18.05 Lecture 2
February 4, 2005

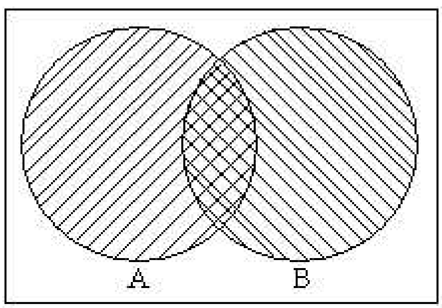
§1.5 Properties of Probability.
1. $\mathbb{P}(A) \in [0, 1]$
2. $\mathbb{P}(S) = 1$
3. $\mathbb{P}(\cup A_i) = \sum P(A_i)$ if disjoint $\rightarrow A_i \cap A_j = \emptyset, i \neq j$
The probability of a union of disjoint events is the sum of their probabilities.

4. $\mathbb{P}(\emptyset), \mathbb{P}(S) = \mathbb{P}(S \cup \emptyset) = \mathbb{P}(S) + \mathbb{P}(\emptyset) = 1$
where S and $\emptyset$ are disjoint by definition, $\mathbb{P}$(S) = 1 by #2., therefore, $\mathbb{P}(\emptyset) = 0$.

5. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
because A, A^c are disjoint, $\mathbb{P}(A \cup A^c) = \mathbb{P}(S) = 1 = \mathbb{P}(A) + \mathbb{P}(A^c)$
the sum of the probabilities of an event and its complement is 1.

6. If $A \subseteq B, \mathbb{P}(A) \leq \mathbb{P}(B)$
by definition, $B = A \cup (B \setminus A)$, two disjoint sets.
$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$

7. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$
must subtract out intersection because it would be counted twice, as shown:



write in terms of disjoint pieces to prove it:
$\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(AB)$
$\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(AB)$
$\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(AB)$

Example: A doctor knows that $\mathbb{P}$(bacterial infection) = 0.7 and $\mathbb{P}$(viral infection) = 0.4
What is $\mathbb{P}$(both) if $\mathbb{P}$(bacterial $\cup$ viral) = 1?
$\mathbb{P}$(both) = $\mathbb{P}$(B $\cap$ V)
1 = 0.7 + 0.4 - $\mathbb{P}$(BV)
$\mathbb{P}$(BV) = 0.1

Finite Sample Spaces
There are a finite # of outcomes $S = \{s_1, ..., s_n\}$
Define $p_i = \mathbb{P}(s_i)$ as the probability function.

$$p_i \geq 0, \sum_{i=1}^{n} p_i = 1$$

$$\mathbb{P}(A) = \sum_{s \in A} \mathbb{P}(s)$$

Classical, simple sample spaces - all outcomes have equal probabilities.
$\mathbb{P}(A) = \frac{\#(A)}{\#(S)}$, by counting methods.
Multiplication rule: $\#(s_1) = m, \#(s_2) = n, \#(s_1 \times s_2) = mn$

Sampling without replacement: one at a time, **order is important**
$s_1 ... s_n$ outcomes
k $\leq$ n (k chosen from n)
$\#$(outcome vectors) $= (a_1, a_2, ..., a_k) = n(n-1) \times ... \times (n-k+1) = P_{n,k}$

Example: order the numbers 1, 2, and 3 in groups of 2. (1, 2) and (2, 1) are different.
$P_{3,2} = 3 \times 2 = 6$
$P_{n,n} = n(n-1) \times ... \times 1 = n!$

$$P_{n,k} = \frac{n!}{(n-k)!}$$

Example: Order 6 books on a shelf = 6! permutations.

Sampling with replacement, k out of n
number of possibilities $= n \times n \times n... = n^k$
Example: Birthday Problem- In a group of k people,
what is the probability that 2 people will have the same birthday?
Assume n = 365 and that birthdays are equally distributed throughout the year, no twins, etc.
$\#$ of different combinations of birthdays= $\#$(S = all possibilities) $= 365^k$
$\#$ where at least 2 are the same $= \#(S) - \#$(all are different) $= 365^k - P_{365,k}$

$$\mathbb{P}(\text{at least 2 have the same birthday}) = 1 - \frac{P_{365,k}}{365^k}$$

Sampling without replacement, k at once
$s_1 ... s_n$
sample a subset of size k, $b_1 ... b_k$, if we aren't concerned with order.

$$\text{number of subsets} = C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

each set can be ordered k! ways, so divide that out of $P_{n,k}$
$C_{n,k}$ - binomial coefficients

Binomial Theorem:

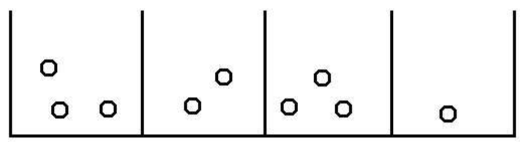
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

There are $\binom{n}{k}$ times that each term will show up in the expansion.

Example: a - red balls, b - black balls.
number of distinguishable ways to order in a row =

$$\binom{a+b}{a} = \binom{a+b}{b}$$

Example: $r_1 + ... + r_k = n; r_i$ = number of balls in each box; n, k given
How many ways to split n objects into k sets?
Visualize the balls in boxes, in a line - as shown:



Fix the outer walls, rearrange the balls and the separators.

If you fix the outer walls of the first and last boxes,
you can rearrange the separators and the balls using the binomial theorem.
There are n balls and k-1 separators (k boxes).
Number of different ways to arrange the balls and separators =

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Example: $f(x_1, x_2, ..., x_k)$, take n partial derivatives:

$$\frac{\partial^n f}{\partial^2 x_1 \partial x_2 \partial^5 x_3 ... \partial x_k}$$

k "boxes" $\rightsquigarrow$ k "coordinates"
n "balls" $\rightsquigarrow$ n "partial derivatives"
number of different partial derivatives = $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$

Example: In a deck of 52 cards, 5 cards are chosen.
What is the probability that all 5 cards have different face values?
total number of outcomes = $\binom{52}{5}$
total number of face value combinations = $\binom{13}{5}$
total number of suit possibilities, with replacement = 4^5

$$\mathbb{P}(\text{all 5 different face values}) = \frac{\binom{13}{5}4^5}{\binom{52}{5}}$$

** End of Lecture 2.