18.05 Lecture 15
March 9, 2005

Review for Exam 1

Practice Test 1:

1. In the set of all green envelopes, only 1 card can be green.
Similarly, in the set of red envelopes, only 1 card can be red.
Sample Space = 10! ways to put cards into envelopes, treating each separately.
You can't have two of the same color matching, as that would be 4 total.
Degrees of Freedom = which envelope to choose (5×5) and which card to select (5×5)
Then, arrange the red in green envelopes (4!), and the green in red envelopes (4!)

$$\mathbb{P} = \frac{5^4(4!)^2}{10!}$$

2. Bayes formula:

$$\mathbb{P}(fair|HHH) = \frac{\mathbb{P}(HHH|fair)\mathbb{P}(fair)}{\mathbb{P}(HHH|fair)\mathbb{P}(fair) + \mathbb{P}(HHH|unfair)\mathbb{P}(unfair)} = \frac{0.5^3 \times 0.5}{0.5^3 \times 0.5 + 1 \times 0.5}$$

3. $f_1(x) = 2xI(0 < x < 1), f_2(x) = 3x^2 I(0 < x < 1)$
$Y = 1, 2 \rightarrow \mathbb{P}(Y = 1) = 0.5, \mathbb{P}(Y = 2) = 0.5$
$f(x, y) = 0.5 \times I(y = 1) \times 2xI(0 < x < 1) + 0.5 \times I(y = 2) \times 3x^2 I(0 < x < 1)$
$f(x) = 0.5 \times 2xI(0 < x < 1) + 0.5 \times 3x^2 I(0 < x < 1) = (x + 1.5x^2)I(0 < x < 1)$

$$\mathbb{P}(Y = 1|X = \frac{1}{4}) = \frac{f_1(\frac{1}{4}) \times \frac{1}{2}}{f_1(\frac{1}{4}) \times \frac{1}{2} + f_2(\frac{1}{4}) \times \frac{1}{2}} = \frac{2 \times 1/4 \times 1/2}{2 \times 1/4 \times 1/2 + 3 \times 1/16 \times 1/2}$$

4. $f(z) = 2e^{-2z} I(Z > 0), T = 1/Z$ we know $t > 0$
$\mathbb{P}(T \le t) = \mathbb{P}(1/Z \le t) = \mathbb{P}(Z \ge 1/t) =$

$$= \int_{1/t}^{\infty} 2e^{-2z} dz, \text{ p.d.f. } f(t) = \frac{\partial F(T \le t)}{\partial t} = -2e^{-2/t} \times -\frac{1}{t^2} = \frac{2}{t^2} e^{-2/t}, t > 0 \text{ (0 otherwise)}$$

$T = r(Z), Z = s(T) = \frac{1}{T} \rightarrow g(t) = |s'(t)| f(1/t)$ by change of variable.

5. $f(x) = e^{-x} I(x > 0)$
Joint p.d.f. $f(x, y) = e^{-x} I(x > 0) e^{-y} I(y > 0) = e^{-(x+y)} I(x > 0, y > 0)$

$$U = \frac{X}{X + Y}; V = X + Y$$

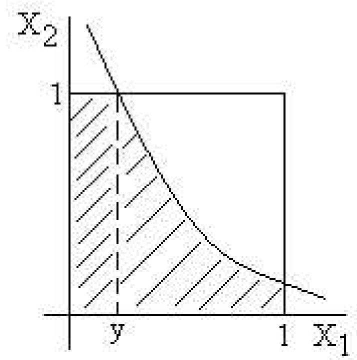
Step 1 - Check values for random variables:
$(0 < V < \infty), (0 < U < 1)$
Step 2 - Account for change of variables:
$X = UV; Y = V - UV = V(1 - U)$
Jacobian:

$$J = \det \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \text{V} & \text{U} \\ \text{-V} & \text{1 - U} \end{vmatrix} = V(1 - U) + UV = V$$

$g(u,v) = f(uv, v(1-u)) \times |v| I(uv > 0, v(1-u) > 0) = e^{-v} v I(v > 0, 0 < u < 1)$

Problem Set #5 (practice pset, see solutions for details):

p. 175 #4
$f(x_1, x_2) = x_1 + x_2 I(0 < x_1 < 1, 0 < x_2 < 1)$
$Y = X_1 X_2 (0 < Y < 1)$
First look at the c.d.f.: $\mathbb{P}(Y \le y) = \mathbb{P}(X_1 X_2 \le y) = \int_{\{x_1 x_2 \le y\} = \{x_2 \le y/x_1\}} f(x_1, x_2) dx_1 dx_2$



Due to the complexity of the limits, you can integrate the area in pieces,
or you can find the complement, which is easier with only 1 set of limits.

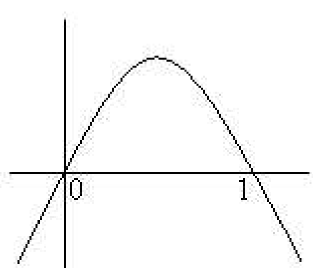
$$f(x_1, x_2) = 1 - \int_{\{x_1 x_2 > y\}} = 1 - \int_y^1 \int_{y/x_1}^1 (x_1 + x_2) dx_2 dx_1 = 1 - (1-y)^2 = 2y - y^2$$

$$f(x_1, x_2) = 0 \text{ for } y < 0; 2y - y^2 \text{ for } 0 < y < 1; 1 \text{ for } y > 1.$$

p.d.f.:

$$g(y) = \{ \frac{\partial \mathbb{P}(Y \le y)}{\partial y} = 2(1-y), y \in (0,1); 0, \text{ otherwise.} \}$$

p. 164 #3
$f(x) = \{\frac{x}{2}, 0 \le x \le 2; 0 \text{ otherwise}\}$
$Y = X(2 - X)$, find the p.d.f. of Y.
First, find the limits of Y, notice that it is not a one-to-one function.



Y varies from 0 to 1 as X varies from 0 to 2.
Look at the c.d.f.:

$$\mathbb{P}(Y \le y) = \mathbb{P}(X(2-X) \le y) = \mathbb{P}(X^2 - 2X + 1 \ge 1 - y) = \mathbb{P}((1-X)^2 \ge 1 - y) =$$

$$= \mathbb{P}(|1 - X| \ge \sqrt{1-y}) = \mathbb{P}(1 - X \ge \sqrt{1-y} \text{ or } 1 - X \le -\sqrt{1-y}) =$$

$$= \mathbb{P}(X \le 1 - \sqrt{1-y} \text{ or } X \ge 1 + \sqrt{1-y}) =$$

$$= \mathbb{P}(0 \le X \le 1 - \sqrt{1-y}) + \mathbb{P}(1 + \sqrt{1-y} \le X \le 2) =$$

$$= \{\int_0^{1-\sqrt{1-y}} \frac{x}{2} dx + \int_{1+\sqrt{1-y}}^{2} \frac{x}{2} dx = 1 - \sqrt{1-y}, 0 \le y \le 1; 0, y < 0; 1, y > 1\}$$

Take derivative to get the p.d.f.:

$$g(y) = \frac{1}{4\sqrt{1-y}}, 0 \le y \le 1; 0, \text{ otherwise.}$$

** End of Lecture 15