18.05 Lecture 18
March 18, 2005

Law of Large Numbers.

$X_1, ..., X_n$ - i.i.d. (independent, identically distributed)

$$\overline{x} = \frac{X_1 + ... + X_n}{n} \rightarrow \text{ as } n \rightarrow \infty, \mathbb{E}X_1$$

Can be used for functions of random variables as well:
Consider $Y_i = r(X_1)$ - i.i.d.

$$\overline{Y} = \frac{r(X_1) + ... + r(X_n)}{n} \rightarrow \text{ as } n \rightarrow \infty, \mathbb{E}Y_1 = \mathbb{E}r(X_1)$$

Relevance for Statistics: Data points x_i, as $n \rightarrow \infty$,
The average converges to the unknown expected value of the distribution which often contains a lot (or all) of information about the distribution.

Example: Conduct a poll for 2 candidates:
$p \in [0, 1]$ is what we're looking for
Poll: choose n people randomly: $X_1, ..., X_n$
$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = 0) = 1 - p$

$$\mathbb{E}X_1 = 1(p) + 0(1 - p) = p \leftarrow \frac{X_1 + ... + X_n}{n} \text{ as } n \rightarrow \infty$$

Other characteristics of distribution:
Moments of the distribution: for each integer, $k \geq 1$, kth moment $\mathbb{E}X^k$
kth moment is defined only if $\mathbb{E}|X|^k < \infty$
Moment generating function: consider a parameter $y \in \mathbb{R}$.
and define $\phi(t) = \mathbb{E}e^{tX}$ where X is a random variable.
$\phi(t)$ - m.g.f. of X

$$\text{Taylor series of } \phi(t) = \sum_{k=0}^{\infty} \frac{\phi^k(0)}{k!} t^k$$

$$\text{Taylor series of } \mathbb{E}e^{tX} = \mathbb{E} \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}X^k$$

$\mathbb{E}X^k = \phi^k(0)$

Example: Exponential distribution $E(\alpha)$ with p.d.f. $f(x) = \{\alpha e^{-\alpha x}, x \geq 0; 0, x < 0\}$
Compute the moments:
$\mathbb{E}X^k = \int_0^{\infty} x^k \alpha e^{-\alpha x} dx$ is a difficult integral.
Use the m.g.f.:

$$\phi(t) = \mathbb{E}e^{tX} = \int_0^{\infty} e^{tx} \alpha e^{-\alpha x} dx = \int_0^{\infty} \alpha e^{(t-\alpha)x} dx$$

(defined if $t < \infty$ to keep the integral finite)

$$= \frac{\alpha e^{(t-\alpha)x}}{t-\alpha}|_0^\infty = 1 - \frac{\alpha}{t-\alpha} = \frac{\alpha}{\alpha - t} = \frac{1}{1-t/\alpha} = \sum_{k=0}^{\infty}(\frac{t}{\alpha})^k = \sum \frac{t^k}{k!}\mathbb{E}X^k$$

Recall the formula for geometric series:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ when k} < 1$$

$$\frac{1}{\alpha^k} = \frac{\mathbb{E}x^k}{k!} \rightarrow \mathbb{E}x^k = \frac{k!}{\alpha^k}$$

The moment generating function completely describes the distribution.
$\mathbb{E}x^k = \int x^k f(x)dx$
If f(x) unknown, get a system of equations for f $\rightarrow$ unique distribution for a set of moments.
M.g.f. uniquely determines the distribution.

X_1, X_2 from $\mathbb{E}(\alpha), Y = X_1 + X_2$.
To find distribution of sum, we could use the convolution formula,
but, it is easier to find the m.g.f. of sum Y:

$$\mathbb{E}e^{tY} = \mathbb{E}e^{t(X_1+X_2)} = \mathbb{E}e^{tX_1}e^{tX_2} = \mathbb{E}e^{tX_1}\mathbb{E}e^{tX_2}$$

Moment generating function of each:

$$\frac{\alpha}{\alpha - t}$$

For the sum:

$$(\frac{\alpha}{\alpha - t})^2$$

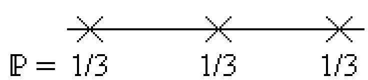
Consider the exponential distribution:

$$E(\alpha) \sim X_1, \mathbb{E}X = \frac{1}{\alpha}, f(x) = \{\alpha e^{-\alpha x}, x \geq 0; 0, x < 0\}$$

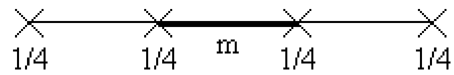
This distribution describes the life span of quality products.
$\alpha = \frac{1}{\mathbb{E}X}$, if α small, life span is large.

Median:
$m \in \mathbb{R}$ such that:

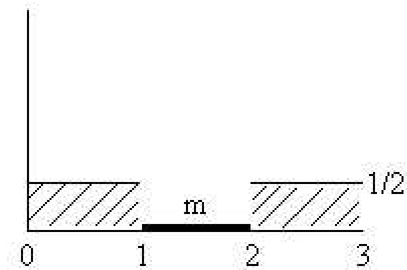
$$\mathbb{P}(X \geq m) \geq \frac{1}{2}, \mathbb{P}(X \leq m) \geq \frac{1}{2}$$



(There are times in discrete distributions when the probability cannot ever equal exactly 0.5)
When you exclude the point itself: $\mathbb{P}(X > m) \leq \frac{1}{2}$
$\mathbb{P}(X \leq m) + \mathbb{P}(X > m) = 1$
The median is not always uniquely defined. Can be an interval where no point masses occur.



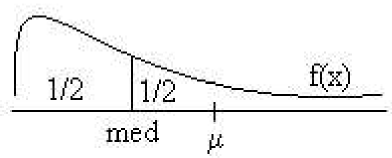
For a continuous distribution, you can define $\mathbb{P} >$ or $< m$ as equal to $\frac{1}{2}$.
But, there are still cases in which the median is not unique!



For a continuous distribution:

$$\mathbb{P}(X \leq m) = \mathbb{P}(X \geq m) = \frac{1}{2}$$

The average measures center of gravity, and is skewed easily by outliers.



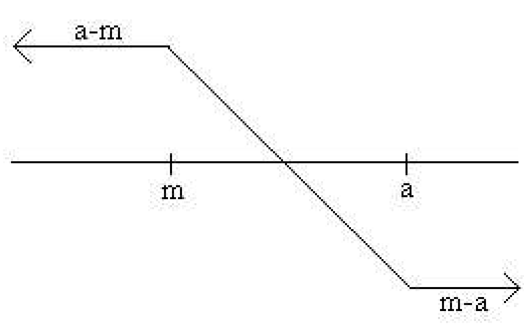
The average will be pulled towards the tail of a p.d.f. relative to the median.

Mean: find $a \in \mathbb{R}$ such that $\mathbb{E}(X-a)^2$ is minimized over a.

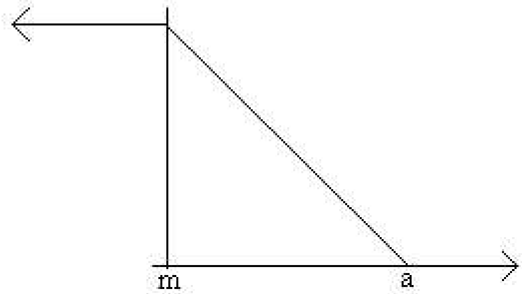
$$\frac{\partial}{\partial a}\mathbb{E}(X-a)^2 = -\mathbb{E}2(X-a) = 0, \mathbb{E}X - a = 0 \to a = \mathbb{E}X$$

expectation - squared deviation is minimized.

Median: find $a \in \mathbb{R}$ such that $\mathbb{E}|X-a|$ is minimized.
$\mathbb{E}|X-a| \geq \mathbb{E}|X-m|$, where m - median
$\mathbb{E}(|X-a| - |X-m|) \geq 0$
$\int(|x-a| - |x-m|)f(x)dx$



Need to look at each part:
1)$a - x - (m - x) = a - m, x \leq m$
2)$x - a - (x - m) = m - a, x \geq m$
3)$a - x - (x + m) = a + m - 2x, m \leq x \leq a$



The integral can now be simplified:

$$\int (|x-a| - |x-m|) f(x) dx \geq \int_{-\infty}^{m} (a-m) f(x) dx + \int_{m}^{\infty} (m-a) f(x) dx =$$

$$= (a-m)\left(\int_{-\infty}^{m} f(x) dx - \int_{m}^{\infty} f(x) dx\right) = (a-m)(\mathbb{P}(X \leq m) - \mathbb{P}(X > m)) \geq 0$$

As both $(a - m)$ and the difference in probabilities are positive.
The absolute deviation is minimized by the median.

** End of Lecture 18