18.05. Test 2.

(1) Let X be the players fortune after one play. Then

$$P(X = 2c) = \frac{1}{2} \text{ and } P(X = \frac{c}{2}) = \frac{1}{2}$$

and the expected value is

$$EX = 2c \times \frac{1}{2} + \frac{c}{2} \times \frac{1}{2} = \frac{5}{4}c.$$

Repeating this n times we get the expected values after n plays $(5/4)^n c$.

(2) Let $X_i, i = 1, \ldots, n = 1000$ be the indicators of getting heads. Then $S_n = X_1 + \ldots + X_n$ is the total number of heads. We want to find k such that $P(440 \leq S_n \leq k) \approx 0.5$. Since $\mu = EX_i = 0.5$ and $\sigma^2 = \text{Var}(X_i) = 0.25$ by central limit theorem,

$$Z = \frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{S_n - 500}{\sqrt{250}}$$

is approximately standard normal, i.e.

$$\begin{aligned} P(440 \leq S_n \leq k) &= P(\frac{440 - 500}{\sqrt{250}} = -3.79 \leq Z \leq \frac{k - 500}{\sqrt{250}}) \\ &\approx \Phi(\frac{k - 500}{\sqrt{250}}) - \Phi(-3.79) = 0.5. \end{aligned}$$

From the table we find that $\Phi(-3.79) = 0.0001$ and therefore

$$\Phi(\frac{k - 500}{\sqrt{250}}) = 0.4999.$$

Using the table once again we get $\frac{k-500}{\sqrt{250}} \approx 0$ and $k \approx 500$.

(3) The likelihood function is

$$\varphi(\theta) = \frac{\theta^n e^{n\theta}}{(\prod X_i)^{\theta+1}}$$

and the log-likelihood is

$$\log \varphi(\theta) = n \log \theta + n\theta - (\theta + 1) \log \prod X_i.$$

We want to find the maximum of log-likelihood so taking the derivative we get

$$\frac{n}{\theta} + n - \log \prod X_i = 0$$

and solving for θ, the MLE is

$$\hat{\theta} = \frac{n}{\log \prod X_i - n}.$$

(4) The prior distribution is

$$f(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

and the joint p.d.f. of $X_1, \ldots, X_n$ is

$$f(X_1, \ldots, X_n|\theta) = \frac{\theta^n e^{n\theta}}{(\prod X_i)^{\theta+1}}.$$

Therefore, the posterior is proportional to (as usual, we keep track only of the terms that depend on θ)

$$\begin{aligned} f(\theta|X_1, \ldots, X_n) &\sim \theta^{\alpha-1} e^{-\beta\theta} \frac{\theta^n e^{n\theta}}{(\prod X_i)^{\theta+1}} = \frac{1}{\prod X_i} \frac{\theta^{\alpha+n-1} e^{-\beta\theta+n\theta}}{(\prod X_i)^\theta} \\ &\sim \theta^{\alpha+n-1} e^{-\beta\theta+n\theta-\theta\log\prod X_i} = \theta^{(\alpha+n)-1} e^{-(\beta-n+\log\prod X_i)\theta}. \end{aligned}$$

This shows that the posterior is again a gamma distribution with parameters

$$\Gamma(\alpha + n, \beta - n + \log \prod X_i).$$

Bayes estimate is the expectation of the posterior which in this case is

$$\hat{\theta} = \frac{\alpha + n}{\beta - n + \log \prod X_i}.$$

(5) The confidence interval for μ is given by

$$\bar{X} - c\sqrt{\frac{1}{n-1}(\overline{X^2} - \bar{X}^2)} \le \mu \le \bar{X} + c\sqrt{\frac{1}{n-1}(\overline{X^2} - \bar{X}^2)}$$

where c that corresponds to 90% confidence is found from the condition

$$t_{10-1}(c) - t_{10-1}(-c) = 0.9$$

or $t_9(c) = 0.95$ and $c = 1.833$.

The confidence interval for σ^2 is

$$\frac{n(\overline{X^2} - \bar{X}^2)}{c_2} \leq \sigma^2 \leq \frac{n(\overline{X^2} - \bar{X}^2)}{c_2}$$

where c_1, c_2 satisfy

$$\chi^2_{10-1}(c_1) = 0.05 \text{ and } \chi^2_{10-1}(c_2) = 0.95,$$

and $c_1 = 3.325, c_2 = 16.92$.