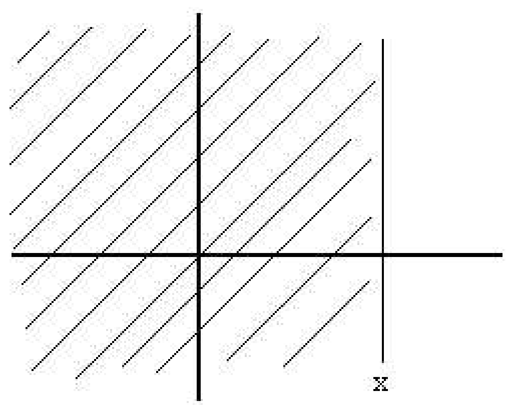
18.05 Lecture 10
February 25, 2005

Review of Distribution Types
Discrete distribution for (X, Y): joint p.f. $f(x, y) = \mathbb{P}(X = x, Y = y)$
Continuous: joint p.d.f. $f(x, y) \geq 0, \int_{\mathbb{R}^2} f(x, y)dxdy = 1$
Joint c.d.f.: $F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$
$F(x) = \mathbb{P}(X \leq x) = \lim_{y\to\infty} F(x, y)$

In the continuous case: $F(x, y) = \mathbb{P}(X \leq x, T \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x, y)dxdy.$
Marginal Distributions
Given the joint distribution of (X, Y), the individual distributions of X, Y are marginal distributions.

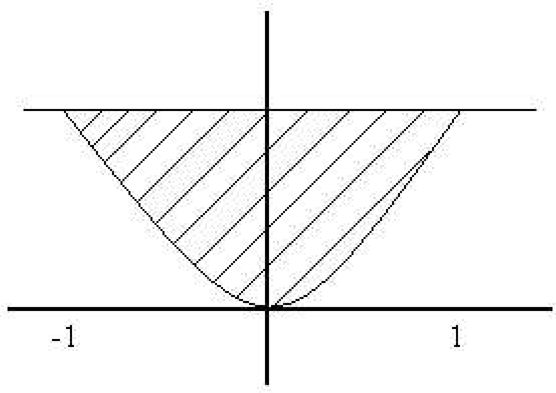
Discrete (X, Y): marginal probability function
$f_1(x) = \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y f(x, y)$
In the table for the previous lecture, of probabilities for each point (x, y):
Add up all values for y in the row x = 1 to determine $\mathbb{P}(X = 1)$

Continuous (X, Y): joint p.d.f. f(x, y); p.d.f. of X: $f_1(x) = \int_{-\infty}^{\infty} f(x, y)dy$
$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y \leq \infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(x, y)dydx$



$f_1(x) = \frac{\partial F}{\partial x} = \int_{-\infty}^{\infty} f(x, y)dy$
Why not integrate over line?
$\mathbb{P}(\{X = x\}) = \int_{-\infty}^{\infty}(\int_{x}^{x} f(x, y)dx)dy = 0$
$\mathbb{P}$(of continuous random variable at a specific point) = 0.

Example: Joint p.d.f.
$f(x, y) = \frac{21}{4}x^2y, x^2 \leq y \leq 1, 0 \leq x \leq 1; 0$ otherwise



What is the distribution of x?
p.d.f. $f_1(x) = \int_{x^2}^{1} \frac{21}{4}x^2 y dy = \frac{21}{4}x^2 \times \frac{1}{2}y^2|_{x^2}^{1} = \frac{21}{8}x^2(1-x^4), -1 \leq x \leq 1$

Discrete values for X, Y in tabular form:

	1	2	
1	0.5	0	0.5
2	0	0.5	0.5
	0.5	0.5	

Note: If all entries had 0.25 values, the two variables would have the same marginal dist.

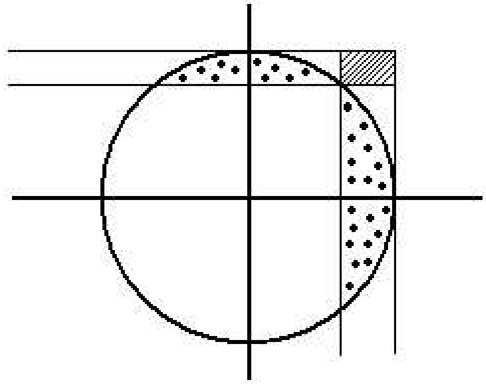
Independent X and Y:

Definition: X, Y independent if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$
Joint c.d.f. $F(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y) = F_1(x)F_2(y)$ (intersection of events)
The joint c.d.f can be factored for independent random variables.

Implication: continuous (X, Y): joint p.d.f. f(x, y), marginal $f_1(x), f_2(y)$
$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dydx = F_1(x)F_2(y) = \int_{-\infty}^{x} f_1(x)dx \times \int_{-\infty}^{y} f_2(y)dy$
Take $\frac{\partial^2}{\partial x \partial y}$ of both sides: $f(x,y) = f_1(x)f_2(y)$
Independent if joint density is a product.

Much simpler in the discrete case:
Discrete (X, Y): $f(x,y) = \mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y) = f_1(x)f_2(y)$ by definition.

Example: Joint p.d.f.
$f(x,y) = kx^2y^2, x^2 + y^2 \leq 1; 0$ otherwise
X and Y are not independent variables.
$f(x,y) \neq f_1(x)f_2(y)$ because of the circle condition.



$\mathbb{P}(\text{square}) = 0 \neq \mathbb{P}(X \in \text{ side}) \times \mathbb{P}(Y \in \text{ side})$

Example: $f(x, y) = kx^2y^2, 0 \leq x \leq 1, 0 \leq y \leq 1; 0$ otherwise
Can be written as a product, as they are independent:
$f(x, y) = kx^2y^2 I(0 \leq x \leq 1, 0 \leq y \leq 1) = k_1x^2 I(0 \leq x \leq 1) \times k_2y^2 I(0 \leq y \leq 1)$
Conditions on x and y can be separated.

Note: Indicator Notation
$I(x \in A) = 1, x \in A; 0, x \notin A$

For the discrete case, given a table of values, you can tell independence:

	b_1	b_2	...	b_m	
a_1	p_{11}	p_{12}	...	p_{1m}	p_{1+}
a_2	...	...	...	...	p_{2+}
...	...	...	...	...	...
a_n	p_{n1}	...	...	p_{nm}	p_{n+}
	p_{+1}	p_{+2}	...	p_{+n}	

$p_{ij} = \mathbb{P}(X = a_i, Y = b_j) = \mathbb{P}(X = a_i)\mathbb{P}(Y = b_j)$
$p_{i+} = \mathbb{P}(X = a_i) = \sum_{j=1}^{m} p_{ij}$
$p_{+j} = \mathbb{P}(Y = b_j) = \sum_{i=1}^{n} p_{ij}$
$p_{ij} = p_{i+} \times p_{+j}$, for every i, j - all points in table.

** End of Lecture 10