18.05 Lecture 16
March 14, 2005

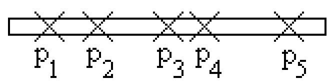
Expectation of a random variable.
X - random variable
roll a die - average value = 3.5
flip a coin - average value = 0.5 if heads = 0 and tails = 1

Definition: If X is discrete, p.f. $f(x)$ = p.f. of X,
Then, expectation of X is $\mathbb{E}X = \sum xf(x)$
For a die:

	1	2	3	4	5	6
f(x)	1/6	1/6	1/6	1/6	1/6	1/6

$\mathbb{E} = 1 \times \frac{1}{6} + ... + 6 \times \frac{1}{6} = 3.5$

Another way to think about it:



Consider each p_i as a weight on a horizontal bar.
Expectation = center of gravity on the bar.

If X - continuous, $f(x)$ = p.d.f. then $\mathbb{E}(X) = \int xf(x)dx$
Example: X - uniform on $[0, 1]$, $\mathbb{E}(X) = \int_0^1 (x \times 1)dx = 1/2$

Consider $Y = r(x)$, then $\mathbb{E}Y = \sum_x r(x)f(x)$ or $\int r(x)f(x)dx$
p.f. $g(y) = \sum_{\{x:y=r(x)\}} f(x)$
$\mathbb{E}(Y) = \sum_y yg(y) = \sum_y y \sum_{\{x:y=r(x)\}} f(x) = \sum_y \sum_{\{x:r(x)=y\}} yf(x) = \sum_y \sum_{\{x:r(x)=y\}} r(x)f(x)$
then, can drop y since no reference to y:
$\mathbb{E}(Y) = \sum_x r(x)f(x)$

Example: X - uniform on $[0, 1]$
$\mathbb{E}X^2 = \int_0^1 X^2 \times 1dx = 1/3$

$X_1, ..., X_n$ - random variables with joint p.f. or p.d.f. $f(x_1...x_n)$
$\mathbb{E}(r(X_1, ..., X_n)) = \int r(x_1, ..., x_n)f(x_1, ..., x_n)dx_1...dx_n$

Example: **Cauchy distribution**
p.d.f.:

$$f(x) = \frac{1}{\pi(1+x^2)}$$

Check validity of integration:

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}tan^{-1}(x)|_{-\infty}^{\infty} = 1$$

But, the expectation is undefined:

$$\mathbb{E}|X| = \int_{-\infty}^{\infty} |x| \frac{1}{\pi(1+x^2)} dx = 2\int_{0}^{\infty} \frac{x}{\pi(1+x^2)} = \frac{1}{2\pi} \ln(1+x^2)|_0^\infty = \infty$$

Note: Expectation of X is defined if $\mathbb{E}|X| < \infty$

Properties of Expectation:

1) $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$
Proof: $\mathbb{E}(aX + b) = \int (aX + b)f(x)dx = a\int xf(x)dx + b\int f(x)dx = a\mathbb{E}(X) + b$

2) $\mathbb{E}(X_1 + X_2 + ... + X_n) = \mathbb{E}X_1 + \mathbb{E}X_2 + ... + \mathbb{E}X_n$
Proof: $\mathbb{E}(X_1 + X_2) = \int\int (x_1 + x_2)f(x_1, x_2)dx_1dx_2 =$
$= \int\int x_1 f(x_1, x_2)dx_1dx_2 + \int\int x_2 f(x_1, x_2)dx_1dx_2 =$
$= \int x_1 \int f(x_1, x_2)dx_2dx_1 + \int x_2 \int f(x_1, x_2)dx_1dx_2 =$
$= \int x_1 f_1(x_1)dx_1 + \int x_2 f_2(x_2)dx_2 = \mathbb{E}X_1 + \mathbb{E}X_2$

Example: Toss a coin n times, "T" on i: $X_i = 1$; "H" on i: $X_i = 0$.
Number of tails = $X_1 + X_2 + ... + X_n$
$\mathbb{E}(\text{number of tails}) = \mathbb{E}(X_1 + X_2 + ... + X_n) = \mathbb{E}X_1 + \mathbb{E}X_2 + ... + \mathbb{E}X_n$
$\mathbb{E}X_i = 1 \times \mathbb{P}(X_i = 1) + 0 \times \mathbb{P}(X_i = 0) = p$, probability of tails
Expectation = p + p + ... + p = np
This is natural, because you expect np of n for p probability.

Y = Number of tails, $\mathbb{P}(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$
$\mathbb{E}(Y) = \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = np$
More difficult to see though definition, better to use sum of expectations method.

Two functions, h and g, such that $h(x) \leq g(x)$, for all $x \in \mathbb{R}$
Then, $\mathbb{E}(h(X)) \leq \mathbb{E}(g(X)) \rightarrow \mathbb{E}(g(X) - h(X)) \geq 0$
$\int (g(x) - h(x)) \times f(x)dx \geq 0$
You know that $f(x) \geq 0$, therefore $g(x) - h(x)$ must also be ≥ 0

If $a \leq X \leq b \rightarrow a \leq \mathbb{E}(X) \leq \mathbb{E}(b) \leq b$
$\mathbb{E}(I(X \in A)) = 1 \times \mathbb{P}(X \in A) + 0 \times \mathbb{P}(X \notin A)$, for A being a set on $\mathbb{R}$
$Y = I(X \in A) = \{1, \text{ with probability } \mathbb{P}(X \in A); 0, \text{ with probability } \mathbb{P}(X \notin A) = 1 - \mathbb{P}(X \in A)$
$\mathbb{E}(I(X \in A)) = \mathbb{P}(X \in A)\}$
In this case, think of the expectation as an indicator as to whether the event happens.

Chebyshev's Inequality
Suppose that $X \geq 0$, consider $t > 0$, then:

$$\mathbb{P}(X \geq t) \leq \frac{1}{t}\mathbb{E}(X)$$

Proof: $\mathbb{E}(X) = \mathbb{E}(X)I(X < t) + \mathbb{E}(X)I(X \geq t) \geq \mathbb{E}(X)I(X \geq t) \geq \mathbb{E}(t)I(X \geq t) = t\mathbb{P}(X \geq t)$

** End of Lecture 16