18.05. Test 1.

(1) Consider events $A = \{\text{HHH at least once}\}$ and $B = \{\text{TTT at least once}\}$. We want to find the probability $P(A \cap B)$. The complement of $A \cap B$ will be $A^c \cup B^c$, i.e. no TTT or no HHH, and

$$P(A \cap B) = 1 - P(A^c \cup B^c).$$

To find the last one we can use the probability of a union formula

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c).$$

Probability of A^c, i.e. no HHH, means that on each toss we don't get HHH. The probability not to get HHH on one toss is 7/8 and therefore,

$$P(A^c) = \left(\frac{7}{8}\right)^{10}.$$

The same for $P(B^c)$. Probability of $A^c \cap B^c$, i.e. no HHH and no TTT, means that on each toss we don't get HHH and TTT. The probability not to get HHH and TTT on one toss is 6/8 and, therefore,

$$P(A^c \cap B^c) = \left(\frac{6}{8}\right)^{10}.$$

Finally, we get,

$$P(A \cap B) = 1 - \left(\left(\frac{7}{8}\right)^{10} + \left(\frac{7}{8}\right)^{10} - \left(\frac{6}{8}\right)^{10}\right).$$

(2) We have

$$P(F) = P(M) = 0.5, \quad P(CB|M) = 0.05 \ \text{ and } \ P(CB|F) = 0.0025.$$

Using Bayes' formula,

$$P(M|CB) = \frac{P(CB|M)P(M)}{P(CB|M)P(M) + P(CB|F)P(F)} = \frac{0.05 \times 0.5}{0.05 \times 0.5 + 0.0025 \times 0.5}$$

(3) We want to find

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

which is defined only when $f(x) > 0$. To find $f_1(x)$ we have to integrate out y, i.e.

$$f_1(x) = \int f(x,y)dy.$$

To find the limits we notice that for a given x, $0 < y^2 < 1 - x^2$ which is not empty only if $x^2 < 1$, i.e. $-1 < x < 1$. Then $-\sqrt{1-x^2} < y < \sqrt{1-x^2}$. So if $-1 < x < 1$ we get,

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} c(x^2+y^2)dy = c(x^2y+\frac{y^3}{3})\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 2c(x^2\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{3/2}).$$

Finally, for $-1 < x < 1$,

$$f(y|x) = \frac{c(x^2+y^2)}{2c(x^2\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{3/2})} = \frac{x^2+y^2}{2x^2\sqrt{1-x^2}+\frac{2}{3}(1-x^2)^{3/2}}$$

if $-\sqrt{1-x^2} < y < \sqrt{1-x^2}$, and 0 otherwise.

(4) Let us find the c.d.f first.

$P(Y \le y) = P(\max(X_1, X_2) \le y) = P(X_1 \le y, X_2 \le y) = P(X_1 \le y)P(X_2 \le y)$.

The c.d.f. of X_1 and X_2 is

$$P(X_1 \le y) = P(X_2 \le y) = \int_{-\infty}^{y} f(x)dx.$$

If $y \le 0$, this is

$$P(X_1 \le y) = \int_{-\infty}^{y} e^x dx = e^x \Big|_{-\infty}^{y} = e^y$$

and if $y > 0$ this is

$$P(X_1 \le y) = \int_{-\infty}^{0} e^x dx = e^x \Big|_{-\infty}^{0} = 1.$$

Finally, the c.d.f. of Y,

$$P(Y \le y) = \begin{cases} e^{2y}, & y \le 0 \\ 1, & y > 0. \end{cases}$$

Taking the derivative, the p.d.f. of Y,

$$f(y) = \begin{cases} 2e^{2y}, & y \le 0 \\ 0, & y > 0. \end{cases}$$

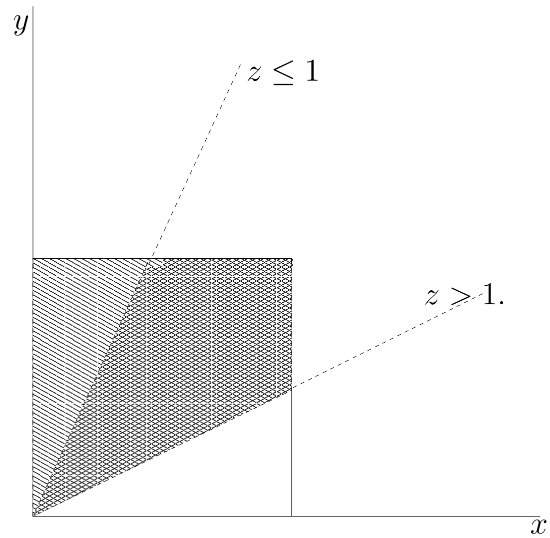

Figure 1: Region $\{x \le zy\}$ for $z \le 1$ and $z > 1$.

(5) Let us find the c.d.f. of $Z = X/Y$ first. Note that for $X, Y \in (0, 1)$, Z can take values only > 0, so let $z > 0$. Then

$$P(Z \le z) = P(X/Y \le z) = P(X \le zY) = \int_{\{x \le zy\}} f(x, y)dxdy.$$

To find the limits, we have to consider the intersection of this set $\{x \le zy\}$ with the square $0 < x < 1, 0 < y < 1$. When $z \le 1$, the limits are

$$\int_0^1 \int_0^{zy} (x + y)dxdy = \int_0^1 (\frac{x^2}{2} + xy)\Big|_0^{zy} dy = \int_0^1 (\frac{z^2}{2} + z)y^2 dy = \frac{z^2}{6} + \frac{z}{3}.$$

When $z \ge 1$, the limits are different

$$\int_0^1 \int_{x/z}^1 (x + y)dydx = \int_0^1 (\frac{y^2}{2} + xy)\Big|_{x/z}^1 dx = 1 - \frac{1}{6z^2} - \frac{1}{3z}.$$

So the c.d.f. of Z is

$$P(Z \le z) = \begin{cases} \frac{z^2}{6} + \frac{z}{3}, & 0 < z \le 1 \\ 1 - \frac{1}{6z^2} - \frac{1}{3z}, & z > 1 \end{cases}$$

The p.d.f. is

$$f(z) = \begin{cases} \frac{z}{3} + \frac{1}{3}, & 0 < z \le 1 \\ \frac{1}{3z^3} + \frac{1}{3z^2}, & z > 1 \end{cases}$$

and zero otherwise, i.e. for $z \le 0$.