18.05 Lecture 14
March 7, 2005

Linear transformations of random vectors: $\overrightarrow{Y} = r(\overrightarrow{X})$

$$\begin{vmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{vmatrix} = \text{A} \begin{vmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{vmatrix}$$

A - n by n matrix, $\overrightarrow{X} = A^{-1}\overrightarrow{Y}$ if det A $\neq 0 \rightarrow A^{-1} = B$
$x_1 = b_1 y_1 + ... + b_{1n} y_n$
J = Jacobian = det $\begin{vmatrix} b_{11}... & b_{1n} \\ ... & ... \\ b_{n1}... & b_{nn} \end{vmatrix}$ where $b_i' s$ are partial derivatives of s_i with respect to y_i
det B = det $A^{-1} = \frac{1}{\text{det} A}$
p.d.f. of Y:

$$g(y) = \frac{1}{|\text{det} A|} f(A^{-1}\overrightarrow{x})$$

Example: $\overrightarrow{X} = (x_1, x_2)$ with p.d.f.:

$$f(x_1, x_2) = \{cx_1x_2, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1; 0 \text{ otherwise}\}$$

To make integral equal 1, c = 4.

$$Y_1 = X_1 + 2X_2, Y_2 = 2X_1 + X_2; A = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \rightarrow det(A) = -3$$

Calculate the inverse functions:

$$X_1 = -\frac{1}{3}(Y_1 - 2Y_2), X_2 = -\frac{1}{3}(Y_2 - 2Y_1)$$

New joint function:

$$g(y_1, y_2) = \{\frac{1}{3} \times 4(-\frac{1}{3}(y_1 - 2y_2))(-\frac{1}{3}(y_2 - 2y_1))$$

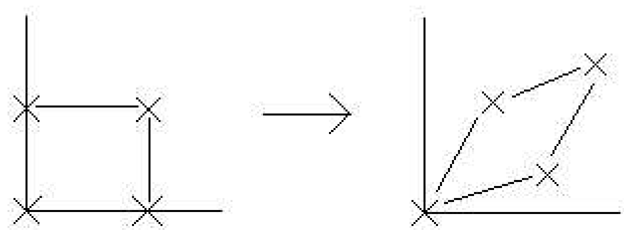
$$\text{for } 0 \leq -\frac{1}{3}(y_1 - 2y_2) \leq 1 \text{ and } 0 \leq -\frac{1}{3}(y_2 - 2y_1) \leq 1;$$

$$0, \text{ otherwise}\}$$

Simplified:

$$f(y_1, y_2) = \{\frac{4}{27}(y_1 - 2y_2)(y_2 - 2y_1) \text{ for } -3 \leq y_1 - 2y_2 \leq 0, -3 \leq y_2 - 2y_1 \leq 0;$$

$$0, \text{ otherwise}\}$$



Linear transformation distorts the graph from a square to a parallelogram.

Note: From Lecture 13, when min() and max() functions were introduced, such functions describe engines in series (min) and parallel (max).
When in series, the length of time a device will function is equal to the minimum life in all the engines (weakest link).
When in parallel, this is avoided as a device can function as long as one engine functions.

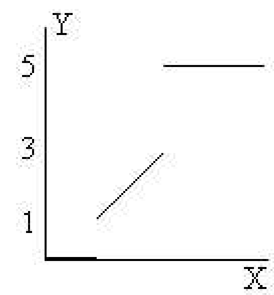
Review of Problems from PSet 4 for the upcoming exam: (see solutions for more details)

Problem 1 - $f(x) = \{ce^{-2x} \text{ for } x \geq 0; 0 \text{ otherwise}\}$
Find c by integrating over the range and setting equal to 1:

$$1 = \int_0^\infty ce^{-2x}dx = -\frac{1}{2}ce^{-2x}|_0^\infty = -\frac{c}{2} \times -1 = 1 \to c = 2$$

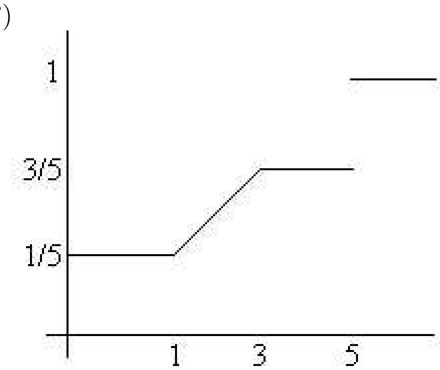
$\mathbb{P}(1 \leq X \leq 2) = \int_1^2 2e^{-2x}dx = e^{-2} - e^{-4}$

Problem 3 - $X \sim U[0,5], Y = 0$ if $X \leq 1; Y = X$ if $1 \leq X \leq 3; Y = 5$ if $3 < X \leq 5$
Draw the c.d.f. of Y, showing $\mathbb{P}(Y \leq y)$



Graph of Y vs. X, not the c.d.f.

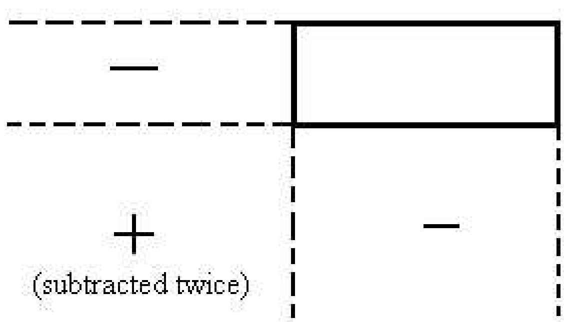
Write in terms of X $\to \mathbb{P}(X-?)$



Cumulative Distribution Function

Cases:
$y < 0 \rightarrow \mathbb{P}(Y \leq y) = \mathbb{P}(\emptyset) = 0$
$0 \leq y \leq 1 \rightarrow \mathbb{P}(Y \leq y) = \mathbb{P}(0 \leq X \leq 1) = 1/5$
$1 < y \leq 3 \rightarrow \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq y) = y/5$
$3 < y \leq 5 \rightarrow \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq 3) = 3/5$
$y > 5 \rightarrow \mathbb{P}(Y \leq 5) = \mathbb{P}(X \geq 5) = 1$
These values over X from 0 to ∞ give its c.d.f.

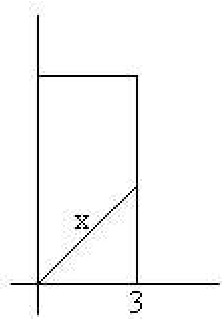
Problem 8 - $0 \leq x \leq 3, 0 \leq y \leq 4$
c.d.f. $F(x,y) = \frac{1}{156}xy(x^2+y)$
$\mathbb{P}(1 \leq x \leq 2, 1 \leq y \leq 2) = F(2,2) - F(2,1) - F(1,2) + F(1,1)$



Rectangle probability algorithm.

or, you can find the p.d.f. and integrate (more complicated):
c.d.f. of Y: $\mathbb{P}(Y \leq y) = \mathbb{P}(X \leq \infty, Y \leq y) = \mathbb{P}(X \leq 3, Y \leq y)$
(based on the domain of the joint c.d.f.)
$\mathbb{P}(Y \leq y) = \frac{1}{156}3y(9+y)$ for $0 \leq y \leq 4$
Must also mention: $y \leq 0, \mathbb{P}(Y \leq y) = 0; y \geq 4, \mathbb{P}(Y \leq y) = 1$
Find the joint p.d.f. of x and y:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \{\frac{1}{156}(3x^2+2y), 0 \leq x \leq 3, 0 \leq y \leq 4; 0 \text{ otherwise}\}$$



$$\mathbb{P}(Y \leq X) = \int_{y \leq x} f(x,y)dxdy = \int_0^3 \int_0^x \frac{1}{156}(3x^2+2y)dydx = \frac{93}{208}$$

** End of Lecture 14