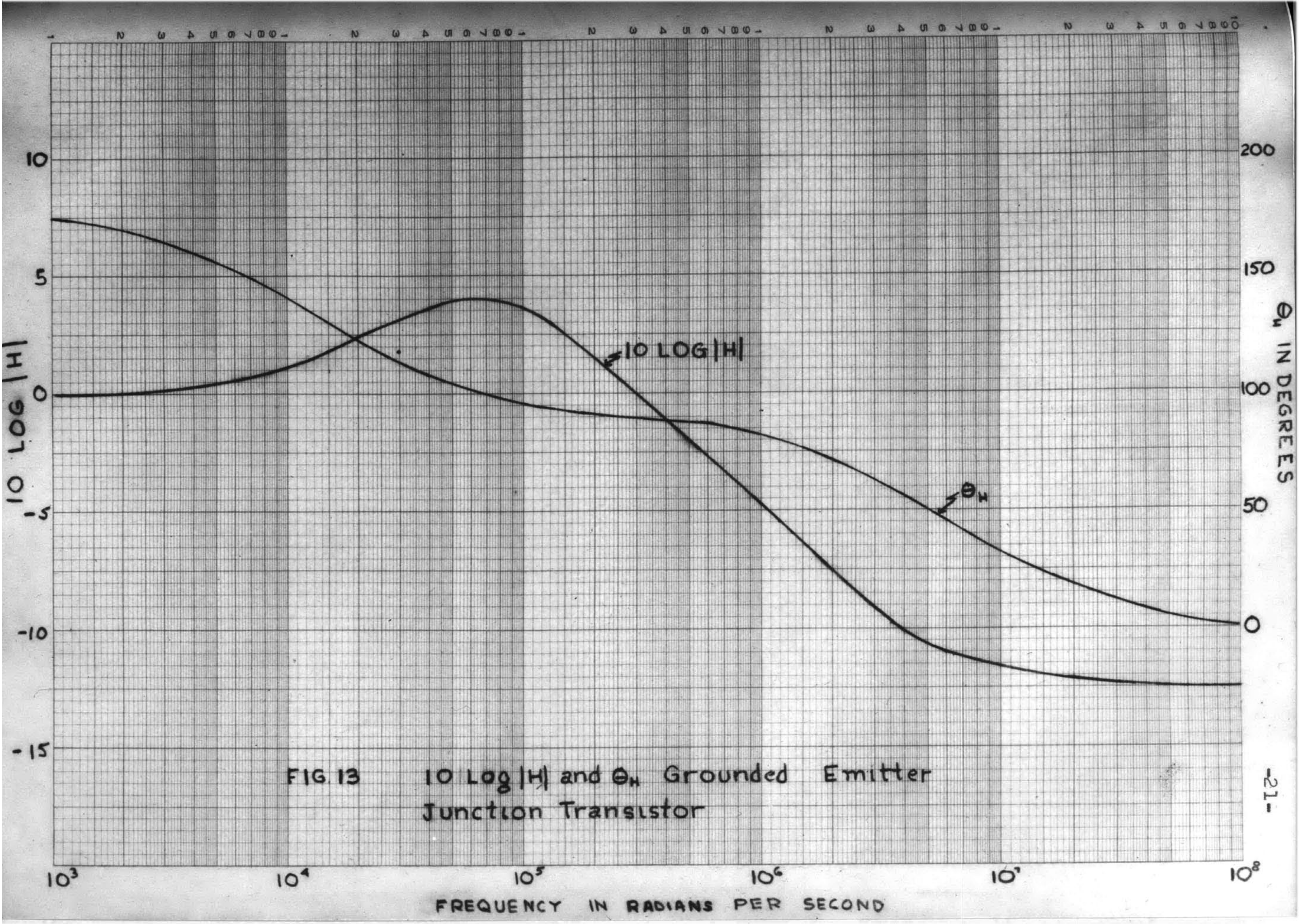


FIG. 13 10 Log |H| and Θ_H Grounded Emitter Junction Transistor the two curves of Figure 13, the curve of 10 log $|H|$ versus θ_H in Figure 14 can be plotted. The method of use of Figure 14 is the same as that described for Figure 9.

From Figure 14 and equation (2-17) a gain curve for the grounded emitter amplifier can be drawn. The equation for the gain, with numerical values substituted is

$$G_P = 296\ r^2 \ \frac{(s+7.75x10^7)(s-7.75x10^7)(s+6.6x10^4)(s-6.6x10^4)}{(s+6.25x10^6)(s-6.25x10^6)(s+1.3x10^4)(s-1.3x10^4)} \qquad (3\text{-}8)$$

Again the variation of r with frequency is not a known function of s, and a point by point method of evaluation is needed. The curve that was plotted (Figure 15) was for the frequency range in which the amplifier was in case two.

One of the big differences in this curve from that of the grounded base gain curve is the short frequency range it covers. However, it can be noted that only over a very small frequency range (9.86 - 19.1 k.c.) does the 10 log$|H|$ curve of Figure 14 lie outside the $m = \infty$ curve. Except for that one octave in the frequency spectrum, a conjugate image impedance match could be used, but stability would become a problem at frequencies bordering on this octave.

This gain curve drops rapidly at high frequencies, resulting from the fact that r is much less than 1 and the ratio of the vector magnitudes is approaching unity. Therefore, at these frequencies, a conjugate image impedance match should be used in order to try to improve the gain. However, at low frequencies the gain is still rising and this would make one other piece of information of interest, this being the zero frequency gain of the amplifier. This gain must be evaluated on a conjugate image impedance match basis and is 30.9 db. Although Figure 15 shows the gain to be increasing, it has fallen off somewhat

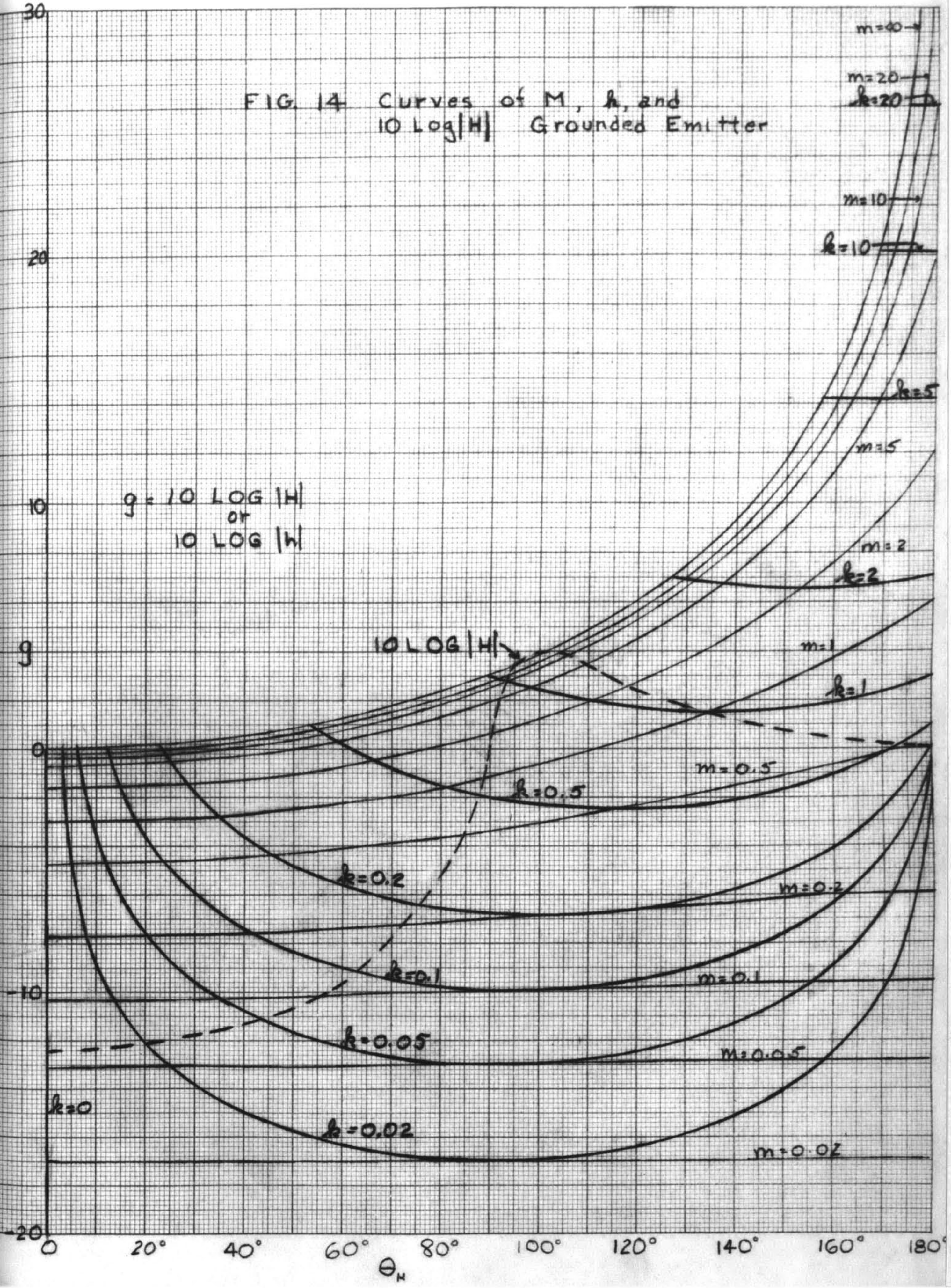


FIG. 14 Curves of M, k, and 10 Log|H| Grounded Emitter
g = 10 LOG |H| or 10 LOG |h|
10 LOG|H|
m=40
m=20
k=20
m=10
k=10
k=5
m=5
m=2
k=2
m=1
k=1
m=0.5
k=0.5
k=0.2
m=0.2
k=0.1
m=0.1
k=0.05
m=0.05
k=0
k=0.02
m=0.02
g
30
20
10
0
-10
-20
0
20°
40°
60°
80°
100°
120°
140°
160°
180°
Θ_H

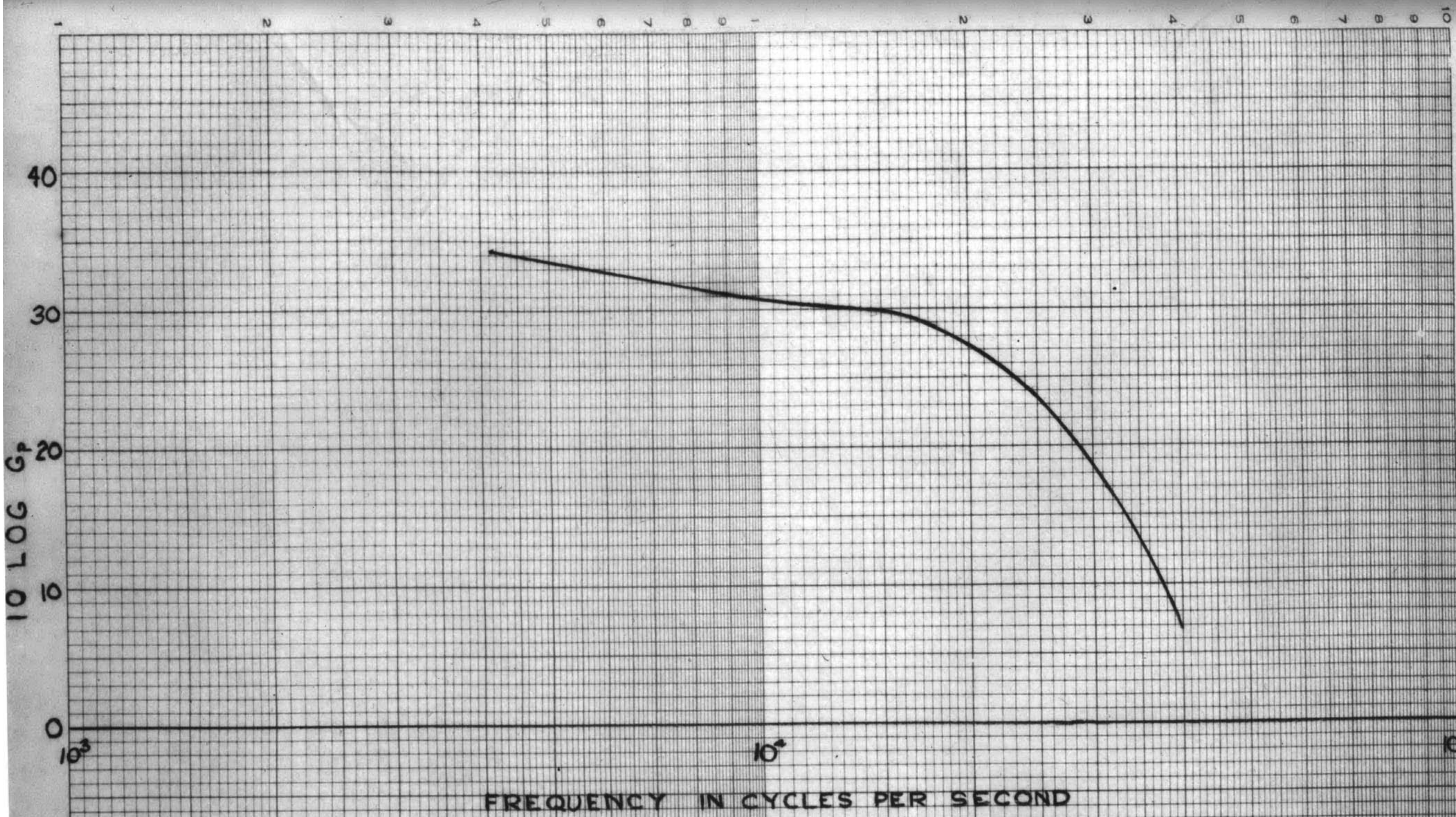


FIG 15 Maximum Permissable Gain Grounded Emitter Junction Transistor Case 2

at zero frequency.

Evaluation of equations (2-16) at two frequencies gives the values needed for Z_G and Z_L to obtain these gains. These values at two frequencies are listed in Table 2.

Frequency cps	Z_G ohms	Z_L ohms	Gain db
4.46×10^3	24+j473	1080+j21,200	34
30.2×10^3	59+j386	832+j5430	18.5

Table 2. Values of Z_G and Z_L for Grounded Emitter Amplifier

Synthesis of these generator and load impedances calls for a series R - L connection if the element values can be practically obtained.

3.4 Determination of U

In Chapter 2, a unilateral amplification factor, U, was defined. Since the value of U is the same regardless of the manner in which the transistor is connected, one curve suffices for both the grounded base and the grounded emitter connection. The numerical value of U is

$$U = .845 \times 10^{14} \frac{1}{(s+9.81 \times 10^4)(s-9.81 \times 10^4)} \quad (3\text{-}9)$$

from which Figure 16 is drawn.

Since U is the gain in a unilateral amplifier, a comparison of the curve of Figure 16 with the gain curves for the grounded base and grounded emitter is in order.

First look at the grounded base connection. Comparison at the lower

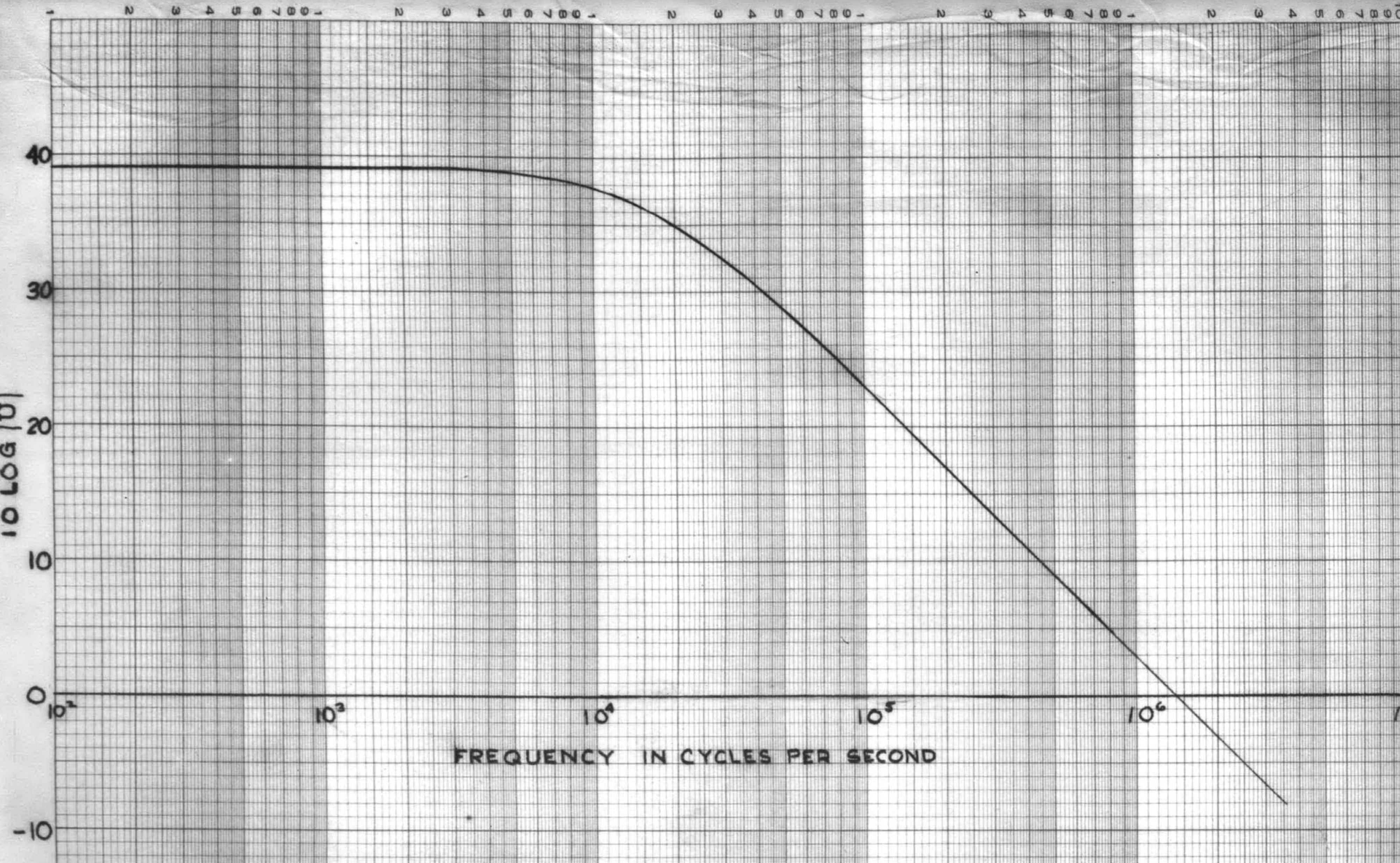


FIG. 16 Unilateral Amplification Factor For Junction Transistor

frequencies shows that the grounded base circuit has some built in negative feedback, while at higher frequencies, where the gain is dropping, there is some built in positive feedback. This means that at the low frequency end, a unilateral grounded base connection realizes the most gain and at high frequencies, a straight grounded base connection is the best.

However, for the grounded emitter connection, the unilateral connection realizes more gain over the whole frequency spectrum, indicating that the grounded emitter transistor has some negative feedback built into it. The calculation of the gain of the grounded emitter circuit at zero frequency showed that even there, this connection could not realize more gain than a unilateral connection.

Concluding Remarks

The method of gain maximization that is indicated in this paper requires considerable work. No simple method has been arrived at so that the solution for any network can be reached quickly. Values of the generator and load impedances must be evaluated, frequency by frequency. However, for an application of this method, a load and source impedance can be synthesized, which will maximize the power gain over the frequency range for which the network is to be used.

In determining the function U, which indicates the amount of gain that can be realized by transforming a feedback network into a unilateral amplifier, no method has been stated for realizing the transforming network. Further work along this line is indicated, since, in some cases, the gain of a network can be increased by the use of such a coupling network.

I <u>Grounded Base</u>

In Chapter 3, several numerical formulas were given without the actual mathematics involved in obtaining them. The mathematics will be outlined here. First, the four circuit parameters, Z_{11}, Z_{12}, Z_{21}, and Z_{22} must be broken down into their real and imaginary parts. For Z_{11} and Z_{12} this is very simple, since they are both real impedances at all frequencies.

$$Z_{11} = r_b + r_\epsilon = 370 \tag{A-1}$$

$$Z_{12} = r_b = 350 \tag{A-2}$$

However, for the remaining two circuit parameters some work is required. Z_{21} is

$$Z_{21} = r_b + \frac{a_o r_c}{(1+\tau_a s)(1+\tau_c s)} \tag{A-3}$$

where a_o is the zero frequency value of a and τ_a is the reciprocal of the a cutoff frequency in radians per second. This function can be written as a product of s-plane poles and zeros or as the sum of its real and imaginary parts. Both methods will be considered.

First, Z_{21} will be written as a quotient of its pole-zero factors. Obtaining a common denominator for equation (A-3) and collecting terms of the same power of s leaves Z_{21} in the form

$$Z_{21} = \frac{r_b \tau_a \tau_c s^2 + r_b(\tau_a + \tau_c)s + r_b + a_o r_c}{(1 + \tau_a s)(1 + \tau_c s)} \tag{A-4}$$

Dividing both numerator and denominator by $r_b \tau_a \tau_c$, substituting the parameter values given in Chapter 3, and then factoring the numerator of Z_{21} gives the relationship

$$Z_{21} = 350\frac{(s+3.12x10^6+j1.82x10^7)(s+3.12x10^6-j1.82x10^7)}{(s+1.3x10^4)(s+6.25x10^6)} \qquad (A-5)$$

Obtaining Z_{21} as the sum of its real and imaginary parts requires that the second term in equation (A-3) be rationalized. Doing this, and then collecting even power terms of s as the real part of Z_{21} and odd power terms of s as the imaginary part of Z_{21} results in

$$Z_{21} = R_{21} + j\,X_{21}$$

$$= r_b + \frac{a_o r_c(1+\Upsilon_a \Upsilon_c s^2)}{(1-\Upsilon_c^2 s^2)(1-\Upsilon_a^2 s^2)} - \frac{a_o r_c(\Upsilon_a + \Upsilon_c)s}{(1-\Upsilon_c^2 s^2)(1-\Upsilon_a^2 s^2)} \qquad (A-6)$$

Substituting in equation (A-6) gives

$$R_{21} = 350 + 1.19 \times 10^{17} \frac{(s^2+8.3x10^{10})}{(s^2-39x10^{12})(s^2-1.69x10^8)} \qquad (A-7a)$$

and

$$X_{21} = -7.63 \times 10^{23} \frac{s}{(s^2-39x10^{12})(s^2-1.69x10^8)} \qquad (A-7b)$$

A similar procedure must be followed for Z_{22}, except only the real and imaginary parts need to be obtained. Z_{22} is

$$Z_{22} = r_b + \frac{r_c}{1+\Upsilon_c s} \qquad (A-8)$$

Rationalizing this gives

$$Z_{22} = r_b + \frac{r_c}{1-\Upsilon_c^2 s^2} - \frac{r_c \Upsilon_c s}{1-\Upsilon_c^2 s^2} \qquad (A-9)$$

Substituting

$$R_{22} = 350 - \frac{2.7x10^{14}}{(s^2-1.69x10^8)} \qquad (A-10a)$$

$$X_{22} = 2x10^{10} \frac{s}{s^2-1.69x10^8} \qquad (A-10b)$$

Because of the form in which H was obtained, it will be necessary to determine R_{22} as a quotient of factors. Evaluating equation (A-10a) in this manner gives

$$R_{22} = 350 \frac{(s+8.75x10^5)(s-8.75x10^5)}{(s+1.3x10^4)(s-1.3x10^4)} \quad \text{(A-11)}$$

Substituting the values for Z_{11}, Z_{12}, Z_{21}, and Z_{22} in equation (2-12a) results in equation (3-3) for H. A similar substitution in equation (2-17), with M = 2, gives equation (3-5) for the maximum permissible gain.

Evaluation of the generator and load impedances was done at two frequencies. The first was $\omega = 10^4$ radians per second. At this frequency r = 0.32 and k = 0.3. From equation (2-16) optimum values of Z_G and Z_L at this frequency are

$$\begin{aligned} Z_G &= r R_{11} + j (1+r) k R_{11} \\ &= .32 \times 370 + j (1.32)(.3)370 \\ &= 118 + j 146 \end{aligned} \quad \text{(A-12)}$$

and

$$\begin{aligned} Z_L &= r R_{22} + j \left[(1+r) k R_{22} - X_{22}\right] \\ &= .32x995x10^3 + j \left[1.32x3x995x10^3 - 7.44x10^5\right] \\ &= 318 \times 10^3 - j 349 \times 10^3 \end{aligned} \quad \text{(A-13)}$$

The second frequency at which the generator and load impedances are calculated is $\omega = 5.5x10^6$ radians per second. At this frequency r = 0.98 and k = 1.3. Optimum values of Z_G and Z_L at this frequency are

$$\begin{aligned} Z_G &= r R_{11} + j (1+r) k R_{11} \\ &= .98x370 + j (1.98) 1.3 \; 370 \\ &= 362 + j 952 \end{aligned} \quad \text{(A-14)}$$

and

$$Z_L = r\,R_{22} + j\left[(1+r)\,k\,R_{22} - X_{22}\right]$$

$$= .98\text{x}366 + j\left[1.98\text{x}1.3\text{x}366 - 3.64\text{x}10^3\right] \qquad \text{(A-15)}$$

$$= 354 - j\,2698$$

II <u>Grounded Emitter</u>

An analysis similar to that for the grounded base connection will be done for the grounded emitter connection. First, the four circuit parameters will be evaluated. As in the grounded base case Z_{11} and Z_{12} are real impedances at all frequencies and are

$$Z_{11} = r_b + r_\epsilon = 370 \qquad \text{(A-16)}$$

$$Z_{12} = r_\epsilon = 20 \qquad \text{(A-17)}$$

Again the remaining two parameters are complex impedances. First look at

$$Z_{21} = r_\epsilon - \frac{a_o r_c}{(1+\tau_c s)(1+\tau_a s)} \qquad \text{(A-18)}$$

This function will be evaluated both as a quotient of its s-plane factors and as the sum of its real and imaginary parts.

Consider first the quotient of factors. Obtaining a common denominator and collecting terms of the same power of s, Z_{21} becomes

$$Z_{21} = \frac{r_\epsilon\,\tau_a\tau_c s^2 + r_\epsilon(\tau_a + \tau_c)s + r_\epsilon - a_o r_c}{(1+\tau_c s)(1+\tau_a s)} \qquad \text{(A-19)}$$

Substitution of the circuit parameter values in equation (A-19) and obtaining the roots of the polynomials results in

$$Z_{21} = 20\frac{(s+8\text{x}10^7)(s-7.4\text{x}10^7)}{(s-1.3\text{x}10^4)(s+6.75\text{x}10^6)} \qquad \text{(A-20)}$$

Next the real and imaginary parts of Z_{21} must be obtained.

$$Z_{21} = R_{21} + j\,X_{21}$$

$$= r_\epsilon - \frac{a_o r_c(1+\tau_a\tau_c s^2)}{(1-\tau_c^2 s^2)(1-\tau_a^2 s^2)} + \frac{a_o r_c(\tau_a + \tau_c)s}{(1-\tau_c^2 s^2)(1-\tau_a^2 s^2)} \qquad \text{(A-21)}$$

Numerical substitution in equation (A-21) gives

$$R_{21} = 20 - 1.19x10^{17} \frac{s^2+8.3x10^{10}}{(s^2-39x10^{12})(s\ -1.69x10^8)} \qquad \text{(A-22a)}$$

and

$$X_{21} = 7.63x10^{23} \frac{s}{(s^2-39x10^{12})(s^2-1.69x10^8)} \qquad \text{(A-22b)}$$

The only remaining parameter to be evaluated is Z_{22}. It will be obtained as the sum of its real and imaginary parts.

$$Z_{22} = R_{22} + j\,X_{22}$$

$$= r_\epsilon + \frac{\left(1 - \frac{a_o}{1+\tau_a s}\right) r_c}{1 + \tau_c s} \qquad \text{(A-23)}$$

This equation can be broken down into its real and imaginary components.

$$R_{22} = \frac{r_\epsilon \tau_c^2 \tau_a^2 s^4 - \left[r_\epsilon(\tau_a^2+\tau_c^2)+\tau_a \tau_c(\tau_a+a_o r_c)\right] s^2 + r_\epsilon - r_c(1-a_o)}{(1-\tau_c^2 s^2)(1-\tau_a^2 s^2)} \qquad \text{(A-24a)}$$

$$X_{22} = \frac{r_c \tau_a^2 \tau_c s^3 + r_c\left[a_o(\tau_a+\tau_c)-\tau_a\right] s}{(1-\tau_c^2 s^2)(1-\tau_a^2 s^2)} \qquad \text{(A-24b)}$$

Substituting in equations (A-24) and factoring gives

$$R_{22} = 20 \frac{(s+7.75x10^7)(s-7.75x10^7)(s+6.6x10^4)(s-6.6x10^4)}{(s-1.3x10^4)(s+1.3x10^4)(s-6.25x10^6)(s+6.25x10^6)} \qquad \text{(A-25a)}$$

$$X_{22} = 2x10^{10} \frac{s(s+j6.17x10^6)(s-j6.17x10^6)}{(s-1.3x10^4)(s+1.3x10^4)(s-6.25x10^6)(s+6.25x10^6)} \qquad \text{(A-25b)}$$

Introduction of relations (A-16), (A-17), (A-20), and (A-24a) into equation (2-12a) results in equation (A-26) for H.

$$H = .056 \frac{(s+8x10^7)(s-7.4x10^7)(s-6.25x10^6)(s-1.3x10^4)}{(s+7.75x10^7)(s-7.75x10^7)(s+6.6x10^4)(s-6.6x10^4)} \qquad \text{(A-26)}$$

The factors which occur at the large values of s (about $s = 7.5x10^7$) have little or no effect on the phase and gain of H at the frequencies of interest.

Therefore, for computation purposes, these factors may be neglected. This is the approximation that is referred to in connection with equation (3-9). A similar substitution of equation (A-24a) in equation (2-17), with $M = 2$, results in equation (3-5).

The generator and load impedances were evaluated at two frequencies. The first was $\omega = 2.8 \times 10^4$ radians per second, at which frequency $r = .065$ and $k = 1.2$. Equations (2-16) then gives for Z_G and Z_L

$$Z_G = r\, R_{11} + j\,(1+r)\, k\, R_{11}$$

$$= .065 \times 370 + j\,(1.065)(1.2)\, 370 \qquad \text{(A-27)}$$

$$= 24 + j\, 473$$

and

$$Z_L = r\, R_{22} + j\left[(1+r)\, k\, R_{22} - X_{22}\right]$$

$$= .065 \times 16{,}600 + j\left[1.065 \times 1.2 \times 16{,}600 - 20\right]$$

$$= 1080 + j\, 21{,}200 \qquad \text{(A-28)}$$

The second frequency is $\omega = 1.9 \times 10^5$ radians per second for which $r = 0.16$ and $k = 0.9$. This leads to the following source and load impedances

$$Z_G = r\, R_{11} + j\,(1+r)\, k\, R_{11}$$

$$= .16 \times 370 + j\, 1.16 \times 9 \times 370$$

$$= 59 + j\, 386 \qquad \text{(A-29)}$$

and

$$Z_L = r\, R_{22} + j\left[(1+r)\, k\, R_{22} - X_{22}\right]$$

$$= .16 \times 5200 + j\left[1.16 \times .9 \times 5200 - 0.5\right]$$

$$= 832 + j\, 5430 \qquad \text{(A-30)}$$

III U

The function U, which is the same for the grounded base and the grounded emitter connections, was determined by using the circuit values for the grounded emitter connection. For the function U, as defined in relationship (2-18)

$$Z_{21} - Z_{12} = r_{\epsilon} - \frac{ar_c}{1+ \mathcal{T}_c s} - r_{\epsilon} = -\frac{ar_c}{1+ \mathcal{T}_c s} \tag{A-31}$$

and

$$R_{11}R_{22}-R_{12}R_{21} = - \left[r_b a_o \mathcal{T}_a \mathcal{T}_c + (r_{\epsilon} + r_b) \mathcal{T}_a\right] s^2$$

$$+ (r_b+r_c)r_{\epsilon} + r_b r_c(1-a_o). \tag{A-32}$$

Substitution of the transistor parameters in these two relations, followed by subsequent division of equation (A-31) by equation (A-32) times 4, results in equation (3-9).

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